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NEXT-GENERATION INVENTORY OPTIMIZATION: ADVANCED INVENTORY MANAGEMENT HARNESSING DEMAND VARIABILITY INTEGRATING FUZZY LOGIC AND GRANULAR DIFFERENTIABILITY

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Abstract. Using fuzzy logic, the paper proposes a special inventory control method, considering the complexity and uncertainty of real supply chain management in a single time frame. The fundamental novelty is the acceptance of a non-linear demand function considering several important variables, including product quality, price, and supply levels. Combining these elements helps the model better explain the complex demand dynamics resulting from initial demand patterns, logistical operations, unpredictability in advertising rates, stock levels, selling prices, and product quality. By using its approaches to maximize inventory levels and price decisions in the face of uncertainty, this model could help to improve inventory management activities by offering a more complete knowledge of demand patterns. Given the complexity of current supply chains, it emphasizes the need to consider inventory management issues outside of traditional linear demand models. Defuzzing will ensure that the model is successful and stable. Our research shows that granular differentiability should be integrated with defuzzification. Granular differentiability generates fuzzy derivatives based on horizontal membership functions, thereby adding a new dimension. Our study is noteworthy for being the first to apply the granular differentiation approach to production inventory systems. Our work addresses both analytical methods and numerical simulations to maximize loosely defined controls using granular differentiation. Using this special technique, we want to increase our knowledge of production inventory systems that operate in fuzzy environments as well as our optimization strategies. This work clarifies how uncertainty affects decision-making procedures in such systems, therefore providing useful information for the field.

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1. Introduction

In the field of organizational management, a basic obstacle arises in the appropriate control of production inventories. This calls for the creation of mathematical models and the application of the best inventory control techniques. Various inventory control models exist, each tailored to the specific needs and constraints of different companies or sectors. These models center on basic characteristics such as stock rates, demand rates, and

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production rates; nevertheless, because of natural uncertainty and unpredictability, they may not always fit exactly the actual situations.

Normal inventory control systems fall apart, especially in cases of notable changes in product sales seasons. This sometimes results in surplus stockpiles at the end of a season, which makes them unable to meet the needs of the next season and forces the use of single-period models. Still, obtaining actual input values for the mathematical inventory model parameters seems to be a big task.

Traditionally, inventory control theories have largely concentrated on deterministic approaches, which might not be able to regulate the complexity of actual surroundings. Because they allow uncertainty inside a more flexible framework, fuzzy models have gained some popularity in the field of production inventories. Two main sources of uncertainty are included by production inventory models: first, demand variations and accompanying knock-on effects on suppliers; and second, manufacturing process-related uncertainty, including component quality control. This opens a fresh path in production theory, mostly in terms of character, given the surprising nature of model formation. Measurement and proof of properties including stability, controllability, and observability inside complex systems depend on sophisticated equations as fundamental mathematical instruments.

Both regular and nonlinear differential equation analysis emphasizes the need of knowing how dynamically systems react. Industrial inventory control systems have attracted a lot of attention recently, particularly modeling both linear and nonlinear systems with fixed parameters and variables.

This ignores the volatility in actual markets, with different demand and changing pricing policies in the meantime. This approach could produce erroneous findings and overlook the evolving aspects of supply chain management. Stochastic modeling – also called fuzzy logic – allows one to include uncertainty and unpredictability into systems of inventory control.

These models become more flexible and sensitive to changes in the market, which at last helps supply lines run more efficiently. In real-world systems, spontaneous volatility makes it hard to make exact forecasts about the properties and variables of dynamical systems. Therefore, it would not be sufficient to show the complexity of political economics based just on linear mathematical models. This is thus because linear models reduce complex system dynamics, so influencing knowledge and prediction capacity for actual events. By using more flexible modeling techniques able to better manage the nonlinear and unpredictable character of political economy, we can overcome this limitation and increase our capacity to understand and control actual events. Real-world events demonstrate the requirement of a more acceptable, nonlinear model representation including declining marginal utility, lowering of marginal returns for enterprises, demand uncertainty, supply changes in reaction to price fluctuations, and swings in inflation rates. Including these features in uncertain linear and nonlinear dynamical systems necessitates nonlinear modeling techniques to improve the efficiency and realism of the proposed models. This approach improves knowledge of policy development and decision-making in manufacturing inventory control by providing a completer and more accurate image of the many linkages in real-world systems.

The objective of the research article is to:

Create a thorough inventory control model that uses fuzzy logic to take into consideration supply chains' intrinsic complexity and unpredictability as well as factors like product quality, pricing strategies, supply levels, and demand uncertainty.

Add granular differentiability to models of inventory management, offering a fresh method of obtaining fuzzy derivatives *via* horizontal membership functions to enhance the ability to make decisions in the face of uncertainty.

To show how granular differentiability-based inventory models perform better than current fuzzy logic and graded differentiability models in controlling erratic demand and supply dynamics, compare their performance.

Examine how important variables affect the demand function, showing how these interact in non-linear and uncertain contexts. Examples of these variables include pricing, product quality, stock levels, and advertising.

Provide a defuzzification strategy that makes the model stable and robust, enabling more accurate optimization of inventory levels and pricing choices in erratic markets.

Use the suggested model in simulation and analytical settings to assess its effectiveness and show how applicable it is to actual production inventory systems, with the ultimate goal being to maximize inventory choices in the face of uncertainty.

One may use the smartphone business as a real-world example to show how the suggested demand model is dependent on time, inventory levels, selling price, and product quality, all of which have a big impact on demand dynamics.

2. Literature review

Research on complex systems displaying nonlinearity and fuzziness – known as FDSs – has gotten substantial attention in various fields, including health, finance, fractional calculus, control theory, and biomathematics. We review nonlinear fuzzy differential equations (FDEs). Notably contributing to this topic are Esmi *et al.* [5], Mazandarani and Xiu [13], Van Nghia *et al.* [32], Ochuba *et al.* [21], and Helmes *et al.* [7]. While the best possible limitation model based on FDSs is clearly relevant for manufacturing supply chain issues, Piegat and Landowski [23] have hardly investigated this field.

Most previous research in fuzzy optimal control has focused on linear dynamical systems defined by fuzzy parameters and unknown starting conditions [14–16]. The researchers have applied the idea of graded differentiability, which consists of various intrinsic restrictions like the potential for the non-existence of solutions, the need for monotonic uncertainty, and the generation of solutions at several degrees instead of exact ones [9]. Inspired by the horizontal membership functions of Mazandarani and Ma [12] and Najariyan and Farahi [14,17,18], the idea of granular differentiability has evolved to help address these restrictions. This method offers solutions outside the boundaries of earlier methods and shows enormous promise in control systems [6,14,24]. Furthermore, while addressing unknown variables requires fuzzy arithmetic, conventional vertical fuzzy sets are prone to constraints and may cause paradoxes [4,6,23]. Presenting more exact solutions than traditional interval mathematics, multidimensional comparison distance-determining interval arithmetic has been proposed to solve these challenges [24,31,33]. In the realm of manufacturing inventory systems, demand is a crucial factor, usually demonstrating nonlinear tendencies [2,10,25]. Recent studies of time-varying demand patterns have given inventory control models exponentially time-dependent demand [2,6,8,10]. Furthermore, elements like advertising, stock availability, product quality, and selling price influence the demand function [26].

