

## STUDY ON EQUILIBRIUM STRATEGIES FOR TRANSBOUNDARY POLLUTION UNDER COMPETITIVE CONDITIONS \*

WENWEN WANG<sup>1</sup>, LINZHAO XUE<sup>1,2</sup> AND MING ZHANG<sup>1,\*</sup>

**Abstract.** Competitive relationships among players plays an increasing role on transboundary pollution management. In this paper, a transboundary pollution game between two competitive regions as a player is constructed to explore the equilibrium strategies of output and pollution abatement efforts. The dynamic of the pollution stocks for regions are modeled separately in terms of a differential equation. Subsequently, the corresponding equilibrium strategies and value functions are derived in two regimes: Nash non-cooperative and Stackelberg leader–follower games. And the expectation and variances of pollution stocks are calculated over time. Our results are summarized as follows. First, an increase in the proportion of transboundary pollution increases the output of pollution upstream region, decreases the output of pollution downstream region, and ultimately raises the aggregate pollution stock of both regions. Second, regional competitive preference can enhance the local abatement effort and reduce the aggregate pollution stock. Finally, the total social welfare and environmental quality are higher in the Nash non-cooperative game, but pollution downstream region seek to be a leader in the Stackelberg game.

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### 1. INTRODUCTION

In multiple transboundary pollution problems, whether regions improve environmental quality or not increasingly depends on their neighbors' behavior [6, 20, 28]. To obtain higher benefits, pollution upstream regions as a net outflow area of transboundary pollution issues are reluctant to strictly comply with emission abatement policies, because they are less or not affected by cross-boundary pollutants emitted due to geographical advantages. Unfortunately, the related downstream regions as a net inflow area are facing a dilemma in which most of environmental regulations are difficult to have a substantial effect. Additionally, there is well documented that the competitive relationships between players exacerbate the difficulty of resolving this issue [27, 31]. Such as, developing countries do not intend to slow down the development of highly polluting industries in order to increase their competitiveness in the international market. Therefore, transboundary pollution management with competitive awareness has already been an inescapable issue of developed countries [11, 15]. Our goal is

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\* *Ming Zhang, Linzhao Xue, and Wenwen Wang are co-lead authors to this paper.*

<sup>1</sup> China University of Mining and Technology, Xuzhou 221116, P.R. China.

<sup>2</sup> Wuhan University, Wuhan 430072, P.R. China.

\*Corresponding author: [zhangmingdlut@163.com](mailto:zhangmingdlut@163.com)

to uncover how competitive regions optimally response to transboundary pollution issues, further revealing the dynamics of pollution stocks and social welfare in different regimes.

Considering the competitive relationship on pollution reduction, we propose a transboundary pollution dynamic game where the interdependent pollution stocks are formulated separately in terms of differential equations. Our game model considers two competitive players divided into pollution upstream and pollution downstream regions. Here regional competitive preference is introduced into the objective functionals to characterize how the regions compete in pollution abatement efforts. Then the equilibrium strategies of output and abatement efforts are captured in two regimes: Nash non-cooperative and Stackelberg leader–follower games. And the value functions under equilibrium are also obtained to figure out which regime is better for the players.

Our analysis reveals that as the proportion of transboundary pollution increases, the equilibrium output of pollution upstream region grows, but the equilibrium output of pollution downstream region shows a downward trend. Moreover, through the impact analysis, an increase in the proportion can raise up the aggregate pollution stock of both regions. On the other side, the equilibrium abatement efforts of regions are improved by their own competitive preference, not by their competitor’s one. Any increase in competitive preference makes the aggregate pollution stock decline. Through a numerical simulation, we find that in the Nash non-cooperative game, the total social welfare and environmental quality of two regions are higher. But pollution downstream region prefers to be a leader in the Stackelberg leader–follower game in order to get higher benefits.

## 2. RELATED LITERATURE

As the concern of transboundary pollution increases, cross-regional pollution governance between adjacent regions has drawn growing attention from scholars around the world [17, 24, 32]. In particular, many scholars are exhibiting great interest in ecological compensation. Huang *et al.* [9] establish a static Bayesian game model between one compensator and one compensated party under incomplete information. Their study explored the effect of the strategy choice of players on the size of ecological compensation. By constructing an evolutionary game, Xu *et al.* [26] analyze the changing outcomes amid conflicted interests in watershed ecological compensation. Besides, Li and Chen [15] discuss how green subsidies affect emissions and welfare in an international transboundary pollution game. Unlike them, we focus on the impacts of competition and game regimes between players in a differential game.

Similar to our method, there is some literature that focuses on differential games to address transboundary pollution issues in a continue horizon. Specifically, a cooperative differential game of transboundary industrial pollution is presented by Yeung [29]. Differential game models have been further expanded. Subsequently, Li [14] explores a differential game of transboundary industrial pollution where emission permit trading was considered. From the view of achievement evaluation of ecological civilization, Hu *et al.* [8] develop a continuous-time different game of water pollution control between local government and enterprises. Li and Guo [16] present a differential game in which the multi-pollutant is derived from transboundary pollution control.

Since stochastic theory was introduced to differential game models, studies on stochastic differential games have sprung up. The most advantage of stochastic differential game models characterizes the cumulative process of state variables which are affected by unpredictable factors, such as extreme weather, natural disasters and human activities. To account for uncertainty in nature’s capability, Jiang *et al.* [11] explore a stochastic differential game to study optimal control for transboundary pollution under ecological compensation and designed a welfare distribution mechanism. Through establishing a stochastic differential game for studying pollution management, Yeung [29] obtains the time consistent solutions under a cooperative form for the first time. Lai *et al.* [13] use finite difference methods to address two dimensional Hamilton–Jacobi–Bellman (HJB) equations and systems which stem from the modelling of transboundary pollution with emission permits trading.

However, the differential game models in the studies mentioned only build on a single differential equation capturing the total amount of pollutants from a third-party perspective. It leads to two problems: First, we don’t know how the stocks of pollutants evolve separately in each region. Second, competitive relationships between players in the evolutionary process of transboundary pollution are inappropriately neglected by the existing

literature above. Therefore, in this paper, we model a differential game of transboundary pollution between competitive regions in which the evolution of pollutants in each region is considered separately. We attempt to find equilibrium output and abatement efforts of competitive regions in different game regimes, and further study the evolutionary paths of pollution stocks as well as welfare over time under the influence of competition and cross-region pollution proportion.

The structure of the remainder of this paper is as follows. The next section outlines the basic structure of a differential model of transboundary pollution in a competitive environment. Then the regions' equilibrium output and abatement effort levels are derived and analyzed in Sections 4 and 5. Under equilibrium strategies, furthermore, the expectation and variances of the pollutant stocks are addressed in Section 6. In Section 6 we investigate the evolutionary paths of pollution stocks and value function over time. Subsequently, sensitivity analysis is set up. Finally, in Section 7 conclusions are drawn.

### 3. THE BASIC MODEL

#### 3.1. Notations

Consider a transboundary pollution game with two competitive regions as a player in a finite time horizon  $[0, T]$ . The regions are indexed by  $h = i, j$ . In particular, region  $j$  represents a pollution upstream region, a net outflow region of pollutants, whereas region  $i$  denotes a downstream region, a net inflow region of pollutants. At any given time  $t \in [0, T]$ , homogeneous products with similar properties are manufactured by these two regions, and generate pollutants as by-products. The control variables of region  $h$  include the output of products and the level of pollution abatement effort. More importantly, they compete in abatement efforts to maximize their discounted stream of net revenues. To be specific, let  $q_h(t)$  represent the output of region  $h$  under a fixed endowment of factors of production and technology at time  $t$ . Based on the analytical framework of Yeung [29], the price function  $P_h$  at time  $t$  has the following form

$$P(t) = \alpha - \sum_{h \in \{i, j\}} \beta_h q_h(t), \quad (1)$$

where  $\alpha$  is a positive constant. And  $\beta_h > 0$  represents the marginal impact coefficient of production on the price. Distinct from the previous studies, we focus on the interaction of regions in production. When goods of the same attribute are produced, the price in a market consisted of both regions is significantly affected by the output of each region [22]. Therefore, the payoffs depend on the game relationship between players. Also, the productive process of products would bring certain costs, such as material, labor costs and selling expenses [21]. Therefore, the production revenue of region  $h$  can be expressed as

$$R_h(t) = \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h(t) \right) q_h(t) - c_h q_h(t), \quad (2)$$

where  $c_h > 0$  represents the marginal cost coefficient of production for region  $h$ . On the other hand, at time  $t$ , each region has to choose an appropriate level of pollution abatement effort  $u_h(t)$  based on own interests. According to [2, 3, 18], we get the cost function of abatement effort

$$C_h(t) = k_h u_h^2(t), \quad (3)$$

where  $k_h > 0$  denotes the cost coefficient of abatement effort, and  $k_h > 0$ . It shows that the cost of pollution reduction that region  $h$  must incur raises sharply as the level of abatement effort increases.

