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ECONOMIC EVALUATION OF POTENTIAL IMPROVEMENT IN TWO-STAGE NETWORK DATA ENVELOPMENT ANALYSIS: A CASE STUDY IN THE BANKING SECTOR

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Abstract. This paper investigates the concept of potential improvement within two-stage network data envelopment analysis (DEA) models, with a focus on its application in the banking sector. Network DEA, which evaluates the internal structure of decision-making units (DMUs), is instrumental in assessing operational efficiency within complex networks. Our study provides a detailed review of these methodologies and explores the concept of normalized potential improvement (NPI), specifically extending it to two-stage network production processes. We conduct a comparative analysis of both radial Farrell and non-radial NPI approaches to offer a comprehensive evaluation. To illustrate the practical application of these concepts, we assess the performance of 35 bank branches operating within two-stage structures. The objective is to provide insights aimed at optimizing efficiency in the banking sector from an economic standpoint.

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1. Introduction

In modern industries, the evaluation of performance is crucial for maintaining competitiveness and achieving sustainable growth. Data Envelopment Analysis (DEA) has emerged as a pivotal tool offering a systematic framework to assess efficiency and productivity across various sectors. DEA enables regulators, analysts, and managers to identify strengths, weaknesses, and improvement opportunities within complex operational environments [20,21]. This analytical approach spans sectors from manufacturing to services, providing insights that drive strategic decisions and operational enhancements.

Within the globalized and competitive business environment, the banking sector plays a critical role as a pillar of the financial system. Regulatory bodies and market analysts leverage DEA to effectively monitor the performance of banking systems and individual institutions. By accessing pertinent data, they pinpoint

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inefficiencies, compare operational efficiencies across different banks, benchmark selection, and advocate for structural reforms that enhance overall system productivity and resilience [2, 3, 9, 33, 38, 42]. Such insights are essential for fostering financial stability and supporting economic growth on national and international scales.

The imperative for banking regulators and market analysts to access pertinent data for monitoring banking system and individual bank performance in today's globalized and competitive business environment is underscored by the potential for structural reforms, enhanced competition, mergers, and acquisitions to improve overall efficiency and productivity. Persistent inefficiencies within the banking sector highlight the need for these measures, which can ultimately foster financial development and economic growth at the national level [27, 30]. The banking sector is a cornerstone of the financial system, playing a pivotal role in driving economic growth [10]. For instance, China's remarkable gross domestic product growth over the past four decades owes, in part, to the development of its financial systems, including the banking industry. However, since China's accession to the World Trade Organization in 2001, heightened competition has necessitated that bank managers scrutinize their internal operations. This urgency is underscored by the intensely competitive landscape. Improving bank performance has become imperative from both policy and operational standpoints [55]. As empirical studies have shown, superior performance is crucial for gaining a competitive edge [4]. Consequently, bank managers must embrace portfolio management practices that can potentially drive superior performance.

Ray [46] and Staub *et al.* [49] harnessed DEA to gauge the efficiency of Indian and Brazilian banks, respectively. In a parallel endeavor, Chang *et al.* [11] introduced a more advanced metric, the Input Slack-Based Productivity Index, to delve into the sources of productivity growth in the Chinese banking sector. Meanwhile, Chortareas *et al.* [17] and Barros *et al.* [6] leveraged DEA models to scrutinize the efficiency of banks in Greece and Brazil, respectively. Wanke *et al.* [52] applied the stochastic DEA method to assess the efficiency of the Angolan bank. While the specific inputs and outputs considered in their study are randomness, their analysis highlights the use of stochastic DEA to address uncertainty and variability in banking performance evaluation.

Traditional DEA models, while effective, often fail to account for the complex, multi-stage processes inherent in banking operations. This limitation has led to the development of two-stage and network DEA models, which decompose the production process into interconnected stages, offering a more granular and insightful evaluation of efficiency. Wang *et al.* [50] and Wank and Barros [51] assessed the performance of Chinese and Brazilian banking sectors using a two-stage DEA method. They demonstrated that by enhancing efficiency at each operational stage, the overall efficiency of banks can be significantly improved. Fernandes *et al.* [24] developed a two-stage network DEA model to evaluate the influence of risk variables on the efficiency and financial development impact of European domestic banks. Their model allows for a comprehensive assessment of how risk factors affect bank performance and contribute to the overall financial development of the European banking sector. Fukuyama *et al.* [26] employed a two-stage model to analyze cost inefficiency levels within Turkish banks between 2007 and 2016. Their approach utilized the Koopmans input efficiency framework, which decomposes the estimated Nerlovian cost inefficiency into its constituent components: slack-based allocative and technical inefficiency levels. Ma and Zhao [41] introduced a hybrid two-stage DEA approach, considering the series-parallel internal structure and cooperative-Stackelberg relations between sub-stages, analyzing data from 19 Chinese commercial banks. Farzadi *et al.* [23] proposed non-parametric models with interval ratio data for evaluating the banking industry.

For further insights and analysis of the impact of performance measurement on network banking with a two-stage structure, refer to the work of Henriques *et al.* [29]. Their research provides valuable insights into the relationship between performance evaluation and the efficiency of network banking operations.

Faults in benchmark selection and measuring potential in the banking sector can lead to significant misjudgments and inefficiencies. Traditional performance measurement tools often rely on aggregate data and simplistic radial models that fail to capture the complex, multifaceted nature of banking operations. These methods might overlook critical factors such as risk management, customer satisfaction, and technological innovation, leading to an incomplete assessment of a bank's true potential. Such inaccuracies can result in poor strategic decisions, misallocation of resources, and ultimately, a loss of competitive edge. Therefore, a more nuanced and comprehensive approach to measuring potential improvements, incorporating advanced analytics and more powerful indicators, is crucial for the accurate assessment and sustainable growth of banks.

The potential improvement in two-stage network DEA holds significant importance for the banking sector. Traditional DEA models assess efficiency by considering the bank as a single unit, potentially overlooking the intricacies of internal processes. However, a two-stage network DEA decomposes the banking operations into interconnected stages, such as resource allocation and service delivery, allowing for a more granular analysis of efficiency. By identifying inefficiencies within each stage, banks can implement targeted strategies to optimize resource utilization and service provision, leading to enhanced overall performance. This methodological advancement facilitates a deeper understanding of operational dynamics, thereby promoting better decision-making, strategic planning, and competitive advantage in the banking industry.

