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GRAPHS WITH UNIQUE MINIMUM SEMITOTAL DOMINATING SETS

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Abstract. In an isolate-free graph G, a subset S of vertices is a *semitotal dominating set* of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The *semitotal domination number* of G, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set in G. We prove that if G is a connected graph with order $n \geqslant 3$ and a unique minimum semitotal dominating set, then $\gamma_{t2}(G) \leqslant \frac{n-1}{2}$, and we characterize the infinite families of graphs that achieve equality in this bound. By strengthening the condition of the degree, we can get stronger upper bounds. Using edge weighting functions on semitotal dominating sets, for a connected graph G of order n with a unique minimum semitotal dominating set S, giving each edge in $E[V(G) \setminus S, S]$ a weight of size within $[0, 1]$, we prove that $\gamma_{t2}(G) \leqslant \frac{2}{5}n$ and this bound is sharp if the minimum degree of G is at least two, and $\gamma_{t2}(G) \leqslant \frac{1}{3}n$ if the minimum degree of G is at least three.

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1. Introduction

In this paper, we consider only finite simple undirected graphs. A subset D of vertices in a graph G is a *dominating set* (resp. *total dominating set*) of G if every vertex of $V(G) \setminus D$ (resp. $V(G)$) is adjacent to a vertex in D. The minimum cardinality of a dominating set (resp. total dominating set) is called the *domination number* (resp. *total domination number*), represented as $\gamma(G)$ (resp. $\gamma_t(G)$) of G. It is worth noting that the study of total dominating set is meaningful only in an isolate-free graph. Since 1997, domination and its variations have been extensively studied, and those who are interested can see [3, 5, 9, 11].

Semitotal domination is a relaxed form of total domination, and it was first introduced by Goddard, Henning and McPillan [1]. A subset D of vertices in an isolate-free graph G is a *semitotal dominating set*, abbreviated semi-TD-set, of G if it is a dominating set of G and every vertex in D is within distance 2 of another vertex of D. The *semitotal domination number* of G, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semi-TD-set in G. We refer to a minimum semi-TD-set of G as a $\gamma_{t2}(G)$-*set*. Since every total dominating set is a semi-TD-set and every semi-TD-set is a dominating set, $\gamma(G) \leqslant \gamma_{t2}(G) \leqslant \gamma_t(G)$. However, the semitotal domination number is very different from the domination and total domination number. For example, the total domination number cannot be compared with the matching number, while the semitotal domination number is comparable with

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the matching number and cannot be greater than the matching number plus one (see [6, 10]). That makes the study of semitotal domination number interesting.

Semitotal domination also has many practical applications. For example, let the vertices of a graph represent the people in some organization. An edge between two people means that they know each other. A person can be supervised by another person if they know each other or there is a mutual acquaintance. Then a minimum semitotal dominating set of this organization represents a committee with the fewest members, in which everyone who is not on the committee knows at least one member of the committee, and everyone on this committee is supervised by at least one other person on the committee.

Up to now, there has been a lot of research work to determine the bound of semitotal domination number of graphs. For example, Goddard *et al.* [1] proved that if G is a connected graph of order $n \geqslant 4$, then $\gamma_{t2}(G) \leqslant \frac{n}{2}$ and characterized the graphs of minimum degree 2 achieving this bound (*i.e.* Thm. 1.1). Henning and Marcon [7], and Zhu *et al.* [12, 13] studied the upper bounds on the semitotal domination number of connected claw-free cubic graphs. In [4], Henning established the tight upper bounds on the upper semitotal domination number of a regular graph using edge weighting functions. For algorithmic aspects of semitotal domination in graphs, Henning and Pandey [8] showed the semitotal domination problem is NP-complete for planar graphs, chordal bipartite graphs and split graphs.

Theorem 1.1. *If G is a connected graph of order $n \geqslant 4$, then $\gamma_{t2}(G) \leqslant \frac{n}{2}$ and this bound is sharp.*

A graph G having a unique $\gamma_{t2}(G)$-set is called a *USTD*-graph. T.W. Haynes and M.A. Henning [2] characterized the USTD-trees. Our aim is to continue studying USTD-graphs. This paper is organized as follows. In Section 2, we give some basic definitions and a useful lemma as preliminaries. In Section 3, we prove that if G is a connected USTD-graph of order $n \geqslant 3$, then $\gamma_{t2}(G) \leqslant \frac{n-1}{2}$, and characterize the graphs attaining this bound. In Sections 4 and 5, using edge weighting functions on semitotal dominating sets, we discuss the upper bounds on the semitotal domination number with the additional condition that the minimum degree of G is at least two and three, respectively.

2. Preliminaries

In this section, we introduce some basic definitions and a useful lemma.

Let $G = (V(G), E(G))$ be a connected graph with vertex set $V(G)$ of order $n = |V(G)|$ and edge set $E(G)$. For a vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ the *neighborhood* of v and by $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighborhood* of v. The *degree* of v is $d_G(v) = |N_G(v)|$ and the number $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ is the *minimum degree* of G. We call a path connecting vertices u and v a (u, v)-*path*. The *distance* $d_G(u, v)$ between u and v is the length of a shortest (u, v)-path in G. If there is no confusion, then the subscript G is omitted in the notation, such as $N(v), d(v), d(u, v)$, etc.. For a positive integer r, $[r]$ denotes $\{1, \ldots, r\}$.

