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AN ODD $[1, b]$-FACTOR IN A GRAPH FROM SIGNLESS LAPLACIAN SPECTRAL RADIUS

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Abstract. An odd $[1, b]$-factor of a graph G is a spanning subgraph F of G such that $d_F(v)$ is odd and $1 \leq d_F(v) \leq b$ for every $v \in V(G)$, where b is a positive odd integer. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G, where $D(G)$ denotes the degree diagonal matrix of G and $A(G)$ denotes the adjacency matrix of G. Let $q_1(G)$ denote the signless Laplacian spectral radius of G. In this paper, we study the existence of an odd $[1, b]$-factor of a graph G and derive a signless Laplacian spectral radius condition for a graph to possess an odd $[1, b]$-factor.

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1. Introduction

In this paper we deal only with finite and undirected graphs which have no loops and no multiple edges. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. The order of G is defined by the number $n = |V(G)|$ of its vertices. For a vertex v of G, we denote by $N_G(v)$ the set of all neighbours of v in G, and $d_G(v) = |N_G(v)|$ is the degree of v in G. For a given vertex subset S of G, we use $G[S]$ and $G - S$ to denote the subgraphs of G induced by S and $V(G) \setminus S$, respectively. The complement graph $\overline{G}$ of G is the graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G.

Let G and H be two graphs. The union $G \cup H$ denotes the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The join $G \vee H$ denotes the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Recall that $V(G) = \{v_1, v_2, \cdots, v_n\}$. The adjacency matrix $A(G)$ of G is an $n \times n$ matrix with the (i, j)-entry equal to 1 if v_i and v_j are adjacent, and 0 otherwise. The diagonal matrix of G is denoted by $D(G) = diag(d_G(v_1), d_G(v_2), \cdots, d_G(v_n))$. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix. Let $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$ be the eigenvalues of $Q(G)$. Particularly, the eigenvalue $q_1(G)$ is called the signless Laplacian spectral radius of G.

A subgraph F of a graph G satisfying $V(F) = V(G)$ is called a spanning subgraph of G. An $[a, b]$-factor of a graph G is defined as a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for every $x \in V(G)$. An odd $[1, b]$-factor of a graph G is a spanning subgraph F of G such that $d_F(v)$ is odd and $1 \leq d_F(v) \leq b$ for every $v \in V(G)$, where b is a positive odd integer. If $b = 1$, then an odd $[1, b]$-factor is a 1-factor or a perfect matching.

Keywords. Graph, signless Laplacian spectral radius, odd $[1, b]$-factor.

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For a positive integer k, a graph is k-regular if its every vertex has the same degree k. If $k = n - 1$, then a k-regular graph is a complete graph of order n which is denoted by K_n.

Brouwer and Haemers [3] claimed sufficient conditions for a graph to possess a perfect matching in terms of its Laplacian eigenvalues and, for a regular graph, obtain an improvement in terms of its third largest adjacency eigenvalue. Cioabă *et al.* [4] presented a sharp upper bound on the third largest adjacency eigenvalue that is sufficient to guarantee the existence of a perfect matching in a regular graph. Suil [17] posed an adjacency spectral radius condition for a graph to admit a perfect matching. Zhang and Lin [24] characterized a graph with a perfect matching according to its distance spectral radius. Liu *et al.* [14] showed a sufficient condition for the existence of a perfect matching in a graph with respect to signless Laplacian spectral radius. Many researchers claimed some sufficient conditions on various parameters for the existence of $[1, 2]$-factors in graphs, such as the degree conditions [2, 19, 21, 26], the neighborhood condition [27, 34], the isolated toughness [7, 9, 28, 33], the spectral condition [36, 37], the sun toughness [31], and the independence number [13]. Wang and Zhang [20] verified some results on the existence of $[1, b]$-factors in graphs. Cui and Kano [5] obtained a sufficient condition for the existence of an odd $[1, b]$-factor in a graph by using neighborhoods. Lu *et al.* [15] gave Laplacian eigenvalue conditions and adjacency eigenvalue conditions for graphs to possess odd $[1, b]$-factors, respectively. Kim *et al.* [10] showed an upper bound for the third largest adjacency eigenvalue in a regular graph to guarantee the existence of an odd $[1, b]$-factor. Fan *et al* [8] established a lower bound for the adjacency spectral radius in a graph to guarantee the existence of an odd $[1, b]$-factor. Some other results on graph factors see [11, 12, 16, 22, 25, 29, 30, 32, 35].