Adapting and extending the NPI index to address two-stage network processes in banking represents a theoretical and practical breakthrough. This customization fills a critical gap in DEA methodologies, tailoring efficiency measurement to intricate banking operations spanning multiple stages. Theoretical refinements include incorporating improvement potentials derived from specific input excesses across operational phases, enhancing efficiency assessment accuracy. Practically, applying the NPI index to two-stage networks equips banks with a sophisticated tool to pinpoint inefficiencies and optimize resource allocation effectively. This pioneering application underscores the NPI index's potential to advance efficiency analysis and operational management in financial services.

Empirically, our study identifies inefficiency sources in Iranian banks in Guilan province. Key findings highlight significant potential to enhance branch efficiency within the banking system as a whole, alongside notable efficiency disparities among different bank types across various dimensions. These findings underscore the value of employing the proposed two-stage normalized multi-directional efficiency analysis approach. Benchmark analyses further validate the robustness of the new NPI approach.

The primary contributions of this paper can be summarized as follows:

– Introducing novel two-stage normalized potential improvement models for assessing efficiency in network structures, encompassing radial and non-radial approaches.
– Applying these models to enhance the efficiency of 35 bank branches in Guilan province, Iran, offering practical insights and recommendations for systemic efficiency improvements.

The structure of this article is organized as follows: Section 2 provides a brief review of the related literature. Section 3 outlines the prerequisites. Section 4 introduces the extended NPI index for two-stage network production processes. The application of our proposed approach in Iranian banking is illustrated in Section 5. Section 6 concludes the study and suggests directions for future research.

2. Literature review

2.1. Data envelopment analysis

Data Envelopment Analysis is a non-parametric technique that combines mathematical programming and multi-criteria decision-making principles. It is widely recognized for its effectiveness in evaluating and ranking decision-making units (DMUs) based on multiple inputs and outputs. DEA has proven its value across various fields, including agriculture, transportation, healthcare, finance, manufacturing, education, and more. Its ability to assess efficiency and performance across diverse contexts has made it a popular tool for researchers and practitioners alike. This popularity is evident in the extensive literature on DEA, with numerous studies exploring its applications and advancements [5, 13, 16, 19, 37, 39, 45].

The DEA approach offers several advantages, including the ability to operate without prior knowledge of production functions and constraints, the capacity to manage multiple inputs and outputs simultaneously, the flexibility to accommodate diverse input and output indicators of varying scales, the capability to directly compare inefficient decision-making units with reference sets, the ability to rank DMUs, and the provision of benchmarks for inefficient units. However, the DEA approach also has certain drawbacks. It focuses on relative efficiency rather than absolute efficiency, faces challenges in solving large problems due to computational complexity, can be affected by measurement errors leading to deviations in results, shows variability in performance evaluation based on changes in inputs and outputs, has limitations in conducting statistical tests like hypothesis testing due to its non-parametric nature, and its performance assessments can be sensitive to fluctuations in sample composition [12, 18].

2.2. Two-stage network DEA

The two-stage network DEA approach divides the production process into two interrelated stages. Each DMU is evaluated based on its ability to convert inputs into intermediate products in the first stage and subsequently transform these intermediates into final outputs in the second stage. This decomposition helps in identifying inefficiencies at different stages of the production process, which are not observable in the traditional single-stage DEA models [31]. Many real-world situations involve DMUs with complex, multi-stage structures, often referred to as "two-stage network structures". These structures involve intermediate products where the outputs of one stage are used as inputs in the next. Treating these complex structures as simple "black boxes" can lead to misleading efficiency assessments. For example, a DMU might appear efficient overall, even if some of its sub-processes are inefficient. Seiford and Zhu [47] highlighted the importance of recognizing these multi-stage structures, particularly in banking. Their work suggests that applying DEA to banks as two-stage systems produces more accurate and reliable efficiency measurements.

Two-stage production systems represent a fundamental and widely studied category of multi-process activities. A significant body of research has focused on analyzing two-stage systems and extending them to more complex scenarios [15]. Zhu [56] examined the efficiency of the top 500 companies by modeling them as two-stage systems. Lewis and Sexton [34] utilized two-stage DEA models to evaluate team performance in Major League Baseball. Fare *et al.* [22], accounting for intermediate products. Chen *et al.* [14] proposed a method to provide efficient projections and benchmarking. These studies demonstrate the wide applicability and ongoing development of DEA methods for analyzing multi-stage processes. Li *et al.* [35] developed a slack-based measure model for a two-stage network production system, employing a scientifically constructed input–intermediate–output index system. Yadollahi and Kazemi Matin [54] considered and developed centralized resource reallocation models for two-stage production systems. Xie *et al.* [53] proposed a novel two-stage method. In the first stage, entropy theory is used to generate a comprehensive efficiency score for each DMU. In the second stage, input and output variables are selected using the Bayesian information criterion, with the comprehensive efficiency score serving as the dependent variable and the input and output variables as explanatory variables. Omid *et al.* [43] assessed the operational and environmental efficiencies of a network structure system by transforming a multi-objective optimization problem into a linear single-objective function. The study proposed a tri-objective function technique, where one objective function is retained while the other two are shifted into the model's constraints, effectively converting the multi-objective problem into a single-objective one.

2.3. Potential improvement in DEA

The concept of potential improvement in Data Envelopment Analysis (DEA), originally introduced by Bogetoft and Hougaard [7], plays a pivotal role in enhancing industry performance by quantifying the input changes required to achieve desired efficiency levels. Building on this foundation, Asmild *et al.* [4] expanded the concept with their Multi-Directional Efficiency Analysis (MEA) inefficiency index, which utilizes Luenberger's excess function [40] to assess potential improvements.

Laplante and Paradi [33] contributed by exploring five models that examine branch growth perspectives within the context of a leading Canadian bank. Their study not only evaluated individual branch growth potential but also provided specific improvement recommendations, demonstrating practical applications of theoretical advancements in DEA.

Further methodological innovations include Yu *et al.*'s [55] development of a two-stage DEA model designed for inverse analysis, which assesses operational efficiencies and potential income gains while considering credit risk. Meanwhile, Li *et al.* [36] introduced a novel two-stage DEA approach focusing on Chinese listed banks from 2014 to 2018, utilizing deposits as a flexible metric to identify potential Pareto improvements. Similar research on this topic can also be found in [39].

Central to efficiency enhancement in banking contexts is the NPI index, which identifies areas where reducing inputs can lead to improved efficiency. This index facilitates pinpointing inefficiencies within the production process, thereby enabling targeted improvement strategies. For instance, within banking operations, the index can highlight branches that underutilize resources, guiding recommendations such as optimizing staff levels or reallocating resources effectively.