In response to globalization and changing market dynamics, industrial companies are prioritizing product innovation techniques [11,21,27,30]. These methods cover creating new products, upgrading existing ones, and providing fresh features for aging ones [1,3,22,34]. In today's global economy, product innovation is clearly important for maintaining competitiveness and satisfying the growing needs of consumers.

We can underline important variations between current work and previous works, especially in areas like fuzzy modelling, granular differentiability, demand functions, and inventory control under uncertainty, so contrasting this study with closely related publications in the literature review and justify its novelty. The organized comparison of closely comparable studies is offered in the following table:

This study's use of granular differentiability, which provides more precise control and flexibility than previous systems based on graded differentiability or conventional fuzzy arithmetic, is its most noteworthy novelty. This work is the first to apply granular differentiation in the context of inventory control, although previous research has used graded differentiability. A more thorough understanding of demand variability is provided by this research, which broadens the demand modelling framework by including variables like stock levels, price, and product quality. Many previous studies have concentrated on time-dependent or exponentially time-dependent demand functions. In contrast to traditional fuzzy models, which concentrate on vertical fuzzy sets, this study makes use of horizontal membership functions to improve the defuzzification process' accuracy. This makes it possible for the model to manage uncertainty in the actual world more skilfully, especially in intricate supply chains with numerous sources of unpredictability. By contrasting the aforementioned works, it is evident that this study represents a breakthrough in the field of fuzzy logic-based inventory control, bringing innovative methods such granular differentiability to address the shortcomings of earlier studies [19,20,29].

Author(s)	Focus of Study	Methodology	Key Contribution	How Current Study Contrasts
Najariyan *et al.* [14]	Fuzzy parameters and uncertain initial conditions in control systems	Fuzzy arithmetic	Handled initial condition uncertainty in fuzzy systems	Our approach leverages granular differentiation to overcome constraints seen in conventional vertical fuzzy sets
Shah and Oaghela [25]	Nonlinear demand modeling in inventory systems	Nonlinear demand function	Modeled time-varying demand patterns	We include additional factors like advertising, product quality, and price sensitivity in demand modeling
Khatua *et al.* [9]	Fuzzy optimal control in production systems	Fuzzy logic with monotonic uncertainty	Highlighted the limitations of monotonic uncertainty	Our model applies granular differentiation, enhancing flexibility in uncertain demand and supply chain control
Mazandarani *et al.* [13]	Fuzzy differential equations using graded differentiability	Graded differentiability	Applied graded differentiability to FDEs	Current study moves beyond graded differentiability by introducing granular differentiation, improving precision in control
Piegat *et al.* [23]	Fuzzy modeling for supply chains	Vertical membership functions	Addressed some limitations of fuzzy sets in control	This study introduces horizontal membership functions and defuzzification for better handling of fuzzy derivatives
	Fuzzy differential systems (FDSs) for uncertain environments	Nonlinear FDEs	Proposed FDEs for inventory control under uncertainty	This study integrates granular differentiability and introduces nonlinear demand functions affected by various variables
Alam *et al.* [2]	Time-dependent demand modeling for inventory systems	Exponentially time-dependent demand	Extended time-varying demand for supply chain control	Our study adds fuzzy parameters and granular differentiation to enhance handling of uncertainty in real-world scenarios
Ochuba *et al.* [21]	Product innovation strategies in response to market changes	Dynamic fuzzy modeling	Proposed dynamic fuzzy models for product innovation	Our study applies fuzzy logic in supply chain optimization with granular differentiation to address broader uncertainties

3. Assumptions

The model describes in this paper has the following key characteristics:

(a) It operates within a single cycle with a restricted time limit, ensuring concise management.
(b) The model enforces no shortages, guaranteeing adequate inventory to fulfill demand.
(c) Demand is influenced by temporal considerations, existing supply levels, selling prices, and product quality.
(d) The advertisement rate is maintained at a consistent level during the entire duration for the sake of simplicity.
(e) Fuzzy parameters, depicted as triangular fuzzy numbers, accommodate uncertainty in input variables.
(f) The model assumes negligible inflation, streamlining the economic framework.
(g) Furthermore, co-state variables are handled as fuzzy entities in the model.

4. Model formulation

4.1. Assumptions

(a) The model operates within a single finite time period.
(b) Shortages are strictly prohibited.
(c) Demand is contingent upon time, inventory levels, selling price, and product quality.
(d) The advertising rate remains constant.
(e) Fuzzy parameters are represented by triangular fuzzy numbers to capture uncertainty.
(f) Negligible inflation is assumed.
(g) Co-state variable quantity is treated as fuzzy variable quantity in the model.

4.2. Symbolization

4.2.1. Constraints

$\tilde{P}(t)$	Manufacture cost, a control variable determined on interval t.
$\tilde{I}(t)$	Uncertain initial investment in product innovation: an extra variable that varies over time.
$\tilde{X}(t)$	Fuzzy stock rate, a state variable dependent on the temporal parameter t.
$\tilde{Q}(t)$	Fuzzy product quality, another time-dependent state variable on time t.
$\tilde{D}(t)$	Fuzzy product demand, a time-dependent random function on time t.
HMFs	Horizontal membership functions.

4.2.2. Parameters

In this model, the following fuzzy parameters are defined:

$\tilde{d_0}(t)$	Fuzzy coefficient of initial demand.
$\tilde{\theta}(t)$	Fuzzy production defect rate.
$\tilde{\gamma}(t)$	Fuzzy rate of imperfection.
$\tilde{\vartheta}(t)$	Fuzzy advertising frequency.
$\tilde{K}(t)$	Fixed Positive value.
$\tilde{\rho}(t)$	Positive constant, with ρ falling inside the interval $[0, 1]$.
$\tilde{S}(t)$	Variable selling price with fuzziness.
$\tilde{\eta}(t)$	Fuzzy quality deterioration rate.
$\tilde{C_h}(t)$	Variable holding rate/entity of supply with fuzziness.
$\tilde{C_p}(t)$	Variable manufacture price/entity with fuzziness.
$\tilde{C_l}(t)$	Variable investment cost for product innovation with fuzziness.
$\tilde{C_v}(t)$	Variable advertising cost per advertisement unit with fuzziness.
$\tilde{S_f}(t)$	Resale value.

4.3. Fuzzy model framework

Constructing a production inventory model involves a detailed analysis of product demand, a vital factor in the dynamics of any production system. However, establishing the demand function is typically problematic due to the multidimensional nature of real-world events. Unlike linear trends, the demand for products changes nonlinearly, impacted by factors such as current supply levels, selling prices, and product quality. Moreover, amidst the intricacy of the current market, uncertainties develop in numerous forms, encompassing both randomness and fuzziness. Consequently, the variables that support the demand function are inherently ambiguous, rendering the demand function itself inherently unpredictable. Considering these complications, we formulate the demand function as follows:

$$\tilde{D}(t) = \frac{\tilde{d}_0 e^t}{1+e^t} + \frac{\tilde{v}}{1+\tilde{v}}\tilde{X}(t) + k\tilde{S}^{-\rho}\tilde{Q}(t). \tag{1}$$

The supply chain deterioration function acts as the basic component in our demand model, offering a realistic picture of market dynamics. We accept the considerable importance of product promotion, with specific emphasis on the advertising rate. In today's landscape, both online and offline media outlets, including newspapers and social media, play significant roles in product promotion, often defining the success of a new product introduction. Employing creative display tactics in showrooms further boosts product exposure and consumer engagement.