Consider S and S_1 to be two disjoint subsets of $V(G)$. Let $E[S, S_1] = \{u_1u_2 \mid u_1 \in S \text{ and } u_2 \in S_1\}$. We denote by $G[S]$ the subgraph of G induced by S and $N_S(u)$ the neighborhood restricted on S of u, where u is a vertex of $V(G)$. For a vertex v of S, the S-*external private neighborhood* of v, denoted by $epn(v, S)$, is the set of all vertices in $V(G)\backslash S$ that are adjacent to v but to no other vertex of S. That is, $epn(v, S) = \{u \in V(G)\backslash S \mid N_S(u) = \{v\}\}$. The S-*internal private* 2-*neighborhood* of v, denoted by $ipn_2(v, S)$, is the set of all vertices in $S\backslash\{v\}$ at distance 2 from v but at distance greater than 2 from every other vertex of S. That is $ipn_2(v, S) = \{u \in S\backslash\{v\} \mid d(u, v) = 2$ and $d(u, w) > 2$ for any vertex $w \in S \setminus \{u, v\}\}$.

The following result in [2] provides conditions that a USTD-graph must satisfy.

Lemma 2.1 ([2]). *If G is a connected USTD-graph of order $n \geqslant 3$ and S is the unique $\gamma_{t2}(G)$-set, then every vertex $v \in S$ satisfies (a) or (b) below.*

(a) $|epn(v, S)| + |ipn_2(v, S)| \geqslant 2$.
(b) *The vertex v is isolated in $G[S]$ and $|epn(v, S)| = 1$. Further, $d(v', S \setminus \{v\}) = 3$ where $epn(v, S) = \{v'\}$.*

3. USTD-graphs with minimum degree at least one

In this section, we establish the tight upper bound on the semitotal domination number of a USTD-graph and characterize all USTD-graphs attaining this upper bound. For a positive integer $r \geqslant 2$, a *non-trivial star* $K_{1,r}$ is a connected graph consisting of a vertex of degree r and r vertices of degree 1. Let $\mathcal{T}$ be the family of all graphs that can be obtained from a non-trivial star by subdividing every edge once. Graph P_7 represents a path of order 7.

Theorem 3.1. *Let G be a connected USTD-graph of order $n \geqslant 3$. Then $\gamma_{t2}(G) \leqslant \frac{n-1}{2}$, with equality if and only if $G \in \{P_7\} \cup \mathcal{T}$.*

Proof. If $n = 3$, then G is not a USTD-graph, a contradiction. Thus $n \geqslant 4$. By Theorem 1.1, $\gamma_{t2}(G) \leqslant \frac{n}{2}$. Let S be the unique $\gamma_{t2}(G)$-set. Then $|S| \leqslant \frac{n}{2}$. For convenience, set $\overline{S} = V(G) \setminus S$. Clearly, $|\overline{S}| = n - |S|$.

Suppose that there exists a vertex $u \in S$ such that $epn(u, S) = \emptyset$. By Lemma 2.1, $|ipn_2(u, S)| \geqslant 2$. Let u_1 and u_2 be two vertices of $ipn_2(u, S)$, let z_1 be a vertex connecting u_1 and u, and let z_2 be a vertex connecting u_2 and u. Then $\{z_1, z_2\} \subseteq \overline{S}$. Observe that $ipn_2(u_1, S) = \emptyset$, $ipn_2(u_2, S) = \emptyset$, $N_S(z_1) = \{u, u_1\}$ and $N_S(z_2) = \{u, u_2\}$. Again by Lemma 2.1, $epn(u_1, S) \neq \emptyset$ and $epn(u_2, S) \neq \emptyset$. In conclusion, every vertex in S that has no S-external private neighbors corresponds to at least two vertices of $\overline{S}$ (u corresponds to z_1 and z_2). Let X be the set of vertices in S that has no S-external private neighbors and $x = |X|$. Then $|\overline{S}| \geqslant 2x + \sum_{w \in S\setminus X} |epn(w, S)| \geqslant 2x + |S| - x$. Combined with $|\overline{S}| = n - |S|$, we have $|S| \leqslant \frac{n-x}{2} \leqslant \frac{n-1}{2}$.

Assume $|S| = \frac{n-1}{2}$. Then $x = 1$, $X = \{u\}$ and any vertex $w \in S \setminus \{u\}$ satisfies $|epn(w, S)| = 1$. Observe that $|\overline{S}| = n - |S| = \frac{n+1}{2} = |S| + 1$. Thus $\overline{S} = \{z_1, z_2\} \cup (\cup_{w \in S\setminus\{u\}} epn(w, S))$. If $S \neq \{u, u_1, u_2\}$, then without loss of generality, let $u_3 \in S \setminus \{u, u_1, u_2\}$ and $epn(u_3, S) = \{z\}$. Observe that $N_{\overline{S}}(u_3) \subseteq \{z, z_1, z_2\}$. Recall that $N_S(z_1) = \{u, u_1\}$ and $N_S(z_2) = \{u, u_2\}$. Thus $N_{\overline{S}}(u_3) = \{z\}$. Since S is a semi-TD-set of G, there is a vertex in $S \setminus \{u\}$ that is within distance 2 of u. It follows that $N_S(u_3) \neq \emptyset$. Note that $ipn_2(u_3, S) = \emptyset$. Thus $(S \setminus \{u_3\}) \cup \{z\}$ is a $\gamma_{t2}(G)$-set, a contradiction. Hence $S = \{u, u_1, u_2\}$. Let $epn(u_1, S) = \{z_3\}$ and $epn(u_2, S) = \{z_4\}$. Then $\overline{S} = \{z_1, z_2, z_3, z_4\}$. Combining Lemma 2.1 and u_1, we note that $d(z_3, S \setminus \{u_1\}) = 3$. It follows that $d(z_3) = 1$. Similarly, $d(z_4) = 1$. If $z_1z_2 \in E(G)$, then $(S \setminus \{u\}) \cup \{z_2\}$ is a semi-TD-set of G, a contradiction. Thus $z_1z_2 \notin E(G)$. Further, $G \cong P_7$.