In the meanwhile, regions maintain a competitive relationship on abatement effort levels. Here assume that changes in one party's effort level will affect the other party's benefits. As a result, let  $T_h(t)$  be the utility

function to characterize the interaction of pollution abatement effort for competitive players

$$\begin{cases} T_i(t) = \Delta_i(u_i(t) - u_j(t)) \\ T_j(t) = \Delta_j(u_j(t) - u_i(t)) \end{cases} \quad (4)$$

where  $\Delta_h$  is the marginal coefficient of pollution abatement variance that depicts the region  $h$ 's preference for competition.

On one hand, the manufacturing process of industrial products leads to the generation of pollutants, and on the other hand, the abatement effort level of region  $h$  reduces the level of actual emissions in own region. Note that, the emission of region  $j$  not only causes environmental damages in local region, but also exerts a negative impact on the environment of adjacent regions due to pollution spillover. Formally, assume that the portion of pollution spillover to neighboring regions received from region  $j$  is denoted as  $\psi$ . Let  $I_h(t)$  represent the increment function of pollutant for region  $h$  at time  $t$ , which satisfies the following form

$$\begin{cases} I_i(t) = v_i q_i(t) - r_i u_i(t) + \psi x_j(t) \\ I_j(t) = v_j q_j(t) - r_j u_j(t) - \psi x_j(t) \end{cases} \quad (5)$$

where  $v_h$  denotes the amount of pollutants from one unit of products and  $r_h u_h(t)$  describes the level of emission reduction triggered by the abatement effort  $u_h(t)$ . Here  $x_h(t)$  characterizes the pollution stock in region  $h$  at time  $t$ .

The accumulation of pollution stocks involves not only the above increment, but also the impacts of local ecological systems [5, 12]. Ecological systems respond to a natural self-purification ability, which can absorb parts of pollutants. To accurately describe the cumulative process of pollutants, the natural self-purification rate is thus introduced to the following differential equation (6) to stimulate the evolution of pollution stocks, for both regions,

$$\begin{cases} dx_i(t) = [v_i q_i(t) - r_i u_i(t) + \psi x_j(t) - \delta_i x_i(t)] dt \\ dx_j(t) = [v_j q_j(t) - r_j u_j(t) - (\psi + \delta_j) x_j(t)] dt \\ x_i(0) = x_0^i, \quad x_j(0) = x_0^j, \quad x_0^j \geq 0, \quad x_0^i \geq 0, \end{cases} \quad (6)$$

where  $\delta_h$  represents the natural self-purification rate of region  $h$ . And  $x_0^h \geq 0$  represents the pollutant stocks of region  $h$  in initial state.

There is a large body of literature showing that pollutants originated from industrial production have negative externalities that cause hazard to the natural environment and human health [1, 4]. Therefore, both regions would suffer economic losses related with their own stock of pollution. According to Li and Guo [16], the linear loss function is

$$L_h(t) = \eta_h x_h(t), \quad (7)$$

where  $\eta_h > 0$  indicates the marginal loss of region  $h$ , and  $\eta_h$  is a constant.

### 3.2. The model

In general, two regions are supposed to make rational decisions with the objective of maximizing the discounted stream of net revenues over a finite time interval  $[0, T]$ . Here  $T$  is a non-negative constant, and any

$t \in [0, T]$ . In conclusion, the revenue functions of region  $i$  and  $j$  can be expressed as:

$$\left\{ \begin{array}{l} J_i(q_i(t), u_i(t)) = \int_0^T e^{-\rho t} [R_i(t) - C_i(t) + T_i(t) - L_i(t)] dt \\ \quad = \int_0^T e^{-\rho t} \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h(t) \right) q_i(t) - c_i q_i(t) - k_i u_i^2(t) \right. \\ \quad \quad \left. + \Delta_i(u_i - u_j) - \eta_i x_i(t) \right] dt, \\ J_j(q_j(t), u_j(t)) = \int_0^T e^{-\rho t} [R_j(t) - C_j(t) + T_j(t) - L_j(t)] dt \\ \quad = \int_0^T e^{-\rho t} \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h(t) \right) q_j(t) - c_j q_j(t) - k_j u_j^2(t) \right. \\ \quad \quad \left. + \Delta_j(u_j - u_i) - \eta_j x_j(t) \right] dt \end{array} \right. \quad (8)$$

where  $\rho$  is the discount rate, common to two regions, and  $\rho \in (0, 1]$ . Next sections, two regions' equilibrium strategies of production and abatement effort level are discussed under different game situations. And we will analyze the value function of region  $h$  under equilibrium.

#### 4. NASH NON-COOPERATIVE GAME

In this section, the assumption is that no cooperative strategies will be taken into consideration by any region in Nash non-cooperative game. In this context, both regions will simultaneously and independently seek to their optimal strategies to maximize their discounted flow of net revenue in a finite time horizon. Based on the background above, the Nash non-cooperative game between region  $i$  and  $j$  can be given as follows

$$\begin{array}{l} \max_{q_i, u_i} \int_0^T e^{-\rho t} \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h(t) \right) q_i(t) - c_i q_i(t) - k_i u_i^2(t) + \Delta_i(u_i - u_j) - \eta_i x_i(t) \right] dt \\ \text{s.t.} \begin{cases} dx_i(t) = [v_i q_i(t) - r_i u_i(t) + \psi x_j(t) - \delta_i x_i(t)] dt, \\ dx_j(t) = [v_j q_j(t) - r_j u_j(t) - (\psi + \delta_j) x_j(t)] dt, \\ x_i(0) = x_0^i, \quad x_j(0) = x_0^j; \quad x_0^j \geq 0, \quad x_0^i \geq 0. \end{cases} \\ \max_{q_j, u_j} \int_0^T e^{-\rho t} \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h(t) \right) q_j(t) - c_j q_j(t) - k_j u_j^2(t) + \Delta_j(u_j - u_i) - \eta_j x_j(t) \right] dt \\ \text{s.t.} \begin{cases} dx_j(t) = [v_j q_j(t) - r_j u_j(t) - (\psi + \delta_j) x_j(t)] dt, \\ x_j(0) = x_0^j, \quad x_0^j \geq 0. \end{cases} \end{array} \quad (9)$$

**Proposition 1.** *In the Nash non-cooperative game, the equilibrium strategies of output and abatement effort for regions are*

$$\begin{aligned} q_i^* &= \frac{-2\eta_i v_i}{3\beta_i(\delta_i + \rho)} + \frac{\eta_j v_j}{3\beta_i(\psi + \delta_j + \rho)} + \frac{(\alpha + c_j - 2c_i)}{3\beta_i}; \\ q_j^* &= \frac{-2\eta_j v_j}{3\beta_j(\psi + \delta_j + \rho)} + \frac{\eta_i v_i}{3\beta_j(\delta_i + \rho)} + \frac{\alpha + c_i - 2c_j}{3\beta_j}; \\ u_i^* &= \frac{\Delta_i}{2k_i} + \frac{\eta_i r_i}{2k_i(\delta_i + \rho)}; \\ u_j^* &= \frac{\Delta_j}{2k_j} + \frac{\eta_j r_j}{2k_j(\psi + \delta_j + \rho)}. \end{aligned} \quad (10)$$