Extensive research into consumer psychology highlights the favorable correlation between displayed stock levels, advertising activities, and sales rates. Additionally, we add the impact of selling price and product quality to our demand function. Consumer tastes differ, with some emphasizing quality and being prepared to pay premium pricing, while others desire quality without major financial expenditure. To account for this variability, we add a reversible selling price, represented by a small Positive constant ρ. Within our suggested production system, the incidence of defective items is inherent. We model the production process to generate defective items at a constant rate. Notably, our method allows for the recycling of some damaged parts, hence altering the total stock level dynamics.

$$\dot{\tilde{X}}(t) = \Big(1-(1-\tilde{\theta})\Big)\tilde{\gamma}P(t) - \tilde{D}(t). \tag{2}$$

Integrating the rate of stock alteration, we define the defective rate as a fuzzy term that represents the proportion of products that do not pass quality testing. The faulty rate is computed using the formula: The faulty rate is calculated by multiplying the number of flaws per output tested by 100. As the manufacturing price of goods varies over time, so does the faulty rate, reflecting fluctuations in production efficiency. Our purpose is to optimize the salvage price.

$$\tilde{Q}(t) = \tilde{I}(t) - \tilde{\eta}\tilde{Q}(t). \tag{3}$$

The objective function is formulated as:

$$\Big(\tilde{X}(t), \tilde{Q}(t), \tilde{P}(t), \tilde{I}(t), t\Big) = \int_0^t \Big(\tilde{S}\tilde{D}(t) - \tilde{C}_h\tilde{X}(t) - \tilde{C}_p P(t)^2 - \tilde{C}_i I(t)^2 - \tilde{C}_v\tilde{v}\Big)\,\mathrm{d}t + \tilde{S}_f\tilde{X}(t). \tag{4}$$

The above formulation for the objective function should be supremum. The restrictions considered as:

$$\begin{cases} \tilde{X}(t) = \Big[1-(1-\tilde{\theta})\Big]\tilde{\gamma}P(t) - \tilde{D}(t) \\ \tilde{Q}(t) = \tilde{I}(t) - \tilde{\eta}\tilde{Q}(t) \end{cases} \tag{5}$$

where the product's inventory level is $\dot{\tilde{X}}(t)$. Furthermore, the product's quality $\tilde{Q}(t)$ exist and serve as constant variables, while rate of producing goods $P(t)$ and the immediate investment in new product development $\tilde{I}(t)$ act as control variables. In this formulation, the inventory of items X, denoted as $\dot{\tilde{X}}(t)$ Furthermore, the product's quality Q, denoted as $\tilde{Q}(t)$, serve as fixed variables. For the moment, the rate of manufacturing inventories $P(t)$ and the instantaneous stock in developed product $\tilde{I}(t)$ are described as limited variables.

5. Stability analysis

Stability analysis is particularly beneficial for industrial inventory control systems because it ensures operational dependability and performance. Stability is the condition whereby the system settles within a certain range or returns to its beginning condition following outside disturbances or changes in input. In an invariant control system, stability must meet two criteria:

The mechanism produces a limited output for every limited input as well. When there is no input, the output remains 0. Stability analysis is more important for systems running across infinite time horizons than for finite periods, where continuous stability is not a major concern. Still, comparing several approaches to evaluating stability in fuzzy dynamical systems (FDS) offers an insightful analysis of their effectiveness.

Various approaches exist in the literature for analyzing the stability of production inventory models, each offering distinct advantages and limitations. To assess the stability of the model, we examine equations (12) and (13), which may be communicated as:

$$\tilde{X}(t) = \begin{cases} -\frac{\tilde{v}}{1+\tilde{v}}\tilde{X}(t) - kS^{-\rho}\tilde{Q}(t) \\ \tilde{Q}(t) = -\eta\tilde{Q}(t). \end{cases} \tag{6}$$

Rewriting equation (6) in the form $\dot{\tilde{X}}(t) = \tilde{A}\tilde{x}(t)$ and $\tilde{A}$ is given by

$$\begin{pmatrix} -\frac{\tilde{v}}{1+\tilde{v}} & kS^{-\rho} \\ 0 & -\tilde{\eta} \end{pmatrix}. \tag{7}$$

The stochastic $\tilde{A}$ is related to the subsequent crisp base by using fuzzy standard interval arithmetic and the symmetrically derivative of interval valued function in the original form:

$$\begin{pmatrix} 0 & -\frac{\overline{v}}{1+\overline{v}} & 0 & -k\overline{S}^{-\rho} \\ -\frac{\underline{v}}{1+\underline{v}} & 0 & -k\underline{S}^{-\rho} & 0 \\ 0 & 0 & 0 & \overline{\eta} \\ 0 & 0 & \underline{\eta} & 0 \end{pmatrix}. \tag{8}$$

Numerical values of the parameters are given as: $\underline{v} = 2, \overline{v} = 5.01, \underline{S} = 39.99, \overline{S} = 65, \eta = 0.099, \overline{\eta} = 0.169, k = 0.701, \rho = 0.59$, we construct the eigenvalues of dynamical system (15) as follows: λ_1 is 1.735, λ_2 is -1.735, λ_3 is 1.061, λ_4 is -1.058. Hence, dynamical structure (15) is deemed variable. The system curve is depicted in Figure 1. Employing the symmetrical derivative of interval valued function techniques, here we can see fuzzy dynamical system (15) mapped onto the crisp dynamical system:

$$\begin{pmatrix} \dot{X}^{gr}(t,\mu,\alpha_X) \\ \dot{Q}^{gr}(t,\mu,\alpha_Q) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{v^{gr}(\mu,\alpha_v)}{1+v^{gr}(\mu,\alpha_v)}kS^{gr}(\mu,\alpha_S)^{-\rho} \\ 0 & -\eta^{gr}(\mu,\alpha_v) \end{pmatrix} \begin{pmatrix} X^{gr}(t,\mu,\alpha_X) \\ Q^{gr}(t,\mu,\alpha_Q) \end{pmatrix}. \tag{9}$$

Since the eigenvalues of the uncontrollable part are:

$$-\frac{v^{gI}(\mu,\alpha_v)}{1+v^{gI}(\mu,\alpha_v)} \quad \text{and} \quad -\eta^{gr}(\mu,\alpha_\eta). \tag{10}$$

The unmanageability aspect is consistent. The granular dynamical system (18) can be stabilized. The supply point and the superiority of the manufactured goods are demonstrated in Figures 2 and 3, respectively.

