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SLACK-BASED MODELS FOR INVERSE DATA ENVELOPMENT ANALYSIS IN MERGING UNITS WITH INTERVAL DATA

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Abstract. The paper proposes non-radial slack-based measure (SBM) models for the inverse data envelopment analysis (InvDEA) when dealing with data involving uncertainty represented by interval values. The models are specifically developed to ascertain the optimal inputs or outputs for the merged decision-making unit (DMU) to attain its desired efficiency level. To achieve the predefined optimistic efficiency, we estimate the input interval for the newly merged DMU while keeping its outputs equal to the sum of the outputs from the merging DMUs. Conversely, to achieve the predefined pessimistic efficiency, we estimate the output interval for the new DMU while maintaining its inputs as the sum of the inputs from the merging DMUs. In addition, we also calculate the minimum and maximum possible values for optimistic and pessimistic efficiency. The advantages of the SBM InvDEA models are highlighted in comparison to the radial BCC model. We apply the proposed models to a real-world scenario involving the 2017 merger of a few public sector banks with one of India's largest banks.

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1. Introduction

Data envelopment analysis (DEA) models assess the efficiency of homogeneous decision-making units (DMUs) utilizing their input and output values. On the other hand, inverse DEA (InvDEA) models are employed subsequent to the DEA study to ascertain the optimal values of inputs and outputs to meet predetermined efficiency targets of units. InvDEA examines the requisite adjustments in outputs (or inputs) if the unit's inputs (or outputs) are varied while maintaining a constant efficiency level.

Zhang and Cui [27] introduced the concept of InvDEA in project evaluations. They considered how much additional input is needed to increase outputs without changing the unit's current efficiency score.

Wei *et al.* [25] discussed the InvDEA problem and transformed it into a multiobjective linear programming problem (MOLP) for inefficient DMUs. The Pareto optimal solutions of MOLP provide the optimal resource levels that the unit needs to maintain efficiency. Since then, their method has been applied in many theoretical and practical studies.

Hadi-Vencheh *et al.* [15] pointed out that the model in [25] for calculating input levels when outputs increase while efficiency stays the same is wrong. They established sufficient conditions for input estimation when output is increased.

Keywords. Inverse data envelopment analysis, slack-based model, optimistic and pessimistic models, interval data, merger.

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Very recently, a review article on InvDEA by Emrouznejad *et al.* [10] provides the current state of the art of InvDEA and its wide range of applications.

Associated with the radial DEA models that quantify radial efficiency are the radial InvDEA models (see, [1, 2, 12, 17, 21, 25, 28]). A notable radial DEA model is the BCC model, introduced by Banker *et al.* [4] for efficiency evaluation under variable returns to scale. The model computes efficiency by proportionally scaling inputs or outputs. The model has an analogous inverse DEA model [13], which identifies the changes required in inputs or outputs to achieve a desired efficiency score. When applied within the BCC framework, this approach is referred to as the inverse BCC model. The BCC models ignore slacks and fail to be unit invariant when estimating possible output (or input) levels with increasing inputs (or outputs) and preserving the efficiency score. As a result, the radial InvDEA estimates become unreliable and unattainable in situations where slacks are significant.

Jahanshahloo [18] suggested an InvDEA model based on the non-radial enhanced Russell model. Zhang and Cui [29] offered a more generic InvDEA model based on non-radial DEA termed inverse non-radial DEA. This model assumes that the total efficiency scores remain constant rather than considering each dimension individually.

Hu *et al.* [16] proposed a revised InvDEA model when the DMU involves no mix efficiency and, hence, excluding slacks correctly measures the input-output changes while preserving the total efficiency.

Conventional DEA models assume precise input and output values. However, in many real-world scenarios, data is uncertain due to estimation errors, missing values, or inherent variability. While all such data can be broadly classified as uncertain, the distinction between fuzzy, interval, stochastic, and robust methods arises based on approaches used to address uncertainty, each suited to specific contexts. Stochastic DEA (SDEA) addresses random errors and variability by assuming probability distributions for the uncertain data, making it ideal for large datasets where statistical assumptions hold true. Robust DEA is designed to evaluate performance under the worst-case scenario within a defined uncertainty set, ensuring stable and reliable efficiency scores even in highly adverse conditions. Fuzzy DEA (FDEA) represents uncertainty using fuzzy sets and membership functions, making it suitable under vagueness or subjectivity in the data and more so in ordinal data. Interval DEA (IDEA) models uncertainty by considering inputs and outputs as interval ranges, aiming to estimate errors and thereby determine efficiency bounds. This approach is realistic and simple, as it avoids distribution or membership function structures. In this study, we adopted IDEA to handle uncertainty.

The interval method's main concern is estimating the unit's efficiencies' lower and upper bounds. Some researchers put forward different findings, including Despotis and Smirlis [9], Entani *et al.* [11], Kao [19], and Wang *et al.* [24].

Entani *et al.* [11] suggested a suitable technique to determine unit efficiency and inefficiency intervals, considering both optimistic and pessimistic views. Despotis and Smirlis [9] introduced a set of linear programming models for estimating upper and lower bounds of efficiency for interval data. They used distinct production methods or production possibility sets (PPS) to determine the unit's efficiency score, making comparison impossible. In order to overcome this drawback, Despotis and Smirlis [9] updated the models to allow each unit to measure the lower and upper bounds of efficiency.

All these are radial models. The underlying issue with radial DEA models is neglecting slacks. The slacks-based measures (SBM) would better estimate the efficiency interval in uncertain data. Lotfi *et al.* [23] created SBM models for judging units' performance with interval data only from an optimistic point of view. Multiple production boundaries used in efficiency analysis make it harder to compare units. Hossein Azizi *et al.* [3] adopted optimistic and pessimistic approaches to evaluate the efficiency of units with imprecise data under a single production possibility set.

One of the applications of inverse DEA models is in merger and acquisition. Many businesses use mergers and acquisitions to enter new markets, strengthen their position in existing ones, or get access to cutting-edge resources and talent. A merger usually happens when at least two DMUs work together to make a new unit with better performance. Also, companies merge with other companies to develop new technologies or to cut down on competition. These businesses could be transformed into more efficient, profitable, and robust entities. The

DEA technique is a potent mathematical instrument that can be used to evaluate the performance of mergers in various fields. Estimating inputs and outputs inherited from merged DMUs is an essential topic.

Gattoufi *et al.* [12] proposed the inverse DEA to identify the input/output levels inherited from merging units to attain the predefined target level. Numerous inverse DEA models have been put out in scholarly literature to assess prospective merger benefits. In [14], the authors introduced inverse DEA models to describe a global improvement in the efficiency of new entities.

Younesi *et al.* [26] proposed the SBM inverse DEA model to handle integer and continuous interval data. These models expand the use of inverse DEA to other forms of data and efficiency objectives, offering decision-makers a wider range of choices for assessing possible mergers and acquisitions.

Ghobadi [13] formulated inverse DEA models for estimating inputs and outputs in merging units with interval data. The model applied the radial DEA model. In addition, the non-radial DEA models have attracted attention for their better-discriminating power by applying slacks. This article aims to develop slack-based inverse DEA models to assess the input and output levels of merged DMUs with interval data.

Section 2 describes the slack-based DEA models for computing optimistic efficiency and pessimistic inefficiency of units with interval data. Section 3 formulates the inverse DEA models to determine the input interval for the merged DMU at specified optimistic efficiency and pessimistic inefficiency levels. Section 4 extends the description to estimate the output intervals of the merged unit. Section 5 examines the minimum and maximum achievable efficiencies for the lower and upper bounds of interval efficiencies. Section 6 offers an illustrative example to demonstrate the significance of non-radial slack-based models. Section 7 presents a case study on the merger of public sector banks in India. Section 8 concludes with suggestions for future research.

2. Preliminaries

2.1. Technologies definition

Performance estimation in DEA models can be approached from both optimistic and pessimistic perspectives. Optimistic models prioritize the efficient frontier, emphasizing the distance between points and this frontier. In contrast, pessimistic models focus on inefficient boundaries, highlighting the distance of DMUs from these boundaries.

The double-frontier study in performance evaluation of the DMUs enabled us to distinguish between efficient and non-efficient units with the inefficient and non-inefficient units, respectively. The idea of double production frontier emerged from the research article [3], wherein the author put forward bounded DEA models for the efficiency evaluation of units from the optimistic and pessimistic perspectives.

Suppose there are set of n homogenous DMUs, $\{\text{DMU}_j : j = 1, \ldots, n\}$, each of which uses m inputs $X_j = (x_{ij},\ i = 1, \ldots, m)$, to produce s outputs $Y_j = (y_{rj},\ r = 1, \ldots, s)$, and each $x_{ij} \geq 0,\ y_{rj} \geq 0$.

Throughout this study, we use the following notations:

$J = \{1, \ldots, n\}$: the set of n DMUs, j: index of J;

$I = \{1, \ldots, m\}$: the set of m inputs, i: index of I;

$O = \{1, \ldots, s\}$: the set of s outputs, r: index of O;

x_{ij} : value of ith input for jth DMU;

y_{rj} : value of rth output for jth DMU;

$\mathbb{R}^p$: p-dimensional Euclidean space;

S_i^- : ith input slack;

S_r^+ : rth output slack;

o : a DMU under consideration.

The production possibility set (PPS) of all n units is described by

$$\big\{(X,Y)\in\mathbb{R}^{m+s}: Y=(y_1,\dots,y_s) \text{ can be produced from } X=(x_1,\dots,x_m)\big\}.$$

The PPS of an optimistic efficient production frontier that includes all possible input-output bundles (X,Y) is defined by

$$P_{\text{eff}}=\left\{(X,Y)\in\mathbb{R}^{m+s}:\sum_{j=1}^{n}\lambda_jX_j\le X,\ \sum_{j=1}^{n}\lambda_jY_j\ge Y,\ \lambda_j\ge 0,\ \ j\in J\right\}.$$

The PPS of the pessimistic inefficient production frontier that includes all possible input-output bundles (X,Y) is defined by:

$$P_{\text{inff}}=\left\{(X,Y)\in\mathbb{R}^{m+s}:\sum_{j=1}^{n}\lambda_jX_j\ge X,\ \sum_{j=1}^{n}\lambda_jY_j\le Y,\ \lambda_j\ge 0,\ \ j\in J\right\}.$$

Here, $\lambda_j\,(j\in J)$, is the dual variable attached to the j-th DMU. The two PPS described above assume the constant returns to scale (CRS). If the production scale changes to the variable returns to scale (VRS), then a convexity constraint $\sum_{j=1}^{n}\lambda_j=1$ is added in the PPS.

A $\text{DMU}_o=(X_o,Y_o)$ is said to be *optimistic efficient* if and only if it lies on the optimistic production frontier, that is, if for any $(X,Y)\in P_{\text{eff}}$ with $X\le X_o$ and $Y\ge Y_o$, then $(X,\,Y)=(X_o,\,Y_o)$.

A DMU in P_{eff} but not on the efficient production frontier is called a *non-efficient* DMU in the optimistic sense.

A DMU_0 is said to be *pessimistic inefficient* if and only if it lies on the pessimistic production frontier, that means, if for any $(X,Y)\in P_{\text{inff}}$ with $X\ge X_o$ and $Y\le Y_o$, then $(X,\,Y)=(X_o,\,Y_o)$.

In the pessimistic sense, a DMU in P_{inff} but not on the inefficient production frontier is called a non-inefficient DMU.

It is important to note here that a non-inefficient DMU is not necessarily efficient, and a non-efficient DMU is not inefficient. The negation of one does not yield the affirmation of the other.

2.2. Multiobjective programming

Multiobjective programming (MOP) involves optimizing two or more objective functions, subject to constraints.

$$\begin{aligned}&\min\ f(x)=(f_1(x),f_2(x),\dots,f_v(x))\\&\text{s.t. } x\in\mathbb{X}\subseteq\mathbb{R}^p.\end{aligned}$$

An $x^*\in\mathbb{X}$ is called a Pareto optimal solution of MOP if there does not exist $x\in\mathbb{X}$ such that: $f_\xi(x)\le f_\xi(x^*),\ \xi\in\{1,\dots,v\}$ and $f_\rho(x)<f_\rho(x^*)$, for at least one $\rho\in\{1,\dots,v\}$. In other words, x^* is the Pareto optimal solution if there is no other feasible solution that improves at least one objective without worsening another.

2.3. DEA with interval data

We assume that inputs and outputs vary in intervals *i.e.*, for $j\in J$, $i\in I$, $r\in O$, we have, $x_{ij}\in[x_{ij}^l,x_{ij}^u]$ and $y_{rj}\in[y_{rj}^l,y_{rj}^u]$, where $x_{ij}^u\ge x_{ij}^l>0$ and $y_{rj}^u\ge y_{rj}^l>0$. Consequently, the technical efficiency of a unit is an interval. Ghobadi [13] proposed four linear programming models to calculate the interval efficiency of DMU_o, using an input-oriented BCC model as follows:

$$\begin{aligned}&\min\ \theta_o^l &&\text{(BCC1)}\\&\text{s.t.}\ \sum_{j=1}^{n}\lambda_jx_{ij}^l\le\theta_o^lx_{io}^u, && i\in I,\end{aligned}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^u \geq y_{ro}^l, \qquad r \in O,$$

$$\sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geq 0, \qquad j \in J.$$

$$\min \ \theta_o^u \tag{BCC2}$$

$$\text{s.t.} \ \sum_{j=1}^{n} \lambda_j x_{ij}^l \leq \theta_o^u x_{io}^l, \qquad i \in I,$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^u \geq y_{ro}^u, \qquad r \in O,$$

$$\sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geq 0, \qquad j \in J.$$

Here, $\theta_o^l \leq \theta_o^u \leq 1$. A DMU is input-efficient if and only if $\theta_o^u = 1$.

The output-oriented version of the models (BCC1) and (BCC2) are as follows:

$$\max \ \phi_o^l \tag{BCC3}$$

$$\text{s.t.} \ \sum_{j=1}^{n} \lambda_j x_{ij}^l \leq x_{io}^l, \qquad i \in I,$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^u \geq \phi_o^l y_{ro}^u, \qquad r \in O,$$

$$\sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geq 0, \qquad j \in J.$$

$$\max \ \phi_o^u \tag{BCC4}$$

$$\text{s.t.} \ \sum_{j=1}^{n} \lambda_j x_{ij}^l \leq x_{io}^u, \qquad i \in I,$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^u \geq \phi_o^u y_{ro}^l, \qquad r \in O,$$

$$\sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geq 0, \qquad j \in J.$$

Here, $1 \leq \phi_o^l \leq \phi_o^u$. A DMU is output-efficient if and only if $\phi_o^l = 1$.

2.4. Merging DMUs: SBM models

Azizi *et al.* [3] proposed four linear models with slack values to determine the interval efficiency using an optimistic and pessimistic approach.

Let $\emptyset \neq \Lambda \subset J$, and $\Pi = J - \Lambda$ (elements of J not in Λ). Suppose that units $\{\text{DMU}_j, \ j \in \Lambda\}$, decided to merge, resulting in a new unit DMU_q.