And the value functions of regions are

$$\begin{aligned}
 V_j(t, x_j) &= \frac{-\eta_j x_j}{\psi + \delta_j - \rho} - \frac{1}{\rho} \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h^* \right) q_j^* - c_j q_j^* - k_j u_j^{*2} + \Delta_j (u_j^* - u_i^*) - \frac{\eta_j (v_j q_j^* - r_j u_j^*)}{(\psi + \delta_j - \rho)} \right]; \\
 V_i(t, x_i, x_j) &= \frac{-\eta_i x_i}{\delta_i - \rho} - \frac{-\eta_i \psi x_j}{(\delta_i - \rho)(\psi + \delta_j - \rho)} - \frac{1}{\rho} \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h^* \right) q_i^* - c_i q_i^* - k_i u_i^{*2} \right. \\
 &\quad \left. + \Delta_i (u_i^* - u_j^*) - \frac{\eta_i (v_i q_i^* - r_i u_i^*)}{(\rho + \delta_i)} - \frac{\eta_i \psi (v_j q_j^* - r_j u_j^*)}{(\rho + \delta_i)(\rho + \psi + \lambda_j)} \right]. \tag{11}
 \end{aligned}$$

*Proof.* See Appendix A. □

- Remark 1.** (1) In the Nash non-cooperative game, a higher pollutant diffusion ability  $\psi$  increases the equilibrium output of region  $j$ . That is, the equilibrium output of region  $j$  is positively proportional to the diffusion coefficient, whereas the equilibrium output of region  $i$  is inversely proportional to it. ( $\partial q_j^* / \partial \psi > 0, \partial q_i^* / \partial \psi < 0$ ). In addition, the output of both regions is negatively correlated with the production cost, emission per unit of product as well as marginal loss of itself. ( $\partial q_h^* / \partial c_h < 0, \partial q_h^* / \partial v_h < 0, \partial q_h^* / \partial \eta_h < 0$ ). However, it is positively correlated with the above factors of competitor. ( $\partial q_h^* / \partial c_{\bar{h}} > 0, \partial q_h^* / \partial v_{\bar{h}} > 0, \partial q_h^* / \partial \eta_{\bar{h}} > 0$ ).
- (2) The equilibrium abatement effort level of both regions in the Nash non-cooperative game is positively influence by the competitive preference, the marginal abatement effect and the marginal loss. ( $\partial u_h^* / \partial \Delta_h > 0, \partial u_h^* / \partial r_h > 0, \partial u_h^* / \partial \eta_h > 0$ ). Conversely, we find that the abatement cost and self-purification rate decrease the level of abatement effort on pollutant stock. ( $\partial u_h^* / \partial k_h < 0, \partial u_h^* / \partial \delta_h < 0$ ). In summary, equilibrium abatement effort is raised by its own competitive preference, not the competitor's one.

## 5. STACKELBERG LEADER-FOLLOWER GAME

In this section, consider a Stackelberg leader-follower scenario [19]. To be specific, we assume region  $i$  as the net inflow area of pollutants takes its priority in determining its output and abatement effort level based on the revenue function behind. According to the region  $i$ 's action, the region  $j$  then decides its optimal strategies. In such a game region  $i$  is the role of a leader, but region  $j$  is a follower. Therefore, a two-stage Stackelberg leader-follower game is constructed for region  $i$  and  $j$ , and is shown as follows.

$$\begin{aligned}
 \text{Leader : } & \max_{q_i, u_i} \int_0^T e^{-\rho t} \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h(t) \right) q_i(t) - c_i q_i(t) - k_i u_i^2(t) + \Delta_i (u_i - u_j) - \eta_i x_i(t) \right] dt \\
 & \text{s.t. } \begin{cases} dx_i(t) = [v_i q_i(t) - r_i u_i(t) + \psi x_j(t) - \delta_i x_i(t)] dt \\ dx_j(t) = [v_j q_j(t) - r_j u_j(t) - (\psi + \delta_j) x_j(t)] dt, \\ x_i(0) = x_0^i, \quad x_j(0) = x_0^j; \quad x_0^j \geq 0, \quad x_0^i \geq 0 \end{cases} \\
 \text{Follower : } & \max_{q_j, u_j} \int_0^T e^{-\rho t} \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h(t) \right) q_j(t) - c_j q_j(t) - k_j u_j^2(t) + \Delta_j (u_j - u_i) - \eta_j x_j(t) \right] dt \\
 & \text{s.t. } \begin{cases} dx_j(t) = [v_j q_j(t) - r_j u_j(t) - (\psi + \delta_j) x_j(t)] dt, \\ x_j(0) = x_0^j, \quad x_0^j \geq 0. \end{cases} \tag{12}
 \end{aligned}$$

**Proposition 2.** *In the Stackelberg leader–follower game, the equilibrium strategies of output and abatement effort for regions are*

$$\begin{aligned} q_i^* &= \frac{\alpha + c_j - 2c_i}{2\beta_i} + \frac{\eta_j v_j}{2\beta_i(\rho + \psi + \delta_j)} - \frac{v_i \eta_i}{\beta_i(\rho + \delta_i)} + \frac{\eta_i \psi v_j}{2\beta_j(\rho + \delta_i)(\rho + \psi + \delta_j)}; \\ q_j^* &= \frac{\alpha - 3c_j + 2c_i}{4\beta_j} + \frac{v_i \eta_i}{2(\rho + \delta_i)\beta_j} - \frac{3\eta_j v_j}{4(\rho + \psi + \delta_j)\beta_j} - \frac{\eta_i \psi v_j \beta_i}{4\beta_j^2(\rho + \delta_i)(\rho + \psi + \delta_j)}; \\ u_i^* &= \frac{\Delta_i}{2k_i} + \frac{\eta_i r_i}{2k_i(\rho + \delta_i)}; \\ u_j^* &= \frac{\Delta_j}{2k_j} + \frac{\eta_j r_j}{2k_j(\rho + \psi + \delta_j)}. \end{aligned} \quad (13)$$

And the related value functions are

$$\begin{aligned} V_j^S(x_j) &= \frac{-\eta_j x_j}{\rho + \psi + \delta_j} + \frac{1}{\rho} \left[ (\alpha - \beta_i q_i^* - \beta_j q_j^* - c_j) q_j^* - k_j u_j^{*2} + \Delta_j (u_j^* - u_i^*) - \frac{\eta_j (v_j q_j^* - r_j u_j^*)}{\rho(\rho + \psi + \delta_j)} \right]; \\ V_i^S(x_i, x_j) &= \frac{-\eta_i x_i}{\rho + \delta_i} - \frac{\eta_i \psi x_j}{(\rho + \delta_i)(\rho + \psi + \delta_j)} \\ &+ \frac{1}{\rho} \left[ (\alpha - \beta_j q_j^* - \beta_i q_i^* - c_i) q_i^* - k_i u_i^{*2} + \Delta_i (u_i^* - u_j^*) - \frac{\eta_i (v_i q_i^* - r_i u_i^*)}{(\rho + \delta_i)} - \frac{\eta_i \psi (v_j q_j^* - r_j u_j^*)}{(\rho + \delta_i)(\rho + \psi + \delta_j)} \right]. \end{aligned} \quad (14)$$

*Proof.* See Appendix B. □

- Remark 2.** (1) In the Stackelberg leader–follower game, Proposition 2 indicates that the equilibrium output of region  $i$  is positively affected by the marginal loss of region  $j$ . ( $\partial q_i^* / \partial \eta_j > 0$ ). However, a higher loss  $\eta_j$  will reduce the equilibrium output of region  $j$ . ( $\partial q_j^* / \partial \eta_j < 0$ ). Besides, the effects of the production cost and emissions per unit of product are conspicuously similar as obtained in the Nash non-cooperative game.
- (2) There is no difference between the Nash non-cooperative and the Stackelberg leader–follower game in equilibrium abatement effort. The reason behind is that the abatement cost functions caused by emission reduction are independent. It implies that throughout this paper, the pollution abatement effort of each region does not change as the regimes change.