This literature review provides a brief overview of DEA and its applications, laying a foundation for understanding current research in efficiency measurement. Highlighting the potential for advancements, our study focuses on refining the NPI index for two-stage production processes in banking applications. This approach aims to enhance efficiency evaluation and benchmarking, offering practical insights to optimize banking operations and drive performance improvements.

3. Prerequisites

In this section, we lay the foundation for introducing a novel potential improvement concept within two-stage network DEA models by outlining the necessary prerequisites. These prerequisites encompass the introduction and exploration of the potential improvement index, initially proposed by Bogetoft and Hagaard [7].

3.1. The potential improvements inefficiency index

Suppose that the production set is denoted by $T = \{(\boldsymbol{x}, \boldsymbol{y}) \epsilon \mathbb{R}_+^{m+s} | \boldsymbol{x}$ can produce $\boldsymbol{y}$ and the input set $L(\mathbf{y}) \subset \mathbb{R}_+^n$ represents a combination of inputs $\mathbf{x} \in \mathbb{R}_+^n$ capable of generating output $\mathbf{y} \in \mathbb{R}_+^m$. We assume that $L(y)$ is characterized as a non-empty, closed and convex set.

Additionally, the strong efficiency subset is defined as follows:

$$\text{Eff } L(\mathbf{y}) = \{\mathbf{x} : \mathbf{x} \in L(\mathbf{y}), \acute{\mathbf{x}} \leq \mathbf{x}, \acute{\mathbf{x}} \neq \mathbf{x} \Rightarrow \acute{\mathbf{x}} \notin L(\mathbf{y})\}. \tag{1}$$

For any given $\mathbf{x} \in L(\mathbf{y})$, Bogetoft and Pruzan [8] provided the following definition for the associated ideal input components:

$$x_i^*(\boldsymbol{x}) = \min\{\acute{x_i} \in \mathbb{R}_+ | (\acute{x_i}, \mathbf{x}_{-i}) \in L(\mathbf{y})\}; \quad \text{for all } i \in \{1, \ldots, m\} \tag{2}$$

where $\mathbf{x}_{-i}$ means all the components of $\boldsymbol{x}$, except the i-th one. The ideal point $\boldsymbol{x}^*(\boldsymbol{x})$ is defined accordingly; which is depends on the given $\mathbf{x} \in L(\mathbf{y})$. Tn Potential Improvements Inefficiency indicator can be defined based on the directional projection of the feasible input point $\mathbf{x} \in L(\mathbf{y})$ into the efficient boundary Eff $L(\mathbf{y})$.

The procedure for calculating the Potential Improvements Inefficiency Index according to the ideal point is illustrated in Figure 1.

In line with Figure 1, $\boldsymbol{g}^{\text{PI}}$ is the reference direction that is proportionate to $\boldsymbol{x} - \boldsymbol{x}^*(\boldsymbol{x})$. The selection of potential improvements or the target point, denoted as $S^{\text{PI}}(\boldsymbol{x})$, is a vector defined as $S^{\text{PI}}(\boldsymbol{x}) = \boldsymbol{x} - e(\boldsymbol{x}, L, \boldsymbol{g}^{\text{PI}})\boldsymbol{g}^{\text{PI}}$. Here, the function $e(\boldsymbol{x}, L, \boldsymbol{g})$ is the excess function introduced by Luenberger [40], as follows:

$$e(\boldsymbol{x}, L, \boldsymbol{g}) = \max\{\beta \in \mathbb{R}^+ | \boldsymbol{x} - \beta \boldsymbol{g} \in L, \boldsymbol{g} \neq \mathbf{0}\}. \tag{3}$$

This leads to the following definition of the Normalized Potential Improvements (NPI) inefficiency indicator by Bogetoft and Pruzan [8]:

$$E^{\text{NPI}}(\mathbf{x}) = \sum_{i=1}^{m} \frac{x_i - S_i^{\text{PI}}}{x_i^+ - x_i^-} = e\big(\boldsymbol{x}, L, \boldsymbol{g}^{\text{PI}}\big) \sum_{i=1}^{m} \frac{g_{\boldsymbol{i}}^{\text{PI}}}{x_i^+ - x_i^-} \cdot \tag{4}$$

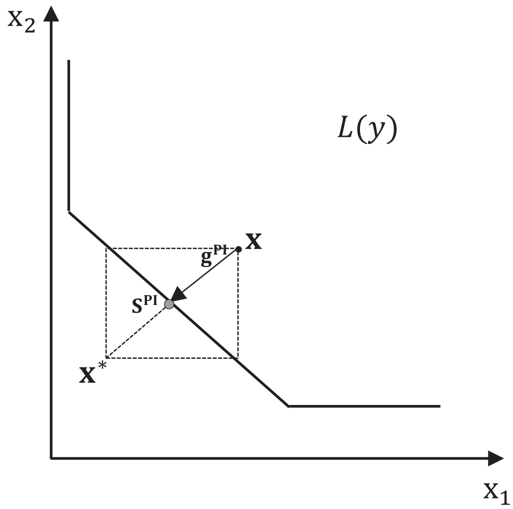

FIGURE 1. The potential improvements reference point S^{PI}.

In this definition $0 < x_i^- < x_i^+ < +\infty$ are arbitrary real numbers. Note that, in a special case, by choosing $\boldsymbol{g}^{\mathrm{PI}} = \boldsymbol{x} - \boldsymbol{x}^*(\boldsymbol{x})$, we have the following NPI indicator:

$$E^{\mathrm{NPI}}(\mathbf{x}) = e\big(\boldsymbol{x}, L, \boldsymbol{g}^{\mathrm{PI}}\big) \sum_{i=1}^{m} \frac{x_i - x_i^*(\boldsymbol{x})}{x_i^+ - x_i^-} \cdot \tag{5}$$

This can be interpreted as a normalized measure of input surplus in the desired improvement direction for the unit under evaluation.

The NPI indicator satisfies the following properties, as outlined by Bogetoft and Hagaard [7]:

(i) $E^{\mathrm{NPI}}(\mathbf{x}) \geq 0$.
(ii) $E^{\mathrm{NPI}}(\mathbf{x})$ increasing as $\mathbf{x}$ gets more inefficient.
(iii) $E^{\mathrm{NPI}}(\mathbf{x}) = 0$ if and only if $\boldsymbol{x} \in \mathrm{Eff}\ L(\boldsymbol{y})$.
(iv) $E^{\mathrm{NPI}}(\mathbf{x})$ is unit invariant.
(v) $E^{\mathrm{NPI}}(\mathbf{x})$ is monotonic in inputs.