The merged unit DMU_q inherits input and output values from its merging DMUs, *i.e.*,

$$x_{iq}^l = \sum_{j \in \Lambda} x_{ij}^l, \ \ x_{iq}^u = \sum_{j \in \Lambda} x_{ij}^u, \ \ y_{rq}^l = \sum_{j \in \Lambda} y_{rj}^l, \ \ y_{rq}^u = \sum_{j \in \Lambda} y_{rj}^u, \qquad i \in I, \ r \in O.$$

Drawing upon the concept proposed by Azizi *et al.* [3], we put forward two models to determine the optimistic efficiency interval $[\theta_q^l, \theta_q^u]$ of the DMU_q, when all units in Π use their best input-outputs (using minimum input $x_{ij}^l\,(i \in I,\ j \in \Pi)$ in producing maximum output $y_{rj}^u\,(r \in O,\ j \in \Pi)$) and the merged unit DMU_q utilizes its best (favourable case) input-outputs for the upper-efficiency score and the worst (unfavourable case) (using maximum input $x_{ij}^u\,(i \in I,\ j \in \Lambda)$ for producing minimum output $y_{rj}^l\,(r \in O,\ j \in \Lambda)$) input-output for its lower efficiency score.

The model computes the optimistic upper bound efficiency of DMU_q:

$$\begin{aligned}
\min\ \theta_q^u &= \frac{1 - \frac{1}{m}\sum_{i=1}^m \frac{s_i^-}{x_{iq}^l}}{1 + \frac{1}{s}\sum_{r=1}^s \frac{s_r^+}{y_{rq}^u}} \\
\text{s.t. } & \sum_{j\in\Pi} \mu_j x_{ij}^l + \mu_q x_{iq}^l + s_i^- = x_{iq}^l, && i \in I, \\
& \sum_{j\in\Pi} \mu_j y_{rj}^u + \mu_q y_{rq}^u - s_r^+ = y_{rq}^u, && r \in O, \\
& \mu_j \geq 0,\ s_i^- \geq 0,\ s_r^+ \geq 0, && j \in \Pi \cup \{q\},\ i \in I,\ r \in O.
\end{aligned}$$

Here, $\mu_j\,(j \in \Pi \cup \{q\})$ is the dual variable attached to the jth DMU, $s_i^-\,(i \in I)$ and $s_r^+\,(r \in O)$, are input and output slacks respectively.

The Charnes and Cooper (CC) transformation is a linearization technique used in fractional programming to convert a fractional optimization problem into a linear programming problem [5]. By applying the CC-transformation and taking $t^{-1} = 1 + \frac{1}{s}\sum_{r=1}^s \frac{s_r^+}{y_{rq}^u}$, the linear form of the above model is as follows:

$$\begin{aligned}
\min\ \theta_q^u &= t - \frac{1}{m}\sum_{i=1}^m \frac{S_i^-}{x_{iq}^l} \qquad (2.1)\\
\text{s.t. } & t + \frac{1}{s}\sum_{r=1}^s \frac{S_r^+}{y_{rq}^u} = 1, \\
& \sum_{j\in\Pi} \lambda_j x_{ij}^l + \lambda_q x_{iq}^l + S_i^- = t x_{iq}^l, && i \in I, \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^u + \lambda_q y_{rq}^u - S_r^+ = t y_{rq}^u, && r \in O, \\
& t > 0,\ \lambda_j \geq 0,\ S_i^- \geq 0,\ S_r^+ \geq 0, && j \in \Pi \cup \{q\},\ i \in I,\ r \in O,
\end{aligned}$$

where, $\lambda_j = t\mu_j\ (j \in \Pi \cup \{q\}),\ S_i^- = t s_i^-\ (i \in I)$, and $S_r^+ = t s_r^+\ (r \in O)$.

Instead of presenting the ratio model each time, we have directly written the simplified linear form, similar to (2.1) formulation, with the variables defined accordingly.

The following linear model gives the optimistic lower bound efficiency of DMU_q:

$$\begin{aligned}
\min\ \theta_q^l &= t - \frac{1}{m}\sum_{i=1}^m \frac{S_i^-}{x_{iq}^u} \qquad (2.2)\\
\text{s.t. } & t + \frac{1}{s}\sum_{r=1}^s \frac{S_r^+}{y_{rq}^l} = 1, \\
& \sum_{j\in\Pi} \lambda_j x_{ij}^l + \lambda_q x_{iq}^u + S_i^- = t x_{iq}^u, && i \in I,
\end{aligned}$$

$$\sum_{j\in\Pi}\lambda_j y_{rj}^u + \lambda_q y_{rq}^l - S_r^+ = t y_{rq}^l, \qquad r\in O,$$
$$t>0,\ \lambda_j\geq 0,\ S_i^-\geq 0,\ S_r^+\geq 0, \qquad j\in\Pi\cup\{q\},\ i\in I,\ r\in O.$$

A merged unit DMU_q is called optimistic efficient if its optimal value $\theta_q^{u*}=1$, else if, $\theta_q^{u*}<1$, then it is called optimistic non-efficient.

Similarly, applying the pessimistic approach, when all units in Π use their worst input-output while the merged unit DMU_q applies its best input-output (for the upper bound) and worst input-output (for the lower bound) of the inefficiency. It requires solving the following two models:

$$\begin{aligned}
\min\ \frac{1}{\phi_q^u} &= t-\frac{1}{s}\sum_{r=1}^{s}\frac{S_r^-}{y_{rq}^u} && (2.3)\\
\text{s.t. } t &+ \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^+}{x_{iq}^l}=1,\\
&\sum_{j\in\Pi}\lambda_j x_{ij}^u + \lambda_q x_{iq}^l - S_i^+ = t x_{iq}^l, && i\in I,\\
&\sum_{j\in\Pi}\lambda_j y_{rj}^l + \lambda_q y_{rq}^u + S_r^- = t y_{rq}^u, && r\in O,\\
&t>0,\ \lambda_j\geq 0,\ S_i^+\geq 0,\ S_r^-\geq 0 && j\in\Pi\cup\{q\},\ i\in I,\ r\in O,
\end{aligned}$$

and

$$\begin{aligned}
\min\ \frac{1}{\phi_q^l} &= t-\frac{1}{s}\sum_{r=1}^{s}\frac{S_r^-}{y_{rq}^l} && (2.4)\\
\text{s.t. } t &+ \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^+}{x_{iq}^u}=1,\\
&\sum_{j\in\Pi}\lambda_j x_{ij}^u + \lambda_q x_{iq}^u - S_i^+ = t x_{iq}^u, && i\in I,\\
&\sum_{j\in\Pi}\lambda_j y_{rj}^l + \lambda_q y_{rq}^l + S_r^- = t y_{rq}^l, && r\in O,\\
&t>0,\ \lambda_j\geq 0,\ S_i^+\geq 0,\ S_r^-\geq 0, && j\in\Pi\cup\{q\},\ i\in I,\ r\in O.
\end{aligned}$$

A merged unit q is called pessimistic inefficient if the optimal value $\phi_q^{l^*}=1$, else if $\phi_q^{l^*}>1$, then it is called pessimistic non-inefficient.

The four models discussed above are feasible and bounded linear programs.

It is important to note that we have assumed the input and output of the merged unit can be set as the sum of the inputs and outputs of all the merging units in Λ. In this setting, the merged unit need not achieve a higher and better efficiency level for itself. We employ the inverse data envelopment analysis (InvDEA) to get better off from the merged unit. We look at the two inverse problems: (i) determine the input interval, lower and upper bounds of the input, of the merged unit q to achieve the preset higher optimistic efficiency score from the unit q when the output of the merged unit is taken as the sum of the outputs of the merging units. (ii) to find the output interval of unit q to achieve the preset higher efficiency for pessimistic case, lower inefficiency) when the input of q is the sum of the inputs of the merging units. We address the first problem in Section 3 and the second one in Section 4.

3. Estimation of input for the merged DMU

In this section, we present a non-radial inverse DEA model to estimate the lower and upper bounds of inherited input values. The model assumes that the outputs of the newly merged DMU are the sum of the outputs of the merging DMUs, in order to achieve a predefined optimistic efficiency target.

Suppose the output of DMU_q is the sum of outputs of the merging DMUs *i.e.*, $y_{rq} \in [y^l_{rq}, y^u_{rq}]$, $y^l_{rq} = \sum_{j\in\Lambda} y^l_{rj}(r \in O)$, and $y^u_{rq} = \sum_{j\in\Lambda} y^u_{rj}(r \in O)$. Let the predefined optimistic efficiency target $\hat{\theta}_q \in [\hat{\theta}^l_q, \hat{\theta}^u_q]$, $\hat{\theta}^u_q \leq 1$, such that $\hat{\theta}^l_q \geq \theta^{l^*}_q$, and $\hat{\theta}^u_q \geq \theta^{u*}_q$, where $\theta^{l^*}_q$ and θ^{u*}_q are the optimal values of the problems (2.1) and (2.2), respectively.

We look for the minimum inherited input values $x_{iq} \in [x^l_{iq}, x^u_{iq}]$ with $x^l_{iq} = \sum_{j\in\Lambda} \alpha^l_{ij}(i \in I)$ and $x^u_{iq} = \sum_{j\in\Lambda} \alpha^u_{ij}(i \in I)$ to reach the efficiency target. Here α^l_{ij}, $\alpha^u_{ij}(i \in I, j \in \Lambda)$, are unknown variables to be estimated and $\alpha^l_{ij} \geq 0$ with at least one $\alpha^l_{ij} > 0$ for some $i \in I$ and $j \in \Lambda$. Here the notation I, J, O, Λ, Π are the same as defined in Section 2.4.

To estimate the lower bound for inputs for achieving the upper bound $\hat{\theta}^u_q$ of efficiency, we proposed the following slack-based multiobjective optimization problem:

$$\min \left(\alpha^l_{ij};\ \ i \in I,\ j \in \Lambda\right) \tag{3.1}$$

$$\text{s.t.}\ \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s^-_i}{x^l_{iq}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s^+_r}{y^u_{rq}}} = \hat{\theta}^u_q, \tag{3.1.1}$$

$$\sum_{j\in\Pi} \mu_j x^l_{ij} + \mu_q x^l_{iq} + s^-_i = x^l_{iq}, \qquad i \in I, \tag{3.1.2}$$

$$\sum_{j\in\Pi} \mu_j y^u_{rj} + \mu_q y^u_{rq} - s^+_r = y^u_{rq}, \qquad r \in O, \tag{3.1.3}$$

$$x^l_{iq} = \sum_{j\in\Lambda} \alpha^l_{ij}, \qquad i \in I, \tag{3.1.4}$$

$$0 \leq \alpha^l_{ij} \leq x^l_{ij}, \qquad i \in I,\ j \in \Lambda, \tag{3.1.5}$$

$$\mu_j \geq 0,\ s^-_i \geq 0,\ s^+_r \geq 0, \qquad j \in \Pi \cup \{q\},\ i \in I,\ r \in O. \tag{3.1.6}$$

Here, $x^l_{iq}\,(i \in I)$, $\alpha^l_{ij}\,(i \in I, j \in \Lambda)$, $\mu_j\,(j \in \Pi \cup \{q\})$, $s^-_i\,(i \in I)$, $s^+_r\,(r \in O)$, are unknown variables, and s^-_i, s^+_r, are input and output slacks respectively.

Theorem 3.1. *If DMU_q lies within the current optimistic PPS for interval-valued data, model (3.1) can be reformulated as a following multiobjective nonlinear model.*

(The constraint (3.1.1), originally in ratio form, is also linearized using the CC-transformation by setting $t^{-1} = 1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s^+_r}{y^u_{rq}}$).

$$\min \left(\alpha^l_{ij};\ \ i \in I,\ j \in \Lambda\right) \tag{3.2}$$

$$\text{s.t.}\ t + \frac{1}{s}\sum_{r=1}^{s} \frac{S^+_r}{y^u_{rq}} = 1, \tag{3.2.1}$$

$$t - \frac{1}{m}\sum_{i=1}^{m} \frac{S^-_i}{x^l_{iq}} = \hat{\theta}^u_q, \tag{3.2.2}$$

$$\sum_{j\in\Pi} \lambda_j x^l_{ij} + S^-_i = t x^l_{iq}, \qquad i \in I, \tag{3.2.3}$$

$$\sum_{j\in\Pi} \lambda_j y^u_{rj} - S^+_r = t y^u_{rq}, \qquad r \in O, \tag{3.2.4}$$

$$x_{iq}^{l} = \sum_{j\in\Lambda} \alpha_{ij}^{l}, \qquad i \in I, \tag{3.2.1}$$

$$0 \le \alpha_{ij}^{l} \le x_{ij}^{l}, \qquad i \in I,\ j \in \Lambda, \tag{3.2.6}$$

$$t > 0,\ \lambda_j \ge 0,\ S_i^- \ge 0,\ S_r^+ \ge 0, \qquad j \in \Pi,\ i \in I,\ r \in O. \tag{3.2.7}$$

Here, $(t,\ x_{iq}^{l}\,(i\in I), \alpha_{ij}^{l}\,(i\in I,\ j\in\Lambda), \lambda_j\,(j\in\Pi), S_i^-(i\in I), S_r^+\,(r\in O))$, are unknown variables, and $S_i^- = t\,s_i^-$, $S_r^+ = t\,s_r^+$, and $\lambda_j = t\,\mu_j$.

Proof. If DMU_q lies in the interior of the PPS, it can be expressed as a linear combination of units that have not participated in the merger. If $(t^*,\ x_{iq}^{l^*}\,(i\in I), \alpha_{ij}^{l^*}\,(i\in I, j\in\Lambda), \lambda_j^*\,(j\in\Pi), S_i^{-^*}\,(i\in I), S_r^{+^*}\,(r\in O))$ is an Pareto optimal solution of (3.2), then, $\lambda_q^* = 0$ in each Pareto solution of (3.1).

If DMU_q is on the frontier of the current PPS, it is efficient or the convex combination of other efficient units. We can continue to take $\lambda_q^* = 0$. It means ignoring one Pareto optimal solution of (3.1). □

In this paper, based on the above theorem, we limit our analysis to the case where the merged unit (DMU_q) lies within the current PPS. Exploring cases where this assumption is not satisfied could be an interesting topic for future research, but it is beyond the scope of this study.

There is typically no feasible solution of MOP that maximizes all objective functions simultaneously. Therefore, the Pareto solution for model (3.2) is defined as follows.

Definition 3.2. $(t^*,\ x_{iq}^{l^*}\,(i\in I), \alpha_{ij}^{l^*}\,(i\in I,\ j\in\Lambda), \lambda_j^*\,(j\in\Pi), S_i^{-^*}\,(i\in I), S_r^{+^*}\,(r\in O))$ is an Pareto optimal solution for (3.2), if for any feasible solution $(t,\ x_{iq}^{l}(i\in I), \alpha_{ij}^{l}, (i\in I,\ j\in\Lambda), \lambda_j(j\in\Pi), S_i^-(i\in I),\ S_r^+(r\in O))$ of (3.2), it is not possible to have, $\alpha_{ij}^{l} \le \alpha_{ij}^{l^*}(i\in I,\ j\in\Lambda)$, with at least one strict inequality.

The following theorem suggests that the model (3.2) may be used to establish a lower bound on DMU_q's input levels.

Theorem 3.3. *If the following assumptions hold:*

(A1) *The merged unit DMU_q is within the current optimistic PPS for the interval data;*
(A2) $S^l = (t^*,\ x_{iq}^{l^*}\,(i\in I), \alpha_{ij}^{l^*}\,(i\in I, j\in\Lambda), \lambda_j^*\,(j\in\Pi), S_i^{-^*}\,(i\in I), S_r^{+^*}\,(r\in O))$ *is a Pareto optimal solution of* (3.2).