As shown above, the value functions of two regions in the games are greatly impacted by the accumulation of pollutants. Notice that, the pollutant stocks in two regions continuously evolve over time. Therefore, this is necessary for decision-makers to reveal the role of expectation and variance of pollutant stocks in shaping the trend of value functions.

To begin with, substituting equilibrium strategies obtained by Proposition 1 or 2 into equation (6), we can get the dynamic of pollution stocks under equilibrium.

$$\begin{cases} dx_j(t) = [\phi_j^N - (\psi + \delta_j)x_j(t)] dt \\ dx_i(t) = [\phi_i^N + \psi x_j - \delta_i x_i] dt \\ x_i(0) = x_0^i, \quad x_j(0) = x_0^j, \end{cases} \quad (15)$$

where  $\phi_h^N = v_h q_h^* - r_h u_h^*$ , and  $t \in [0, T]$ .

**Proposition 3.** *In the games proposed, the expectations of the variable  $x_i$  and  $x_j$  during the interval  $[0, t]$  are*

$$E[x_j(t)] = \frac{\phi_j^N}{\psi + \delta_j} + \left( x_0^j - \frac{\phi_j^N}{\psi + \delta_j} \right) e^{-(\psi + \delta_j)t},$$

$$E[x_i(t)] = \gamma - \kappa e^{-(\psi+\delta_j)t} + [x_0^i + \kappa - \gamma]e^{-\delta_i t}, \quad (16a)$$

where  $\gamma = \frac{\phi_i^N}{\delta_i} + \frac{\psi\phi_j^N}{\delta_i(\psi+\delta_j)}$ ,  $\kappa = \frac{\psi x_0^j}{\psi+\delta_j-\delta_i} - \frac{\psi\phi_j^N}{(\psi+\delta_j)(\psi+\delta_j-\delta_i)}$ .

And their evolutionary trajectories are

$$\begin{aligned} x_j(t) &= \frac{\phi_j^N}{\psi + \delta_j} + \left( x_0^j - \frac{\phi_j^N}{\psi + \delta_j} \right) e^{-(\psi+\delta_j)t}, \\ x_i(t) &= \frac{A_1}{\delta_i} + \frac{A_2}{\delta_i - \psi - \delta_j} e^{-(\psi+\delta_j)t} + \left( x_0^i - \frac{A_1}{\delta_i} - \frac{A_2}{\delta_i - \psi - \delta_j} \right) e^{-\delta_i t}, \end{aligned} \quad (16b)$$

where  $A_1 = \phi_i^N + \frac{\psi\phi_j^N}{\psi+\delta_j}$ ,  $A_2 = \varphi(x_0^j - \frac{\phi_j^N}{\psi+\delta_j})$ .

*Proof.* See Appendix C. □

The expectation and trajectories of the pollution stocks during the interval  $[0, t]$  are calculated in Proposition 3. Thus, through Proposition 3, we can forecast the theoretical values of the pollutant stock stored in any region, so that pollution abatement effect can be monitored to achieve a given environmental regulation objective and detect the implementation of emission reductions in both regions.

## 6. NUMERICAL SIMULATION

In this section a numerical simulation is implemented based on an actual transboundary river pollution issue in China. The results serve to investigate how pollutant stocks and value functions of regions change in different games.

### 6.1. Background

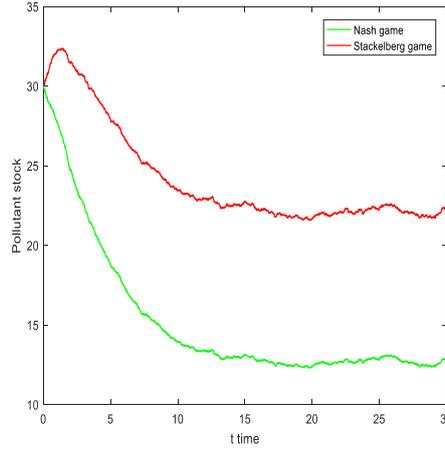
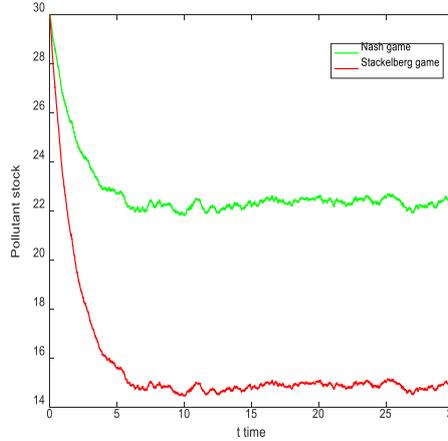
Xin'an River Basin is an important river system in China, spans from Anhui to Zhejiang province. It gave birth to Qiandao lake, a strategic water reserve in Zhejiang province, which is used to provide drinking water for surrounding residents. But pollutants from the upstream basin have caused increasingly serious levels of eutrophication in Qingdao lake [11]. More importantly, the increase of pollutants is harming the local environmental and living quality. According to the data from China statistical yearbook in 2019, moreover, the output value of the secondary industry in the two provinces accounted for more than 40% of the total output value. They are both manufacturing provinces and are adjacent to each other, so there is inevitably competition between them. With the above background of this case, we now start assigning values to parameters.

Based on the relationships of the players consisted of net outflow region  $j$  (*i.e.*, pollution upstream region) and net inflow region  $i$  (*i.e.*, pollution downstream region), the portion of transboundary pollutants originated from region  $j$  is set at  $\psi = 0.4$  [10]. And the marginal losses of regions are  $\eta_i = 3$  and  $\eta_j = 2$  [30]. According to Heutel [7], the natural self-purification rates are set at  $\delta_j = 0.02$  and  $\delta_i = 0.03$ , respectively. According to Jiang *et al.* [11], we set the cost coefficients of abatement effort at:  $k_i = 0.7$ ,  $k_j = 0.5$ . Referring to Wang and Ewald [25], Jiang *et al.* [11] and Yeung [29], other parameters are held at:  $\alpha = 50$ ,  $\beta_i = 1$ ,  $\beta_j = 0.5$ ,  $c_i = 1.2$ ,  $c_j = 1$ ,  $\Delta_i = 2$ ,  $\Delta_j = 4$ ,  $v_i = 0.5$ ,  $v_j = 0.6$ ,  $r_i = 0.4$ ,  $r_j = 0.5$ ,  $\sigma_i = 0.07$ ,  $\sigma_j = 0.06$ .

### 6.2. Path analysis of pollution stock

Consider some unpredictable factors [23], such as extreme weather, natural disaster and human activities, may cause short-term local influence on pollution stocks. Thus, a stochastic term is introduced to equation (15) to stimulate the evolution of the pollution stocks for two regions. The detail is as follows.

$$\begin{cases} dx_j(t) = [\phi_j^N - (\psi + \delta_j)x_j(t)] dt + \sigma_j(x_j(t)) dw_j(t) \\ dx_i(t) = [\phi_i^N + \psi x_j - \delta_i x_i] dt + \sigma_i(x_i(t)) dw_i(t) \\ x_i(0) = x_0^i, \quad x_j(0) = x_0^j, \end{cases} \quad (17)$$

FIGURE 1. The pollutant stock of region  $i$ .FIGURE 2. The pollutant stock of region  $j$ .

where  $w_h(t)$  is a standard Wiener process. Considering regional heterogeneity, we assume that  $w_i(t)$  and  $w_j(t)$  are independent of each other. Based on the stochastic analysis theory, let  $\sigma_h(x_h)dw_h(t) = \sigma_h\sqrt{x_h}dw_h(t)$ , where  $\sigma_h$  is a non-negative random factor.