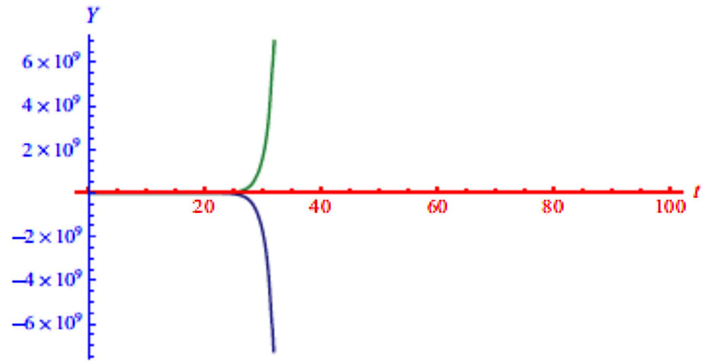

FIGURE 1. The unstable fuzzy system.

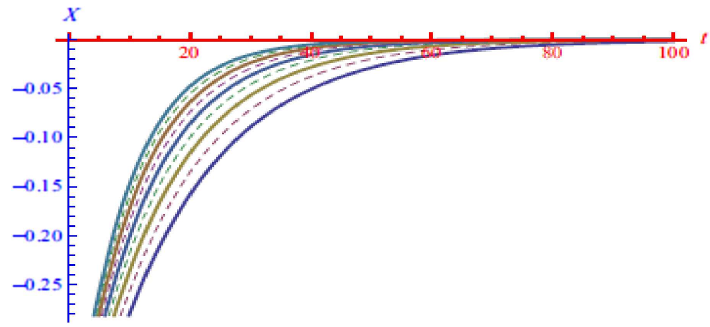

FIGURE 2. The stable μ-level set of the stock rate.

6. Optimality tests

The stochastic Hamiltonian objective is given as follows:

$$H\Big(\tilde{X}(t),\tilde{Q}(t),P(t),\tilde{I}(t)\Big)=\tilde{S}\Big(\tilde{d}_0\frac{e^t}{1+e^t}+\frac{\tilde{v}}{1+\tilde{v}}\tilde{X}(t)+k\tilde{S}^{\alpha}\tilde{Q}(t)\Big)-\tilde{C}_h\tilde{X}(t)-\tilde{C}_pP(t)^2-\tilde{C}_i\tilde{I}(t)^2-\tilde{C}_v\tilde{v}. \quad (11)$$

Using the conclusions achieved by Shin *et al.* [28] and Abdali *et al.* [1] as a background, the fuzzy operate technique.

Based on symmetrically derivative of interval valued function, the objective function can be repented as following:

$$\begin{aligned} h\big(x^{*\mathrm{grd}},Q^{*\mathrm{grd}},p^*,i^{*\mathrm{grd}}\big)=S^{\mathrm{gr}}\Big(\frac{d_0^{\mathrm{gr}}e^t}{1+e^t}+\frac{v^{\mathrm{gr}}}{1+v^{\mathrm{gr}}}X^{*\mathrm{gr}}+kS^{\mathrm{gr}\,\rho}Q^{*\mathrm{gr}}\Big)\\ -C_h^{\mathrm{gr}}X^{*\mathrm{gr}}-C_p^{\mathrm{gr}}P(t)^2-C_i^{\mathrm{gr}}I^{*\mathrm{gr}2}-C_v^{\mathrm{gr}}v^{\mathrm{gr}}. \end{aligned} \quad (12)$$

So, afterwards implementing the technique of differentiability on the obtained matrix operate, equation (13) is transformed to calculation which is like recognized as crisp equation (13) and same to resolve the coordination conditions and fuzzy dominate functions. So, for evaluating the restricted varying value, we get,

$$\begin{cases} \dfrac{agfH}{3P}=0\\ \dfrac{agfH}{3T^{*\varrho r}}=0 \end{cases}$$

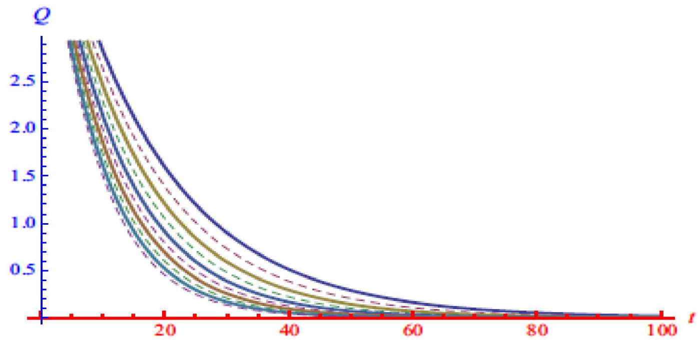

FIGURE 3. The stable μ-level set of the quality of the product.

$$\rightarrow\begin{cases} P^{*gf}(t,\mu,\alpha_p) = \dfrac{(1-(1-\theta)^{gf}(\mu,\alpha_b))\gamma^{gf}(\mu,\alpha_\gamma)}{2C_p^{gf}}\lambda_1^{*gf}(t,\mu,\alpha_{\lambda_1}) \\ I^{*gf}(t,\mu,\alpha_I) = \dfrac{1}{2C^{gf}(\mu,\alpha_{C_l})}\lambda_2^{*gf}(t,\mu,\alpha_{\lambda_2}). \end{cases} \tag{13}$$

For Lagrange's multiplier, we have,

$$\begin{cases} \dot{\lambda}_1^{*\mathrm{gr}}(t,\mu,\alpha_{\lambda_1}) = -\dfrac{\partial^{\mathrm{gr}}H}{\partial X^{*\mathrm{gr}}} \\ \dot{\lambda}_2^{*\mathrm{gr}}(t,\mu,\alpha_{\lambda_2}) = -\dfrac{\partial^{\mathrm{gr}}H}{\partial Q^{*\mathrm{gr}}} \end{cases} \tag{14}$$

$$\rightarrow\begin{cases} \dot{\lambda}_1^{*\mathrm{gr}}(t,\mu,\alpha_{\lambda_1}) = -S_f\dfrac{\vartheta_f(\mu,\alpha_v)}{1+\vartheta_f^{\mathrm{gf}}(\mu,\alpha_v)} + C_h^{\mathrm{gf}}(\mu,\alpha_{C_h}) - \dfrac{\vartheta_f(\mu,\alpha_0)}{1+\vartheta_f^{\mathrm{gf}}(\mu,\alpha_0)}\lambda_1^{(*\mathrm{gf})}(t,\mu,\alpha_{\lambda_1}) \\ \dot{\lambda}_2^{*\mathrm{gr}}(t,\mu,\alpha_{\lambda_2}) = -kS_f^{(1-\rho)}(\mu,\alpha_S) + kS^{\mathrm{gf}-\rho}(\mu,\alpha_S)\lambda_1^{*\mathrm{gf}}(t,\mu,\alpha_{\lambda_1}) - \eta^{\mathrm{gf}}(\mu,\alpha_\eta)\lambda_2^{*\mathrm{gf}}(t,\mu,\alpha_{\lambda_2}). \end{cases} \tag{15}$$