Then, the optimal value of model (2.1), *with* $x_{iq}^{l} = \sum_{j\in\Lambda}\alpha_{ij}^{l^*}(i\in I)$, *and* $t = t^*$, *equals* $\hat{\theta}_q^u$.

Proof. In our arguments, we consider model (2.1) by setting $x_{iq}^{l} = \sum_{j\in\Lambda}\alpha_{ij}^{l^*}$ and $t = t^*$. Let $(\lambda_j^{**}\,(j\in\Pi),\ \lambda_q^{**},\ S_i^{-^{**}}\,(i\in I),\ S_r^{+^{**}}\,(r\in O),\ \theta_q^{u**})$ be the optimal solution of (2.1). Since S^l is feasible for (3.2), it is feasible for (2.1), hence, $\theta_q^{u**} \le \hat{\theta}_q^u$.

Furthermore, by assumption, $A1$, the current optimistic PPS, and the efficient frontier remain unaltered on merging the units, and the inputs decrease $\alpha_{ij}^{l} \le x_{ij}^{l}(i\in I,\ j\in\Lambda)$, while the outputs remain the same in models (3.2) and (2.1), hence at their optimal solutions, $S_i^{-^{**}} \ge S_i^{-^*}(i\in I)$.

Suppose $\theta_q^{u**} < \hat{\theta}_q^u$. Applying optimal solutions of (2.1), and (3.2), we have,

$$\begin{aligned} t^* y_{rq}^u &= \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^u + \sum_{j\in\Lambda}\lambda_q^{**}y_{rj}^u - S_r^{+^{**}} \\ &= \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^u + \lambda_q^{**}\left(\sum_{j\in\Pi}\left((\lambda_j^* y_{rj}^u)/t^*\right) - \left(S_r^{+^*}/t^*\right)\right) - S_r^{+^{**}} \\ &= \sum_{j\in\Pi}\left(\lambda_j^{**} + \lambda_q^{**}\lambda_j^* t^{*-1}\right)y_{rj}^u - \left(\lambda_q^{**}S_r^{+^*}t^{*-1}\right) - S_r^{+^{**}} \end{aligned}$$

$$= \sum_{j\in\Pi} \bar{\lambda}_j y^u_{rj} - \bar{S}^+_r,$$

where $\bar{\lambda}_j = \lambda^{**}_j + \lambda^{**}_q \lambda^*_j t^{*-1},\ j \in \Pi$, and $\bar{S}^+_r = \lambda^{**}_q S^{+^*}_r t^{*-1} + S^{+^{**}}_r,\ r \in O$. For this $\bar{S}^+_r$,

$$\begin{aligned} t^* + \frac{1}{s}\sum_{r=1}^{s} \frac{\bar{S}^+_r}{y^u_{rq}} &= t^* + \frac{1}{s}\sum_{r=1}^{s} \frac{\frac{\lambda^{**}_q S^{+^*}_r}{t^*} + S^{+^{**}}_r}{y^u_{rq}} \\ &= t^* + \frac{1}{s}\sum_{r=1}^{s} \frac{S^{+^{**}}_r}{y^u_{rq}} + \frac{\lambda^{**}_q}{st^*}\sum_{r=1}^{s} \frac{S^{+^*}_r}{y^u_{rq}} \\ &= 1 + \lambda^{**}_q \frac{1-t^*}{t^*}, \quad \text{(using constraints of (2.1) and (3.2))} \\ &= 1 \text{ if and only if } \lambda^{**}_q = 0. \end{aligned}$$

Hence, $\bar{\lambda}_j = \lambda^{**}_j\ (j \in \Pi)$ and $\bar{S}^+_r = S^{+^{**}}_r\ (r \in O)$. Furthermore, for $\bar{S}^-_i = S^{-^{**}}_i (i \in I)$, we need to show that

$$t^* x^l_{iq} = \sum_{j\in\Pi} \lambda^{**}_j x^l_{ij} + \lambda^{**}_q x^l_{iq} + S^{-^{**}}_i = \sum_{j\in\Pi} \bar{\lambda}_j x^l_{ij} + \bar{S}^-_i,\ i \in I.$$

Since $\theta^{u**}_q < \hat{\theta}^u_q$, it implies

$$\begin{aligned} t^* - \frac{1}{m}\sum_{i=1}^{m} \frac{S^{-^{**}}_i}{x^l_{iq}} &< t^* - \frac{1}{m}\sum_{i=1}^{m} \frac{S^{-^*}_i}{x^l_{iq}} \\ \implies \sum_{i=1}^{m} \frac{S^{-^{**}}_i}{x^l_{iq}} &> \sum_{i=1}^{m} \frac{S^{-^*}_i}{x^l_{iq}} \\ \implies \sum_{i=1}^{m} \frac{S^{-^{**}}_i - S^{-^*}_i}{x^l_{iq}} &> 0. \end{aligned}$$

There exist some p for which $S^{-^{**}}_p > S^{-^*}_p$. Let $\Omega = \{i \in I : S^{-^{**}}_i > S^{-^*}_i\} \neq \emptyset$. For $i \in I - \Omega,\ S^{-^{**}}_i = S^{-^*}_i$. It follows that for $i \in \Omega$,

$$\begin{aligned} &\sum_{j\in\Pi} \lambda^{**}_j x^l_{ij} + S^{-^{**}}_i > \sum_{j\in\Pi} \lambda^{**}_j x^l_{ij} + S^{-^*}_i \\ &\implies t^* \sum_{j\in\Lambda} \alpha^{l^*}_{ij} > \sum_{j\in\Pi} \lambda^{**}_j x^l_{ij} + S^{-^*}_i \qquad \text{(using a constraint of model (2.1) and } \lambda^{**}_q = 0). \end{aligned}$$

For $i \in \Omega$, we can choose $k(i) \in \Lambda$ such that

$$t^* \sum_{j\in\Lambda} \bar{\alpha}^l_{ij} = \sum_{j\in\Pi} \lambda^{**}_j x^l_{ij} + S^{-^*}_i,$$

where

$$\bar{\alpha}^l_{ij} = \begin{cases} \alpha^{l^*}_{ik(i)} - \mu_i & j = k(i), \\ \alpha^{l^*}_{ij} & j \neq k(i), \end{cases} \tag{A}$$

for some $\mu_i > 0$ with $\alpha^{l^*}_{ik(i)} > 0$. Taking $\bar{S}^-_i = S^{-^*}_i (i \in \Omega),\ \bar{S}^-_i = S^{-^{**}}_i (i \in I - \Omega)$, then

$$t^* - \frac{1}{m}\sum_{i=1}^{m} \frac{\bar{S}^-_i}{x^l_{iq}} = t^* - \frac{1}{m}\left(\sum_{i\in\Omega} \frac{S^{-^*}_i}{x^l_{iq}} + \sum_{i\in I-\Omega} \frac{S^{-^{**}}_i}{x^l_{iq}}\right)$$

$$= t^* - \frac{1}{m}\left(m(t^* - \hat{\theta}_q^u) - \sum_{i\in I-\Omega}\frac{S_i^{-^*}}{x_{iq}^l} + \sum_{i\in I-\Omega}\frac{S_i^{-^{**}}}{x_{iq}^l}\right)$$

(using a constraint of model (3.2))

$$= \hat{\theta}_q^u + \sum_{i\in I-\Omega}\frac{S_i^{-^{**}} - S_i^{-^*}}{x_{iq}^l} = \hat{\theta}_q^u. \qquad (\text{as } S_i^{-^{**}} = S_i^{-^*},\ i \in I-\Omega).$$

Taking $\bar{S}_i^- = S_i^{-^*}\,(i\in\Omega)$, $\bar{S}_i^- = S_i^{-^{**}}\,(i\in I-\Omega)$, $\bar{S}_r^+ = S_r^{+^{**}}\,(r\in O)$, $\bar{\lambda}_j = \lambda_j^{**}\,(j\in\Pi)$, $\bar{\lambda}_q = 0$, it yields that $(\bar{\alpha}_{ij}^l\,(i\in\Omega,\ j\in\Lambda),\ \alpha_{ij}^{l^*}\,(i\in I-\Omega,\ j\in\Lambda),\ \bar{S}_i^-\,(i\in I),\ \bar{S}_r^+\,(r\in O),\ \bar{\lambda}_j\,(j\in\Pi), \bar{\lambda}_q)$ is feasible for (3.2) with (A); contradicting the Pareto optimal of $\alpha_{ij}^{l^*}\,(i\in I,\ j\in\Lambda)$ for (3.2). Thus, $\theta_q^{u**} = \hat{\theta}_q^u$, completing the proof. □

We put forward the following non-radial multiobjective model to estimate the upper bound of inputs for DMU_q to reach the lower efficiency level $\hat{\theta}_q^l$.

$$\begin{aligned}
&\min\ \left(\alpha_{ij}^u;\ \ i\in I,\ j\in\Lambda\right) && && (3.3)\\
&\text{s.t. } t + \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^+}{y_{rq}^l} = 1, && && (3.3.1)\\
&t - \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^-}{x_{iq}^u} = \hat{\theta}_q^l, && && (3.3.2)\\
&\sum_{j\in\Pi}\lambda_j x_{ij}^l + \lambda_q x_{iq}^u + S_i^- = t x_{iq}^u, && i\in I, && (3.3.3)\\
&\sum_{j\in\Pi}\lambda_j y_{rj}^u + \lambda_q y_{rq}^l - S_r^+ = t y_{rq}^l, && r\in O, && (3.3.4)\\
&x_{iq}^u = \sum_{j\in\Lambda}\alpha_{ij}^u, && i\in I, && (3.3.5)\\
&0 \le \alpha_{ij}^{l^*} \le \alpha_{ij}^u \le x_{ij}^u, && i\in I,\ j\in\Lambda && (3.3.6)\\
&t > 0,\ \lambda_j \ge 0,\ S_i^- \ge 0,\ S_r^+ \ge 0, && j\in\Pi\cup\{q\},\ i\in I,\ r\in O. && (3.3.7)
\end{aligned}$$

In the above model, $t,\ x_{iq}^u\,(i\in I)$, $\alpha_{ij}^u\,(i\in I,\ j\in\Lambda)$, $\lambda_j\,(j\in\Pi\cup\{q\})$, $S_i^-\,(i\in I)$, $S_r^+\,(r\in O)$ are unknown variables, and $\alpha_{ij}^{l^*}\,(i\in I,\ j\in\Lambda)$ is one of the Pareto optimal value of model (3.2) Here, we have directly written the constraint in linear form by taking $t^{-1} = 1 + \frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{rq}^l}$ and variables S_i^-, S_r^+, and λ_j are defined accordingly.

Analogous to Theorem 3.1, we have the following result for model (3.3).

Theorem 3.4. *If DMU_q is within the current optimistic PPS for interval data, then model (3.3) can be converted into the following multiobjective problem.*

$$\begin{aligned}
&\min\ \left(\alpha_{ij}^u;\ \ i\in I,\ j\in\Lambda\right) && && (3.4)\\
&\text{s.t. } (3.3.1),\ (3.3.2),\ (3.3.5),\ (3.3.6),\ (3.3.7), && &&\\
&\qquad\sum_{j\in\Pi}\lambda_j x_{ij}^l + S_i^- = t x_{iq}^u, && i\in I, &&\\
&\qquad\sum_{j\in\Pi}\lambda_j y_{rj}^u - S_r^+ = t y_{rq}^l, && r\in O. &&
\end{aligned}$$

Proof. The proof is analogous to the proof of Theorem 3.1. □

In the spirit of Theorems 3.3 and 3.4, we have the following theorem ensuring the lower bound on the efficiency of DMU_q.

Theorem 3.5. *Let the following hold:*

(A3) *The merged unit DMU_q is within the current optimistic PPS for interval data;*
(A4) $S^u = (t^*,\ x^{u*}_{iq}\,(i \in I),\ \alpha^{u*}_{ij}\,(i \in I, j \in \Lambda), \lambda^*_j\,(j \in \Pi), S^{-^*}_i\,(i \in I), S^{+^*}_r\,(r \in O))$ *is a Pareto optimal solution of* (3.4).

Then, the optimal value of model (2.2), *with* $x^u_{iq} = \sum_{j\in\Lambda} \alpha^{u*}_{ij}(i \in I)$, *and* $t = t^*$, *equals* $\hat{\theta}^l_q$.

Proof. See Appendix A for the proof. □

We next aim to estimate output values' lower and upper bounds to reduce inefficiency. The inefficiency decreases as the value of ϕ_q increases. Therefore, we estimate output values for a predefined inefficiency target, ensuring they are not lower than the inefficiency values obtained from models (2.3) and (2.4).

4. Estimation of output for the merged DMU

In this section, we present a non-radial inverse DEA model to estimate the lower and upper bounds of inherited output values. The model assumes that the inputs of the newly merged DMU are the sum of the inputs of the merging DMUs, in order to achieve a predefined efficiency target.

Suppose the input of DMU_q is the sum of inputs of merging DMUs *i.e.*, $x^l_{iq} = \sum_{j\in\Lambda} x^l_{ij}$, and $x^u_{iq} = \sum_{j\in\Lambda} x^u_{ij}$, $(i \in I)$. Let the predefined pessimistic inefficiency target $\hat{\phi}_q \in [\hat{\phi}^l_q, \hat{\phi}^u_q]$, such that $\hat{\phi}^l_q \geq \phi^{l^*}_q$, and $\hat{\phi}^u_q \geq \phi^{u*}_q$, where $\phi^{l^*}_q$, and ϕ^{u*}_q, are optimal values of problems (2.3) and (2.4), respectively.

We look out for the maximum inherited output $y_{rq} \in [y^l_{rq}, y^u_{rq}]$ with $y^l_{rq} = \sum_{j\in\Lambda} \beta^l_{rj}$ and $y^u_{rq} = \sum_{j\in\Lambda} \beta^u_{rj}$ to reach the predefined inefficiency target. Here $\beta^l_{rj},\ \beta^u_{rj}(r \in O, j \in \Lambda)$, are unknown variable with $\beta^l_{rj} \geq 0$ with at least one $\beta^l_{rj} > 0$ for some $r \in O$ and $j \in \Lambda$.

To estimate the upper bound for outputs for achieving $\hat{\phi}^u_q$, we propose the following slack-based multiobjective optimization problem:

$$\min\ \left(\beta^u_{rj};\ \ r \in O,\ j \in \Lambda\right) \tag{4.1}$$

$$\text{s.t.}\ \frac{1 - \frac{1}{s}\sum_{r=1}^{s} \frac{s^-_r}{y^u_{rq}}}{1 + \frac{1}{m}\sum_{i=1}^{m} \frac{s^+_i}{x^l_{iq}}} = \frac{1}{\hat{\phi}^u_q}, \tag{4.1.1}$$

$$\sum_{j\in\Pi} \mu_j x^u_{ij} + \mu_q x^l_{iq} - s^+_i = x^l_{iq}, \qquad i \in I \tag{4.1.2}$$

$$\sum_{j\in\Pi} \mu_j y^l_{rj} + \mu_q y^u_{rq} + s^-_r = y^u_{rq}, \qquad r \in O, \tag{4.1.3}$$

$$y^u_{rq} = \sum_{j\in\Lambda} \beta^u_{rj}, \qquad r \in O, \tag{4.1.4}$$

$$\beta^u_{rj} \geq y^u_{rj}, \qquad r \in O,\ j \in \Lambda, \tag{4.1.5}$$

$$\mu_j \geq 0,\ s^+_i \geq 0,\ s^-_r \geq 0, \qquad j \in \Pi \cup \{q\},\ r \in O. \tag{4.1.6}$$

Here, $x^l_{iq}\,(i \in I)$, α^l_{ij}, $(i \in I, j \in \Lambda)$, $\mu_j\,(j \in \Pi \cup \{q\})$, $s^-_i\,(i \in I)$, $s^+_r\,(r \in O)$ are unknown variables, and s^-_i, s^+_r, are input and output slacks respectively.