Based on the form of equation (17), the numerical approximation follows

$$\begin{aligned} x_j(t + \Delta t) &= x_j(t) + [\phi_j^N - (\psi + \delta_j)x_j(t)]\Theta + \sigma_j\sqrt{x_j}\sqrt{\Theta}\zeta_1(t) \\ x_i(t + \Delta t) &= x_i(t) + [\phi_i^N + \psi x_j(t) - \delta_i x_i(t)]\Theta + \sigma_i\sqrt{x_i}\sqrt{\Theta}\zeta_2(t) \end{aligned}$$

where both  $\zeta_1(t)$  and  $\zeta_2(t)$  are independent random variables and are assigned to different standard normal variables with a time step of  $\Theta = 0.01$ , *i.e.*,  $\zeta_1(t) : N_1(0, 1)$  and  $\zeta_2(t) : N_2(0, 1)$ . Next, numerical simulations are conducted to study how paths of pollutant stocks vary over time in two regions under different scenarios.

Figures 1 and 2 display the changes in the amount of pollutants in the two games during  $[0, 30]$ . The reason why the pollution stocks keep on fluctuating continuously is the introduction of the Brownian motion, which makes the evolutionary paths more visually reflect the interference of external factors. As can be seen from Figure 1, in the Stackelberg game, the pollutant stock of region  $i$  increases then decreases as  $t$  increases. Finally,

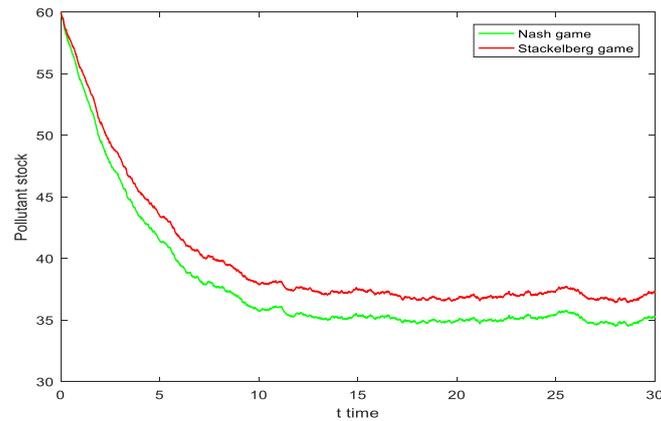


FIGURE 3. Total amount of pollutant in two regions.

the pollutant stock fluctuates in a small range around the value 22. On the other side, the amount of pollutants falls directly and then fluctuates slightly around the value 13 in the Nash game. Figure 1 shows that for region  $i$ , the effect of pollution control in the Nash game is greater than that in the Stackelberg game.

As for region  $j$ , Figure 2 shows the performance of pollution stocks in the Stackelberg game is best. We see that the amount of pollutants removed in the Stackelberg game is almost twice that in the Nash game.

As shown in Figure 3, we find that the total amount of pollutants in two regions has significantly decreased in both game situations. But the total pollutant stock in the Stackelberg game is lower than that in the Nash game during  $[0, 30]$ . Therefore, the level of pollution reduction is the highest in the Nash game. Figure 3 demonstrates that the comprehensive environmental quality of two regions in the Nash game is better than that in the Stackelberg game.

### 6.3. Analysis for value function

Compared with the results from Figures 1 to 3, we are now able to understand the advantages of pollution abatement between the Nash and Stackelberg games. Next, we further conduct numerical simulations to learn how value functions change over time. Based on the simulation results of the value functions derived from Propositions 1 and 2, we can judge the game tendency of these two regions.

Figure 4 reveals that the benefit of region  $i$  in the Stackelberg game is higher than that in the Nash game. As a result, region  $i$  will get a higher return if choosing to play the Stackelberg game. However, Figure 5 finds that region  $j$  is more likely to play the Nash game, because the benefit of region  $j$  in the Nash game is much higher than that in the Stackelberg game. Nevertheless, combining Figures 4 and 5, we find that the total social welfare of two regions is maximized in the Nash non-cooperative game.

### 6.4. Impact analysis

In this section we will reveal how various parameters influences the pollutant stocks in the region  $i$ , region  $j$  and the total region. To investigate the effects on the variation of pollutant stocks, we vary one of critical parameters over a given range according to the pattern of  $-50%$ ,  $-25%$ ,  $+25%$ ,  $+50%$ , and fix other parameters. The results are calculated in the Nash and Stackelberg games, and are then displayed in Tables 1 and 2, respectively. If the pollution stocks are increasing (or declining), we denote a plus (or minus) in the corresponding column. The sign of ( $\times$ ) indicates that there is no significant change in pollutant stocks. Thus, the sensitivity analysis in the Nash non-cooperative game is presented in Table 1.

According to Table 1, any increase in the abatement cost  $c_i$  and marginal loss  $\eta_i$  causes a decrease in the pollutant stocks of region  $i$  and the total region but an increase in the pollution stock of region  $j$ . Compared

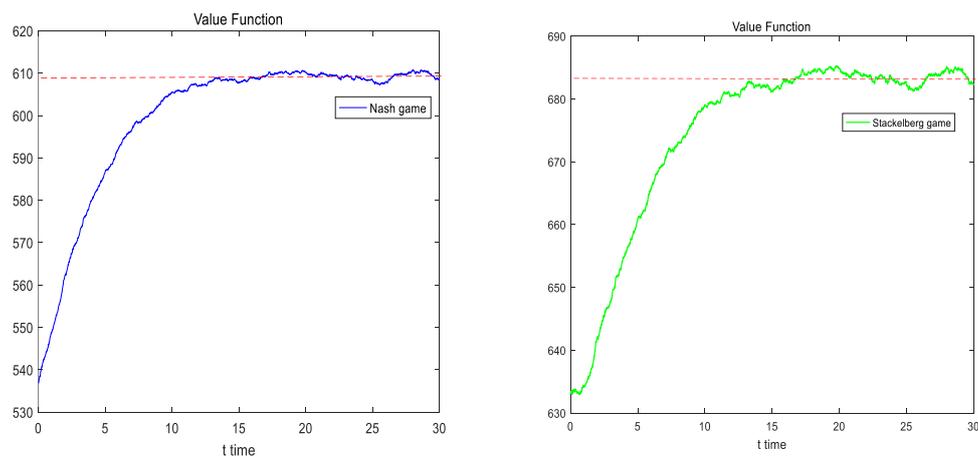


FIGURE 4. The value function of region  $i$ .

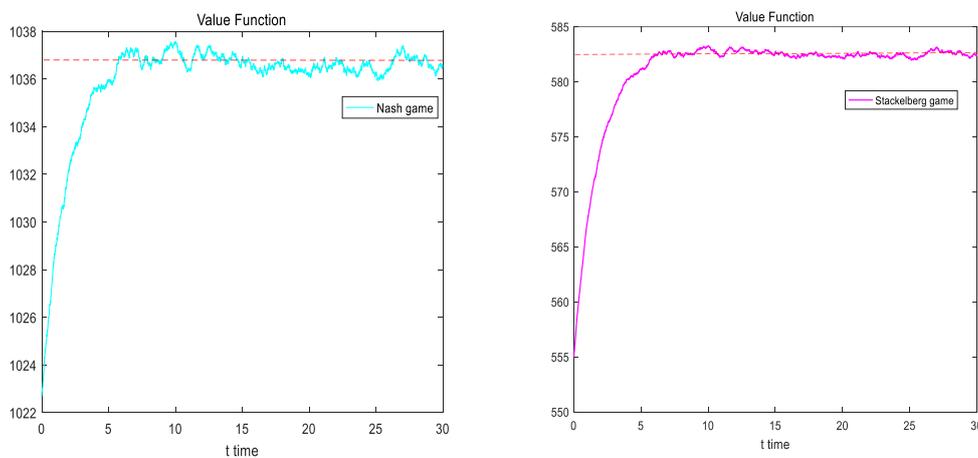


FIGURE 5. The value function of region  $j$ .

TABLE 1. Effect of the parameters in the Nash non-cooperative game.

Parameters	$x_i$	$x_j$	$x_{\text{total}} = x_i + x_j$
$c_i \eta_i$	-	+	-
$k_i$	+	×	+
$\psi \delta_i$	+	-	+
$k_j v_i v_j$	+	+	+
$r_i \Delta_i$	-	×	-
$c_j \Delta_j \eta_j r_j \delta_j$	-	-	-

TABLE 2. Effect of the parameters in the Stackelberg game.