Next, suppose that every vertex of S has at least one S-external private neighbor. Then $|\overline{S}| \geqslant |S|$. If $|\overline{S}| \geqslant |S| + 2$, then $|S| \leqslant \frac{n-2}{2}$, as desired. Thus, we may assume that $|\overline{S}| = |S|$ or $|S| + 1$. If S is not an independent set of G, then there exists two adjacent vertices u_1, u_2 in S. Let $z_1 \in epn(u_1, S)$ and $z_2 \in epn(u_2, S)$. Then $N_{\overline{S}}(u_1) \subseteq \{z_1\} \cup N_{\overline{S}}(u_2)$ or $N_{\overline{S}}(u_2) \subseteq \{z_2\} \cup N_{\overline{S}}(u_1)$, otherwise $|\overline{S}| \geqslant |N_{\overline{S}}(u_1) \setminus (\{z_1\} \cup N_{\overline{S}}(u_2))| + |N_{\overline{S}}(u_2) \setminus (\{z_2\} \cup N_{\overline{S}}(u_1))| + |\{z_1, z_2\}| + \sum_{u \in S\setminus\{u_1,u_2\}} |epn(u, S)| \geqslant 2 + |S|$, a contradiction. Without loss of generality, consider $N_{\overline{S}}(u_1) \subseteq \{z_1\} \cup N_{\overline{S}}(u_2)$. It follows that $(S \setminus \{u_1\}) \cup \{z_1\}$ is a $\gamma_{t2}(G)$-set, a contradiction.

Hence, S is an independent set of G. Since S is a semi-TD-set of G, there exist two vertices v_1, v_2 in S such that $d(v_1, v_2) = 2$. Let w_1 be a vertex connecting v_1 and v_2. Clearly, $w_1 \in \overline{S}$. Then $|\overline{S}| = |S| + 1$, $\overline{S} = \{w_1\} \cup (\cup_{w \in S} epn(w, S))$ and any vertex w of S satisfies $|epn(w, S)| = 1$. Note that $|S| = \frac{n-1}{2}$. Let v_3 be a vertex of $S \setminus \{v_1, v_2\}$. Since S is a semi-TD-set of G, $v_3w_1 \in E(G)$. That is, for any vertex w of S, $N_{\overline{S}}(w) = \{w_1\} \cup epn(w, S)$. We claim that $\overline{S}$ is an independent set of G. To the contrary, we may assume that $w_2w_1 \in E(G)$ or $w_2w_3 \in E(G)$, where $\{w_2\} = epn(v_1, S)$ and $\{w_3\} = epn(v_2, S)$. Then $(S \setminus \{v_1\}) \cup \{w_2\}$ is a $\gamma_{t2}(G)$-set, a contradiction. Thus, $\overline{S}$ is an independent set of G and further $G \in \mathcal{T}$. □

4. USTD-graphs with minimum degree at least two

In this section, we consider USTD-graphs with minimum degree at least two. Let $\mathcal{G}_1$ be an infinite family of graphs obtained by adding edges to a path with $5t$ vertices, where $t \geqslant 3$ is an integer, between each vertex of degree 1 and the vertex with a distance of 5 from it.

Lemma 4.1. *Let G be a graph of order n in $\mathcal{G}_1$. Then $\gamma_{t2}(G) = \frac{2}{5}n$ and G is a USTD-graph with minimum degree $\delta(G) \geqslant 2$.*

Proof. Let G be the graph obtained from P_{5t} by adding two edges v_1v_6 and $v_{5t}v_{5t-5}$, where $P_{5t} = v_1v_2v_3v_4v_5\cdots v_{5t-4}v_{5t-3}v_{5t-2}v_{5t-1}v_{5t}$ and $t \geqslant 3$ is an integer. Then $\delta(G) \geqslant 2$ and $n = 5t$. Let S be a $\gamma_{t2}(G)$-set. If there exists $i_0 \in \{0, 1, 2, \ldots, t-1\}$ such that $|\{v_{5i_0+1}, v_{5i_0+2}, v_{5i_0+3}, v_{5i_0+4}, v_{5i_0+5}\} \cap S| \leqslant 1$, then $\{v_{5i_0+1}, \ldots, v_{5i_0+5}\} \cap S = \{v_{5i_0+3}\}$ in order to dominate v_{5i_0+2}, v_{5i_0+3} and v_{5i_0+4}. However, there does not exist a vertex within distance 2 from v_{5i_0+3} in $S \setminus \{v_{5i_0+3}\}$, contradicting that S is a semi-TD-set of G. Thus, $|\{v_{5i+1}, \ldots, v_{5i+5}\} \cap S| \geqslant 2$ for any $i \in \{0, 1, 2, \ldots, t-1\}$ and further $\gamma_{t2}(G) = |S| \geqslant 2t$. Observe that $\{v_2, v_4, v_7, v_9, \ldots, v_{5t-3}, v_{5t-1}\}$ is a semi-TD-set of G. Thus $\gamma_{t2}(G) \leqslant 2t$. Hence $\gamma_{t2}(G) = 2t = \frac{2}{5}n$.