In the context of traditional radial Farrell measurement, which is defined as: $E^F(\mathbf{x}) = \min\{\theta \in \mathbb{R} | \theta\mathbf{x} \in L(\mathbf{y})\}$ for, $\mathbf{x} \in L(\mathbf{y})$. It represents the minimum scaling factor θ such that the scaled input $\theta\boldsymbol{x}$ belongs to the input set $L(\boldsymbol{y})$ and varies in the range $(0, 1]$.

In this case, by considering $S^F = E^F(\boldsymbol{x})\boldsymbol{x}$ as the selected target point, the Farrell NPI indicator can be stated as follows:

$$E^{\text{NPI-F}}(\mathbf{x}) = \sum_{i=1}^{m} \frac{x_i - E^F(\boldsymbol{x})x_i}{x_i^+ - x_i^-} = \big(1 - E^F(\mathbf{x})\big) \sum_{i=1}^{m} \frac{x_i}{x_i^+ - x_i^-} \cdot \tag{6}$$

A critical distinction between the NPI inefficiency index and Farrell's efficiency index lies in their benchmark selection methodologies. The NPI method, detailed by Bogetoft and Hougaard [7], separates benchmark selection from efficiency measurement, providing a more nuanced assessment. It establishes benchmarks based on input-specific excesses, focusing on improvement potentials rather than past production levels, as traditional Farrell-based approaches do. This approach ensures that the NPI index comprehensively accounts for the configuration of the dominating set, thereby avoiding the oversight of significant input excesses typical in conventional methods. Consequently, comparisons between these indices often reveal substantial disparities in inefficiency values and production rankings.

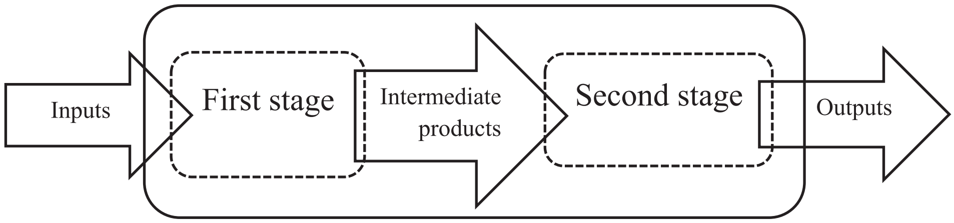

FIGURE 2. A typical two-stage production process.

While Farrell's index measures relative efficiency along a fixed improvement direction towards the origin in the input set, the NPI index provides an absolute measure of inefficiency towards a specified direction $\boldsymbol{g}^{\mathrm{PI}}$. This distinction offers a robust framework for identifying and addressing inefficiencies in complex operational environments, making the NPI indicator particularly suitable for comprehensive efficiency analysis [7].

In other words, the Farrell index does not account for the configuration of the dominating set, which may lead to overlooking significant input excesses. In contrast, the potential improvements approach considers the entirety of the dominating set by scaling down inputs proportionally based on their improvement potentials derived from input excesses.

3.2. Basic two-stage DEA model

In practical applications of production processes, two-stage systems are commonplace in DEA. This subsection begins by computing efficiency using a radial model based on Farrell's definition. This involves assessing the efficiency of the first stage by scaling down the inputs of the entire system while considering the intermediate products. Subsequently, overall system efficiency is calculated using a linear programming (LP) optimization model. This model not only incorporates inputs and outputs but also integrates intermediate products, providing a comprehensive assessment of efficiency.

3.2.1. Radial two-stage DEA efficiency

Consider a scenario where we have n Decision-Making Units (DMUs) that transform input vector $\boldsymbol{x} \in \mathbb{R}_+^m$ into output vector $\boldsymbol{y} \in \mathbb{R}_+^s$ through intermediary vector represented by $\mathbf{z} \in \mathbb{R}_+^p$ in a two-stage process, as depicted in Figure 2. Traditional DEA models view inputs as producers of final outputs, maintaining a black-box structure. However, network models address this limitation. The following section outlines a comprehensive two-stage network structure.

To construct the production set T, we assume that there are n observed production units where DMU_j is shown as $(\boldsymbol{x}_j, \boldsymbol{z}_j, \boldsymbol{y}_j)$ for $j = 1, \ldots, n$. Under the assumption of Variable Returns to Scale (VRS), the associated production set is defined as follows:

$$T_{\text{two-stage}}^{\text{VRS}} = \left\{ (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in R_+^{m+p+s} \middle| \boldsymbol{x} \geq \sum_{j=1}^n \lambda_j^1 \boldsymbol{x}_j, \quad \boldsymbol{z} \leq \sum_{j=1}^n \lambda_j^1 \boldsymbol{z}_j, \quad \boldsymbol{z} \geq \sum_{j=1}^n \lambda_j^2 \boldsymbol{z}_j, \quad \boldsymbol{y} \leq \sum_{j=1}^n \lambda_j^2 \boldsymbol{y}_j, \quad \sum_{j=1}^n \lambda_j = 1, \quad \forall j : \lambda_j \geq 0 \right\}.$$

Here, λ_j^1 and λ_j^2 are structural intensity variables used to form non-negative combinations of DMUs in stages 1 and 2, respectively.

Now, following Kao and Hwang [31], we employ the following radial (Farrell) model to evaluate the overall efficiency of DMU_o:

$$E_{\text{Two-Stage}}^F(\boldsymbol{x}_o) = \min \theta$$

$$\begin{aligned}
\text{s.t. } & \sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq \theta x_{io}, && i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j^1 z_{gj} \geq z_{go}, && g = 1, \ldots, p \\
& \sum_{j=1}^{n} \lambda_j^2 z_{gj} \leq z_{go}, && g = 1, \ldots, p \\
& \sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}, && r = 1, \ldots, s \\
& \lambda_j^1, \lambda_j^2 \geq 0, && j = 1, \ldots, n.
\end{aligned} \tag{7}$$

Here, θ represents represents the reduction ratio of the original inputs in this two-stage process.

4. New normalized potential improvement index in two-stage DEA

In this section, we introduce a novel methodology to assess NPI inefficiencies within a two-stage production process in the DEA framework. Building on the black-box NPI index introduced in the preceding subsection, we now present an extended non-radial NPI index specifically designed for two-stage production systems.