Parameters	$x_i$	$x_j$	$x_{\text{total}}$
$c_i \Delta_i$	-	+	-
$k_i$	+	×	+
$k_j v_i v_j \delta_i$	+	+	+
$c_j \Delta_j \eta_i \eta_j r_j \delta_j$	-	-	-
$r_i$	-	×	-
$\psi$	+	-	-

with them, the self-purification rate  $\delta_i$  of region  $i$  and the portion  $\psi$  of transboundary pollution have an opposite effect. In addition, when the abatement cost  $k_i$  of region  $i$  increases, the pollutant stocks of region  $i$  and the total region also raise. Instead, larger  $\Delta_i$  and  $r_i$  lead the pollutant stocks of region  $i$  and the total region to decrease. Here the pollution stock of region  $j$  does not depend significantly on  $k_i$  and  $r_i$ .

Besides, an increase in the abatement cost  $k_i$  or the marginal impact coefficient  $v_h$  causes the stocks of pollutants in both regions to ascend. In contrast, higher  $c_j$ ,  $\Delta_j$ ,  $\eta_j$ ,  $r_j$  and  $\delta_j$  cause lower stocks of pollutants in both regions. Next, results of sensitivity analysis in the Stackelberg game are summarized in Table 2.

Here we focus on the differences between Tables 1 and 2. In the Stackelberg game, higher  $\eta_i$  causes the pollutant stock of region  $j$  to decrease. But the stock of pollutants in region  $i$  is raised by increasing  $\delta_i$ . And a larger  $\psi$  leads to a larger pollutant stock  $x_{\text{total}}$ . The effect of other parameters is consistent with that in the Nash game. According to the above analysis, we see that higher  $c_j$ ,  $\Delta_j$ ,  $\eta_j$ ,  $r_j$  and  $\delta_j$  would lead to a decrease in the pollutant stock of each region in the two games. And lower  $k_j$ ,  $v_j$  and  $v_j$  can achieve the same effect. It is not hard to find that most of these parameters are associated with region  $j$ . Therefore, strengthening pollution management in pollution upstream regions has a great opportunity to achieve satisfactory results for all parties, especially when game information of both parties is not clear.

## 7. CONCLUSIONS

Considering that transboundary pollution management is increasingly dependent of competitive relationships among players, a transboundary pollution dynamic game model between two competitive players (*i.e.*, regions) who compete in pollution abatement efforts is proposed. On one hand, the accumulation of the associated pollution stocks is formulated in virtue of a differential equation, and on the other hand, regional competitive preference is embedded into the discounted net revenues of regions. Subsequently, the equilibrium output and pollution abatement efforts of regions are captured in two specific regimes: Nash non-cooperative and Stackelberg leader–follower games. We also provide the expectation and variances of the pollution stocks over time.

Our study finds that growth in the proportion of transboundary pollution increases the equilibrium output of pollution upstream region, but reduces the equilibrium output of pollution downstream region. Moreover, equilibrium output is constrained by region’s own headwinds, but it is also encouraged by competitors’ ones. Through the impact analysis, enlarged proportion of transboundary pollution raise the aggregate pollution stock of both regions.

Secondly, the equilibrium strategies of pollution abatement efforts are affected by the local competitive preference, not by competitors’ one. Also, an increasing in regional competitive preference can significantly reduce the aggregate pollution stock. Note that competitive preference is not anticipated in shaping regional output. Comparing the simulation results based on the Nash non-cooperative and Stackelberg leader–follower games, finally, we reveal that the total social welfare and environmental quality in the Nash non-cooperative game are higher than in the Stackelberg leader–follower game. But pollution downstream region pursues to be a leader in the Stackelberg leader–follower game, due to a higher benefit.

Throughout this paper, we just discuss the competitive relationship among players in abatement efforts. Considering the public's growing concern for residential environments and global climate, competition on environmental quality is now promising in a dynamic game. Additionally, the role of network structure on transboundary pollution games is neglected in our work, when the pollution transmission paths are embedded into a network. Thus, incorporating these features into our game framework will be a critical direction for future research.

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## DATA AVAILABILITY STATEMENT

The research data associated with this article are included in the article.

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APPENDIX A. PROOF OF PROPOSITION 1

According to the optimal control theory, there exists current value functions in regions  $i$  and  $j$  respectively, denoted by  $V_i(t, x_i, x_j)$  and  $V_j(t, x_j)$ . They are all continuous differentiable functions satisfying the following Hamilton–Jacobi–Bellman(HJB) equations

$$-\frac{\partial V_j(t, x_j)}{\partial t} = \max_{q_j, u_j} \left\{ \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h \right) q_j - c_j q_j - k_j u_j^2 + \Delta_j (u_j - u_i) - \eta_j x_j \right] \right\}$$

$$\begin{aligned}
& + \lambda'_{jj} [v_j q_j - r_j u_j - (\psi + \delta_j) x_j] \Big\}, \\
-\frac{\partial V_i(t, x_i, x_j)}{\partial t} = & \max_{q_i, u_i} \left\{ \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h \right) q_i - c_i q_i - k_i u_i^2 + \Delta_i (u_i - u_j) - \eta_i x_i \right] \right. \\
& \left. + \lambda'_{ii} [v_i q_i(t) - r_i u_i(t) + \psi x_j(t) - \delta_i x_i(t)] + \lambda'_{ij} [v_j q_j(t) - r_j u_j(t) - (\psi + \lambda_j) x_j(t)] \right\} \quad (\text{A.1})
\end{aligned}$$

where  $\lambda'_{jj} = \frac{\partial V_j(t, x_j)}{\partial x_j}$ ,  $\lambda'_{ij} = \frac{\partial V_i(t, x_i, x_j)}{\partial x_j}$  and  $\lambda'_{ii} = \frac{\partial V_i(t, x_i, x_j)}{\partial x_i}$  are the dynamic variables related with the state equations for  $\dot{x}_j(t)$  and  $\dot{x}_i(t)$ , separately. By differentiating the above HJB in equation (A.1) with respect to  $q_h$  and  $u_h$  respectively, the optimal strategies of production and effort level can be expressed as follows

$$\begin{cases} q_i^* = \frac{2\lambda'_{ii} v_i - \lambda'_{ij} v_j + \alpha + c_j - 2c_i}{3\beta_i} \\ q_j^* = \frac{2\lambda'_{jj} v_j - \lambda'_{ij} v_i + \alpha + c_i - 2c_j}{3\beta_j} \\ u_h^* = \frac{\Delta_h - \lambda'_{hh} r_h}{2k_h}. \end{cases} \quad (\text{A.2})$$

Substituting the optimal strategies (Eq. (A.2)) into equation (A.1) gives

$$\begin{aligned}
-\frac{\partial V_j(t, x_j)}{\partial t} = & \left\{ \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h^* \right) q_j^* - c_j q_j^* - k_j u_j^{*2} + \Delta_j (u_j^* - u_i^*) - \eta_j x_j \right] \right. \\
& \left. + \lambda'_{jj} [v_j q_j^* - r_j u_j^* - (\psi + \delta_j) x_j] \right\}, \\
-\frac{\partial V_j(t, x_i, x_j)}{\partial t} = & \left\{ \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h^* \right) q_i^* - c_i q_i^* - k_i u_i^{*2} + \Delta_i (u_i^* - u_j^*) - \eta_i x_i \right] \right. \\
& \left. + \lambda'_{ii} [v_i q_i^* - r_i u_i^* + \psi x_j(t) - \delta_i x_i(t)] + \lambda'_{ij} [v_j q_j^* - r_j u_j^* - (\psi + \lambda_j) x_j(t)] \right\}. \quad (\text{A.3})
\end{aligned}$$