In this paper, we put forward a sufficient condition to ensure the existence of an odd $[1, b]$-factor in a graph by using the signless Laplacian spectral radius.

Theorem 1.1. *Let b be a positive odd integer, and let G be a connected graph of even order n.*

(i) *For $n \geq b+5$ and $n \neq 2b+4$, if $q_1(G) > r(n)$, then G contains an odd $[1, b]$-factor, where $r(n)$ is the largest root of $x^3 - (4n - 3b - 10)x^2 + n(2n - 2b - 5)x - 2(n - b - 3)(n - b - 2) = 0$.*
(ii) *For $n = b + 3$, if $q_1(G) > b + 3$, then G contains an odd $[1, b]$-factor.*
(iii) *For $n = 2b + 4$, if $q_1(G) > b + 3 + \sqrt{(b + 1)(b + 5)}$, then G contains an odd $[1, b]$-factor.*

2. Some preliminaries

In this section, we pose some necessary preliminary lemmas, which will be used to justify our main results.

Lemma 2.1 ([6])**.** *Let $n \geq 2$ be an integer, and K_n be a complete graph of order n. Then $q_1(K_n) = 2n - 2$.*

Lemma 2.2 ([18])**.** *Let G be a connected graph. If H is a subgraph of G and $H \neq G$, then $q_1(H) < q_1(G)$.*

Next, we explain the concepts of equitable quotient matrices and equitable partitions.

Definition 2.3. Let M be the following matrix of order n

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1r} \\ M_{21} & M_{22} & \cdots & M_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rr} \end{pmatrix},$$

whose rows and columns are partitioned into subsets $X_1, X_2, \cdots, X_r$ of $\{1, 2, \cdots, n\}$. Let M_{ij} denote the submatrix (called a block) of M by deleting the rows in $\{1, 2, \cdots, n\} - X_i$ and deleting the columns in $\{1, 2, \cdots, n\} - X_j$. The quotient matrix B of M is the $r \times r$ matrix whose entries are the average row sums of the blocks M_{ij} of M. The partition is equitable and B is called an equitable quotient matrix of M if every block M_{ij} of M admits constant row sum.

Lemma 2.4 ([23]). *Let B be an equitable quotient matrix of M as defined in Definition 2.3, and M be a nonnegative matrix. Then the spectral radius of the equitable quotient matrix B equals to the spectral radius of M.*

A component of a graph is said to be odd or even by virtue of whether its order is odd or even. We use $o(G)$ to denote the number of odd components of G. Amahashi [1] obtained a well-known necessary and sufficient condition in a graph to guarantee the existence of an odd $[1, b]$-factor.

Lemma 2.5 ([1]). *Let b be a positive odd integer and let G be a graph. Then G has an odd $[1, b]$-factor if and only if*

$$o(G - S) \leq b|S|$$

for every vertex subset S of G.

3. The proof of Theorem 1.1

Proof of Theorem 1.1. Let $\varphi(x) = x^3 - (4n - 3b - 10)x^2 + n(2n - 2b - 5)x - 2(n - b - 3)(n - b - 2)$ and let $r(n)$ be the largest root of $\varphi(x) = 0$. Suppose to the contrary that G contains no odd $[1, b]$-factor. Then by Lemma 2.5, there exists a nonempty subset $S \subseteq V(G)$ such that $o(G - S) \geq b|S| + 1$. Note that since n is even, b is odd, and $o(G - S) \equiv b|S| \pmod 2$, we admit $o(G - S) \geq b|S| + 2$. Select a connected graph G of order n such that its signless Laplacian spectral radius is as large as possible.