This new index accounts for both overall and individual stage efficiencies of the observed unit $(\mathbf{x}_o, \mathbf{z}_o, \mathbf{y}_o)$ of DMU_o is expressed as:

$$E_{\text{two-stage}}^{\text{NPI}}(\boldsymbol{x}_{\text{o}}) = E_1^{\text{NPI}}(\boldsymbol{x}_{\text{o}}) + E_2^{\text{NPI}}(\boldsymbol{x}_{\text{o}}) \tag{8}$$

where:

$$E_1^{\text{NPI}}(\boldsymbol{x}_{\text{o}}) = e\big(\boldsymbol{x}_{\mathbf{o}}, L_1, \boldsymbol{g}_1^{\text{PI}}\big) \sum_{i=1}^{m} \frac{x_{io} - x_i^*(\boldsymbol{x}_{\mathbf{o}})}{x_i^+ - x_i^-} \tag{9}$$

$$E_2^{\text{NR}}(\boldsymbol{x}_{\text{o}}) = e\big(\boldsymbol{z}_{\mathbf{o}}, L_2, \boldsymbol{g}_2^{\text{PI}}\big) \sum_{g=1}^{p} \frac{z_{go} - z_g^*(\boldsymbol{z}_{\mathbf{o}})}{z_g^+ - z_g^-} \cdot \tag{10}$$

Here, $x_i^*(\boldsymbol{x}_{\mathbf{o}})$ and $z_g^*(\boldsymbol{z}_{\mathbf{o}})$ represent the ideal input components derived from the structure of the production set $T_{\text{two-stage}}^{\text{VRS}}$, and Luenberger excess functions $e\big(\boldsymbol{x}_{\mathbf{o}}, L_1, \boldsymbol{g}_1^{\text{PI}}\big)$ and $e\big(\boldsymbol{z}_{\mathbf{o}}, L_2, \boldsymbol{g}_2^{\text{PI}}\big)$ quantify the inefficiencies of DMU_o in the first and second stages, respectively.

Note that the new NPI index for measuring inefficiency in two-stage production systems retains the desired properties of the black-box NPI index. Additionally, it accounts for the potential improvement of intermediate products. Furthermore, a unit is classified as efficient in the extended NPI approach, *i.e.* $E_{\text{two-stage}}^{\text{NPI}} = 0$, if and only if it performs efficiently in both stages, $E_1^{\text{NPI}}(\boldsymbol{x}_{\text{o}}) = 0$ and $E_2^{\text{NPI}}(\boldsymbol{x}_{\text{o}}) = 0$.

To calculate $x_i^*(\boldsymbol{x}_{\mathbf{o}})$, the following LPs are solved for $i = 1, \ldots, m$ to find the ideal reference point $\boldsymbol{x}^*(\boldsymbol{x}_{\mathbf{o}})$:

$$\begin{aligned}
\min_{\lambda_1, \lambda_2, \bar{x}} \ & \bar{x}_i \\
\text{s.t. } & \sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq \bar{x}_i, \\
& \sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq x_{(-i)o}, && -i = 1, \ldots, i-1, i+1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j^1 z_{gj} \geq z_{go}, && g = 1, \ldots, p
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{n} \lambda_j^2 z_{gj} \leq z_{go}, && g=1,\dots,p\\
&\sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}, && r=1,\dots,s\\
&\sum_{j=1}^{n} \lambda_j^1 = 1, \quad \sum_{j=1}^{n} \lambda_j^2 = 1, &&\\
&\lambda_j^1, \lambda_j^2 \geq 0, && j=1,\dots,n.
\end{aligned} \tag{11}$$

At optimality the ideal point of the first stage is given by $\boldsymbol{x}^*(\boldsymbol{x_o}) = (\bar{x}_1^*, \dots, \bar{x}_m^*)$. Similar LPs also need to be solved to calculate $\boldsymbol{z}^*(\boldsymbol{z}_o)$ by treating $\boldsymbol{z_o}$ as the input for the second stage.

Now, with access to the components of the ideal input points $\boldsymbol{x}^*(\boldsymbol{x_o})$ and $\boldsymbol{z}^*(\boldsymbol{z}_o)$, we suggest to solve the following LP to calculate the Luenberger excess functions of DMU_o in the both stages:

$$\begin{aligned}
\max_{\lambda_1,\lambda_2,\beta_1,\beta_1} \quad & \beta_1 + \beta_2 &&\\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq x_{io} - \beta_1 (x_{io} - \bar{x}_i^*), && i=1,\dots,m\\
& \sum_{j=1}^{n} \lambda_j^1 z_{gj} \geq z_{go}, && g=1,\dots,p\\
& \sum_{j=1}^{n} \lambda_j^2 z_{gj} \leq z_{go} - \beta_2 \big(z_{go} - \bar{z}_g^*\big), && g=1,\dots,p\\
& \sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}, && r=1,\dots,s\\
& \sum_{j=1}^{n} \lambda_j^1 = 1, \quad \sum_{j=1}^{n} \lambda_j^2 = 1, &&\\
& \lambda_j^1, \lambda_j^2 \geq 0, && j=1,\dots,n.
\end{aligned} \tag{12}$$

At optimality, we have $e\big(\boldsymbol{x_o}, L_1, \boldsymbol{g}_1^{\mathrm{PI}}\big) = \beta_I^*$ and $e\big(\boldsymbol{z_o}, L_2, \boldsymbol{g}_2^{\mathrm{PI}}\big) = \beta_2^*$. Note that both LP models (11) and (12) are feasible and bounded. Also, $E_{\text{two-stage}}^{\mathrm{NPI}}$ and its components are always non-negative, and if DMU_o at any stage or in the overall context is efficient, its corresponding potential improvement is zero.

The overall radial (Farrell) potential improvement, while considering internal indicators for DMU_o could is expressed as:

$$E_{\text{two-stage}}^{\text{NPI-F}}(\boldsymbol{x}_{\mathrm{o}}) = \big(1 - E_{\text{Two-Stage}}^{F}\big) \sum_{i=1}^{m} \frac{x_{io}}{x_i^+ - x_i^-}. \tag{13}$$

Here, $E_{\text{Two-Stage}}^{F}$ represents the radial efficiency value obtained from the radial model (7) in evaluating DMU_o. Based on the derived inefficiency indices, we introduce potential improvements for the two-stage production context, encompassing both radial and non-radial methodologies.

4.1. Illustrative example

To demonstrate the effectiveness of the proposed two-stage NPI approaches, we analyze six hypothetical DMUs, each characterized by one input, one intermediate product, and one output. The details of these six DMUs are presented in Table 1.

Table 1. Input, intermediate, and output data for the six DMUs.