The following theorem follows on the similar lines of Theorem 3.1.

Theorem 4.1. *If DMU_q lies within the current optimistic PPS for interval-valued data, model (4.1) can be reformulated as following multiobjective nonlinear model. (The first constraint, originally in ratio form, is linearized by setting $t^{-1} = 1 + \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^+}{x_{iq}^l}$.*

$$\begin{aligned}
&\min\ \left(\beta_{rj}^u;\ \ r \in O,\ j \in \Lambda\right) && (4.2)\\
&\text{s.t. } t - \frac{1}{s}\sum_{r=1}^{s} \frac{S_r^-}{y_{rq}^u} = \frac{1}{\hat{\phi}_q^u},\\
&\quad t + \frac{1}{m}\sum_{i=1}^{m} \frac{S_i^+}{x_{iq}^l} = 1,\\
&\quad \sum_{j\in\Pi} \lambda_j x_{ij}^u - S_i^+ = t x_{iq}^l, \qquad i \in I,\\
&\quad \sum_{j\in\Pi} \lambda_j y_{rj}^l + S_r^- = t y_{rq}^u, \qquad r \in O,\\
&\quad (4.1.4),\ (4.1.5),\ (4.1.6).
\end{aligned}$$

Here, t, $y_{rq}^u\,(r \in O)$, $\beta_{rj}^u\,(r \in O, j \in \Lambda)$, $\lambda_j\,(j \in \Pi)$, $S_i^+\,(i \in I)$, $S_r^-(r \in O)$ are unknown variables, and $S_i^+ = t\,s_i^+$, $S_r^- = t\,s_r^-$, and $\lambda_j = t\,\mu_j$.

Proof. The proof is analogous to the proof of Theorem 3.1. □

Theorem 4.2. *If the following assumptions hold:*

(B1) *The merged unit DMU_q is within the current pessimistic PPS for interval data;*
(B2) *$\Delta^u = (t^*,\ y_{rq}^{u*}\,(r \in O),\ \beta_{rj}^{u*}\,(r \in O, j \in \Lambda),\ \lambda_j^*\,(j \in \Pi),\ S_i^{-^*}\,(i \in I),\ S_r^{+^*}\,(r \in O))$ is a Pareto optimal solution of (4.2).*

Then, the optimal value of model (2.3), with $y_{rq}^u = \sum_{j\in\Lambda} \beta_{rj}^{u*}(r \in O)$, and $t = t^*$, equals $\hat{\phi}_q^u$.

Proof. Here, in our arguments, we consider model (2.3) by setting $y_{rq}^u = \sum_{j\in\Lambda} \beta_{rj}^{u*}(r \in O)$ and $t = t^*$. Let $(\lambda_j^{**}\,(j \in \Pi),\ \lambda_q^{**},\ S_i^{+^{**}}\,(i \in I),\ S_r^{-^{**}}\,(r \in O),\ \phi_q^{u**})$ be the optimal solution of (2.3). Since Δ^u is feasible for (4.2), it is feasible for (2.3), hence, $\phi_q^{u**} \geq \hat{\phi}_q^u$.

Furthermore, by assumption, $B1$, the current pessimistic PPS, and the inefficient frontier remain unaltered on merging the units, and the outputs increase $\beta_{rj}^u \geq y_{rj}^u(r \in O,\ j \in \Lambda)$, while the inputs remain the same in models (4.2) and (2.3), hence at their optimal solutions, $S_r^{-^{**}} \geq S_r^{-^*}(r \in O)$.

Suppose $\phi_q^{u**} > \hat{\phi}_q^u$. Applying optimal solutions of (2.3), and (4.2), we have

$$\begin{aligned}
t^* x_{iq}^l &= \sum_{j\in\Pi} \lambda_j^{**} x_{ij}^u + \sum_{j\in\Lambda} \lambda_q^{**} x_{rj}^l - S_i^{+^{**}}\\
&= \sum_{j\in\Pi} \lambda_j^{**} x_{ij}^u + \lambda_q^{**} \sum_{j\in\Pi} \lambda_j^* \left(\frac{x_{ij}^u - S_i^{+^*}}{t^*}\right) - S_i^{+^{**}}\\
&= \sum_{j\in\Pi} \left(\lambda_j^{**} + \lambda_q^{**}\lambda_j^* t^{*-1}\right) x_{ij}^u - \left(\lambda_q^{**} S_i^{+^*}\right) t^{*-1} - S_i^{+^{**}}\\
&= \sum_{j\in\Pi} \bar{\lambda}_j x_{ij}^l - \bar{S}_i^+,
\end{aligned}$$

where $\bar{\lambda}_j = \lambda_j^{**} + \lambda_q^{**}\lambda_j^{*}t^{*-1}\ (j \in \Pi)$, and $\bar{S}_i^{+} = \lambda_q^{**}S_i^{+^*}t^{*-1} + S_i^{+^{**}}\ (i \in I)$. It follows from here that

$$\begin{aligned} t^* + \frac{1}{m}\sum_{i=1}^{m}\frac{\bar{S}_i^{+}}{x_{iq}^{l}} &= t^* + \frac{1}{m}\sum_{i=1}^{m}\frac{\left(\lambda_q^{**}S_i^{+^*}\right)t^{*-1} + S_i^{+^{**}}}{x_{iq}^{l}} \\ &= t^* + \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^{+^{**}}}{x_{iq}^{l}} + \frac{\lambda_q^{**}}{mt^*}\sum_{i=1}^{m}\frac{S_i^{+^{**}}}{x_{iq}^{l}} \\ &= 1 + \lambda_q^{**}\frac{1-t^*}{t^*} = 1 \text{ if and only if } \lambda_q^{**} = 0, \end{aligned}$$

hence, $\bar{\lambda}_j = \lambda_j^{**}(j \in \Pi)$ and $\bar{S}_i^{+} = S_i^{+^{**}}(i \in I)$. Furthermore, for $\bar{\lambda}_j = \lambda_j^{**}(j \in \Pi)$ and $\bar{S}_r^{-} = S_r^{-^{**}}(r \in O)$, we can work out to show that $t^* y_{rq}^{u} = \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^{u} + y_{rq}^{u}\lambda_q^{**} + S_r^{-^{**}} = \sum_{j\in\Pi}\bar{\lambda}_j y_{rj}^{u} + \bar{S}_r^{-},\ r \in O$.

Since $\phi_q^{u**} > \hat{\phi}_q^{u}$, it implies

$$\begin{aligned} &\frac{1}{t^* - \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^{-^{**}}}{y_{rq}^{u}}} > \frac{1}{t^* - \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^{-^{*}}}{y_{rq}^{u}}} \\ &\implies \sum_{r=1}^{s}\frac{S_r^{-^{**}}}{y_{rq}^{u}} > \sum_{r=1}^{s}\frac{S_r^{-^{*}}}{y_{rq}^{u}} \\ &\implies \sum_{r=1}^{s}\left(\frac{S_r^{-^{**}}}{y_{rq}^{u}} - \frac{S_r^{-^{*}}}{y_{rq}^{u}}\right) > 0. \end{aligned}$$

There exists at least one $k \in O$ such that $S_k^{-^{**}} > S_k^{-^{*}}$.

Let $\Omega_1 = \{r \in O : S_r^{-^{**}} > S_r^{-^{*}}\} \neq \phi$.

For $r \in O - \Omega_1, S_r^{-^{**}} = S_r^{-^{*}}$. It follows that for $r \in \Omega_1$,

$$\begin{aligned} &\implies S_r^{-^{**}} > S_r^{-^{*}} \\ &\implies \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^{l} + S_r^{-^{**}} > \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^{l} + S_r^{-^{*}} \\ &\implies t^*\sum_{j\in\Lambda}\beta_{rj}^{u*} > \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^{l} + S_r^{-^{*}}. \end{aligned}$$

For $r \in \Omega_1$, we can choose $k(r) \in \Lambda$ such that

$$\begin{aligned} t^*\sum_{j\in\Lambda}\bar{\beta}_{rj}^{u} &= \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^{u} + S_r^{-^{*}}, \\ \bar{\beta}_{rj}^{u} &= \begin{cases} \beta_{rk(r)}^{u*} - \mu_r, & j = k(r), \\ \beta_{rj}^{u*}, & j \neq k(r), \end{cases} \end{aligned} \tag{C}$$

for some $\mu_r > 0$ with $\beta_{rk(r)}^{u*} > 0$. Taking $\bar{S}_r^{-} = S_r^{-^{*}}(r \in \Omega_1),\ \bar{S}_r^{-} = S_r^{-^{**}}(r \in O - \Omega_1)$, then

$$\begin{aligned} t^* - \frac{1}{s}\sum_{r=1}^{s}\frac{\bar{S}_r^{-}}{y_{rq}^{u}} &= t^* - \frac{1}{s}\left(\sum_{r\in\Omega_1}\frac{S_r^{-^{*}}}{y_{rq}^{u}} + \sum_{r\in O-\Omega_1}\frac{S_r^{-^{**}}}{y_{rq}^{u}}\right) \\ &= t^* - \frac{1}{s}\left(s\left(t^* - \frac{1}{\hat{\phi}_q^{u}}\right) - \sum_{r\in O-\Omega_1}\frac{S_r^{-^{*}}}{y_{rq}^{u}} + \sum_{r\in O-\Omega_1}\frac{S_r^{-^{**}}}{y_{rq}^{u}}\right) \end{aligned}$$

$$(\text{using a constraint of model (4.2)})$$
$$= \frac{1}{\hat{\phi}_q^u} + \sum_{r\in O-\Omega_1} \frac{S_r^{-^{**}} - S_r^{-^*}}{y_{rq}^u} = \frac{1}{\hat{\phi}_q^u} \qquad (\text{as } S_r^{-^{**}} = S_r^{-^*},\ r\in O-\Omega_1).$$

Taking $\bar{S}_i^+ = S_i^{+^{**}}\ (i\in I), \bar{S}_r^+ = S_r^{+^*}\ (r\in\Omega_1),\ \bar{S}_r^- = S_r^{-^{**}}\ (r\in O-\Omega_1),\ \bar{\lambda}_j = \lambda_j^{**}\ (j\in\Pi),\ \bar{\lambda}_q = 0$, it yields that $(\bar{\beta}_{rj}^u,\ (r\in\Omega_1,\ j\in\Lambda),\ \beta_{rj}^{u*}\ (r\in O-\Omega_1,\ j\in\Lambda),\ \bar{S}_i^+\ (i\in I),\ \bar{S}_r^-\ (r\in O),\ \bar{\lambda}_j\ (j\in\Pi),\ \bar{\lambda}_q)$ is feasible for (4.2) with (C); contradicting the Pareto optimality of β_{rj}^{u*}, $(r\in O,\ j\in\Lambda)$ for (4.2). Thus, $\phi_q^{u**} = \hat{\phi}_q^u$, completing the proof. □

To estimate the lower bound of output values of DMU_q for reaching the lower inefficiency level $\hat{\phi}_q^l$, we proposed the following non-radial model:

$$\min\ \left(\beta_{rj}^l,\quad r\in O,\ j\in\Lambda\right) \tag{4.3}$$
$$\text{s.t. } t - \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^-}{y_{rq}^l} = \frac{1}{\hat{\phi}_q^l}, \tag{4.3.1}$$
$$t + \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^+}{x_{iq}^u} = 1, \tag{4.3.2}$$
$$\sum_{j\in\Pi}\lambda_j x_{ij}^u + \lambda_q x_{iq}^u - S_i^+ = t x_{iq}^u, \qquad i\in I, \tag{4.3.3}$$
$$\sum_{j\in\Pi}\lambda_j y_{rj}^l + \lambda_q y_{rq}^l + S_r^- = t y_{rq}^l, \qquad r\in O, \tag{4.3.4}$$
$$y_{rq}^l = \sum_{j\in\Lambda}\beta_{rj}^l, \qquad r\in O, \tag{4.3.5}$$
$$y_{rj}^l \le \beta_{rj}^l \le \beta_{rj}^{u*}, \qquad r\in O,\ j\in\Lambda, \tag{4.3.6}$$
$$t>0,\ \lambda_j\ge 0,\ S_i^+\ge 0,\ S_r^-\ge 0 \qquad j\in\Pi\cup\{q\},\ i\in I,\ r\in O. \tag{4.3.7}$$

In the above model, $t,\ y_{rq}^l(r\in O),\ \beta_{rj}^l\ (r\in O,\ j\in\Lambda),\ \lambda_j\ (j\in\Pi\cup\{q\}),\ S_i^+\ (i\in I),\ S_r^-\ (r\in O)$ are unknown variables, and $\beta_{rj}^{u*}(\,r\in O,\ j\in\Lambda)$ is one of the Pareto optimal value of model (4.2). Here we have directly written the constraint in linear form by taking $t^{-1} = 1 + \frac{1}{m}\sum_{i=1}^{m}\frac{s_i^+}{x_{iq}^u}$, and variables $S_i^+,\ S_r^-, \lambda_j$ are defined accordingly.

Theorem 4.3. *If DMU_q is within the current pessimistic PPS for interval data, then the model (4.3) can be converted into the following model:*

$$\min\ \left(\beta_{rj}^l,\quad r\in O,\ j\in\Lambda\right) \tag{4.4}$$
$$\text{s.t. } (4.3.1),\ (4.3.2),\ (4.3.5),\ (4.3.6),\ (4.3.7),$$
$$\sum_{j\in\Pi}\lambda_j x_{ij}^u - S_i^+ = t x_{iq}^u, \qquad i\in I,$$
$$\sum_{j\in\Pi}\lambda_j y_{rj}^l + S_r^- = t y_{rq}^l, \qquad r\in O.$$

Proof. The proof is analogous to the proof of Theorem 3.1. □

Theorem 4.4. *If the following assumptions hold:*

(B3) *The merged DMU_q is within the current pessimistic PPS for interval data;*

(B4) $\Delta^l = (t^*,\ y^{l^*}_{rq}\,(r \in O),\ \beta^{l^*}_{rj}(r \in O,\ j \in \Lambda),\ \lambda^*_j\,(j \in \Pi),\ S^{-^*}_i\,(i \in I),\ S^{+^*}_r\,(r \in O))$ *is a Pareto optimal solution of* (4.4).

Then, the optimal value of model (2.4)*, with* $y^l_{rq} = \sum_{j\in\Lambda} \beta^{l^*}_{rj}(r \in O)$*, and* $t^* = t$*, equals* $\hat{\phi}^l_q$.

Proof. The proof is in Appendix A. □

In the next section, we determine the minimum and maximum achievable efficiency targets to predict the position of the merged unit.

5. Minimum and maximum achievable efficiency target

5.1. Case of optimistic efficiencies

Understanding the lowest and highest achievable efficiency targets is crucial for the decision-maker in a merger. To achieve these goals, let the maximum values of the lower and upper bounds of the achievable efficiency targets for the merged DMU be denoted by $\theta^{l-\max}_q$ and $\theta^{u-\max}_q$, respectively. Additionally, assume that the minimum values of the lower and upper bounds of achievable efficiency targets for the merged DMU are denoted by $\theta^{l-\min}_q$ and $\theta^{u-\min}_q$ respectively. In other words, $[\theta^{l-\min}_q,\ \theta^{l-\max}_q]$ and $[\theta^{u-\min}_q,\ \theta^{u-\max}_q]$ represents the minimum and maximum levels of attainable efficiency through this merger. The decision-maker is encouraged to merge if these minimum and maximum levels are satisfactory. This section identifies the attainable minimum and maximum levels through the merged DMU.