Together with Lemma 2.2 and the choice of G, we derive that all the components of $G - S$ are odd, and the induced subgraph $G[S]$ (resp. every connected component of $G - S$) is a complete subgraph. Furthermore, $G = G[S] \vee (G - S)$.

Let $o(G - S) = k$ and $|S| = s$, then $k \geq bs + 2$. Let $G_1, G_2, \cdots, G_k$ be all the components of $G - S$ with $|V(G_1)| \geq |V(G_2)| \geq \cdots \geq |V(G_k)|$. Note that $n = s + n_1 + n_2 + \cdots + n_k$, where $n_i = |V(G_i)|$ for $1 \leq i \leq k$. Recall that $G = G[S] \vee (G - S)$. Thus, we have $G = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k})$.

Claim 1. $n_2 = n_3 = \cdots = n_k = 1$.

Proof. Assume that $n_2 \geq 3$. Then we let $G' = K_s \vee (K_{n_1+2} \cup K_{n_2-2} \cup K_{n_3} \cup \cdots \cup K_{n_k})$. Obviously, $o(G' - S) = o(G - S) = k \geq bs + 2$ and G' is a connected graph of order n. Denote the vertex set of G by $V(G) = V(K_s) \cup V(K_{n_1}) \cup V(K_{n_2}) \cup \cdots \cup V(K_{n_k})$. Let X be the Perron vector of $Q(G)$, and let $X(v)$ denote the entry of X corresponding to the vertex $v \in V(G)$. According to symmetry, it is clear that all vertices of $V(K_s)$ (resp. $V(K_{n_1})$, $V(K_{n_2})$, $\cdots$,$V(K_{n_k})$) possess the same entries in X. Thus we may assume $X(v_0) = x_0$ for any $v_0 \in V(K_s)$, $X(v_1) = x_1$ for any $v_1 \in V(K_{n_1})$, $X(v_2) = x_2$ for any $v_2 \in V(K_{n_2})$, $\cdots$, $X(v_k) = x_k$ for any $v_k \in V(K_{n_k})$. Then we infer

$$\begin{cases} q_1(G)x_1 = sx_0 + (s + 2n_1 - 2)x_1, \\ q_1(G)x_2 = sx_0 + (s + 2n_2 - 2)x_2. \end{cases} \tag{3.1}$$

From (3.1), we deduce

$$(q_1(G) - s - 2n_1 + 2)x_1 = (q_1(G) - s - 2n_2 + 2)x_2. \tag{3.2}$$

Note that K_{s+n_1} and K_{s+n_2} are two proper subgraphs of G. By virtue of Lemmas 2.1 and 2.2, we get

$$q_1(G) > q_1(K_{s+n_1}) = 2(s + n_1) - 2 > s + 2n_1 - 2 \tag{3.3}$$

and

$$q_1(G) > q_1(K_{s+n_2}) = 2(s + n_2) - 2 > s + 2n_2 - 2. \tag{3.4}$$

It follows from (3.2) to (3.4) and $n_1 \geq n_2$ that $x_1 \geq x_2$. In terms of the Rayleigh quotient, we deduce

$$\begin{aligned} q_1(G') - q_1(G) &\geq X^T(Q(G') - Q(G))X \\ &= 2n_1x_1(x_1+x_2) + 2n_1x_2(x_1+x_2) - 8(n_2-2)x_2^2 \\ &\geq 8n_1x_2^2 - 8(n_2-2)x_2^2 \\ &= 8(n_1 - n_2 + 2)x_2^2 \\ &> 0, \end{aligned}$$

which implies $q_1(G') > q_1(G)$, which contradicts the choice of G. Hence, we admit $n_2 = 1$. Recall that $n_2 \geq n_3 \geq \cdots \geq n_k \geq 1$. Thus, $n_2 = n_3 = \cdots = n_k = 1$. This completes the proof of Claim 1. □

Recall that $G = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k})$ and $n = s + n_1 + n_2 + \cdots + n_k$. Combining these with Claim 1, we obtain $n_1 = n - s - (k-1)$ and $G = K_s \vee (K_{n_1} \cup (k-1)K_1)$.