Solving equation (15), we get,

$$\lambda_1^{(*\mathrm{gr})}(t,\mu,\alpha_{\lambda_1}) = (S^{\mathrm{gr}}(\mu,\alpha_S) - C_h^{\mathrm{gr}}(\mu,\alpha_{C_h}))\left(\frac{1+v^{\mathrm{gr}}(\mu,\alpha_v)}{v^{\mathrm{gr}}(\mu,\alpha_v)}\right)\exp\left(\frac{1+v^{\mathrm{gr}}(\mu,\alpha_v)}{v^{\mathrm{gr}}(\mu,\alpha_v)}(T-t)\right) - 1 \tag{16}$$

and

$$\begin{aligned} \lambda_2^{*\mathrm{gf}}(t,\mu,\alpha_{\lambda_2}) = &-\Bigg(\frac{kS^{(1-\rho)gr}(\mu,\alpha s)}{\eta \mathrm{gr}(\mu,\alpha_\eta)} - kS^{(-\rho)gr}(\mu,\alpha s)\Bigg(S^{\mathrm{gr}}(\mu,\alpha_s) - C_h^{\mathrm{gr}}(\mu,\alpha_{Ch}) \\ &\times\frac{1+v^{\mathrm{gr}}(\mu,\alpha_v)}{v^{\mathrm{gr}}(\mu,\alpha_v)}\Bigg)\left(\frac{\exp\left(\frac{1+v^{\mathrm{gr}}(\mu,\alpha_b)}{v^{\mathrm{gr}}(\mu,\alpha_b)}(T-t)\right)}{\eta^{\mathrm{gr}}(\mu,\alpha_\eta) - \frac{v^{\mathrm{gr}}(\mu,\alpha_b)}{\Gamma+v^{\mathrm{gr}}(\mu,\alpha_b)}} - \frac{1}{\eta^{\mathrm{gr}}(\mu,\alpha_\eta)}\right)\Bigg)\Bigg)\left(1-e^{\eta(T-t)}\right). \end{aligned} \tag{17}$$

Shadow tolls provide a solution for countries experiencing supply limits, growing market pricing, and foreign currency shortages. Through the implementation of economic, financial, and additional governmental strategies, the market costs of commodities, materials, and foreign currency may be matched with their shadow prices, supporting effective investment planning. Shadow pricing, as established by J. Tinbergen, indicates the inherent or genuine worth of a product or factor, typically varying between time periods, geographic areas, and job types.

TABLE 1. Index-I: Statistical information for the fuzzy constraints.

Constraints	Fuzzy estimates	HMFs
$\overline{C}_k$	$(2.01, 2.499, 2.99)$	$C_h^{zr} = 2.01 + \mu + 0.61(0.99 - \mu)\alpha_{c_4}$
c_P	$(5, 7, 7.2)$	$c_p^{ar} = 5 + 7\mu + 0.2(1 - \mu)\alpha_{c_p}$
$\tilde{c}_i$	$(2, 2.1, 4.1)$	$c_i^{gr} = 2 + 0.1\mu + 2.1(1 - \mu)\alpha_{c_i}$
s	$(40, 50, 65)$	$S8r = 40 + 10\mu + 15(1 - \mu)\alpha_S$
s_f	$(20, 25, 26)$	$S_f^{\theta r} = 20 + 5\mu + (1 - \mu)\alpha_{s_f}$
$\ddot{\theta}$	$(0.1, 0.3, 0.4)$	$\theta^{zf} = 0.1 + 0.2\mu + 0.1(1 - \mu)\alpha_\theta$
$\overline{\gamma}$	$(0.6, 0.61, 0.66)$	$\gamma^{gr} = 0.6 + 0.01\mu + 0.05(1 - \mu)\alpha_\gamma$
$\overline{n}$	$(0.02, 0.12, 0.17)$	$\eta^{2r} = 0.02 + 0.01\mu + 0.05(1 - \mu)\alpha_\eta$
$\hat{v}$	$(2, 4, 4, 3)$	$v^{sr} = 2 + 2\mu + 0.3(1 - \mu)\alpha_b$
i_0	$(0.15, 0.55, 0.75)$	$d_0^{\mathrm{gr}} = 0.15 + 0.4\mu + 0.3(1 - \mu)\alpha_{d0}$

These shadow prices, represented by $\lambda_1(t)$, serve as a critical feature in strengthening the realism of models. When it comes to state variables,

$$\begin{cases} \dot{X}^{*\mathrm{gr}}(t, \mu, \alpha_X) = -\dfrac{\partial^{\mathrm{gr}} H}{\partial \lambda_1^{*\mathrm{gr}}} \\ \dot{Q}^{*\mathrm{gr}}(t, \mu, \alpha_Q) = -\dfrac{\partial^{\mathrm{gr}} H}{\partial \lambda_2^{*\mathrm{gr}}} \end{cases} \tag{18}$$

$$\rightarrow \begin{cases} \dot{X}^{*\mathrm{gr}}(t, \mu, \alpha_X) = (1 - (1 - \theta^{\mathrm{gr}}(\mu, \alpha_\theta)))\gamma^{\mathrm{gr}}(\mu, \alpha_\gamma) p^{*\mathrm{grd}}(t, \mu, \alpha_P) - D^{*\mathrm{gr}}(t, \mu, \alpha_D) \\ q^{*\mathrm{grd}}(t, \mu, \alpha_Q) = I^{*\mathrm{gr}}(t, \mu, \alpha_I) - \eta^{\mathrm{gr}}(\mu, \alpha_\eta) Q^{*\mathrm{gr}}(t, \mu, \alpha_O). \end{cases} \tag{19}$$

The proposed model's numerical and graphical solutions were obtained using Mathematica software, utilizing the data provided in the subsequent section.

7. Numerical results

This portion looks at the conclusions gained *via* specific differentiability techniques, where we applied the numerical data from Table 1 to our proposed model. Additionally, Index-I presents the specific differentiability method of the fuzzy factors related in the study. Employing the data presented in Index-I, with $\alpha_{C_h} = 0.5, \alpha_{C_p} = 0.8, \alpha_{C_i} = 0.4, \alpha_s = 0.5, \alpha_{S_f} = 0.5, \alpha_\theta = 0.6, \alpha_\gamma = 0.5, \alpha_\eta = 0.6, \alpha_v = 0.6$, and $\alpha_{d_0} = 0.5$, we offer the graphical portrayal of the local constraints and the restricted inconstant in diagram four, five, six and eight correspondingly.

In diagram four, the production rate graph depicts the control variables for different values of μ, demonstrating a well-regulated manufacturing process. Production eventually halts near the conclusion of the period (Fig. 4).

Figure 5 illustrates the supply frequency or stock intensity of the element for various values of μ. Initially, the stock rate grows, stabilizing ultimately in conjunction with the x-axis. Towards the completion of the manufacturing cycle, the stock rate starts to fall. Notably, over the whole length, the stock rate never approaches zero; a residual stock, dubbed salvage stock, exists, which may be sold for a salvage price.

Figure 6 demonstrates the fluctuation in product quality over time. As illustrated in the graph, product quality degrades with time.

Additionally, it's apparent that the product's quality does not reach zero at the completion of the production time. From Figure 7, it is obvious that the demand function is substantially influenced by the stock level of the product. Initially, demand has an upward trend, showing increased demand at the commencement of production.

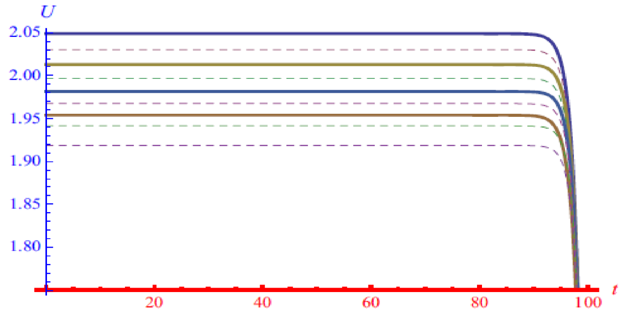

FIGURE 4. The stable μ-level set for the production rate.