Without loss of generality, set $S = \{v_2, v_4, v_7, v_9, \ldots, v_{5t-3}, v_{5t-1}\}$. Suppose to the contrary that G is not a USTD-graph. Among all $\gamma_{t2}(G)$-sets that are different from S, select D such that $|D \cap S|$ is maximized. Note that $|\{v_{5i+1}, \ldots, v_{5i+5}\} \cap D| \geqslant 2$ for any $i \in \{0, 1, 2, \ldots, t-1\}$ and $|D| = \gamma_{t2}(G) = 2t$. Thus, $|\{v_{5i+1}, \ldots, v_{5i+5}\} \cap D| = 2$ for any $i \in \{0, 1, 2, \ldots, t-1\}$. Let i_1 be the minimum label of $\{0, 1, 2, \ldots, t-1\}$ such that $\{v_{5i_1+1}, \ldots, v_{5i_1+5}\} \cap D \neq \{v_{5i_1+2}, v_{5i_1+4}\}$ and $X = \{v_{5i_1+1}, \ldots, v_{5i_1+5}\}$. Then $|D \cap X| = 2$ and $D \cap X \neq \{v_{5i_1+2}, v_{5i_1+4}\}$.

Suppose $v_{5i_1+1} \in D$. In order to dominate v_{5i_1+3} and v_{5i_1+4}, we note that $X \cap D = \{v_{5i_1+1}, v_{5i_1+3}\}$ or $\{v_{5i_1+1}, v_{5i_1+4}\}$. If $X \cap D = \{v_{5i_1+1}, v_{5i_1+3}\}$, then $N_D(v_{5i_1+5}) \setminus \{v_{5i_1+4}\} \neq \emptyset$ in order to dominate v_{5i_1+5}. When $v_{5i_1+1} \notin ipn_2(v_{5i_1+3}, D)$, replacing v_{5i_1+3} in D with v_{5i_1+4} produces a $\gamma_{t2}(G)$-set D' with $|D' \cap S| > |D \cap S|$, contradicting the choice of D. When $v_{5i_1+1} \in ipn_2(v_{5i_1+3}, D)$, we have $i_1 = 0$, otherwise, it follows from the choice of i_1 that $\{v_{5i_1-4}, v_{5i_1-3}, v_{5i_1-2}, v_{5i_1-1}, v_{5i_1}\} \cap D = \{v_{5i_1-3}, v_{5i_1-1}\}$, contradicting $v_{5i_1+1} \in ipn_2(v_{5i_1+3}, D)$. Thus, $\{v_1, v_2, v_3, v_4, v_5\} \cap D = \{v_1, v_3\}$ and $v_6 \in D$. Recall that $v_1v_6 \in E(G)$. Further, $v_1 \notin ipn_2(v_3, D)$, a contradiction.

Hence $X \cap D = \{v_{5i_1+1}, v_{5i_1+4}\}$. Since D is a semi-TD-set of G, either $v_{5i_1+6} \in D$ in the case of $i_1 < t-1$ or $v_{5i_1} = v_{5t-5} \in D$ in the case of $i_1 = t-1$. Let $D' = (D \setminus \{v_{5i_1+1}\}) \cup \{v_{5i_1+2}\}$. Then $|D' \cap S| > |D \cap S|$ and $D' \neq S$. When $i_1 = 0$, we note that $v_6 \in D$ and further D' is a $\gamma_{t2}(G)$-set. When $i_1 \geqslant 1$, $\{v_{5i_1-4}, v_{5i_1-3}, v_{5i_1-2}, v_{5i_1-1}, v_{5i_1}\} \cap D = \{v_{5i_1-3}, v_{5i_1-1}\}$ by the choice of i_1. Then D' is also a $\gamma_{t2}(G)$-set. In both two cases, it contradicts the choice of D.

Suppose $v_{5i_1+1} \notin D$. If $v_{5i_1+2} \in D$, then $X \cap D = \{v_{5i_1+2}, v_{5i_1+3}\}$ or $\{v_{5i_1+2}, v_{5i_1+5}\}$. When $X \cap D = \{v_{5i_1+2}, v_{5i_1+3}\}$, we note that $N_D(v_{5i_1+5}) \setminus \{v_{5i_1+4}\} \neq \emptyset$ in order to dominate v_{5i_1+5}. Then $D' = (D \setminus \{v_{5i_1+3}\}) \cup \{v_{5i_1+4}\}$ is a $\gamma_{t2}(G)$-set with $|D' \cap S| > |D \cap S|$ and $D' \neq S$, a contradiction. When $X \cap D = \{v_{5i_1+2}, v_{5i_1+5}\}$, since D is a semi-TD-set of G, we note that either $v_{5i_1} \in D$ in the case of $i_1 \geqslant 1$ or $v_6 \in D$ in the case of $i_1 = 0$. Combined with the choice of i_1, we have that $i_1 = 0$ and $v_6 \in D$. Replacing v_5 in D with v_4 produces a $\gamma_{t2}(G)$-set D' of G with $|D' \cap S| > |D \cap S|$. Since $v_6 \in D$, we note that $D' \neq S$, contradicting the choice of D. Hence, $v_{5i_1+2} \notin D$. In order to dominate v_{5i_1+2}, we have $v_{5i_1+3} \in D$. In order to dominate v_{5i_1+1}, either $v_{5i_1} \in D$ in the case of $i_1 \geqslant 1$ or $v_6 \in D$ in the case of $i_1 = 0$. Similarly, $i_1 = 0$ and $v_6 \in D$. Let $D' = (D \setminus \{v_3\}) \cup \{v_2\}$. Observe that D' is a $\gamma_{t2}(G)$-set with $|D' \cap S| > |D \cap S|$ and $D' \neq S$, a contradiction. □

Theorem 4.1. *If G is a USTD-graph of order $n \geqslant 3$ with minimum degree $\delta(G) \geqslant 2$, then $\gamma_{t2}(G) \leqslant \frac{2}{5}n$, and this bound is sharp.*