Claim 2. $k = bs + 2$.

Proof. Note that $k \geq bs + 2$ and $k \equiv bs \pmod 2$. Hence, we may let $k \geq bs + 4$. We create a new graph $G'' = K_s \vee (K_{n_1+2} \cup (k-3)K_1)$. Thus, we infer $o(G'' - S) = o(G - S) - 2 = k - 2 \geq bs + 2$ and G is a proper subgraph of G''. In view of Lemma 2.2, $q_1(G) < q_1(G'')$, which contradicts the choice of G. Hence, $k \leq bs + 2$. Thus, we deduce $k = bs + 2$. This completes the proof of Claim 2. □

Recall that $n_1 = n - s - (k-1)$ and $G = K_s \vee (K_{n_1} \cup (k-1)K_1)$. Combining these with Claim 2, we derive $n_1 = n - (b+1)s - 1$ and $G = K_s \vee (K_{n_1} \cup (bs+1)K_1)$. Note that n_1 is odd. Next, we consider two cases by the value of n_1.

Case 1. $n_1 \geq 3$.

In this case, $n = n_1 + (b+1)s + 1 \geq (b+1)s + 4$. The equitable quotient matrix B_1 of the signless Laplacian matrix $Q(G)$ of the graph $G = K_s \vee (K_{n_1} \cup (bs+1)K_1)$ with the vertex partition $\{V((bs+1)K_1), V(K_{n-(b+1)s-1}), V(K_s)\}$ can be expressed as

$$B_1 = \begin{pmatrix} s & 0 & s \\ 0 & 2n-(2b+1)s-4 & s \\ bs+1 & n-(b+1)s-1 & n+s-2 \end{pmatrix}.$$

By a simple calculation, the characteristic polynomial of the matrix B_1 is equal to

$$\begin{aligned} f_{B_1}(x) = x^3 &- (3n-(2b-1)s-6)x^2 + (2n^2 - 8n - (2b-3)sn + 4(b-2)s - 4bs^2 + 8)x \\ &- s(2n^2 - 10n - 4bsn + 10bs + 2b^2s^2 + 12). \end{aligned}$$

Using Lemma 2.4, the largest root, say r_1, of $f_{B_1}(x) = 0$ is equal to the signless Laplacian spectral radius of G. Thus, we see $f_{B_1}(r_1) = 0$ and $q_1(G) = r_1$.

It is obvious that $K_s \vee (n-s)K_1$ is a proper subgraph of G. In terms of Lemma 2.2, $r_1 = q_1(G) > q_1(K_s \vee (n-s)K_1)$. Consider an equitable partition $\{V(K_s), V((n-s)K_1)\}$ of the graph $K_s \vee (n-s)K_1$. The corresponding equitable quotient matrix B_2 of the signless Laplacian matrix $Q(K_s \vee (n-s)K_1)$ of the graph $K_s \vee (n-s)K_1$ admits the following form

$$B_2 = \begin{pmatrix} n+s-2 & n-s \\ s & s \end{pmatrix}.$$

The characteristic polynomial $f_{B_2}(x)$ of the matrix B_2 is

$$f_{B_2}(x) = x^2 - (n+2s-2)x + 2s(s-1).$$

By virtue of Lemma 2.4, the largest root, say r_2, of $f_{B_2}(x) = 0$ equals the signless Laplacian spectral radius of $K_s \vee (n-s)K_1$, that is, $q_1(K_s \vee (n-s)K_1) = r_2$. Therefore, we deduce $r_1 = q_1(G) > q_1(K_s \vee (n-s)K_1) = r_2$. By a simple computation, we get

$$\begin{aligned} r_1 > r_2 &= \frac{(n+2s-2) + \sqrt{(n+2s-2)^2 - 8s(s-1)}}{2} \\ &= \frac{(n+2s-2) + \sqrt{(n-2)^2 + 4s(n-s)}}{2} \\ &> \frac{(n+2s-2) + (n-2)}{2} \\ &= n+s-2. \end{aligned} \tag{3.5}$$