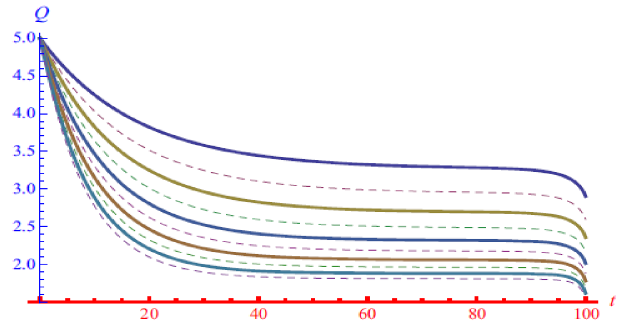

FIGURE 5. Supply frequency for the stable μ-level.

Subsequently, the demand function stabilizes and remains essentially consistent from the middle to near the end of the production period.

Index four displays the μ-mark array of the manufacture price. The 10 curves depict the pre and post extremes of the fuzzy function's μ-mark sets, relating to few estimations of μ. The thickest line at the top signifies a larger output rate, whereas the dashed bottom line indicates a lower manufacturing rate.

Figure 5 depicts the μ-mark positioned of the supply price. The 10 arcs depict the lower and upper ends of the fuzzy utility's μ-marks circles, correspondingly, for $\mu \in [0.01, 0.99]$. The topmost shredded line signifies the greater mark of the supply cost $Xu(t)$, while the filled lowest line displays the drop end of the supply rate $X_l(t)$.

Figure 6 depicts the μ-mark circle representing the characteristic of the consequence. The 10 arcs illustrate the left and right ends of the μ-mark points to fuzzy operate, correspondingly, and $\mu \in [0.01, 0.99]$. The highest wide segment displays the greater near to eminence amount $Q_u(t)$, while the lowest shattered line shows the reduce mark of attribute degree $Q_l(t)$.

Index 7 depicts the demand rate's μ-mark positioned. The 10 curves depict the lower and upper ends of the fuzzy function's μ-mark positions, correspondingly, for $\mu \in [0, 1]$. The highest shattered line reflects the top end of the order value $Di(t)$.

Figure 8 depicts the μ-mark position of the stock price. The 10 arcs represent the lower and upper ends of fuzzy function's μ-mark positions, correspondingly, for $\mu \in [0.01, 0.99]$. The tallest shredded line represents higher mark of the financing price $(Iu)(t)$, while the thickest lowest line represents the smaller mark $I(t)$. Index 8 demonstrates the performance of additional dominate constraints for immediate stock in consequence invention, carefully monitored.

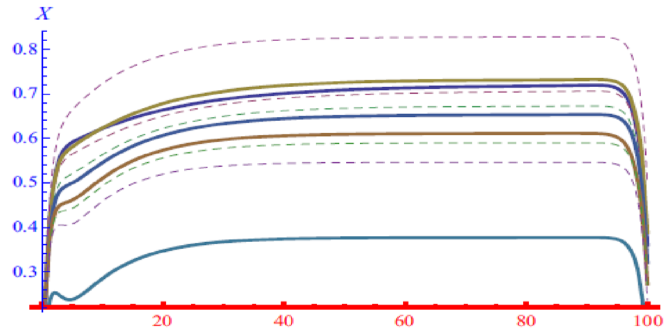

FIGURE 6. Fluctuation in product quality over time for the stable μ-level.

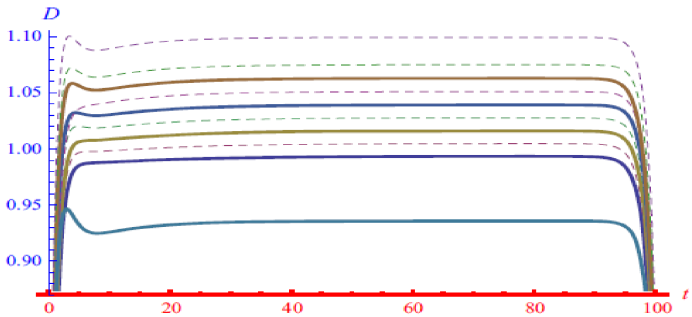

FIGURE 7. Fluctuation in demand function over influenced by stock level.

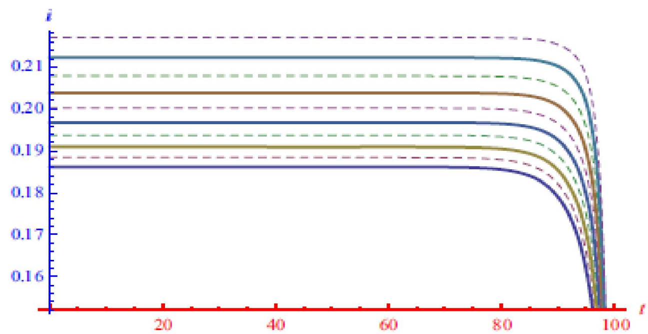

FIGURE 8. The μ-level position of the stock price.

To show the fuzzy yield for a straight point of fuzzy involvement, by use a fuzzy trilateral integer as an entered evaluate. To find solutions to a fuzzy mathematical model, it must be de-fuzzified. For defuzzification, the horizontal membership function is used.

As seen in Index 9, initially, a constraint (variable quantity) is given applying a deltoid fuzzy relationship operate. Subsequently, the variable is signified *via* the level relationship work as $P' = f(\mu, \alpha, P)P' = f(\mu, \alpha, P)$. This function delivers multiple crisp values of $P'P'$ when α is constant μ and vaable μ. These sharp values are indicated $P1'P1', P2'P2'2'P2', \ldots, P10'P10'$ (provided in each photo in the text) Finally, the constraint output is stand a stochastic constraint in a given assortment $[\max(Pi'Pi'), \min(Pi'P')] \cdot P')]$. *E.g.*, in a construction

TABLE 2. Index-2: Mathematical Information for Stochastic constraints.