DMUs	Input (x)	Intermediate product (z)	Output (y)
A	2	2	0.5
B	4	2	3
C	5	0.5	2
D	6	5	5.5
E	2	1	5.5
F	8	3	3

Let's consider DMU_A. Initially, we compute the radial Farrell NPI for this unit by solving the LP model (7) and then applying the relation (13):

$$E^{\text{NPI-F}}_{\text{two-stage}}(A) = (1-1)\times\frac{2}{1} = 0.$$

Next, we compute the proposed non-radial two-stage NPI for DMU_A. Firstly, we determine the ideal target inputs $x^*(x_A) = 2$ and $z^*(z_A) = 0.5$ along with the Luenberger excess function values $e\big(\boldsymbol{x_o}, L_1, \boldsymbol{g}_1^{\text{PI}}\big) = 0$ and $e\big(\boldsymbol{z_o}, L_2, \boldsymbol{g}_2^{\text{PI}}\big) = 1$ from the LP models (11) and (12), respectively. Subsequently, by applying the equations (9) and (10) to obtain:

$$E_1^{\text{NPI}}(A) = 0\times\frac{(2-2)}{1} = 0$$
$$E_2^{\text{NPI}}(A) = 1\times\frac{(2-0.5)}{1} = 1\times 1.5 = 1.5.$$

By substituting these results into the non-radial improvement potential formula (8), we derive the values of the overall and its stages improvement potential inefficiencies:

$$E^{\text{NPI}}_{\text{two-stage}}(A) = 0 + 1.5 = 1.5.$$

Based on these values, we note that the extended NPI for the two-stage system identifies inefficiency sources for DMU A, while the radial and black-box indicators fail and classify this unit as efficient, *i.e.* $E_1^{\text{NPI}}(A) = 0$.

Using the outlined procedure, we computed the NPI values for each individual stage and the overall values for all DMUs. The results from the radial Farrell and non-radial models are shown in Table 2. For comparison, the NPI values for the black-box case, where intermediate values are ignored, are also included in the last column of Table 2.

The most important observation in the results is that the extended $E^{\text{NPI}}_{\text{two-stage}}$ index shows the higher values for potential inefficiencies in both the radial and black-box cases. This indicates a greater discriminative power for identifying inefficiency sources.

Notably, since there is only one input variable, the NPI values of the radial Farrell $E^{\text{NPI-F}}_{\text{two-stage}}$ and the first stage, E_1^{NPI} indices are equal. However, due to differences in technology sets and by ignoring intermediate products, the computed NPI for the black-box case E^{NPI}, and the radial Farrell may differ, as seen with DMU_F.

Furthermore, $E^{\text{NPI-F}}_{\text{two-stage}}$ identifies DMU_E as the only efficient unit in this sample classifying the other units as inefficient. It is important to note that the production set for evaluating E_1^{NPI} is a subset of the black-box case's production set for evaluating E^{NPI}; hence $E_1^{\text{NPI}} \leq E^{\text{NPI}}$. The results also show the relationship $\max\big\{E^{\text{NPI}}, E_1^{\text{NPI}}, E_2^{\text{NPI}}\big\} \leq E^{\text{NPI}}_{\text{two-stage}}$ among potential inefficiency values, providing a structured way to compare the discrimination power gains of the developed approach over the two-stage production process. This means the overall potential improvement $E^{\text{NPI}}_{\text{two-stage}}$ for the two-stage production system is at least equal to the highest

TABLE 2. Calculation of NPIs for the six DMUs.

DMUs	$E^{\text{NPI-F}}_{\text{two-stage}}$	E^{NPI}_1	E^{NPI}_2	$E^{\text{NPI}}_{\text{two-stage}}$	E^{NPI}
A	0	0	1.5	1.5	0
B	2	2	1.355	3.355	2
C	3	3	0	3	3
D	0	0	4.5	4.5	4
E	0	0	0	0	0
F	4.667	4.667	2.357	7.024	6

potential improvement observed in either individual stage or the black-box case. Thus, $E^{\text{NPI}}_{\text{two-stage}}$ serves as a more powerful measure to identify inefficiencies across the entire production process.

Based on the $E^{\text{NPI}}_{\text{two-stage}}$ values, the extended NPI approach suggests the following rank order for the units:

$$\text{DMU}_F \prec \text{DMU}_D \prec \text{DMU}_B \prec \text{DMU}_C \prec \text{DMU}_A \prec \text{DMU}_E.$$

In the subsequent section, we will further illustrate the significance of this subject with a real-world example by analyzing the values and benchmarks.

5. Empirical application

Guilan province's banking system plays a crucial role in advancing economic stability and progress in the northern region of Iran. Beyond its role in financial intermediation, it serves as a vital catalyst for economic vitality and resilience. With abundant agricultural and tourism resources, the banking sector facilitates local trade and attracts investments essential for sustainable growth. By efficiently managing funds and financial services, these institutions not only support entrepreneurship but also foster broader socio-economic advancement. Despite grappling with regulatory challenges and the necessity for modernization to meet national banking standards, Guilan's banks play an indispensable role in implementing regional monetary policies and ensuring market liquidity. Leveraging the potential improvement concept presents significant opportunities to enhance operational efficiencies and optimize resource allocation. This strategic approach strengthens their pivotal role in bolstering economic resilience and driving inclusive development across the province.

5.1. Data and variables

We assessed the efficiency and performance of 35 banking branches in Guilan province using the developed two-stage Normalized Potential Improvement (NPI) approach. Data from the last six months of 2021 provided inputs such as "funds from customers" and "number of cheque accounts". These inputs reflect resources used by branches, yielding outputs like "deposits". In the second stage, outputs such as "number of transactions", "loans", and "profits" were evaluated, representing final services and financial gains derived from processed inputs.

The selection of inputs and outputs for this study was informed by a thorough review of the literature and discussions with banking managers to ensure relevance and applicability. This process ensured that the variables chosen accurately represent the operational processes and performance metrics of interest in the banking sector of Guilan province.

The process flow is illustrated in Figure 3, depicting the sequential transformation of inputs to intermediate product and final outputs across the two stages of our analysis.

Table 3 summarizes the inputs, intermediate products, and outputs used in this paper along with their corresponding references.

Accordingly, the variables are categorized into three types as follows:

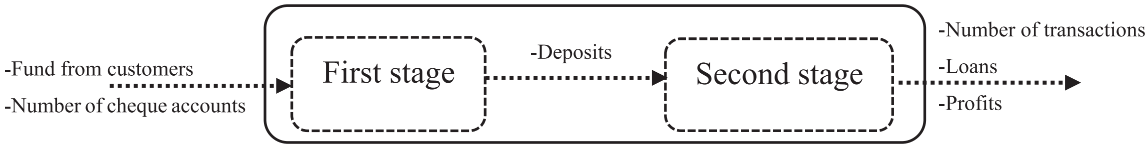

FIGURE 3. Process flow in Guilan bank branches.

TABLE 3. The factors used in the banking case study.