By plugging the value r_1 into $\varphi(x) - f_{B_1}(x)$, it follows from (3.5), $n \geq (b+1)s + 4$ and $f_{B_1}(r_1) = 0$ that

$$\begin{aligned} \varphi(r_1) &= \varphi(r_1) - f_{B_1}(r_1) \\ &= -(n-b-5+(2b-1)(s-1))r_1^2 + (s-1)((2b-3)n + 4bs + 8)r_1 \\ &\quad + (s-1)(2n^2 - 4bn(s+1) - 10n + 10b(s+1) + 2b^2(s^2+s+1) + 12) \\ &\leq -((b+1)s - b - 1 + (2b-1)(s-1))r_1^2 + (s-1)((2b-3)n + 4bs + 8)r_1 \\ &\quad + (s-1)(2n^2 - 4bn(s+1) - 10n + 10b(s+1) + 2b^2(s^2+s+1) + 12) \\ &= (s-1)(-3br_1^2 + ((2b-3)n + 4bs + 8)r_1 + 2n^2 - 4bn(s+1) - 10n \\ &\quad + 10b(s+1) + 2b^2(s^2+s+1) + 12) \\ &\leq (s-1)(-3b(n+s-2)r_1 + ((2b-3)n + 4bs + 8)r_1 + 2n^2 - 4bn(s+1) \\ &\quad - 10n + 10b(s+1) + 2b^2(s^2+s+1) + 12) \\ &= (s-1)((-(b+3)n + bs + 6b + 8)r_1 + 2n^2 - 4bn(s+1) \\ &\quad - 10n + 10b(s+1) + 2b^2(s^2+s+1) + 12) \\ &\leq (s-1)((-(b+3)n + bs + 6b + 8)(n+s-2) + 2n^2 - 4bn(s+1) \\ &\quad - 10n + 10b(s+1) + 2b^2(s^2+s+1) + 12) \\ &= (s-1)(-(b+1)n^2 - (4b+3)sn + 4(b+1)n + b(2b+1)s^2 \\ &\quad + (2b^2 + 14b + 8)s + 2b^2 - 2b - 4) \\ &\leq (s-1)(-(b+1)((b+1)s+4)n - (4b+3)sn + 4(b+1)n + b(2b+1)s^2 \\ &\quad + (2b^2 + 14b + 8)s + 2b^2 - 2b - 4) \\ &= (s-1)(-(b^2+6b+4)sn + b(2b+1)s^2 + (2b^2+14b+8)s + 2b^2 - 2b - 4) \\ &\leq (s-1)(-(b^2+6b+4)((b+1)s+4)s + b(2b+1)s^2 \\ &\quad + (2b^2 + 14b + 8)s + 2b^2 - 2b - 4) \\ &= (s-1)(-(b+1)(b^2+6b+4)s^2 - 4(b^2+6b+4)s + b(2b+1)s^2 \\ &\quad + (2b^2 + 14b + 8)s + 2b^2 - 2b - 4) \\ &\leq (s-1)(-2(b^2+6b+4)s^2 - 4(b^2+6b+4)s + b(2b+1)s^2 \\ &\quad + (2b^2 + 14b + 8)s + 2b^2 - 2b - 4) \\ &= (s-1)(-(11b+8)s^2 - (2b^2+10b+8)s + 2b^2 - 2b - 4) \\ &\leq 0, \end{aligned}$$

which implies $q_1(G) = r_1 \leq r(n)$, which contradicts $q_1(G) > r(n)$.

Case 2. $n_1 = 1$.