Proof. By Lemma 4.1, any graph of $\mathcal{G}_1$ can achieve the upper bound. Suppose that G is a USTD-graph of order $n \geqslant 3$ with minimum degree $\delta(G) \geqslant 2$. Let S be the unique $\gamma_{t2}(G)$-set, $\overline{S} = V(G) \setminus S$ and $D = \{v \mid v \in S$ and $|epn(v, S)| \leqslant 1\}$. We first define an edge weighting function w in G: $E[\overline{S}, S] \to [0, 1]$. For each vertex $z \in \overline{S}$, the function w assigns weight for each edge $e \in E[\{z\}, S]$ as follows:

- If z is an S-external private neighbor, then for the unique edge $e \in E[\{z\}, S]$, $w(e) = 1$.
- If z is not an S-external private neighbor and $E[\{z\}, D] = \emptyset$, then for each edge $e \in E[\{z\}, S]$, $w(e) = \frac{1}{|N_S(z)|}$.
- If z is not an S-external private neighbor and $E[\{z\}, D] \neq \emptyset$, then $w(e) = \frac{1}{|N_D(z)|}$ for each edge $e \in E[\{z\}, D]$ and $w(e) = 0$ for each edge $E[\{z\}, S \setminus D]$.

From the definition of the edge weighting function, we observe that the sum of the weights assigned to the edges joining z to S is 1. Now for any subset S_1 of S, we define a weighting function f on S_1 with $f(S_1) = \sum_{e \in E[\overline{S}, S_1]} w(e)$. Thus $f(S) = \sum_{e \in E[\overline{S}, S]} w(e) = \sum_{v \in S} f(\{v\}) = |\overline{S}| = n - |S|$. The strategy of proof is to show that each vertex of S has a weight of at least $\frac{3}{2}$. Then $\frac{3}{2}|S| \leqslant f(S) = n - |S|$. It follows that $|S| \leqslant \frac{2}{5}n$.

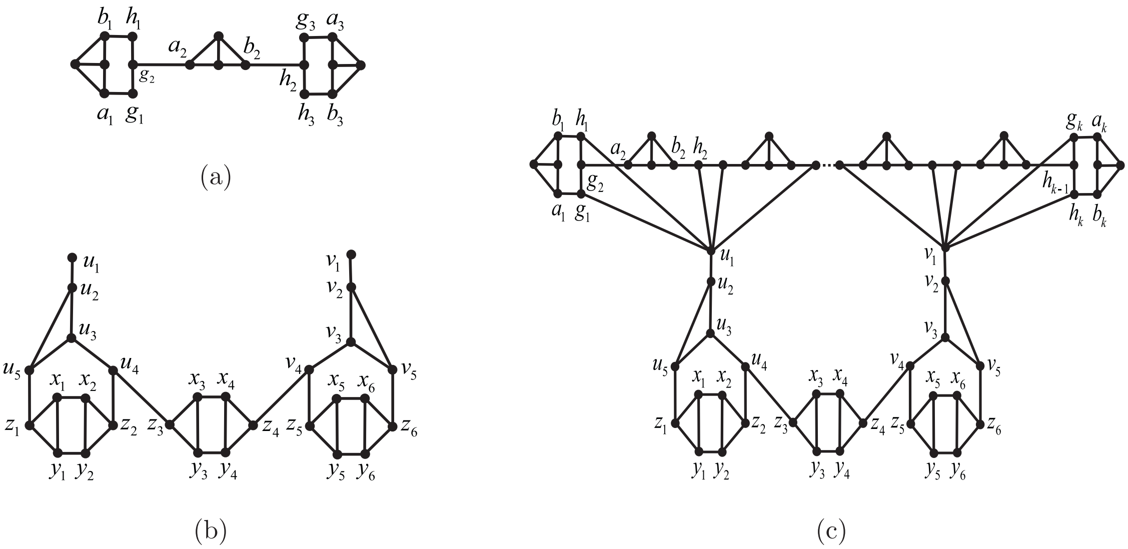

FIGURE 1. Three related graphs. (a) Graph N_3'. (b) Graph G_1. (c) A sample of the graph in $\mathcal{G}_2$.

Claim 1. For any vertex v of S, if $|epn(v,S)| = 1$, then $|ipn_2(v,S)| \geqslant 1$.

Proof. Let v be a vertex of S with $|epn(v,S)| = 1$ and $epn(v,S) = \{v'\}$. Since $\delta(G) \geqslant 2$, there exists a vertex $u' \neq v$ adjacent to v'. Then $u' \in \overline{S}$ as $\{v'\} = epn(v,S)$. Further, there exists a vertex u adjacent to u' in $S \setminus \{v\}$. Thus $d(v', S \setminus \{v\}) = 2$. This implies that v does not satisfy (b) in the statement of Lemma 2.1 but satisfies (a). It follows that $|epn(v,S)| + |ipn_2(v,S)| \geqslant 2$. Combined with $|epn(v,S)| = 1$, $|ipn_2(v,S)| \geqslant 1$. □