The following theorem determines the upper and lower bounds of θ^u_q for the merged DMU.

Theorem 5.1. *Suppose that the model* (3.2) *is feasible for the efficiency target* $\hat{\theta^u_q}$.

(a) *If* $x_{iq} = \sum_{j\in\Lambda} x_{ij}\,(i \in I)$ *and* $y_{rq} = \sum_{j\in\Lambda} y_{rj}\,(r \in O)$ *is in the current PPS, and* θ^{u*}_q *is optimal value of model* (2.1) *then* $\theta^{u*}_q \le \hat{\theta^u_q}$.

(b) *Model* (3.2) *remains feasible for each efficiency upper bound target* $\hat{\theta^u_q} \le \bar{\theta^u_q} < 1$*, and equal to 1 only if the input and output values of new merged DMU can be written as a positive linear combination of inputs and outputs of the non-merging DMUs respectively.*

Proof. Let $S^l = (t^*, \alpha^{l^*}_{ij}\,(i \in I,\, j \in \Lambda),\ \lambda^*_j\,(j \in \Pi),\ S^{-^*}_i\,(i \in I),\ S^{+^*}_r\,(r \in O))$ be a Pareto optimal solution of model (3.2). We have

$$
\begin{aligned}
& t^* - \frac{1}{m}\sum_{i=1}^{m} \frac{S^{-^*}_i}{x^l_{iq}} = \hat{\theta^u_q}, && \\
& t^* + \frac{1}{s}\sum_{r=1}^{s} \frac{S^{+^*}_r}{y^u_{rq}} = 1, && \\
& \sum_{j\in\Pi} \lambda^*_j x^l_{ij} + S^{-^*}_i = t^* x^l_{iq}, && i \in I, \\
& \sum_{j\in\Pi} \lambda^*_j y^u_{rj} - S^{+^*}_r = t^* y^u_{rq}, && r \in O, \\
& \alpha^{l^*}_{ij} \le x^l_{ij}, && i \in I,\ j \in \Lambda, \\
& x^l_{iq} = \sum_{j\in\Lambda} \alpha^{l^*}_{ij}, && i \in I, \\
& t^* > 0,\ \lambda^*_j \ge 0,\ S^{-^*}_i \ge 0,\ S^{+^*}_r \ge 0, && j \in \Pi,\ i \in I,\ r \in O.
\end{aligned}
$$

It implies that $(t^*,\ \hat{\theta_q^u},\ \lambda_j^*\,(j\in\Pi),\ {S_i^-}^*\,(i\in I),\ {S_r^+}^*\,(r\in O))$ is feasible for (2.1) and hence ${\theta_q^u}^* \leq \hat{\theta_q^u}$, yielding the minimum value of $\hat{\theta_q^u}$ is ${\theta_q^u}^*$.

For part (b), note that model (3.2) is nonlinear; an increase in the value of $\hat{\theta}_q^u$ leads to a decrease in x_{iq}^l and $S_i^-(i\in I)$, while the value of t increases. Consequently, under CRS technology, the value of $\lambda_j(j\in\Pi)$ will change to achieve the desired efficiency level. This efficiency will increase to 1 when, for all i and r, $S_i^-=0$ and $S_r^+=0$. Thus, $\hat{\theta}_q^u=1$ if the input and output values of the newly merged DMU can be expressed as a positive linear combination of the inputs and outputs of the non-merging DMUs, respectively. □

The following theorem determines the upper and lower bounds of $\hat{\theta_q^l}$ for the merged DMU.

Theorem 5.2. *Suppose that model (3.4) is feasible for the efficiency target* $\hat{\theta_q^l}$.

(i) *If* $x_{iq}=\sum_{j\in\Lambda}x_{ij}\,(i\in I)$ *and* $y_{rq}=\sum_{j\in\Lambda}y_{rj}\,(r\in O)$ *is in the current PPS, and* ${\theta_q^l}^*$ *is optimal solution for* (2.2), *then* ${\theta_q^l}^*\leq\hat{\theta_q^l}$.

(ii) *If*

$$
\begin{aligned}
&\min\ \theta_q^{l-\max} = t-\frac{1}{m}\sum_{i=1}^{m}\frac{S_i^-}{x_{iq}^u} && (5.2)\\
&\text{s.t. } t+\frac{1}{s}\sum_{r=1}^{s}\frac{S_r^+}{y_{rq}^l}=1,\\
&\quad\sum_{j\in\Pi}\lambda_j x_{ij}^l+S_i^-=tx_{iq}^u, && i\in I,\\
&\quad\sum_{j\in\Pi}\lambda_j y_{rj}^u-S_r^+=ty_{rq}^l, && r\in O,\\
&\quad x_{iq}^u=\sum_{j\in\Lambda}{\alpha_{ij}^l}^*, && i\in I,\\
&\quad t>0,\ \lambda_j\geq 0,\ S_i^-\geq 0,\ S_r^+\geq 0, && j\in\Pi,\ i\in I,\ r\in O.
\end{aligned}
$$

Here, ${\alpha_{ij}^l}^*\,(i\in I,\ j\in\Lambda)$ *is Pareto optimal solution of* (3.2) *for maximum possible value of* $\hat{\theta_q^u}$. *Then* (3.4) *remains feasible for* $\hat{\theta_q^l}\leq\bar{\theta}_q^l\leq\theta_q^{l-\max}$.

Proof. (i) can be proved on similar lines of part (a) Theorem 5.1.

For (ii), let $(\hat{t},\ \hat{\alpha_{ij}^u}\,(i\in I,\ j\in\Lambda),\ \hat{\lambda_j}\,(j\in\Pi),\ \hat{S_i^-}\,(i\in I),\ \hat{S_r^+}\,(r\in O))$ be Pareto optimal solution of (3.4) and $(t^*,\ \theta_q^{l-\max},\ {\lambda_j}^*(j\in\Pi),\ {S_i^-}^*(i\in I),\ {S_r^+}^*(r\in O))$ be optimal solution of (5.2). Then, it is easy to see that ${S_i^-}^*\leq\hat{S_i^-}(i\in I),\ {S_r^+}^*=\hat{S_r^+}\,(r\in O),\ \lambda_j^*=\hat{\lambda}_j\,(j\in\Pi),\ \hat{t}=t^*$, and

$$
\begin{aligned}
&{\alpha_{ij}^l}^*\leq\hat{\alpha_{ij}^u}, && i\in I,\ j\in\Lambda,\\
\Longrightarrow\ &\left(\sum_{j\in\Lambda}{\alpha_{ij}^l}^*\right)^{-1}\geq\left(\sum_{J\in\Lambda}\hat{\alpha_{ij}^u}\right)^{-1}, && i\in I,\\
\Longrightarrow\ &t^*-\frac{1}{m}\sum_{i=1}^{m}\frac{\hat{S_i^-}}{\sum_{J\in\Lambda}\hat{\alpha_{ij}^u}}\leq t^*-\frac{1}{m}\sum_{i=1}^{m}\frac{{S_i^-}^*}{\sum_{j\in\Lambda}{\alpha_{ij}^l}^*}, && i\in I,\\
\Longrightarrow\ &\hat{\theta_q^l}\leq\theta_q^{l-\max}
\end{aligned}
$$

which shows that the maximum value of $\hat{\theta_q^l}$ is $\theta_q^{l-\max}$. □

5.2. Case of pessimistic inefficiencies

Let the maximum values of the lower and upper bounds of the achievable inefficiency targets for the merged DMU be denoted by $\phi_q^{l-\max}$ and $\phi_q^{u-\max}$, respectively. Additionally, assume that the minimum values of the lower and upper bounds of achievable inefficiency targets for the merged DMU are denoted by $\phi_q^{l-\min}$ and $\phi_q^{u-\min}$, respectively.

It is assumed that the minimum and maximum levels of attainable inefficiency through this merger are represented by $[\phi_q^{l-\min}, \phi_q^{l-\max}]$ and $[\phi_q^{u-\min}, \phi_q^{u-\max}]$, respectively. The decision-maker is encouraged to merge if these minimum and maximum levels are satisfactory. This section identifies the attainable minimum and maximum levels through the merged DMU.

The following theorem determines the upper and lower bounds of ϕ_q^u for the merged DMU.

Theorem 5.3. *Suppose that model (4.2) is feasible for the efficiency target* $\hat{\phi}_q^u$.

(1) *If* $x_{iq} = \sum_{j\in\Lambda} x_{ij}$ $(i \in I)$ *and* $y_{rq} = \sum_{j\in\Lambda} y_{rj}$ $(r \in O)$ *is in the current PPS, and* ϕ_q^{u*} *is optimal solution for (2.3), then* $\phi_q^{u*} \le \hat{\phi}_q^u$.

(2) *If*

$$\min \frac{1}{\phi_q^{u-\max}} = t - \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^-}{y_{rq}^u} \tag{5.3}$$

$$\text{s.t. } t + \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^+}{x_{iq}^l} = 1, \tag{5.3.1}$$

$$\sum_{j\in\Pi}\lambda_j x_{ij}^u - S_i^+ = t x_{iq}^l, \quad i \in I, \tag{5.3.2}$$

$$\sum_{j\in\Pi}\lambda_j y_{rj}^l + S_r^- = t y_{rq}, \quad r \in O, \tag{5.3.3}$$

$$t' - \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^-}{x_{iq}^l} \le 1, \tag{5.3.4}$$

$$t' + \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^+}{y_{rq}^u} = 1, \tag{5.3.5}$$

$$\sum_{j\in\Pi}\lambda'_j x_{ij}^l + S_i^- = t' x_{iq}^l, \quad i \in I, \tag{5.3.6}$$

$$\sum_{j\in\Pi}\lambda'_j y_{rj}^u - S_r^+ = t' y_{rq}^u, \quad r \in O, \tag{5.3.7}$$

$$x_{iq}^l = \sum_{j\in\Lambda} x_{ij}^l, \quad i \in I, \tag{5.3.8}$$

$$y_{rq}^u = \sum_{j\in\Lambda}\beta_{rj}^u, \quad r \in O, \tag{5.3.9}$$

$$\beta_{rj}^u \ge y_{rj}^u \quad r \in O,\ j \in \Lambda, \tag{5.3.10}$$

$$t > 0,\ \lambda_j \ge 0,\ S_i^+ \ge 0,\ S_r^- \ge 0, \quad j \in \Pi,\ i \in I,\ r \in O. \tag{5.3.11}$$

$$t' > 0\ \lambda'_j \ge 0,\ \ S_i^- \ge 0,\ S_r^+ \ge 0, \quad j \in \Pi,\ i \in I,\ r \in O. \tag{5.3.12}$$

Then (4.2) remains feasible for $\hat{\phi}_q^u \le \bar{\phi}_q^u \le \phi_q^{u-\max}$.

Proof. Assume that the model (4.2) is feasible for the inefficiency upper bound target $\hat{\phi}_q^u$. Let ϕ_q^{u*} be an optimal solution of (2.3). We need to show that $\phi_q^{u*} \le \hat{\phi}_q^u$. Now, if $S^l = (t^*,\ \beta_{rj}^{u*}\ (r \in O,\ j \in \Lambda),\ \lambda_j^*\ (j \in \Pi),\ S_i^{+^*}\ (i \in$

I), $S_r^{-^*}$ ($r \in O$)) is a Pareto optimal solution of model (4.2), then

$$
\begin{aligned}
& t^* + \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^{+^*}}{x_{iq}^l} = 1 & \\
& t^* - \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^{-^*}}{y_{rq}^u} = \frac{1}{\hat{\phi}_q^u}, & \\
& \sum_{j\in\Pi}\lambda_j^* x_{ij}^u - S_i^{+^*} = t^* x_{iq}^l, & i \in I, \\
& \sum_{j\in\Pi}\lambda_j^* y_{rj}^l + S_r^{-^*} = t^* y_{rq}^u, & r \in O, \\
& y_{rq}^u = \sum_{j\in\Lambda}\beta_{rj}^{u*}, & r \in O, \\
& \beta_{rj}^{u*} \geq y_{rj}^u & r \in O,\ j \in \Lambda, \\
& t^* > 0,\ \lambda_j^* \geq 0,\ S_i^{+^*} \geq 0,\ S_r^{-^*} \geq 0, & j \in \Pi,\ i \in I,\ r \in O.
\end{aligned}
$$

It implies (t^*, $\hat{\phi}_q^l$, λ_j^* ($j \in \Pi$), $S_i^{+^*}$ ($i \in I$), $S_r^{-^*}$ ($r \in O$)) is feasible for (2.3), giving $\phi_q^{u*} \leq \hat{\phi}_q^u$.

For the next part, model (4.2) remains feasible for any $\bar{\phi}_q^u \geq \hat{\phi}_q^u$. However, increase in $\hat{\phi}_q^u$ increases $\beta_{rj}^u (r \in O, j \in \Pi)$, which in turn raises the output. This increase may cause the merged unit to fall outside the optimistic PPS. Constraints (5.3.4)–(5.3.7) and (5.3.12) ensure that the merged unit stays within the optimistic PPS. □

The following theorem provides the upper and lower bounds of ϕ_q^l for the merged DMU.

Theorem 5.4. *Suppose that the model (4.4) is feasible for the inefficiency target $\hat{\phi}_q^l$. Then*

(1) *If $x_{iq} = \sum_{j\in\Lambda} x_{ij}$ ($i \in I$) and $y_{rq} = \sum_{j\in\Lambda} y_{rj}$ ($r \in O$) is in the current PPS, and $\phi_q^{l^*}$ is optimal solution of model (2.4) then $\phi_q^{l^*} \leq \hat{\phi}_q^l$.*

(2) *If*

$$
\begin{aligned}
\min\ & \frac{1}{\phi_q^{l-\max}} = t - \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^-}{y_{rq}^l} & \\
\text{s.t. } & t + \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^+}{x_{iq}^u} = 1, & \\
& \sum_{j\in\Pi}\lambda_j x_{ij}^u - S_i^+ = t x_{iq}^u, & i \in I, \\
& \sum_{j\in\Pi}\lambda_j y_{rj}^l + S_r^- = t y_{rq}^l, & r \in O, \\
& x_{iq}^u = \sum_{j\in\Lambda} x_{ij}^u, & i \in I, \\
& y_{rq}^l = \sum_{j\in\Lambda}\beta_{rj}^{u*}, & r \in O, \\
& t > 0,\ \lambda_j \geq 0,\ S_i^+ \geq 0,\ S_r^- \geq 0, & j \in \Pi,\ i \in I,\ r \in O
\end{aligned}
\tag{5.4}
$$

where β_{rj}^{u*} ($r \in O$, $j \in \Lambda$) is a Pareto optimal solution of (4.2) for maximum value of $\hat{\phi}_q^u$. Then Model (4.4) remains feasible for each inefficiency lower bound target $\hat{\phi}_q^l \leq \phi_q^{l-\max}$.

TABLE 1. Input-output data of eleven DMUs and their interval efficiency scores.