According to the structure of the above HJB functions, let us conjecture that the value functions  $V_j(x_j)$  and  $V_i(x_i, x_j)$  are linear with  $x_h$  as an independent variable

$$\begin{cases} V_j^N(x_j) = e^{-\rho t} (m_1 x_j + m_2) \\ V_i^N(x_i, x_j) = e^{-\rho t} (n_1 x_i + n_2 x_j + n_3) \end{cases} \quad (\text{A.4})$$

where  $m_1, m_2, n_1, n_2$  and  $n_3$  are constant coefficients. Substituting equation (A.4) into equation (A.2), we have the following results

$$\begin{cases} q_i^* = \frac{2e^{-\rho t} n_1 v_i - e^{-\rho t} m_1 v_j + \alpha + c_j - 2c_i}{3\beta_i} \\ q_j^* = \frac{2e^{-\rho t} m_1 v_j - e^{-\rho t} n_1 v_i + \alpha + c_i - 2c_j}{3\beta_j} \\ u_i^* = \frac{\Delta_i - e^{-\rho t} n_1 r_i}{2k_i} \\ u_j^* = \frac{\Delta_j - e^{-\rho t} m_1 r_j}{2k_j}. \end{cases} \quad (\text{A.5})$$

The results of substituting equations (A.4) and (A.5) into equation (A.3) can be expressed as follows.

$$\begin{cases} \rho e^{-\rho t} (m_1 x_j + m_2) = -[e^{-\rho t} m_1 (\psi + \delta_j) + \eta_j] x_j + \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h^* \right) q_j^* - c_j q_j^* - k_j u_j^{*2} \\ \quad + \Delta_j (u_j^* - u_i^*) + e^{-\rho t} m_1 (v_j q_j^* - r_j u_j^*), \\ \rho e^{-\rho t} (n_1 x_i + n_2 x_j + n_3) = -(\eta_i + e^{-\rho t} n_1 \delta_i) x_i + e^{-\rho t} [n_1 \psi - n_2 (\psi + \lambda_j)] x_j + \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h^* \right) q_i^* \\ \quad - c_i q_i^* - k_i u_i^{*2} + \Delta_i (u_i^* - u_j^*) + e^{-\rho t} [n_1 (v_i q_i^* - r_i u_i^*) + n_2 (v_j q_j^* - r_j u_j^*)]. \end{cases} \quad (\text{A.6})$$

From the above equations, the parameters of equation (A.4) can be determined

$$\begin{cases} m_1 = \frac{-\eta_j e^{\rho t}}{\psi + \delta_j + \rho} \\ n_1 = \frac{-\eta_i e^{\rho t}}{\delta_i + \rho} \\ n_2 = \frac{-\eta_i \psi e^{\rho t}}{(\delta_i + \rho)(\psi + \delta_j + \rho)} \\ m_2 = \frac{e^{\rho t}}{\rho} \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h^* \right) q_j^* - c_j q_j^* - k_j u_j^{*2} + \Delta_j (u_j^* - u_i^*) - \frac{\eta_j (v_j q_j^* - r_j u_j^*)}{(\psi + \delta_j - \rho)} \right] \\ n_3 = \frac{e^{\rho t}}{\rho} \left[ \left( \alpha - \sum_{h \in \{i, j\}} \beta_h q_h^* \right) q_i^* - c_i q_i^* - k_i u_i^{*2} + \Delta_i (u_i^* - u_j^*) - \frac{\eta_i (v_i q_i^* - r_i u_i^*)}{(\rho + \delta_i)} - \frac{\eta_i \psi (v_j q_j^* - r_j u_j^*)}{(\rho + \delta_i)(\rho + \psi + \lambda_j)} \right]. \end{cases} \quad (\text{A.7})$$

By substituting the above results into equations (A.2) and (A.4), we can get the conclusions of Proposition 1. The proof thus is complete.

#### APPENDIX B. PROOF OF PROPOSITION 2

With the inverse induction method, the strategies of region  $i$  are attained as  $q_i$  and  $u_i$  that are the knowledges of region  $j$ . According to the optimal theory, there exists a current optimal value function  $V_j^s(t, x_i, x_j)$  for region  $j$ , a continuous differentiable function. And  $V_j^s(t, x_i, x_j)$  for any  $x_j \geq 0$  satisfies the HJB equation below

$$\begin{aligned} & - \frac{\partial V_j^s(t, x_j)}{\partial t} \\ & = \max_{q_j, u_j} \{ (\alpha - \beta_i q_i - \beta_j q_j) q_j - c_j q_j - k_j u_j^2 + \Delta_j (u_j - u_i) - \eta_j x_j + \lambda'_j [v_j q_j - r_j u_j - (\psi + \delta_j) x_j] \} \end{aligned} \quad (\text{B.1})$$

where  $\lambda'_j = \frac{\partial V_j^s(t, x_j)}{\partial x_j}$ .

By setting the first partial derivative equal to zero with respect to  $q_j$  and  $u_j$ , the optimal strategies of region  $i$  are then computed as

$$\begin{cases} q_j^* = \frac{\lambda'_j v_j + \alpha - \beta_i q_i - c_j}{2\beta_j} \\ u_j^* = \frac{\Delta_j - \lambda'_j r_j}{2k_j}. \end{cases} \quad (\text{B.2})$$

Next, we calculate the optimal strategy for region  $i$  while  $q_j^*$  and  $u_j^*$  are the acknowledgements of region  $i$ . Abiding by the optimal control theory, the current value function  $V_i^s(t, x_i, x_j)$  for region  $i$  can also be expressed as the following HJB equation associated with such optimal control problem

$$\begin{aligned} & - \frac{\partial V_i^s(t, x_i, x_j)}{\partial t} = \max_{q_i, u_i} \{ (\alpha - \beta_i q_i - \beta_j q_j^* - c_i) q_i - k_i u_i^2 + \Delta_i (u_i - u_j^*) - \eta_i x_i + \lambda'_{ii} [v_i q_i \\ & \quad - r_i u_i + \psi x_j(t) - \delta_i x_i(t)] + \lambda'_{ij} [v_j q_j^* - r_j u_j^* - (\psi + \lambda_j) x_j(t)] \} \end{aligned} \quad (\text{B.3})$$

where  $\lambda'_{ij} = \frac{\partial V_i^s(t, x_i, x_j)}{\partial x_j}$  and  $\lambda'_{ii} = \frac{\partial V_i^s(t, x_i, x_j)}{\partial x_i}$ . By making use of the first-order condition, the outcomes of  $q_i^*$  and  $u_i^*$  are

$$\begin{cases} q_i^* = \frac{(\alpha + c_j - 2c_i - v_j \lambda'_{ij} + 2v_i \lambda'_{ii}) \beta_j - \lambda'_{ij} \beta_i v_j}{2\beta_i \beta_j} \\ u_i^* = \frac{\Delta_i - \lambda'_{ii} r_i}{2k_i}. \end{cases} \quad (\text{B.4})$$

Substituting equation (B.4) into equation (B.2), the formula (B.2) can be obtained as follow:

$$\begin{cases} q_j^* = \frac{(\alpha - 3c_j + 3\lambda'_{ij} v_j + 2c_i - 2v_i \lambda'_{ii}) \beta_j - \lambda'_{ij} \beta_i v_j}{2\beta_i \beta_j} \\ u_j^* = \frac{\Delta_j - \lambda'_{ij} r_j}{2k_j}. \end{cases} \quad (\text{B.5})$$

Substituting equations (B.4) and (B.5) into (B.3) and (B.1), respectively, one can get the following HJB equations.

$$\begin{aligned}
-\frac{\partial V_j^s(t, x_j)}{\partial t} &= [(\alpha - \beta_i q_i^* - \beta_j q_j^* - c_j) q_j^* - k_j u_j^{*2} + \Delta_j(u_j^* - u_i^*) - \eta_j x_j] + \lambda'_{jj} [v_j q_j^* - r_j u_j^* - (\psi + \delta_j) x_j], \\
-\frac{\partial V_i^s(t, x_i, x_j)}{\partial t} &= [(\alpha - \beta_j q_j^* - \beta_i q_i^* - c_i) q_i^* - k_i u_i^{*2} + \Delta_i(u_i^* - u_j^*) - \eta_i x_i] \\
&\quad + \lambda'_{ii} [v_i q_i^* - r_i u_i^* + \psi x_j(t) - \delta_i x_i(t)] + \lambda'_{ij} [v_j q_j^* - r_j u_j^* - (\psi + \delta_j) x_j].
\end{aligned} \tag{B.6}$$