Factor	Factor type	References
Fund from customers	Input	[25, 46]
Number of cheque accounts	Input	[28]
Deposits	Intermediate	[32, 49]
Number of transactions	Output	[48]
Loans	Output	[45, 50]
Profits	Output	[32, 47]

Input variables

- x_1 : fund from customers (in millions of Iranian Rials (IRR));
- x_2 : number of cheque accounts.

Intermediate variable

- z_1 : deposits (in millions of Iranian Rials (IRR)).

Outputs variables

- y_1 : number of transactions;
- y_2 : loans (in millions of Iranian Rials (IRR));
- y_3 : profits (in millions of Iranian Rials (IRR)).

The data on the input, output, and intermediate factors of the bank branches are detailed in Table 4. Using the developed Normalized Potential Improvement (NPI) indices, we will now evaluate these branches systematically for ranking and benchmark analysis purposes.

5.2. Results and analysis

We applied the three models presented in this study to compute and analyze the results of evaluating 35 bank branches. Table 5 presents the results for calculating the NPI indicators, including Farrell's radial model for the two-stage case ($E^{\text{NPI-F}}_{\text{two-stage}}$), the newly developed non-radial model overall ($E^{\text{NPI}}_{\text{two-stage}}$) and for each stage ($E^{\text{NPI}}_1$ and E^{NPI}_2), and the traditional black-box model (E^{NPI}). The benchmark or target units for evaluating each branch are also identified and reported in the table.

The first two columns indicate that, unlike the simple one-input case, in multidimensional input spaces, the results of E^{NPI}_1 and $E^{\text{NPI-F}}_{\text{two-stage}}$ may not coincide. This discrepancy arises because the unit-specific input ideal point $\boldsymbol{x}^*(\boldsymbol{x_o})$ significantly influences the improvement direction g^{PI}, which differs from the radial Farrell input direction pointing toward the origin for all units. Both E^{NPI}_1 and $E^{\text{NPI-F}}_{\text{two-stage}}$ classify units as efficient or inefficient similarly, except for DMU_6. The radial approach classifies this unit as efficient with $E^{\text{NPI-F}}_{\text{two-stage}} = 0$, while $E^{\text{NPI}}_1 = 145.4591$, revealing significant inefficiency in the input consumption of this unit in the first stage. The NPI value of the second stage further amplifies this difference.

TABLE 4. The data of input, output and intermediate indicators of the bank branches.

Branches	x_1	x_2	z_1	y_1	y_2	y_3
1	15 624.49	18 038	12 508.77	4432.467	287 033	1742.924
2	5970.642	17 036	12 610.26	4116.026	281 588	2208.833
3	15 707.09	18 404	9463.958	1769.922	751 067	1725.173
4	10 850.51	13 484	7666.07	5467.603	472 437	1020.575
5	5792.73	13 028	1583.038	5370.296	137 687	1419.301
6	6317.994	12 846	4990.327	5050.79	543 154	2183.981
7	10 695.9	15 761	9573.36	5682.139	742 585	1268.55
8	9194.238	15 305	10 734.61	5582.533	626 103	940.2833
9	8942.196	14 850	11 940.67	7662	865 675	1208.194
10	7044.468	14 486	6122.575	5889.779	602 035	1847.522
11	6800.898	14 303	5373.894	4848.514	326 761	2438.51
12	15 668.96	12 573	7882.238	4984.131	416 514	1232.5
13	8986.674	13 392	8594.012	2930.715	506 958	2482.206
14	9096.81	15 215	12 359.82	3241.792	951 483	2731
15	10 367.61	16 034	3864.669	3282.401	216 690	2416.116
16	10 022.38	10 751	4960.01	2867.12	367 002	2026.675
17	7321.926	18 038	4853.244	2911.56	371 539	1145.108
18	11 458.38	19 769	9234.609	3578.92	744 755	1801.095
19	18 278.34	23 141	10 647.61	2451.84	708 557	2081.568
20	5860.506	19 406	12 434.96	3598.075	986 300	1657.717
21	7576.086	18 404	12 489	3503.833	598 092	1868.55
22	15 755.8	15 579	7783.381	2593.587	402 804	1647.612
23	21 180	17 036	10 536.89	5601.688	550 947	1639.419
24	7864.134	18 131	7851.922	3503.833	761 818	1258.172
25	13 142.19	12 663	6511.414	3257.882	617 226	1875.651
26	5570.34	14 394	5499.113	3973.513	466 815	1550.662
27	6019.356	13 575	6834.349	4512.152	413 555	1198.363
28	5095.908	15 761	8178.811	3654.008	565 544	852.6182
29	16 293.77	11 297	9429.687	3372.046	377 851	1133.638
30	6559.446	18 038	13 181	4272.331	351 911	603.551
31	8037.81	18 311	12 035.57	3636.385	882 442	887.3019
32	7514.664	19 223	9065.892	3706.109	602 234	1350.206
33	7607.856	18 677	6607.635	3447.9	351 813	547.2924
34	5481.384	13 392	8213.081	4169.66	471 845	467.5472
35	8302.56	12 573	9612.903	5737.306	645 730	609.013

The results of $E^{\text{NPI}}_{\text{two-stage}}$ and E^{NPI} reveal that considering intermediate products in the newly developed NPI indices shows $E^{\text{NPI}}_{\text{two-stage}} \geq E^{\text{NPI}}$, indicating higher inefficiency sources in the two-stage NPI indices compared to the black-box NPI. The black-box NPI classifies 14 units as efficient ($E^{\text{NPI}} = 0$), while the two-stage NPI indices indicate that only three units are efficient in the new non-radial approach. This discrepancy implies about 76% error rate in the black-box approach's classification of efficient branches. The two-stage NPI approach classifies only three branches as efficient ($E^{\text{NPI}}_{\text{two-stage}} = 0$): DMU_5, DMU_{14}, and DMU_{20}, consistent with all other NPI evaluations. As the result, the two approaches are different in ranking the bank branches.

In both of the discussed cases, Figures 4 and 5 depict the differences between the calculated NPI values for all observations.

For inefficient units, the NPI values show considerable differences, indicating significant inefficiency sources ignored by the black-box approach compared to the developed two-stage NPIs. In both the black-box and two-stage DEA models for computing Luenberger excess functions, target units are identified by their positive

TABLE 5. Evaluation of NPI values and benchmarks for Bank Branches.