In this case, $n=(b+1)s+2$ and $G=K_s\vee(bs+2)K_1$. The equitable quotient matrix B_3 of the signless Laplacian matrix $Q(G)$ of the graph G by means of the vertex partition $\{V(K_s),V((bs+2)K_1)\}$ is

$$B_3=\begin{pmatrix} n+s-2 & bs+2 \\ s & s \end{pmatrix}.$$

The characteristic polynomial $f_{B_3}(x)$ of the matrix B_3 is equal to

$$f_{B_3}(x)=x^2-(n+2s-2)x+s(n-(b-1)s-4).$$

In view of Lemma 2.4, the largest root, say r_3, of $f_{B_3}(x)=0$ equals the signless Laplacian spectral radius of $G=K_s\vee(bs+2)K_1$. Thus, we see $f_{B_3}(r_3)=0$ and $r_3=q_1(G)$. By a simple computation, we derive

$$\begin{aligned} r_3 &= \frac{(n+2s-2)+\sqrt{(n+2s-2)^2-4s(n-(b-1)s-4)}}{2} \\ &= \frac{(b+3)s+\sqrt{(b^2+6b+1)s^2+8s}}{2} \\ &> (b+2)s. \end{aligned} \tag{3.6}$$

If $s=1$, then $n=b+3$ and $q_1(G)=r_3=b+3$, which contradicts $q_1(G)>b+3$. If $s=2$, then $n=2b+4$ and $q_1(G)=r_3=b+3+\sqrt{(b+1)(b+5)}$, which contradicts $q_1(G)>b+3+\sqrt{(b+1)(b+5)}$. In what follows, we consider $s\geq 3$.

By plugging the value r_3 into $\varphi(x)-xf_{B_3}(x)$, it follows from (3.6), $n=(b+1)s+2\geq 3b+5$ and $f_{B_3}(r_3)=0$ that

$$\begin{aligned} \varphi(r_3) &= \varphi(r_3)-r_3f_{B_3}(r_3) \\ &= (-3n+3b+2s+8)r_3^2+(n(2n-2b-5)-s(n-(b-1)s-4))r_3 \\ &\quad -2(n-b-3)(n-b-2) \\ &= (-(3b+1)s+3b+2)r_3^2+(((b+1)s+2)(2(b+1)s-2b-1)-s(2s-2))r_3 \\ &\quad -2((b+1)s-b-1)((b+1)s-b) \\ &< (-(3b+1)s+3b+2)(b+2)sr_3 \\ &\quad +(((b+1)s+2)(2(b+1)s-2b-1)-s(2s-2))r_3 \\ &\quad -2((b+1)s-b-1)((b+1)s-b) \\ &= (-(b^2+3b+2)s^2+(b^2+9b+7)s-4b-2)r_3 \\ &\quad -2((b+1)s-b-1)((b+1)s-b) \\ &\leq (-3(b^2+3b+2)s+(b^2+9b+7)s-4b-2)r_3 \\ &\quad -2((b+1)s-b-1)((b+1)s-b) \\ &= -((2b^2-1)s+4b+2)r_3-2((b+1)s-b-1)((b+1)s-b) \\ &< 0, \end{aligned}$$

which implies $q_1(G)=r_3<r(n)$, which contradicts $q_1(G)>r(n)$. This completes the proof of Theorem 1.1. □

4. Sharpness

For $t\geq 3$, the sequential join $G_1\vee G_2\vee\cdots\vee G_t$ of graphs $G_1,G_2,\cdots,G_t$ is the graph with vertex set $V(G_1)\cup V(G_2)\cup\cdots\cup V(G_t)$ and edge set $E(G_1)\cup E(G_2)\cup\cdots\cup E(G_t)\cup\{e=x_ix_{i+1}:x_i\in V(G_i),x_{i+1}\in V(G_{i+1}),1\leq i\leq t-1\}$.

In this section, we claim that the signless Laplacian spectral radius conditions in Theorem 1.1 are sharp.