DMUs	Input	Output	SBMOE	SBNPE
A	$[1,3]$	$[0.5,2]$	$[0.08,1.00]$	$[1.00,12.00]$
B	$[2,4]$	$[2,4]$	$[0.25,1.00]$	$[3.00,12.00]$
C	$[2,4]$	$[1,3]$	$[0.13,0.75]$	$[1.50,9.00]$
D	$[3,5]$	$[2,5]$	$[0.20,0.83]$	$[2.40,10.00]$
E	$[4,6]$	$[3,5]$	$[0.25,0.63]$	$[3.00,7.50]$
F	$[4,6]$	$[1,3]$	$[0.08,0.38]$	$[1.00,4.50]$
G	$[5,7]$	$[2,4]$	$[0.14,0.40]$	$[1.71,4.80]$
H	$[7,9]$	$[4,6]$	$[0.22,0.43]$	$[2.67,5.14]$
I	$[7,9]$	$[6,8]$	$[0.33,0.57]$	$[4.00,6.86]$
J	$[8,10]$	$[5,7]$	$[0.25,0.44]$	$[3.00,5.25]$
K	$[10,12]$	$[6,8]$	$[0.25,0.40]$	$[3.00,4.80]$

Proof. The proof of the first part of the theorem is same as the first part of Theorem 5.3. For the proof of the second part let $(\hat{t},\, \hat{\beta^l_{rj}}\,(r\in O,\, j\in\Lambda),\ \hat{\lambda_j}\,(j\in\Pi),\ \hat{S^+_i}\,(i\in I),\ \hat{S^-_r}\,(r\in O))$ be Pareto optimal solution of (4.4) and $(t^*,\, \phi_q^{l-\max},\ {\lambda_j}^*\,(j\in\Pi), {S_i^+}^*\,(i\in I),\ {S_r^-}^*\,(r\in O))$ be the optimal solution of (5.3). Then ${S_i^+}^* = \hat{S_i^+}(i\in I)$, ${S_r^-}^* \geq \hat{S_r^-}\,(r\in O)$, $\lambda_j^* = \hat{\lambda_j}\,(j\in\Pi)$, $\hat{t} = t^*$ and

$$\begin{aligned}
&\hat{\beta^l_{rj}} \leq \beta^{u*}_{rj}, && r\in O,\ j\in\Lambda,\\
&\implies \left(\sum_{j\in\Lambda}\hat{\beta^l_{rj}}\right)^{-1} \geq \left(\sum_{J\in\Lambda}\beta^{u*}_{rj}\right)^{-1}, && r\in O,\\
&\implies t^* - \frac{1}{s}\sum_{r=1}^{s}\frac{{S_r^-}^*}{\sum_{J\in\Lambda}\beta^{u*}_{rj}} \leq t^* - \frac{1}{s}\sum_{r=1}^{s}\frac{\hat{S_r^-}}{\sum_{j\in\Lambda}\hat{\beta^l_{rj}}}, && r\in O,\\
&\implies \phi_q^{l-\max} \geq \hat{\phi^l_q}.
\end{aligned}$$

□

We illustrate the above concept with a numerical example. Consider eleven DMUs, labeled A to K, each with one input and one output recorded in Table 1. We apply the optimistic and pessimistic SBM models, and the empirical outcomes are reported in Table 1, where SBMOE represents the SBM optimistic efficiency, and SBMPE denotes the SBM pessimistic inefficiency obtained from the models (2.1)–(2.4).

Figure 1 illustrates the optimistic and pessimistic data points and their respective frontiers. Suppose DMUs F and G merge to create a new DMU L. If we simply add the inputs and outputs of DMUs F and G, the input-output combination for DMU L will be $[9,13]$ and $[3,7]$, respectively, which belong to both the optimistic and pessimistic PPS. By applying models (2.1)–(2.4), we get the optimistic efficiency and pessimistic inefficiency intervals for DMU L, as $[0.115,0.388]$ and $[1.38,4.66]$, respectively.

Using the models of Section 5, for unit $L\,(:=q)$, the range of $\hat{\theta^l_q}$ and $\hat{\theta^u_q}$ are $[0.115,0.428]$, and $[0.388,1]$. The range for $\hat{\phi^l_q}$ and $\hat{\phi^u_q}$ are $[1.38,8.3]$ and $[4.66,12]$.

Remark 5.5. The upper bound of $\hat{\theta^u_q}$ is 1, as $x^l_{iq}=3.5$ and $y^u_{rq}=7$ can be represented as a linear combination of inputs and outputs of the non-merging DMUs with $\lambda_2=1.75$ and $\lambda_j=0,\ j\,(\neq 2)\in\Pi$.

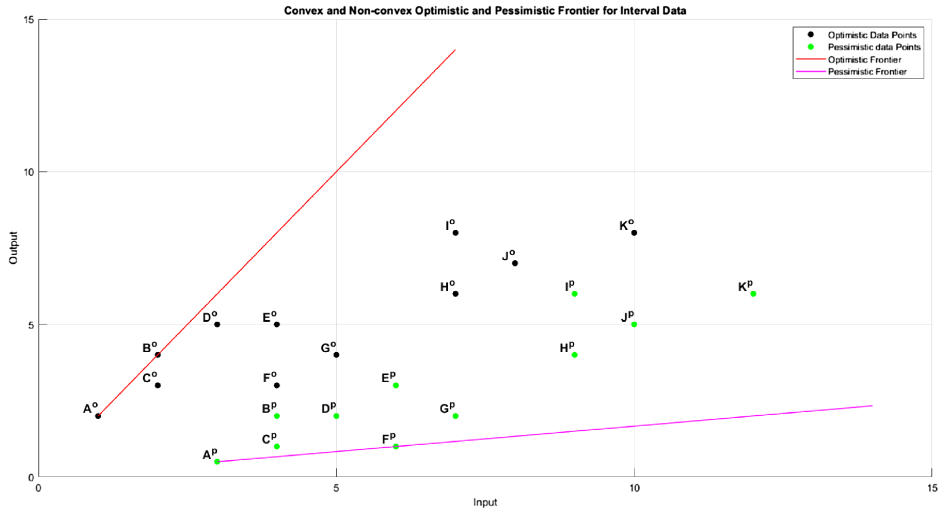

FIGURE 1. Optimistic efficient and pessimistic inefficient frontier.

TABLE 2. Efficiency by the input-oriented BCC models (BCC1) and (BCC2) and the optimistic SBM models (7) and (8) of [3].

Bank	BCCIOE	SBMOME	Bank	BCCIOE	SBMOME
B01	$[0.88, 1.00]$	$[0.31, 1.00]$	B11	$[0.95, 1.00]$	$[0.42, 1.00]$
B02	$[0.79, 0.90]$	$[0.13, 0.26]$	B12	$[0.62, 0.67]$	$[0.07, 0.10]$
B03	$[0.73, 1.00]$	$[0.14, 0.64]$	B13	$[0.68, 0.85]$	$[0.08, 0.12]$
B04	$[0.90, 1.00]$	$[0.30, 1.00]$	B14	$[0.64, 0.71]$	$[0.04, 0.07]$
B05	$[0.96, 1.00]$	$[0.03, 0.05]$	B15	$[0.47, 1.00]$	$[0.17, 1.00]$
B06	$[0.90, 1.00]$	$[0.17, 1.00]$	B16	$[0.57, 0.66]$	$[0.07, 0.16]$
B07	$[0.77, 1.00]$	$[0.21, 1.00]$	B17	$[0.99, 1.00]$	$[0.52, 1.00]$
B08	$[0.90, 1.00]$	$[0.47, 1.00]$	B18	$[0.98, 1.00]$	$[0.02, 0.04]$
B09	$[0.87, 1.00]$	$[0.34, 1.00]$	B19	$[0.86, 1.00]$	$[0.36, 1.00]$
B10	$[0.98, 1.00]$	$[0.19, 1.00]$	B20	$[0.50, 0.66]$	$[0.07, 0.11]$

Remark 5.6. The maximum possible value of $\phi_q^u = 12$ from model (5.3). We set $\hat{\phi_q^u} = 12.5 > 12$; it yields the estimated upper bound of output equals 18.7, with input interval [9, 13]. Thus, the optimistic point (9, 18.7) for unit L falls outside the optimistic PPS.

6. The need for SBM model

We illustrate how the proposed non-radial slack-based models exhibit conceptual advantage over the radial-based BCC models for interval-valued inverse DEA forwarded by Ghobadi [13].

We use the same data set of 20 bank branches with three inputs to produce five outputs as in [13]. Table 2 records the interval efficiency scores of the twenty DMUs by the BCC models (BCC1) and (BCC2) and the SBM models given in [3] for the interval data. Column 2 reports the BCC input-oriented efficiency (BCCIOE), and column 3 depicts the slack-based optimistic efficiency (SBMOME).

TABLE 3. The input and output data for B_q as sum of inputs and outputs of the merging units B12 and B14.

Inputs	$[11874.02, 23490.35]$	$[25.70, 47.15]$	$[34166, 39317]$		
Outputs	$[762840, 824541]$	$[47364, 55725]$	$[369914, 402647]$	$[43901.62, 54731.8]$	$[1311.69, 2028.30]$

TABLE 4. Efficiency of the merged unit B_q by input-oriented BCC and proposed SBM models (2.1) and (2.2).

DMU	BCCIOE	SBMOME
B_q	$[0.37, 0.43]$	$[0.06, 0.09]$

TABLE 5. The inherited inputs of B12 and B14 by inverse BCC model of [13] and proposed inverse SBM models (3.2) and (3.4).

Branches	input1	input2	input3
B12-Old	$[7303.27, 14178.11]$	$[22.87, 23.19]$	$[16148, 21353]$
B14-Old	$[4540.75, 9312.24]$	$[22.83, 23.96]$	$[17918, 17964]$
B12-BCC	$[7303.27, 14178.11]$	$[22.87, 23.19]$	$[16148, 21353]$
B14-BCC	$[4540.75, 9312.24]$	$[22.83, 23.96]$	**$[11401.06, 17302.58]$**
B12-SBM	$[7303.27, 14178.11]$	$[22.87, 23.19]$	$[16148, 21353]$
B14-SBM	$[4540.75, 9312.24]$	$[22.83, 23.96]$	$[17918, 17964]$

The DMUs B12 and B14 merge to form a new unit, B_q; we can apply inverse DEA models to estimate the inherited inputs and outputs required for B_q to achieve the pre-set efficiency target. Initially, we set the input and output values of B_q equal to the sum of the inputs and outputs of B12 and B14, as reported in Table 3.

Following the BCC models (BCC1) and (BCC2) and the SBM models (2.1) and (2.2) from Section 2, to calculate the interval efficiency of B_q, reported in Table 4.

Next, we reverse the process. For a given efficiency target, we estimate the input values while keeping the outputs fixed. Specifically, for the new merged DMU, the outputs are the sum of the outputs of the merging DMUs. For the given optimistic efficiency $[0.37, 0.43]$ of B_q and the inherited output values, $y_{rq} = \sum_{j\in\Lambda} y_{rj} (r \in O)$, we recalculate the inherited input values using models (3.2) and (3.4), and values are reported in Table 5 (here we use the weighted sum approach with uniform weights for all inputs to generate Pareto optimal solutions for the two models (3.2) and (3.4)). We observed that the estimated input values derived from the inverse SBM model align with expectations, while the inputs calculated from the inverse BCC models differ significantly, particularly in the third input interval for unit B14. The same is highlighted in bold in Table 5.

Inverse DEA models should yield consistent input and output data (*i.e.*, $x_{iq} = \sum_{j\in\Lambda} x_{ij}\ (i \in I)$ and $y_{rq} = \sum_{j\in\Lambda} y_{rj}\ (r \in O)$) for the efficiency target $[\theta_q^{l^*}, \theta_q^{u*}]$, obtained from DEA models (2.1) and (2.2). However, the inverse BCC model does not follow this principle. This discrepancy likely arises because the BCC model is radial-based and does not account for slacks, leading to inefficiencies in evaluating DMUs. This underscores the need for a new non-radial model to more accurately assess the input and output values of merged DMUs in an interval setting.

7. Application to Indian Banks

In this section, we implement our proposed models to find the effect of the merger of the five of its associate banks with the State Bank of India (SBI). The operational integration happened in 2017 with the merging of the State Bank of Travancore, State Bank of Mysore, State Bank of Bikaner and Jaipur, State Bank of Hyderabad, and State Bank of Patiala, with the SBI. The data of seventeen public sector Indian banks are collected from the statistical tables and annual reports of the Reserve Bank of India (RBI) for 2017.

Many researchers endeavoured to employ DEA in previous studies to assess the efficiency of Indian commercial banks. These studies have employed diverse sets of inputs and outputs in order to ascertain the banks' efficiency levels. Kumar and Gulati [20] utilized advances and investment as output variables and physical capital (value of fixed assets), labor (number of workers), and loanable money (deposits and borrowings) as input factors in the Indian environment. In this study, we follow Dar *et al.* [7], and took borrowing (borrowings by the banks), labor expenses (payments to and provisions for employees), equity (shares of the bank), net NPA, and total assets of banks as five inputs and investments (Investments by the banks), advances (advances by the bank or loans), and non-interest income (income earned other than interest) as three outputs to calculate the efficiency of banks by our DEA models.

Following Denizer *et al.* [8], the bank's input and output variables are normalized by dividing them by their branch count. The data collected from RBI annually may not always be exact but could vary in range.

To convert the input-output data into the interval form, we took the data of the same seventeen banks on the same input and output features for four years from 2014 to 2017, and for the next five years 2018–2022 on post-merging twelve banks, including the SBI, from the RBI annual reports. So, we have nine-year data for twelve banks and four-year data for the five banks that merged with SBI in 2017. The data is processed and normalized. Standard deviation is calculated for each feature and for each bank. The lower and upper bounds of the data are obtained by subtracting and adding the respective standard deviation of the features from their exact values. The same interval data is reported in Tables B.1 and B.2.

The optimization models of [3] are applied to determine seventeen banks' optimistic efficiency and pessimistic inefficiencies intervals reported in Table 6.

We first set the input and output values of the merged unit SBI_q equal to the sum of the input and output values of the merging units SBI and its associated banks.

In our context,

$$\begin{aligned}\Lambda = \{&\text{State Bank of Travancore, State Bank of Mysore, State Bank of Bikaner and}\\ &\text{Jaipur, State Bank of Hyderabad, State Bank of Patiala, State Bank of India}\},\\ \text{DMU}_q = \;&\text{State Bank of India (SBI).}\end{aligned}$$

We apply the SBM optimistic and pessimistic models (2.1)–(2.4) to calculate the interval efficiency of twelve banks when the inputs and outputs of the SBI (on post merging) are set at $x_{iq}^l = \sum_{j\in\Lambda} x_{ij}^l,\ x_{iq}^u = \sum_{j\in\Lambda} x_{ij}^u$ $(i \in I)$ and $y_{rq}^l = \sum_{j\in\Lambda} y_{rj}^l,\ y_{rq}^u = \sum_{j\in\Lambda} y_{rj}^u$ $(r \in O)$. Table 7 presents these efficiency intervals. From Tables 6 and 7, we observe that the efficiency of eleven non-merging banks remains unaltered while post-merging, the optimistic efficiency and pessimistic inefficiency intervals of SBI decreased, indicating that the non-efficiency of SBI (post-merging) has increased, or we can say that the SBI (post merging) has moved closer to the inefficient production frontier. This indicates the SBI (post-merging) is in the PPS of both the efficient and inefficient production frontiers formed by the remaining eleven banks.