Consider a vertex u of S. If $|epn(u,S)| \geqslant 2$, then $f(\{u\}) \geqslant 2$, as desired. If $|epn(u,S)| = 1$, then $|ipn_2(u,S)| \geqslant 1$ by Claim 1. Let u_1 be a vertex of $ipn_2(u,S)$ and z_1 be a vertex connecting u and u_1. Then $z_1 \in \overline{S}$ and z_1 is adjacent to no vertex of $S \setminus \{u, u_1\}$. Further, $w(z_1u) \geqslant \frac{1}{2}$ and $f(\{u\}) \geqslant 1 + \frac{1}{2} = \frac{3}{2}$, as desired. Thus, we may assume that $epn(u,S) = \emptyset$. It follows that u does not satisfy (b) of Lemma 2.1. Further, u satisfies (a) and $|epn(u,S)| + |ipn_2(u,S)| \geqslant 2$. Thus $|ipn_2(u,S)| \geqslant 2$. Let u_1 and u_2 be two vertices of $ipn_2(u,S)$, let z_1 be a vertex connecting u_1 and u, and let z_2 be a vertex connecting u_2 and u. Observe that $ipn_2(u_1,S) = \emptyset$ and $ipn_2(u_2,S) = \emptyset$. By Lemma 2.1, $epn(u_1,S) \neq \emptyset$ and $epn(u_2,S) \neq \emptyset$. Combined with Claim 1, $|epn(u_1,S)| \geqslant 2$ and $|epn(u_2,S)| \geqslant 2$. Thus $w(z_1u) = 1$ and $w(z_2u) = 1$. Further, $f(\{u\}) \geqslant 2$, as desired. □

5. USTD-GRAPHS WITH MINIMUM DEGREE AT LEAST THREE

In this section, we consider USTD-graphs with minimum degree at least three. The graph obtained from K_4 by deleting an edge is called a *diamond*. Given $k \geqslant 3$ disjoint copies $D_1, \ldots, D_k$ of a diamond and $2k$ vertices $g_1, h_1, \ldots, g_k, h_k$, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and a_ib_i is the missing edge in D_i for $i \in [k]$, form the graph N_k' by adding the edges $\{a_ig_i, b_ih_i \mid i \in [k]\}$, $\{h_ig_{i+1} \mid i \in [k-1]\}$, g_1g_2 and $h_{k-1}h_k$. An example is illustrated in Figure 1a. Define the graph G_1 as shown in Figure 1b. A graph is in $\mathcal{G}_2$ if it is obtained from N_k' and G_1 by adding edges between the vertices of degree 2 in N_k' and the vertices of degree 1 in G_1 such that the resulting graph is a connected graph with minimum degree at least three. Figure 1c shows a sample of the graph in $\mathcal{G}_2$.

Lemma 5.1. *If $G \in \mathcal{G}_2$ has an order $n = 6k + 28$, where $k \geqslant 3$ is an integer, then $\gamma_{t2}(G) = 2k + 8$ and G is a USTD-graph with minimum degree $\delta(G) \geqslant 3$.*

Proof. Let $G \in \mathcal{G}_2$ and $n = 6k+28$, where $k \geqslant 3$ is an integer. Then G is constructed from N'_k and a copy H of G_1. For convenience, we use the same vertex symbols for graph H as for graph G_1 shown in Figure 1b. Observe that $\{a_1, b_1, \ldots, a_k, b_k, u_2, z_1, \ldots, z_6, v_2\}$ is a semi-TD-set of G. Thus $\gamma_{t2}(G) \leqslant 2k+8$. Let D be a $\gamma_{t2}(G)$-set. If there exists $i_0 \in [k]$ such that $|\{a_{i_0}, b_{i_0}, c_{i_0}, d_{i_0}, g_{i_0}, h_{i_0}\} \cap D| \leqslant 1$, then $\{a_{i_0}, b_{i_0}, c_{i_0}, d_{i_0}, g_{i_0}, h_{i_0}\} \cap D = \{c_{i_0}\}$ or $\{d_{i_0}\}$ in order to dominate vertices $a_{i_0}, b_{i_0}, c_{i_0}$ and d_{i_0}. Without loss of generality, consider $\{a_{i_0}, b_{i_0}, c_{i_0}, d_{i_0}, g_{i_0}, h_{i_0}\} \cap D = \{c_{i_0}\}$. However, there does not exist a vertex within distance 2 from c_{i_0} in $D \setminus \{c_{i_0}\}$, contradicting that D is a semi-TD-set of G. Hence $|\{a_i, b_i, c_i, d_i, g_i, h_i\} \cap D| \geqslant 2$ for any $i \in [k]$.

In order to dominate u_2 and u_3, we note that $N[u_2] \cap D \neq \emptyset$ and $N[u_3] \cap D \neq \emptyset$. Similarly, $N[v_2] \cap D \neq \emptyset$ and $N[v_3] \cap D \neq \emptyset$. For $i \in [6]$, let $T_i = G[\{x_i, y_i, z_i\}]$. Observe that if there exists $i_1 \in [6]$ such that $V(T_{i_1}) \cap D = \emptyset$, then $\{x_{i_2}, y_{i_2}\} \subseteq D$ in order to dominate x_{i_1} and y_{i_1}, where $i_2 \in [6] \setminus \{i_1\}$, $\{x_{i_2}\} = N(x_{i_1}) \setminus V(T_{i_1})$ and $\{y_{i_2}\} = N(y_{i_1}) \setminus V(T_{i_1})$. Thus $|(\cup_{i \in [6]} V(T_i)) \cap D| \geqslant 6$. Further, $\gamma_{t2}(G) = |D| \geqslant 2k+2+6 = 2k+8$. It follows from $\gamma_{t2}(G) \leqslant 2k+8$ that $\gamma_{t2}(G) = 2k+8$.

The above discussions imply that any $\gamma_{t2}(G)$-set D satisfies $|\{a_i, b_i, c_i, d_i, g_i, h_i\} \cap D| = 2$ for $i \in [k]$, $|N[u_2] \cap D| + |N[u_3] \cap D| = 1$, $|N[v_2] \cap D| + |N[v_3] \cap D| = 1$ and $|(\cup_{j \in [6]} V(T_i)) \cap D| = 6$. Recall that $N[u_2] \cap D \neq \emptyset$ and $N[u_3] \cap D \neq \emptyset$. Thus $N[u_2] \cap D = N[u_3] \cap D$ and $|N[u_2] \cap D| = 1$. That is, $u_1 \notin D$, $u_4 \notin D$ and further $\{u_1, u_2, u_3, u_4, u_5\} \cap D = \{u_2\}$ or $\{u_3\}$ or $\{u_5\}$. Similarly, $\{v_1, v_2, v_3, v_4, v_5\} \cap D = \{v_2\}$ or $\{v_3\}$ or $\{v_5\}$.