Let $G_1 = K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}$, and let $r(n)$ be the largest root of $x^3 - (4n - 3b - 10)x^2 + n(2n - 2b - 5)x - 2(n - b - 3)(n - b - 2) = 0$, where b is a positive odd integer and $n = b + 5$. The equitable quotient matrix of the signless Laplacian matrix $Q(G_1)$ of the graph $G_1 = K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}$ with the vertex partition $\{V(\overline{K_{b+1}}), V(K_{n-b-2}), V(K_1)\}$ can be expressed as

$$B(K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2n-2b-5 & 1 \\ b+1 & n-b-2 & n-1 \end{pmatrix}.$$

By a simple calculation, the characteristic polynomial of the matrix equals $x^3 - (4n - 3b - 10)x^2 + n(2n - 2b - 5)x - 2(n - b - 3)(n - b - 2)$ by $n = b + 5$. In light of Lemma 2.4, the largest root $r(n)$ of $x^3 - (4n - 3b - 10)x^2 + n(2n - 2b - 5)x - 2(n - b - 3)(n - b - 2) = 0$ is equal to the signless Laplacian spectral radius of the graph $G_1 = K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}$. Thus, we derive $q_1(K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}) = r(n)$. We write $S = V(K_1)$. Then we possess $o(K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}} - S) = b + 2 > b = b|S|$. By virtue of Lemma 2.5, $K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}$ has no odd $[1, b]$-factor.

Given $G_2 = K_1 \vee \overline{K_{b+2}}$, where b is a positive odd integer. The equitable quotient matrix of the signless Laplacian matrix $Q(G_2)$ of the graph $G_2 = K_1 \vee \overline{K_{b+2}}$ with the vertex partition $\{V(K_1), V(\overline{K_{b+2}})\}$ is equal to

$$B(K_1 \vee \overline{K_{b+2}}) = \begin{pmatrix} b+2 & b+2 \\ 1 & 1 \end{pmatrix}.$$

By a simple computation, the characteristic polynomial of the matrix is $x^2 - (b+3)x$. Utilizing Lemma 2.4, the largest root of $x^2 - (b+3)x = 0$ is equal to the signless Laplacian spectral radius of the graph $G_2 = K_1 \vee \overline{K_{b+2}}$. That is to say, $q_1(K_1 \vee \overline{K_{b+2}}) = b+3$. Assume that $S = V(K_1)$. Then we infer $o(K_1 \vee \overline{K_{b+2}} - S) = b+2 > b = b|S|$. By means of Lemma 2.5, $K_1 \vee \overline{K_{b+2}}$ contains no odd $[1, b]$-factor.

Let $G_3 = K_2 \vee \overline{K_{2b+2}}$, where b is a positive odd integer. The equitable quotient matrix of the signless Laplacian matrix $Q(G_3)$ of the graph $G_3 = K_2 \vee \overline{K_{2b+2}}$ with the vertex partition $\{V(K_2), V(\overline{K_{2b+2}})\}$ can be written as

$$B(K_2 \vee \overline{K_{2b+2}}) = \begin{pmatrix} 2b+4 & 2b+2 \\ 2 & 2 \end{pmatrix}.$$

By a simple calculation, the characteristic polynomial of the matrix equals $x^2 - 2(b+3)x + 4$. Applying Lemma 2.4, the largest root of $x^2 - 2(b+3)x + 4 = 0$ equals the signless Laplacian spectral radius of the graph $G_3 = K_2 \vee \overline{K_{2b+2}}$. Thus, we get $q_1(K_2 \vee \overline{K_{2b+2}}) = b + 3 + \sqrt{(b+1)(b+5)}$. Set $S = V(K_2)$. Then we possess $o(K_2 \vee \overline{K_{2b+2}} - S) = 2b + 2 > 2b = b|S|$. According to Lemma 2.5, $K_2 \vee \overline{K_{2b+2}}$ has no odd $[1, b]$-factor.

5. Concluding remarks

In this paper, we consider a spectral condition to ensure the existence of some substructure in the graph. We establish signless Laplacian spectral radius conditions to guarantee that a connected graph G contains an odd $[1, b]$-factor and claim that the signless Laplacian spectral radius conditions are sharp. Along the above line, it is interesting to find some sufficient spectral conditions to guarantee the connected graph or the connected bipartite graph G contains some other type factors. We will do them in the near future.

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Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

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