From Theorems 5.1 to 5.4, the achievable efficiency interval for $\hat{\theta}_q^l$ is $[0.29, 0.506]$, and for $\hat{\theta}_q^u$ is $[0.66, 1]$. The achievable pessimistic inefficiency interval for $\hat{\phi}_q^l$ is $[1.66, 2.822]$, and for $\hat{\phi}_q^u$ is $[3.93, 4.498]$.

Since all associate banks of SBI are merged into SBI, we consider the newly merged DMU as SBI only. Assume the post-merged SBI efficiency is set to $\hat{\theta} = [0.45, 0.9]$. The suggested models (3.2) and (3.4) are used to determine the bounds on the inherited input levels of the SBI post-merging units. We use the weighted sum approach with uniform weights for all inputs to generate Pareto optimal solutions for the two models. To meet

TABLE 6. Optimistic and pessimistic efficiencies intervals by the models (7)–(10) of Section 3 in [3].

Bank name	Optimistic efficiency	Pessimistic efficiency
Bank of Baroda	$[0.30, 1.00]$	$[1.00, 5.55]$
Bank of India	$[0.29, 0.71]$	$[1.34, 18.34]$
Bank of Maharashtra	$[0.20, 1.00]$	$[1.00, 8.04]$
Canra Bank	$[0.34, 1.00]$	$[1.78, 11.45]$
Central Bank of India	$[0.25, 1.00]$	$[1.00, 8.32]$
Indian Bank	$[0.36, 1.00]$	$[2.20, 13.71]$
Indian Overseas Bank	$[0.24, 1.00]$	$[1.00, 3.07]$
Punjab and Sind Bank	$[0.25, 0.73]$	$[1.00, 16.34]$
Punjab National Bank	$[0.34, 0.84]$	$[1.56, 5.79]$
UCO Bank	$[0.26, 1.00]$	$[1.00, 34.71]$
Union Bank of India	$[0.29, 1.00]$	$[1.00, 5.73]$
State Bank of India	$[0.33, 1.00]$	$[1.00, 4.56]$
State Bank of Bikaner and Jaipur	$[0.36, 1.00]$	$[3.01, 5.94]$
State Bank of Hydrabad	$[0.36, 0.66]$	$[4.77, 11.51]$
State Bank of Mysore	$[0.25, 0.69]$	$[1.00, 4.93]$
State Bank of Patiala	$[0.29, 1.00]$	$[1.00, 130.79]$
State Bank of Travancore	$[0.31, 1.00]$	$[1.49, 7.74]$

TABLE 7. Optimistic and pessimistic efficiencies intervals by the proposed models (2.1)–(2.4) in Section 2.

Bank name	Optimistic efficiency	Pessimistic efficiency
Bank of Baroda	$[0.30, 1.00]$	$[1.00, 5.55]$
Bank of India	$[0.29, 0.71]$	$[1.34, 18.34]$
Bank of Maharashtra	$[0.20, 1.00]$	$[1.00, 8.04]$
Canra Bank	$[0.34, 1.00]$	$[1.78, 11.45]$
Central Bank of India	$[0.25, 1.00]$	$[1.00, 8.32]$
Indian Bank	$[0.36, 1.00]$	$[2.20, 13.71]$
Indian Overseas Bank	$[0.24, 1.00]$	$[1.00, 3.07]$
Punjab and Sind Bank	$[0.25, 0.73]$	$[1.00, 16.34]$
Punjab National Bank	$[0.34, 0.84]$	$[1.56, 5.79]$
UCO Bank	$[0.26, 1.00]$	$[1.00, 34.71]$
Union Bank of India	$[0.29, 1.00]$	$[1.00, 5.73]$
State Bank of India	$[0.29, 0.66]$	$[1.66, 3.93]$

the predetermined efficiency target, we have determined the minimum and maximum input levels for the SBI inherited from the merging units (SBI and associates five banks of SBI). These values are reported in Table 8.

Next, suppose the post-merged SBI pessimistic inefficiency is set to $\hat{\phi} = [2.0, 4.4]$. The inherited output levels of the SBI post-merging units are determined by applying models (4.2) and (4.4). We use the weighted sum approach with uniform weights for all outputs to generate Pareto optimal solutions for the two models. To meet the predetermined non-efficiency target, we have determined the minimum and maximum output levels for the

TABLE 8. The inherited inputs of the merging banks and the post-merged SBI bank by proposed inverse DEA models (3.2) and (3.4).

Bank name	Borrowing	Labor expenses	Equity	Net NPA	Total assets
State Bank of India	$[11.662, 11.662]$	$[1.076, 1.808]$	$[2.630, 2.630]$	$[1.852, 1.926]$	$[116.795, 116.795]$
State Bank of B and J	$[0.97, 0.977]$	$[0.829, 1.121]$	$[0.558, 0.558]$	$[2.825, 4.971]$	$[76.801, 76.801]$
State Bank of Hyderabad	$[1.944, 1.944]$	$[0.718, 0.835]$,	$[0.156, 0.156]$	$[3.003, 3.299]$	$[68.878, 68.878]$
State Bank of Mysore	$[0.863, 1.969]$	$[0.990, 1.345]$	$[0.566, 0.566]$	$[3.029, 3.029]$	$[68.854, 68.878]$
State Bank of Patiala	$[0.022, 0.022]$	$[0.924, 1.123]$	$[1.615, 1.615]$	$[3.384, 3.680]$	$[68.854, 68.878]$
State Bank of Travancore	$[0.841, 0.840]$	$[1.097, 1.448]$	$[0.460, 0.46]$	$[2.449, 2.745]$	$[68.854, 68.878]$
State Bank of India (merged SBI unit)	$[16.308, 17.415]$	$[5.635, 7.679]$	$[5.986, 5.986]$	$[16.544, 19.949]$	$[469.037, 469.037]$

TABLE 9. The inherited outputs of the merging banks and the post-merged SBI bank by proposed inverse DEA models (4.2) and (4.4).

Bank name	Investments	Advances	Non-interest income
State Bank of India	$[29.87, 53.54]$	$[76.09, 98.35]$	$[3.84, 3.84]$
State Bank of B and J	$[19.76, 28.58]$	$[42.52, 47.69]$	$[1.49, 1.49]$
State Bank of Hyderabad	$[19.83, 22.30]$	$[31.26, 45.87]$	$[1.86, 1.86]$
State Bank of Mysore	$[18.90, 22.35]$	$[21.13, 38.94]$	$[1.65, 1.65]$
State Bank of Patiala	$[20.14, 25.23]$	$[44.83, 52.77]$	$[1.85, 1.85]$
State Bank of Travancore	$[26.98, 37.96]$	$[30.47, 47.42]$	$[2.04, 2.04]$
State Bank of India (merged SBI unit)	$[135.48, 164.73]$	$[246.30, 331.04]$	$[12.73, 12.73]$

SBI, inherited from the merging units (SBI and associates five banks of SBI). These values are reported in Table 9.

This example, along with its validation, has a limitation. We don't have access to data immediately following bank mergers about what and how much the SBI keeps from the merging banks' units and what is written down or discarded instead of being carried forward. We could not compare and view our findings in Table 9. Moreover, the data from the bank next available to us to view the post-merger analysis is from 2018. However, utilizing the data of the year 2018 will not help us to perform the post-merger analysis for SBI. The efficiency frontier and the inefficiency frontier, efficiency values, and the overall analysis of units will change. The 2018 data cannot determine Table 9 obtained values. This observation also emphasizes studying InvDEA models to investigate merging unit analysis.

8. Concluding remarks

Merging DMUs is a key strategy for enhancing efficiency, synergy, and resource optimization. By consolidating multiple DMUs, organizations can boost performance, reduce inefficiencies, and achieve economies of scale. Inverse DEA plays a crucial role in identifying inefficiencies within DMUs and guiding resource optimization in mergers. InvDEA improves operational efficiency, optimizes resource allocation, and enhances the performance of the merged unit. However, incorporating slacks into InvDEA is essential to prevent inaccurate estimations.

This study explores the use of non-radial InvDEA models for merging units with interval data, particularly when the merged unit is within the current PPS. Comparing radial and non-radial models, especially in the context of bank data, shows that non-radial slack-based models are more effective. A real-world case study of the 2017 merger between the State Bank of India and its associated banks highlights the practical benefits of these models within the Indian public sector banking industry.

One observation on the models proposed in the paper to estimate the lower and upper bounds for input and output values are nonlinear, resulting in local optimal solutions of the models.

We applied InvDEA to assess the input-output values of merging units, assuming the merged unit lies within the PPS. A potential future direction is to develop an SBM inverse DEA model for cases where the merged unit falls outside the PPS using the ideas in [6, 22].

Mergers and acquisitions represent a significant application of InvDEA. Evaluating unit performance before and after the merger, as well as during the transition period and stabilization phase, is crucial. Another promising avenue for future research is introducing a dynamic framework for InvDEA, which could provide deeper insights into the long-term effects of mergers. Additionally, organizations may pursue collaborative partnerships rather than full mergers. Amin [1] proposed DEA models that allocate resources among collaborating entities to improve efficiency. Extending this research to interval data could further enhance the applicability of these models.

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Data availability statement

The data used in Section 6 is available as text in [13] `https://doi.org/10.1051/ro/2020029` on page S1629, Tables E.1 and E.2, and the other data generated and analyzed in this study are included in the text. All the proposed optimization models are solved using licensed professional extended LINGO/win64 software.

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Appendix A.

Theorem A.1. *Let the following hold:*

(1) *The merged unit DMU_q is within the current optimistic PPS for interval data;*
(2) $S^u = (t^*,\ x_{iq}^{u*}(i \in I),\ \alpha_{ij}^{u*}\,(i \in I,\ j \in \Lambda),\ \lambda_j^*(j \in \Pi),\ S_i^{-*}\,(i \in I),\ S_r^{+*}\,(r \in O))$ *is a Pareto optimal solution of* (3.4).

Then, the optimal value of model (2.2)*, with* $x_{iq}^u = \sum_{j\in\Lambda} \alpha_{ij}^{u*}(i \in I)$*, and* $t = t^*$*, equals* $\hat{\theta}_q^l$.

Proof. We consider model (2.2) by setting $x^u_{iq} = \sum_{j\in\Lambda} \alpha^{u*}_{ij}(i \in I)$ and $t = t^*$. Let $(\theta^{l**}_q,\ \lambda^{**}_j\ (j \in \Pi),\ \lambda^{**}_q,\ S^{-^{**}}_i(i \in I),\ S^{+^{**}}_r(r \in O))$, be the optimal solution of (2.2). Since S^u is feasible for (3.4), it is feasible for (2.2), hence, $\theta^{l**}_q \leq \hat{\theta}^l_q$.

Furthermore, by assumption, $A3$, the current optimistic PPS, and the efficient frontier remain unaltered on merging the units, and the inputs decrease $\alpha^u_{ij} \leq x^u_{ij}(i \in I,\ j \in \Lambda)$, while the outputs remain the same in models (3.4) and (2.2), hence at their optimal solutions, $S^{-^{**}}_i \geq S^{-^{*}}_i(i \in I)$.

Applying optimal solution of (3.2), we have,

$$\begin{aligned}
t^* y^l_{rq} &= \sum_{j\in\Pi} \lambda^{**}_j y^u_{rj} + \lambda^{**}_q y^l_{rq} - S^{+^{**}}_r \\
&= \sum_{j\in\Pi} \lambda^{**}_j y^u_{rj} + \lambda^{**}_q \left(\sum_{j\in\Pi} \left(\left(\lambda^*_j y^l_{rj}\right)/t^*\right) - \left(S^{+^{*}}_r/t^*\right)\right) - S^{+^{**}}_r \\
&= \sum_{j\in\Pi} (\lambda^{**}_j + \lambda^{**}_q \lambda^*_j t^{*-1}) y^l_{rj} - \left(\lambda^{**}_q S^{+^{*}}_r t^{*-1} + S^{+^{**}}_r\right) \\
&= \sum_{j\in\Pi} \bar{\lambda}_j y^l_{rj} - \bar{S}^+_r.
\end{aligned}$$

where $\bar{\lambda}_j = \lambda^{**}_j + \lambda^{**}_q \lambda^*_j t^{*-1}$ and $\bar{S}^+_r = \lambda^{**}_q S^{+^{*}}_r t^{*-1} + S^{+^{**}}_r$, For this $\bar{S}^+_r(r \in O)$,

$$\begin{aligned}
t^* + \frac{1}{s}\sum_{r=1}^{s} \frac{\bar{S}^+_r}{y^l_{rq}} &= t^* + \frac{1}{s}\sum_{r=1}^{s} \frac{\left(\lambda^{**}_q S^{+^{*}}_r\right) t^{*-1} + S^{+^{**}}_r}{y^l_{rq}} \\
&= t^* + \frac{1}{s}\sum_{r=1}^{s} \frac{S^{+^{**}}_r}{y^l_{rq}} + \frac{\lambda^{**}_q}{st^*}\sum_{r=1}^{s} \frac{S^{+^{*}}_r}{y^l_{rq}} \\
&= 1 + \frac{\lambda^{**}_q}{st^*} s(1-t^*) \\
&= 1 + \lambda^{**}_q \frac{1-t^*}{t^*} \\
&= 1 \text{ if and only if } \lambda^{**}_q = 0.
\end{aligned}$$

Hence, $\bar{\lambda}_j = \lambda^{**}_j(j \in \Pi)$ and $\bar{S}^+_r = S^{+^{**}}_r(r \in O)$. Let $\theta^{l**}_q < \hat{\theta}^l_q$. Then,

$$\begin{aligned}
& t^* - \frac{1}{m}\sum_{i=1}^{m} \frac{S^{-^{**}}_i}{x^u_{iq}} < t^* - \frac{1}{m}\sum_{i=1}^{m} \frac{S^{-^{*}}_i}{x^u_{iq}} \\
\implies & \sum_{i=1}^{m} \frac{S^{-^{**}}_i}{x^u_{iq}} > \sum_{i=1}^{m} \frac{S^{-^{*}}_i}{x^u_{iq}} \\
\implies & \sum_{i=1}^{m} \left(\frac{S^{-^{**}}_i}{x^u_{iq}} - \frac{S^{-^{*}}_i}{x^u_{iq}} \right) > 0,
\end{aligned}$$

implying that there exists $p \in I$ such that $S^{-^{**}}_p > S^{-^{*}}_p$. Let $\Omega = \{i \in I : S^{-^{**}}_i > S^{-^{*}}_i\}$. It follows that $\Omega \neq \emptyset$, and for any $i \in \Omega$,

$$\begin{aligned}
& \sum_{j\in\Pi} \lambda^{**}_j x^l_{ij} + S^{-^{**}}_i > \sum_{j\in\Pi} \lambda^{**}_j x^l_{ij} + S^{-^{*}}_i \\
\implies & t \sum_{j\in\Lambda} \alpha^{u*}_{ij} > \sum_{j\in\Pi} \lambda^{**}_j x^l_{ij} + S^{-^{*}}_i \\
\implies & t^* \sum_{j\in\Lambda} \alpha^{u*}_{ij} - t^*\nu_i = \sum_{j\in\Pi} \lambda^{**}_j x^l_{ij} + S^{-^{*}}_i, \quad \text{for some } \nu_i > 0 \\
\implies & t^* \sum_{j\in\Lambda} \bar{\alpha}^u_{ij} = \sum_{j\in\Pi} \lambda^{**}_j x^l_{ij} + S^{-^{*}}_i,
\end{aligned}$$

where, for $i \in \Omega$,

$$\bar{\alpha}^{u}_{ij} = \begin{cases} \alpha^{u*}_{iq} - \nu_i & j = k(i),\ \text{for some } k(i) \in \Lambda \\ \alpha^{u*}_{ij} & \text{otherwise.} \end{cases} \tag{B}$$