Let $S = \{a_1, b_1, \ldots, a_k, b_k, u_2, z_1, \ldots, z_6, v_2\}$. Suppose to the contrary that G is not a USTD-graph. Among all $\gamma_{t2}(G)$-sets that are different from S, let S_1 be chosen to have as many vertices in common with S as possible; that is, $|S_1 \cap S|$ is maximized.

Suppose $S_1 \cap V(N'_k) = \{a_1, b_1, \ldots, a_k, b_k\}$. In order to dominate u_1 and v_1, we have $\{u_1, u_2, u_3, u_4, u_5\} \cap S_1 = \{u_2\}$ and $\{v_1, v_2, v_3, v_4, v_5\} \cap S_1 = \{v_2\}$, respectively. Since S is a semi-TD-set of G, we note that $\{z_1, z_6\} \subseteq S_1$. If $z_2 \notin S_1$, then in order to dominate z_2 and u_4, there are $\{x_2, y_2\} \cap S_1 \neq \emptyset$ and $z_3 \in S_1$, respectively. Since S_1 is a semi-TD-set of G, we note that $\{x_3, y_3, x_4, y_4\} \cap S_1 \neq \emptyset$. Recall that $|(\cup_{j \in [6]} V(T_i)) \cap S_1| = 6$. In order to dominate v_4 and z_5, we have $z_5 \in S_1$. However, there does not exist a vertex within distance 2 from z_5 in $S_1 \setminus \{z_5\}$, a contradiction. Thus $z_2 \in S_1$. Similarly, $z_5 \in S_1$. Since S_1 is a semi-TD-set of G, we note that $\{z_3, z_4\} \subseteq S_1$. Now, $S_1 = \{a_1, b_1, \ldots, a_k, b_k, u_2, z_1, \ldots, z_6, v_2\} = S$, a contradiction.

Suppose $S_1 \cap V(N'_k) \neq \{a_1, b_1, \ldots, a_k, b_k\}$. Let i_3 be the minimum label of $[k]$ such that $\{a_{i_3}, b_{i_3}, c_{i_3}, d_{i_3}, g_{i_3}, h_{i_3}\} \cap S_1 \neq \{a_{i_3}, b_{i_3}\}$. If $\{c_{i_3}, d_{i_3}\} \subseteq S_1$, then $b_{i_3} \notin S_1$, otherwise $S_1 \setminus \{d_{i_3}\}$ is a semi-TD-set of G, contradicting the minimality of S_1. However, $S_2 = (S_1 \setminus \{d_{i_3}\}) \cup \{b_{i_3}\}$ is a $\gamma_{t2}(G)$-set with $|S_2 \cap S| > |S_1 \cap S|$, contradicting the choice of S_1. Thus $c_{i_3} \notin S_1$ or $d_{i_3} \notin S_1$. Renaming the vertices if necessary, we may assume that $d_{i_3} \notin S_1$. Let $X = \{a_{i_3}, b_{i_3}, c_{i_3}, d_{i_3}, g_{i_3}, h_{i_3}\}$. Then $|X \cap S_1| = 2$ and $X \cap S_1 \neq \{a_{i_3}, b_{i_3}\}$.

Assume $c_{i_3} \in S_1$. If $\{g_{i_3}, a_{i_3}\} \cap S_1 = \emptyset$, then $|\{b_{i_3}, h_{i_3}\}| \cap S_1 = 1$. In order to dominate g_{i_3}, we have $N_{S_1}(g_{i_3}) \setminus \{a_{i_3}\} \neq \emptyset$. Recall that $u_1 \notin S_1$ and $v_1 \notin S_1$. When $i_3 > 1$, $h_{i_3-1} \in S_1$, contradicting the choice of i_3. When $i_3 = 1$, $g_2 \in S_1$. However, replacing c_1 in S with a_1 produces a $\gamma_{t2}(G)$-set S_2 different from S with $|S_2 \cap S| > |S_1 \cap S|$, a contradiction. Thus $\{g_{i_3}, a_{i_3}\} \cap S_1 \neq \emptyset$. It follows that $\{b_{i_3}, h_{i_3}\} \cap S_1 = \emptyset$. In order to dominate h_{i_3}, we have $N_{S_1}(h_{i_3}) \setminus \{b_{i_3}\} \neq \emptyset$. Let $S_3 = (S_1 \setminus c_{i_3}) \cup \{b_{i_3}\}$. Then S_3 and S_1 are different. When $a_{i_3} \in S_1$ or $g_{i_3} \notin ipn_2(c_{i_3}, S_1)$, we note that S_2 is a $\gamma_{t2}(G)$-set with $|S_2 \cap S| > |S_1 \cap S|$, a contradiction. When $g_{i_3} \in ipn_2(c_{i_3}, S_1)$, we have $i_3 = 1$, otherwise, it follows from the choice of i_3 that $b_{i_3-1} \in S_1$, implying that $g_{i_3} \notin ipn_2(c_{i_3}, S_1)$, a contradiction. Further, $g_2 \in S_1$ in order to dominate h_1. Now, S_3 is a $\gamma_{t2}(G)$-set with $|S_3 \cap S| > |S_1 \cap S|$, a contradiction.