Constraints	Probabilistic Numerals	HMFs
$\tilde{q}_0$	$(8.5, 12.5, 20)$	$q_0^{\mathrm{gr}} = 8.5 + 4\mu + 11.5(1-\mu)\alpha_{q0}$
$\tilde{c}_0$	$(28.5, 31.5, 35)$	$c_0^{\mathrm{gr}} = 28.5 + 3\mu + 6.5(1-\mu)\alpha_{c+0}$
$\tilde{k}_{po}$	$(1.1, 1.5, 2)$	$k_{po}^{\mathrm{gr}} = 1.1 + 0.4\mu + 0.9(1-\mu)\alpha_{k_{po}}$
$\tilde{k}_{pr}$	$(1.4, 1.5, 2)$	$k_{pr}^{\mathrm{gr}} = 1.4 + 0.5\mu + 1.1(1-\mu)\alpha_{k_{pr}}$
$\tilde{d}_0$	$(15, 20, 25)$	$d_0^{\mathrm{gr}} = 15 + 5\mu + 10(1-\mu)\alpha_{d0}$
$\tilde{\delta}$	$(0.01, 0.03, 0.04)$	$\delta^{\mathrm{gr}} = 0.01 + 0.02\mu + 0.03(1-\mu)\alpha_{\delta}$
$\tilde{\sigma}$	$(0.005, 0.01, 0.02)$	$\sigma^{\mathrm{gr}} = 0.005 + 0.005\mu + 0.05(1-\mu)\alpha_{\sigma}$
$\tilde{\phi}$	$(1, 1.2, 1.5)$	$\phi^{\mathrm{gr}} = 1 + 0.2\mu + 0.5(1-\mu)\alpha_{\phi}$
$\tilde{\xi}$	$(1.2, 1.5, 1.6)$	$\xi^{\mathrm{gr}} = 1.2 + 0.3\mu + 0.4(1-\mu)\alpha_{\xi}$

TABLE 3. Index-2: Significance of the crisp constraints.

Constraints	Numerals	Constraints	Numerals	Constraints	Numerals	Constraints	Numerals
δ_1	0.001	σ_1	0.001	a_1	0.45	a_2	0.05
η	1	β	2	l_1	0.4	l_2	0.7

supply chain dynamic, if we consider the manufacturing price of a invention as a probabilistic item and need to adjust the cost of obtained items, given the probabilistic input constraint, we shall obtain to find the result to the stochastic dynamic system, we must de-fuzzified. Introducing the conventional fuzzy interval, we may acquire the maximum interlude or section of the product's cost. Utilizing the horizontal membership function (gr-differentiability technique) at this point, we acquire the crisp consequence of the value for changes of μ and α, as well as the area or period of the fuzzy charge of the consequence.

8. Comparative study

Supply chain model for governing the amount, speculation on artefact and development invention implemented to [30].

$$\max J = \int_0^T \left[(\tilde{p}(t) - \tilde{c}(t))\tilde{D}(t) - \tilde{C}\left(\tilde{i}_{po}(t), \tilde{k}_{po}(t)\right) - \tilde{C}\left(\tilde{i}_{pr}(t), \tilde{k}_{pr}(t)\right)\right] \mathrm{d}t. \tag{20}$$

Sub to constraints:

$$\begin{cases} \tilde{q}(t) = (1 + \delta_1\mu)\tilde{i}_{po}(t) - \delta\tilde{q}(t) \\ \tilde{c}(t) = -(1 + \sigma_1\xi)\tilde{i}_{pr}(t) + \sigma\tilde{c}(t) \\ \tilde{k}_{po}(t) = \tilde{\varphi}\tilde{i}_{po}(t) \\ \tilde{k}_{pr}(t) = \tilde{\xi}\tilde{i}_{pr}(t). \end{cases} \tag{21}$$

To make the formulation and model development further convincing, certain steady parameters are required as fuzzy trilateral quantities. They resolved the challenge by creating the pseudo-marked-wise procedure, which is grounded to specific concept of rate of change. This research paper, the structure is work out by applying the concept of fuzzified differentiability techniques. The arithmetic facts that are incorporated in the purported mathematical invention are presented in index two and three, correspondingly. Index-II also gives the specific

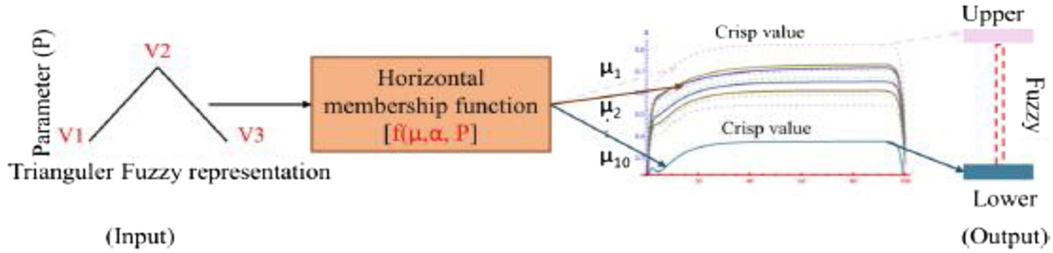

FIGURE 9. Conversion of Stochastic input into fuzzified output.

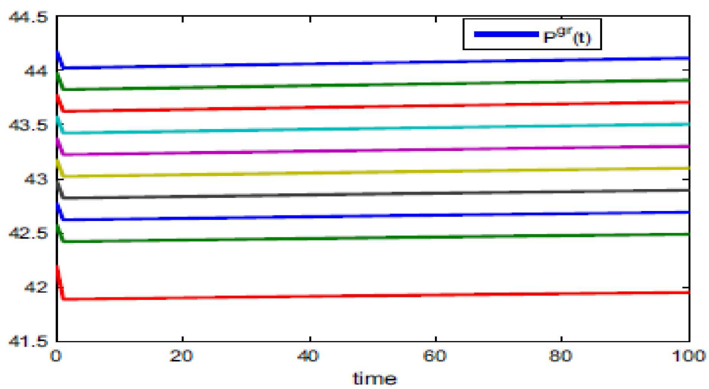

FIGURE 10. The μ-level value of the price of item.

fuzzified differentiability constraints. Grounded to fuzzified rate in change technique, we built the fuzzy number-valued functions with the values of parameters stated in Tables 2 and 3, which are depicted graphically in the research. In the diagram ten, eleven and twelve, the ideal interlude of artefact price is ($\$39.76, \$42.02$), the perfect interlude of manufacture price is ($\$29.76, \$35.01$), and the optimum interlude of product attribute is ($\$09.98, \$11.29$). The ideal results of other aspects are also displayed through diagram (diagrams thirteen, fourteen, fifteen and sixteen). We conclude that expanding possessions and uncommon performance dissipated. Kumar [10] previously noticed the outcome of the simulation provided in the current study by applying a pseudo-mark-wise procedure to the "e" and "g" operatives. In this research, we discovered on the diagram for the trade amount that the leading segment reflects the infimum mark, and the tail segment displays the broader horizontal of the stochastic trading cost. Furthermore, the probabilistic interlude of the inconstant in that inquiry was much larger. In the current research, the aberrant execution in the outcome of the consequence price found in [10] is not evident at all. Also, it offers a point-bipoint answer, as opposed to fuzzy-constructed techniques whose yield is interim resolutions. Finally, the probabilistic cost of selling item interval is additional exact, right, and efficient, which is accomplished by utilizing this strategy (Fig. 9).

Figure 10 depicts the μ-mark value of pricing of the piece. The graphical representation depicts arches for ten contains of μ. The topmost colored arc for $\mu = 1$ represents the greatest limit, while the lowest colored arc for $\mu = 0$ denotes the minimal constraint of consequence cost.

Figure 11 depicts the μ-mark point of the price of the point. The topmost blue arch for $\mu = 1$ implies the greatest limit, while the lowest auburn arch for $\mu = 0$ denotes the inferior constraint of consequence price.