For $i \in I - \Omega,\ \bar{S}^-_i = {S^-_i}^{**}$, hence

$$\begin{aligned} t^* - \frac{1}{m}\sum_{i=1}^{m}\frac{\bar{S}^-_i}{x^u_{iq}} &= t^* - \frac{1}{m}\left(\sum_{i\in\Omega}\frac{{S^-_i}^*}{x^u_{iq}} + \sum_{i\in I-\Omega}\frac{{S^-_i}^{**}}{x^u_{iq}}\right) \\ &= t^* - \frac{1}{m}\left(m\left(t^* - \hat{\theta}^l_q\right) - \sum_{i\in I-\Omega}\frac{{S^-_i}^*}{x^u_{iq}} + \sum_{i\in I-\Omega}\frac{{S^-_i}^{**}}{x^u_{iq}}\right) \\ &= \hat{\theta}^l_q + \sum_{i\in I-\Omega}\frac{{S^-_i}^{**} - {S^-_i}^*}{x^l_{iq}} \\ &= \hat{\theta}^l_q. \end{aligned}$$

Taking $\bar{S}^-_i = {S^-_i}^*,\ i \in \Omega,\ \bar{S}^-_i = {S^-_i}^{**},\ i \in I - \Omega,\ \bar{S}^+_r = {S^+_r}^{**}(r \in O),\ \bar{\lambda}_j = \lambda^{**}_j(j \in \Pi),\ \bar{\lambda}_q = 0$, we get that $(\bar{\alpha}^u_{ij},\ (i \in \Omega,\ j \in \Lambda),\ \alpha^{u*}_{ij},\ (i \in I - \Omega,\ j \in \Lambda),\ \bar{\lambda}_j,\ (j \in \Pi),\ \bar{\lambda}_q,\ \bar{S}^-_i,\ (i \in I),\ \bar{S}^+_r,\ (r \in O))$ is feasible for (3.4) with at least one $\bar{\alpha}^u_{ij} < \alpha^{u*}_{ij}(i \in \Omega \subseteq I,\ j \in \Lambda)$, from (B), contradicting the Pareto optimality for model (3.4). This completes the requisite proof. □

Theorem A.2. *If the following assumptions hold:*

(1) *The merged* DMU_q *is within the current pessimistic PPS for interval data;*
(2) $\Delta^l = (t^*,\ y^{l^*}_{rq}(r \in O),\ \beta^{l^*}_{rj}(r \in O, j \in \Lambda),\ \lambda^*_j(j \in \Pi),\ {S^-_i}^*(i \in I),\ {S^+_r}^*(r \in O))$ *is a Pareto optimal solution of* (4.4).

Then, the optimal value of model (2.4), *with* $y^l_{rq} = \sum_{j\in\Lambda}\beta^{l^*}_{rj}(r \in O)$, *and* $t^* = t$, *equals* $\hat{\phi}^l_q$.

Proof. Here, in our arguments, we consider model (2.4) by setting $y^l_{rq} = \sum_{j\in\Lambda}\beta^{l^*}_{rj}(r \in O)$, and $t^* = t$. Since Δ_l is feasible for (4.4), it is feasible for (2.4), hence $\phi^{l**}_q \geq \hat{\phi}^l_q$.

Furthermore, by assumption, $B3$, the current pessimistic PPS, and the efficient frontier remain unaltered on merging the units, and the outputs increase $\beta^l_{rj} \geq y^l_{rj}(r \in O,\ j \in \Lambda)$, while the inputs remain the same in models (4.4) and (2.4), hence at their optimal solutions, ${S^-_r}^{**} \geq {S^-_r}^*(r \in O)$.

Suppose $\phi^{l**}_q > \hat{\phi}^l_q$. Applying optimal solutions of (2.4) and (4.4), we have

$$\begin{aligned} t^* x^u_{iq} &= \sum_{j\in\Pi}\lambda^{**}_j x^u_{ij} + \lambda^{**}_q \sum_{j\in\Lambda} x^u_{ij} - {S^-_i}^{**} \\ &= \sum_{j\in\Pi}\lambda^{**}_j x^u_{ij} + \lambda^{**}_q\left(\frac{\sum_{j\in\Pi}\lambda^*_j x^u_{ij} - {S^+_i}^*}{t^*}\right) - {S^+_i}^{**} \\ &= \sum_{j\in\Pi}\left(\lambda^{**}_j + \frac{\lambda^{**}_q\lambda^*_j}{t^*}\right)x^u_{ij} - \frac{\left(\lambda^{**}_q {S^+_i}^*\right)}{t^*} - {S^+_i}^{**} \\ &= \sum_{j\in\Pi}\bar{\lambda}_j x^u_{ij} - \bar{S}^+_i, \end{aligned}$$

where $\bar{\lambda}_j = \lambda^{**}_j + \lambda^{**}_q\lambda^*_j t^{*-1}$ and $\bar{S}^+_i = (\lambda^{**}_q {S^+_i}^*)t^{*-1} + {S^+_i}^{**}$. It follows from here that

$$t^* + \frac{1}{m}\sum_{i=1}^{m}\frac{\bar{S}^+_i}{x^u_{iq}} = t^* + \frac{1}{m}\sum_{i=1}^{m}\frac{\frac{\left(\lambda^{**}_q {S^+_i}^*\right)}{t^*} + {S^+_i}^{**}}{x^u_{iq}}$$

$$= t^* + \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^{+^{**}}}{x_{iq}^l} + \frac{\lambda_q^{**}}{mt}\sum_{i=1}^{m}\frac{S_i^{+^{**}}}{x_{iq}^u}$$
$$= 1 + \frac{\lambda_q^{**}}{mt^*}m(1-t^*) = 1 \text{ if and only if } \lambda_q^{**} = 0.$$

Hence, $\bar{\lambda}_j = \lambda_j^{**}(j \in \Pi)$ and $\bar{S}_i^+ = S_i^{+^{**}}(i \in I)$. Furthermore, for $\bar{\lambda}_j = \lambda_j^{**}$ and $\bar{S}_r^- = S_r^{-^{**}}(r \in O)$, we can work out to show that

$$t^* y_{rq}^l = \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^u + y_{rq}^l\lambda_q^{**} + S_r^{-^{**}} = \sum_{j\in\Pi}\bar{\lambda}_j y_{rj}^u + \bar{S}_r^-,\ r \in O.$$

Since $\phi_q^{l**} > \hat{\phi}_q^l$, it implies

$$\frac{1}{t^* - \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^{-^{**}}}{y_{rq}^l}} > \frac{1}{t^* - \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^{-^{*}}}{y_{rq}^l}}$$
$$\implies t^* - \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^{-^{**}}}{y_{rq}^l} < t^* - \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^{-^{*}}}{y_{rq}^l}$$
$$\implies \sum_{r=1}^{s}\frac{S_r^{-^{**}}}{y_{iq}^l} > \sum_{r=1}^{s}\frac{S_r^{-^{*}}}{y_{iq}^l}$$
$$\implies \sum_{r=1}^{s}\left(\frac{S_r^{-^{**}}}{y_{rq}^l} - \frac{S_r^{-^{*}}}{y_{iq}^l}\right) > 0,$$

there exists at least one $k \in O$ such that $S_k^{-^{**}} > S_k^{-^{*}}$. Let $\Omega = \{r \in O : S_r^{-^{**}} > S_r^{-^{*}}\} \neq \emptyset$. For $r \in O - \Omega, S_r^{-^{**}} = S_r^{-^{*}}$. It follows that for $r \in \Omega$,

$$\implies S_r^{-^{**}} > S_r^{-^{*}}$$
$$\implies \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^l + S_r^{-^{**}} > \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^l + S_r^{-^{*}}$$
$$\implies t^*\sum_{j\in\Lambda}\beta_{rj}^{l^*} > \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^l + S_r^{-^{*}}.$$

For $r \in \Omega$, we can choose $k(r) \in \Lambda$ such that, $t^*(\sum_{j\in\Lambda}\bar{\beta}_{rj}^l) = \sum_{j\in\Pi}\lambda_j^{**}y_{rj}^l + S_r^{-^{*}}$. Where

$$\bar{\beta}_{rj}^l = \begin{cases}\beta_{rk(r)}^{l^*} - \mu_r & j = k(r),\\ \beta_{rj}^{l^*} & j \neq k(r),\end{cases} \tag{D}$$

for some $\mu_r > 0$ with $\beta_{rk(r)}^{l^*} > 0$. Taking $\bar{S}_r^- = S_r^{-^{*}},\ r \in \Omega,\ \bar{S}_r^- = S_r^{-^{**}},\ r \in O - \Omega$, then

$$t^* - \frac{1}{s}\sum_{r=1}^{s}\frac{\bar{S}_r^-}{y_{rq}^l} = t^* - \frac{1}{s}\left(\sum_{r\in\Omega}\frac{S_r^{-^{*}}}{y_{rq}^l} + \sum_{r\in O-\Omega}\frac{S_r^{-^{**}}}{y_{rq}^l}\right)$$
$$= t^* - \frac{1}{s}\left(s\left(t^* - \frac{1}{\hat{\phi}_q^l}\right) - \sum_{r\in O-\Omega}\frac{S_r^{-^{*}}}{y_{rq}^l} + \sum_{r\in O-\Omega}\frac{S_r^{-^{**}}}{y_{rq}^l}\right)$$
(using a constraint of model (4.2))
$$= \frac{1}{\hat{\phi}_q^l} + \sum_{r\in O-\Omega}\frac{S_r^{-^{**}} - S_r^{-^{*}}}{y_{rq}^l} = \frac{1}{\hat{\phi}_q^l} \qquad (\text{as } S_r^{-^{**}} = S_r^{-^{*}},\ r \in O - \Omega).$$

Taking $\bar{S}_i^+ = S_i^{+^{**}}(i \in I), \bar{S}_r^+ = S_r^{+^{*}},\ r \in \Omega,\ \bar{S}_r^- = S_r^{-^{**}},\ r \in O - \Omega,\ \bar{\lambda}_j = \lambda_j^{**}(j \in \Pi),\ \bar{\lambda}_q = 0$; it yields that $(\bar{\beta}_{rj}^l,\ (r \in \Omega,\ j \in \Lambda),\ \beta_{rj}^{l^*},\ (r \in I - \Omega,\ j \in \Lambda),\ \bar{\lambda}_j,\ (j \in \Pi), \bar{\lambda}_q,\ \bar{S}_i^+,\ (i \in I),\ \bar{S}_r^-,\ (r \in O))$ is feasible for (4.4) with (D); contradicting the Pareto optimality of $(\beta_{rj}^{l^*},\ (r \in O,\ j \in \Lambda))$ for (4.4). Thus, $\phi_q^{l**} = \hat{\phi}_q^l$, completing the proof. □

Appendix B.

Table B.1. The input intervals of the seventeen public sector banks of India for 2017.

Bank	Borrowing	Labor expenses	Equity	Net NPA	Total
Bank of Baroda	[2.915, 8.054]	[0.593, 1.069]	[0.743, 3.102]	[2.237, 4.242]	[114.500, 134.514]
Bank of India	[5.824, 9.052]	[0.868, 1.169]	[0.129, 4.105]	[3.309, 6.243]	[109.970, 126.462]
Bank of Maharashtra	[2.690, 5.537]	[0.808, 1.019]	[0.273, 2.697]	[3.910, 7.445]	[68.593, 92.503]
Canara Bank	[5.233, 6.950]	[0.553, 0.963]	[0.199, 1.880]	[2.289, 4.388]	[79.628, 100.332]
Central Bank of India	[0.288, 3.537]	[0.795, 0.942]	[0.985, 3.905]	[2.155, 3.704]	[61.753, 75.647]
Indian Bank	[2.449, 6.802]	[0.592, 0.866]	[0.153, 0.817]	[1.646, 2.459]	[67.679, 92.082]
Indian Overseas Bank	[2.060, 7.245]	[0.735, 1.025]	[1.750, 2.672]	[3.723, 7.693]	[65.397, 77.474]
Punjab and Sind Bank	[1.666, 2.162]	[0.479, 0.802]	[0.075, 1.630]	[2.017, 3.642]	[58.032, 66.991]
Punjab National Bank	[3.943, 7.572]	[0.627, 0.904]	[0.588, 3.636]	[3.042, 6.196]	[91.373, 112.110]
UCO Bank	[1.657, 4.402]	[0.401, 0.846]	[0.047, 3.377]	[2.175, 4.627]	[68.345, 78.677]
Union Bank of India	[7.143, 11.477]	[0.647, 0.904]	[0.583, 2.608]	[2.889, 5.617]	[87.724, 116.750]
State Bank of India	[14.615, 19.979]	[1.076, 1.808]	[2.630, 3.405]	[2.067, 4.279]	[116.795, 177.861]
State Bank of B and J	[0.977, 3.127]	[0.829, 1.121]	[0.558, 0.641]	[3.040, 6.413]	[76.801, 84.159]
State Bank of Hyderabad	[1.944, 3.483]	[0.718, 0.835]	[0.156, 0.179]	[3.218, 6.627]	[78.257, 79.338]
State Bank of Mysore	[0.863, 3.716]	[0.990, 1.345]	[0.566, 0.634]	[3.243, 6.825]	[74.823, 79.016]
State Bank of Patiala	[0.022, 5.670]	[0.924, 1.123]	[1.615, 3.449]	[4.795, 10.240]	[81.472, 88.887]
State Bank of Travancore	[0.841, 3.992]	[1.097, 1.448]	[0.460, 1.101]	[2.663, 5.245]	[93.965, 106.539]

Table B.2. The output intervals of seventeen public sector banks of India for 2017.

Bank	Investments	Advances	Non-interest income
Bank of Baroda	[17.855, 28.599]	[60.977, 76.367]	[0.981, 1.441]
Bank of India	[19.802, 28.453]	[63.165, 75.183]	[1.018, 1.538]
Bank of Maharashtra	[12.560, 26.459]	[42.344, 54.234]	[0.455, 1.070]
Canara Bank	[21.300, 25.043]	[47.093, 58.384]	[0.876, 1.454]
Central Bank of India	[13.414, 24.540]	[25.156, 32.292]	[0.464, 0.721]
Indian Bank	[21.010, 28.443]	[39.854, 53.630]	[0.596, 1.023]
Indian Overseas Bank	[16.911, 24.508]	[35.630, 45.560]	[0.595, 1.355]
Punjab and Sind Bank	[14.598, 21.558]	[35.241, 40.224]	[0.245, 0.503]
Punjab National Bank	[21.822, 30.925]	[55.417, 63.084]	[1.065, 1.464]
UCO Bank	[19.878, 27.163]	[31.422, 44.666]	[0.408, 0.936]
Union Bank of India	[18.923, 31.732]	[59.718, 69.670]	[0.906, 1.336]
State Bank of India	[29.873, 53.537]	[72.730, 98.346]	[1.643, 2.218]
State Bank of B and J	[19.758, 28.578]	[42.039, 47.691]	[0.836, 1.123]
State Bank of Hyderabad	[19.831, 22.302]	[30.787, 45.868]	[0.765, 1.080]
State Bank of Mysore	[18.900, 22.347]	[20.653, 38.940]	[0.739, 0.986]
State Bank of Patiala	[20.137, 25.225]	[44.348, 52.765]	[0.865, 1.189]
State Bank of Travancore	[26.975, 37.957]	[29.997, 47.420]	[0.981, 1.379]