Next assume $c_{i_3} \notin S_1$. If $a_{i_3} \in S_1$, then $X \cap S_1 = \{a_{i_3}, h_{i_3}\}$ in order to dominate b_{i_3}. When $i_3 > 1$, $h_{i_3-1} \in S_1$ as S_1 is a semi-TD-set of G and $\{u_1, v_1\} \cap S_1 = \emptyset$, contradicting the choice of i_3. When $i_3 = 1$, $g_2 \in S_1$. Further, $S_3 = (S_1 \setminus \{h_1\}) \cup \{b_1\}$ is a $\gamma_{t2}(G)$-set different from S with $|S_3 \cap S| > |S_1 \cap S|$, a contradiction. Thus $a_{i_3} \notin S_1$. In order to dominate a_{i_3}, c_{i_3} and d_{i_3}, we have $X \cap S_1 = \{g_{i_3}, b_{i_3}\}$.

Since S_1 is semi-TD-set of G and $\{u_1, v_1\} \cap S_1 = \emptyset$, either $g_{i_3+1} \in S_1$ in the case of $i_3 < k$ or $h_{k-1} \in S_1$ in the case of $i_3 = k$. Combined with the choice of i_3, we note that $i_3 < k$ and $g_{i_3+1} \in S_1$. If $\{c_{i_3+1}, d_{i_3+1}\} \cap S_1 \neq \emptyset$, then without loss of generality, consider $c_{i_3+1} \in S_1$. Thus $\{a_{i_3+1}, b_{i_3+1}, c_{i_3+1}, d_{i_3+1}, g_{i_3+1}, h_{i_3+1}\} \cap S_1 = \{g_{i_3+1}, c_{i_3+1}\}$. In order to dominate h_{i_3+1}, we have $N_{S_1}(h_{i_3+1}) \setminus \{b_{i_3+1}\} \neq \emptyset$. Replacing c_{i_3+1} in S_1 with b_{i_3+1} produces a $\gamma_{t2}(G)$-set S_4 different from S with $|S_4 \cap S| > |S_1 \cap S|$, a contradiction. Hence, $\{c_{i_3+1}, d_{i_3+1}\} \cap S_1 = \emptyset$. We note that

$b_{i_3+1} \in S_1$ in order to dominate c_{i_3+1}, d_{i_3+1} and b_{i_3+1}. That is $\{a_{i_3+1}, b_{i_3+1}, c_{i_3+1}, d_{i_3+1}, g_{i_3+1}, h_{i_3+1}\} \cap S_1 = \{g_{i_3+1}, b_{i_3+1}\}$. Continue this process, we can obtain that $\{a_i, b_i, c_i, d_i, g_i, h_i\} \cap S_1 = \{g_i, b_i\}$ for any $i \in \{i_3, \ldots, k\}$. However, there dose not exist vertices within distance 2 from b_k in $S_1 \setminus \{b_k\}$, a contradiction. □

Observation 5.1. If $G \in \mathcal{G}_2$ has an order n, then $lim_{n\to\infty} \frac{\gamma_{t2}(G)}{n} = \frac{1}{3}$.

Theorem 5.1. *If G is a USTD-graph of order $n \geqslant 3$ with minimum degree $\delta(G) \geqslant 3$, then $\gamma_{t2}(G) \leqslant \frac{1}{3}n$.*

Proof. Let G be a USTD-graph of order $n \geqslant 3$ with minimum degree $\delta(G) \geqslant 3$ and S be the unique $\gamma_{t2}(G)$-set. We define two weighting functions just like we did in Theorem 4.1. The difference from the previous proof is that the strategy of this proof is to show that each vertex of S has a weight of at least 2. Further $2|S| \leqslant n - |S|$ and then $|S| \leqslant \frac{1}{3}n$.

Similarly, we can obtain that Claim 1 and $f(\{u\}) \geqslant 2$ for a vertex u of S with $|epn(u, S)| \geqslant 2$ or $|epn(u, S)| = 0$. It remains to consider the case when $|epn(u, S)| = 1$. Combined with Claim 1, $|ipn_2(u, S)| \geqslant 1$. Let u_1 be a vertex of $ipn_2(u, S)$ and $\{z_1\} = epn(u, S)$. Let z_2 be a vertex connecting u and u_1. Then $z_2 \in \overline{S}$ and z_2 is adjacent to no vertex of $S \setminus \{u, u_1\}$. Recall that $D = \{v \mid v \in S$ and $|epn(v, S)| \leqslant 1\}$. Clearly, $u \in D$. If $u_1 \notin D$, then $w(z_2u) = 1$ by the definition of edge weighting functions. Further $f(\{u\}) \geqslant w(z_1u) + w(z_2u) = 2$, as desired. Thus we may assume that $u_1 \in D$ and then $|epn(u_1, S)| \leqslant 1$. Since $\delta(G) \geqslant 3$, we note that $N(u_1) \setminus (\{z_2\} \cup epn(u_1, S)) \neq \emptyset$. Let z_3 be a vertex in $N(u_1) \setminus (\{z_2\} \cup epn(u_1, S))$. Then $z_3 \in \overline{S}$ and $uz_3 \in E(G)$. Hence $f(\{u\}) \geqslant w(z_1u) + w(z_2u) + w(z_3u) = 1 + \frac{1}{2} + \frac{1}{2} = 2$, as desired. □

Remark. To show that the bound in Theorem 5.1 is sharp, we need to find a graph G of order n such that $\frac{\gamma_{t2}(G)}{n} = \frac{1}{3}$. So far, we have not found such a graph. However, we can show that the bound in Theorem 5.1 is sharp if n is sufficiently large (see Observation 5.1).

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Data availability statement

The research data associated with this article are included in the article.

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