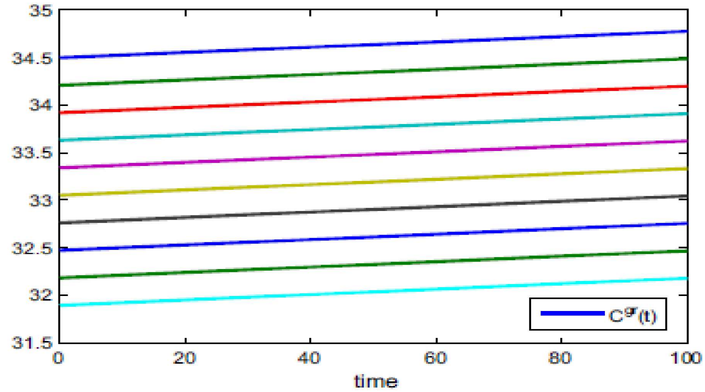

FIGURE 11. The μ-level value of the price of the point.

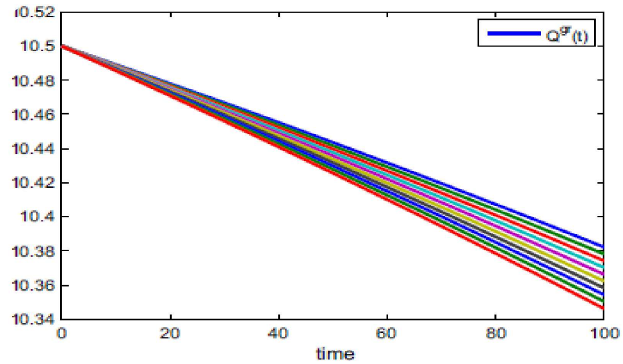

FIGURE 12. The μ-level set of the quality rate of the item.

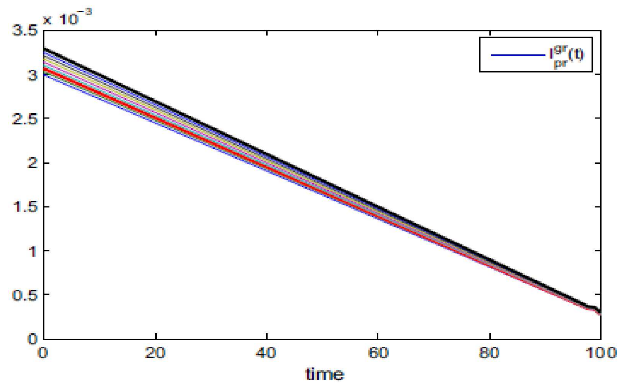

FIGURE 13. The μ-level set of the investment rate for process innovation.

9. Conclusion

A novel stochastic manufactured optimized supply chain inventory model has been created to meet circular probabilistic stocks and fuzzy eminence-reliant changeable demand. By adopting trilateral probabilistic numerals as uncertain supply chain constraints and expressing stochastic derivatives, the approach becomes greatly

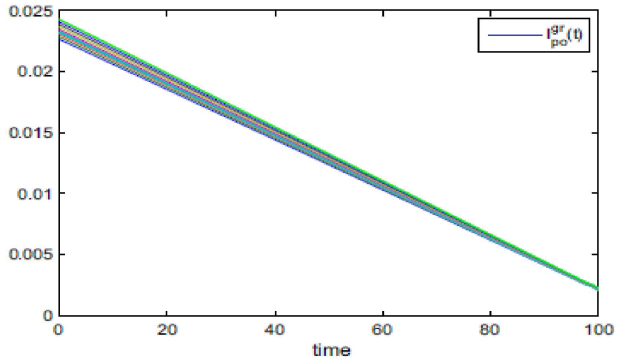

FIGURE 14. The μ-level set of the investment rate for product innovation.

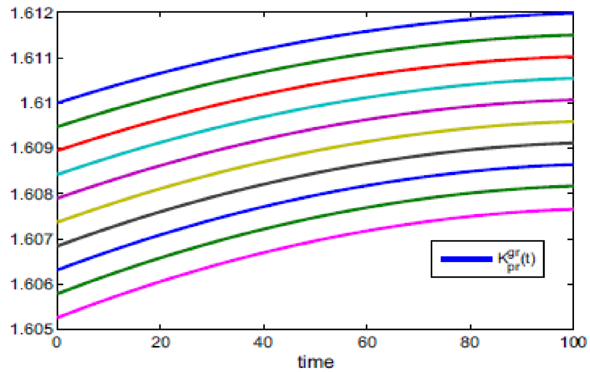

FIGURE 15. The μ-mark point of facts bunching for progression modernization.

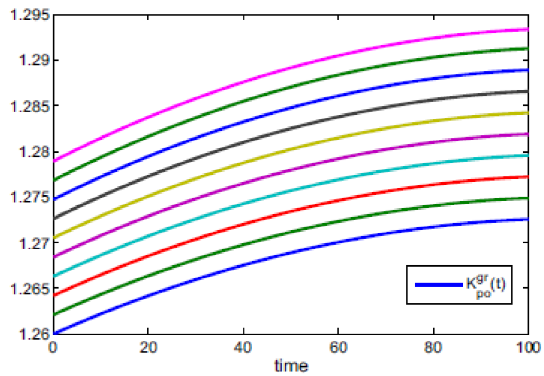

FIGURE 16. The μ-level set of the knowledge gathering for product innovation.

simplified, allowing the derivation of single, crisp formulations for the variables. This technique additionally illustrates the granular sterilizability of the unknown FDS, overcoming challenges linked with interim arithmetic or pseudo-mark-prudent systems grounded to some specific operatives to de-fuzzifying the fixed differentiability states. Unlike standard generalized-differentiability or generalized-differentiability techniques, which often offer level-wise solutions or intervals of fuzzy variables, our method delivers a single solution in a compact form. Additionally, the gr-differentiability approach does not represent the unusual behavior of nonlinear FDS reported in prior methodologies. By employing gr-differentiability, the FDS may be readily turned into a crisp dynamical system for identifying unknown variables. Notably, our technique eliminates the doubling feature of the solution, which is a barrier in SGH-differentiability or gH-differentiability processes. Furthermore, when examined using granular differentiability, the stability of the crisp dynamical system remains intact, while methodologies grounded to SGH-differentiability or gH-differentiability may not give stability. Furthermore, our suggested technique may be refined to analyze stability and handle industrial inventory management challenges in a type-2 fuzzy environment (Figs. 12–16).

Limitation and future scope

One-Time-Frame Model Limitation: Because the model is only intended to be used for one specific time frame, it cannot be used in long-term inventory management scenarios where ongoing replenishment cycles are necessary.

It is suggested that future studies expand the model to incorporate rolling planning horizons or multi-period cycles, where decisions about inventory and demand change over time. As a result, the model would be able to manage more intricate supply chain settings with continuous inventory adjustments.

Constant Advertisement Rate Limitation: The paper makes the assumption that the rate of advertisements will not change while the model runs. But in practical terms, marketing initiatives frequently change according on demand, seasonality, or tactical choices.

One suggestion is to include dynamic advertisement rates in future iterations of the model. These rates would change over time in response to feedback from sales data or market conditions. This would improve the model's application in marketing-driven businesses and increase its realism.

The model's application in high-inflation situations, when inflation has a large impact on pricing, demand, and inventory costs, may be limited by the assumption of low inflation. This assumption simplifies the economic framework.

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