Similarly, let us conjecture that the value functions  $V_j^s(t, x_j)$  and  $V_i^s(t, x_i, x_j)$  are linear with  $x_h$ . That is,

$$\begin{cases} V_j^S(t, x_j) = e^{-\rho t} (m_1 x_j + m_2) \\ V_i^S(t, x_i, x_j) = e^{-\rho t} (n_1 x_i + n_2 x_j + n_3) \end{cases} \tag{B.7}$$

where  $m_1, m_2, n_1, n_2$  and  $n_3$  are constant. Substituting equation (B.7) into equation (B.6), the parameter values can be calculated as

$$\begin{cases} m_1 = \frac{-\eta_j e^{\rho t}}{\rho + \psi + \delta_j} \\ n_1 = \frac{-\eta_i e^{\rho t}}{\rho + \delta_i} \\ n_2 = \frac{-\eta_i \psi e^{\rho t}}{(\rho + \delta_i)(\rho + \psi + \delta_j)} \end{cases} \tag{B.8}$$

$$\begin{cases} m_2 = \left\{ \frac{e^{\rho t}}{\rho} [(\alpha - \beta_i q_i^* - \beta_j q_j^* - c_j) q_j^* - k_j u_j^{*2} + \Delta_j(u_j^* - u_i^*) - \eta_j (v_j q_j^* - r_j u_j^*)] / (\rho + \psi + \delta_j) \right\} \\ n_3 = \left\{ \frac{e^{\rho t}}{\rho} [(\alpha - \beta_j q_j^* - \beta_i q_i^* - c_i) q_i^* - k_i u_i^{*2} + \Delta_i(u_i^* - u_j^*) - \eta_i (v_i q_i^* - r_i u_i^*)] / (\rho + \delta_i) - \eta_i \psi (v_j q_j^* - r_j u_j^*) / (\rho + \delta_i)(\rho + \psi + \delta_j) \right\}. \end{cases} \tag{B.9}$$

Substituting equations (B.8) and (B.9) into equation (B.7), the results of Proposition 2 can be acquired by the value functions. Therefore, this proof is completed.

### APPENDIX C. PROOF OF PROPOSITION 3

According to Fundamental Theorem of Calculus, if  $f(t)$  is a continuous differential function on the interval  $[0, T]$ , then  $t \in [0, T]$  meets the following equation

$$f(t) = f(0) + \int_0^t f'(s) ds. \tag{C.1}$$

Firstly, substituting equation (6) into equation (C.1), we have

$$\begin{cases} x_j(t) = x_0^j + \int_0^t \phi_j^N - (\psi + \delta_j) x_j(s) ds \\ x_i(t) = x_0^i + \int_0^t [\phi_i^N + \psi x_j(s) - \delta_i x_i(s)] ds \\ x_i(0) = x_0^i; \quad x_j(0) = x_0^j \end{cases} \tag{C.2}$$

where  $\phi_h^N = v_h q_h^* - r_h u_h^*$ ,  $h \in \{i, j\}$ .

Thus, the expectation of  $x_j(t)$  can be expressed as

$$E[x_j(t)] = x_0^j + \int_0^t [\phi_j^N - (\psi + \delta_j) E[x_j(s)]] ds. \tag{C.3}$$

Therefore, the solution of the above equation is

$$E[x_j(t)] = \frac{\phi_j^N}{\psi + \delta_j} + \left( x_0^j - \frac{\phi_j^N}{\psi + \delta_j} \right) e^{-(\psi + \delta_j)t} \tag{C.4}$$

and

$$E[x_i(t)] = x_0^i + \int_0^t \left[ \phi_i^N + \psi E[x_j(t)] - \delta_i E[x_i(t)] \right] ds. \quad (\text{C.5})$$

From equations (C.4) and (C.5), we also get the solution

$$E[x_i(t)] = \gamma - \kappa e^{-(\psi+\delta_j)t} + [x_0^i + \kappa - \gamma] e^{-\delta_i t} \quad (\text{C.6})$$

where  $\gamma = \frac{\phi_i^N}{\delta_i} + \frac{\psi \phi_j^N}{\delta_i(\psi+\delta_j)}$ ,  $\kappa = \frac{\psi x_0^j}{\psi+\delta_j-\delta_i} - \frac{\psi \phi_j^N}{(\psi+\delta_j)(\psi+\delta_j-\delta_i)}$ . Next, the optimal trajectory of the pollution stock will be computed. According to equation (6), similarly, we have

$$\begin{cases} \frac{dx_i(t)}{dt} = \phi_i^N + \psi x_j(t) - \delta_i x_i(t), \\ \frac{dx_j(t)}{dt} = \phi_j^N - (\psi + \delta_j)x_j(t), \\ x_i(0) = x_0^i, \quad x_j(0) = x_0^j; \quad x_0^i \geq 0, \quad x_0^j \geq 0, \end{cases} \quad (\text{C.7})$$

where  $\phi_h^N = v_h q_h^* - r_h u_h^*$ . Here we start to calculate the general solution of the following differential equation with respect to  $x_j(t)$

$$\frac{dx_j(t)}{dt} = -(\psi + \delta_j)x_j(t). \quad (\text{C.8})$$

Equation (C.8) is a homogeneous linear differential equation. By solving it, we have

$$x_j(t) = c_1 e^{-(\psi+\delta_j)t}, \quad (\text{C.9})$$

where  $c_1$  is arbitrary constant. By make use of method of variation of constant, we assume that

$$x_j(t) = c_1(t) e^{-(\psi+\delta_j)t}. \quad (\text{C.10})$$

Substituting equation (C.10) into equation (C.7),  $c_1(t)$  can be computed as

$$c_1(t) = \int \phi_j^N e^{(\psi+\delta_j)t} dt = \frac{\phi_j^N}{\psi + \delta_j} e^{(\psi+\delta_j)t} + c_2, \quad (\text{C.11})$$

where  $c_2$  is a constant. Substituting equation (C.11) into equation (C.10) gives

$$x_j(t) = \frac{\phi_j^N}{\psi + \delta_j} + c_2 e^{-(\psi+\delta_j)t}. \quad (\text{C.12})$$

Based on the initial condition  $x_j(0) = x_0^j$ , the parameter  $c_2$  can be determined. In conclusion, the optimal trajectory of pollution stock  $x_j(t)$  can be obtained

$$x_j(t) = \frac{\phi_j^N}{\psi + \delta_j} + \left( x_0^j - \frac{\phi_j^N}{\psi + \delta_j} \right) e^{-(\psi+\delta_j)t}. \quad (\text{C.13})$$

Substituting equation (C.13) into equation (C.7), the differential equation with respect to  $x_i(t)$  can be expressed as

$$\frac{dx_i(t)}{dt} = \phi_i^N + \frac{\psi \phi_j^N}{\psi + \delta_j} + \psi \left( x_0^j - \frac{\phi_j^N}{\psi + \delta_j} \right) e^{-(\psi+\delta_j)t} - \delta_i x_i(t). \quad (\text{C.14})$$

Similarly, by solving the above equation (Eq. (C.14)), the optimal trajectory of pollution stock  $x_i(t)$  is

$$x_i(t) = \frac{A_1}{\delta_i} + \frac{A_2}{\delta_i - \psi - \delta_j} e^{-(\psi+\delta_j)t} + \left( x_0^i - \frac{A_1}{\delta_i} - \frac{A_2}{\delta_i - \psi - \delta_j} \right) e^{-\delta_i t}, \quad (\text{C.15})$$

where  $A_1 = \phi_i^N + \frac{\psi \phi_j^N}{\psi + \delta_j}$ ,  $A_2 = \psi \left( x_0^j - \frac{\phi_j^N}{\psi + \delta_j} \right)$ . Therefore, we get the conclusion of Proposition 3, which finished the proof.