Branches	$E_{\text{two-stage}}^{\text{NPI-F}}$	E_1^{NPI}	Benchmarks units	E_2^{NPI}	Benchmarks units	$E_{\text{two-stage}}^{\text{NPI}}$	E^{NPI}	Benchmarks units
1	22 346.9	81 807.81	14. 30, 2	12 990.742	5, 6	94 798.55	94 583.26	5, 6
2	0	0	2	8754.074	5, 6, 14, 15	8754.074	0	2
3	59 145.04	86 424.78	16, 26, 35	1376.318	6, 14, 15	87 801.1	86 484.75	14, 16, 35
4	15 739.28	32 521.96	16, 29, 35	4309.74	5, 6, 10	36 831.7	36 715.2	6, 16, 35
5	0	0	5	0	5	0	0	5
6	0	145.4591	5, 16, 34	0	6	145.4591	0	6
7	25 174.08	37 652.88	16, 29, 35	947.556	9, 10, 24	38 600.43	30 108.11	9, 14, 16, 35
8	10 458.36	19 470.9	2, 14, 35	4324.069	9, 10, 24	23 794.97	21 514.69	6, 9, 14, 35
9	125.1461	229.79	2, 14, 35	0	9	229.79	0	9
10	9360.856	13 127.35	5, 16, 34	0	10	13 127.35	0	10
11	7878.529	11 122.92	5, 16, 34	0	11	11 122.92	0	11
12	19 248.47	52 935.77	16, 29, 35	23 956.119	5, 6	76 891.88	76 220.53	6, 16, 35
13	8138.102	16 953.75	16, 29, 35	1788.357	6, 14, 15	18 742.11	0	13
14	0	0	14	0	14	0	0	14
15	30 183.95	38 768.08	5, 16, 34	0	15	38 768.08	31 272.07	6, 13, 16
16	0	0	16	1116.221	5, 6, 15	1116.221	0	16
17	23 109.94	22 750.93	5, 16, 34	1305.068	5, 6	24 056	20 999.14	5, 26, 28, 34
18	46 977.43	55 487.91	16, 24, 35	1187.028	6, 14, 24	56 674.94	27 992.21	6, 14, 35
19	87 532.54	119 145.2	16, 29, 35	2960.498	6, 14, 24	122 105.7	119 154.9	14, 16, 35
20	0	0	20	0	20	0	0	20
21	9766.384	14 019.16	2, 14, 35	6779.065	6, 24	20 798.22	18 710.3	6, 20, 26, 28
22	45 802.49	81 475.66	16, 29, 35	3972.468	6, 24	85 448.13	83 008.89	16, 35
23	57 219.1	131 436.3	16, 29	15 880.042	5, 6, 10	147 316.34	146 129.4	6, 9, 35
24	26 132.53	27 712.13	16, 24, 35	0	24	27 712.13	16 784.82	6, 14, 20
25	18 345.14	42 749.73	16, 29, 35	546.9753	6, 14, 24	43 296.7	38 816.58	14, 16, 35
26	2236.433	2115.314	5, 34	1150.294	5, 6	3265.608	0 26	
27	2287.163	3858.148	5, 16, 34	2932.802	5, 6	6790.95	2688.685	5, 6, 34
28	0	0	28	2895.194	6, 24	2895.194	0	28
29	0	0	29	69 236.884	5, 6	69 236.884	66 700.15	16, 35
30	0	0	30	16 341.76	5, 6	16 341.76	15 097.07	5, 28, 34
31	12 989.55	18 318.74	2, 14, 35	1681.78	9, 20, 24	20 000.52	13 324.33	6, 14, 20
32	24 587.05	23 437.37	2, 34, 35	3302.075	6, 24	26 739.44	20 540.42	6, 20, 28, 34
33	26 582.99	25 992.75	5, 16, 34	3224.901	5, 6	29 217.66	25 857.7	5, 28, 34
34	0	0	34	3821.611	5, 6	3821.611	0	34
35	0	0	35	2757.789	9, 10, 24	2757.789	0	35

optimal lambda values. These units are reported in Table 5. Comparing the benchmarks of the first stage evaluation of the new approach with the black-box approach reveals differences. Except for fully efficient units (DMU_5, DMU_{14}, and DMU_{20}), the benchmark sets of the black-box and two-stage approaches differ for other units. The most similarity in benchmarking is for DMU_{22}, where the black-box approach suggests $\{\text{DMU}_{16}, \text{DMU}_{35}\}$ and the two-stage approach suggests $\{\text{DMU}_{16}, \text{DMU}_{29}, \text{DMU}_{35}\}$ for the bench sets, respectively. For the other 31 units (about 89% of observations), the benchmarks are different and even disjoint, highlighting substantial differences in decision-making support between the two approaches.

These findings emphasize the superiority of the two-stage NPI approach in accurately classifying and benchmarking units, revealing inefficiencies overlooked by traditional black-box models. This comprehensive evaluation

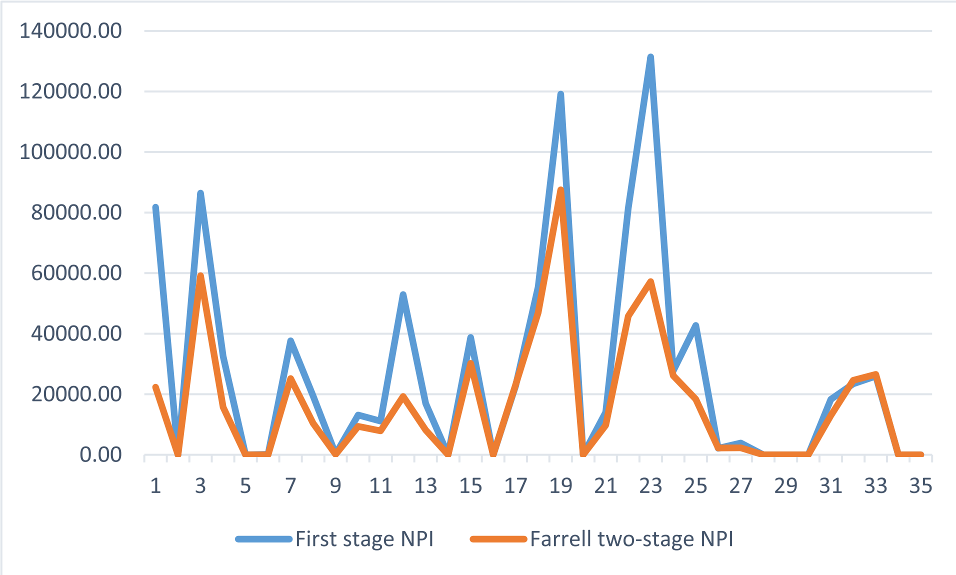

FIGURE 4. First stage and Radial Farrell Two-stage NPIs.

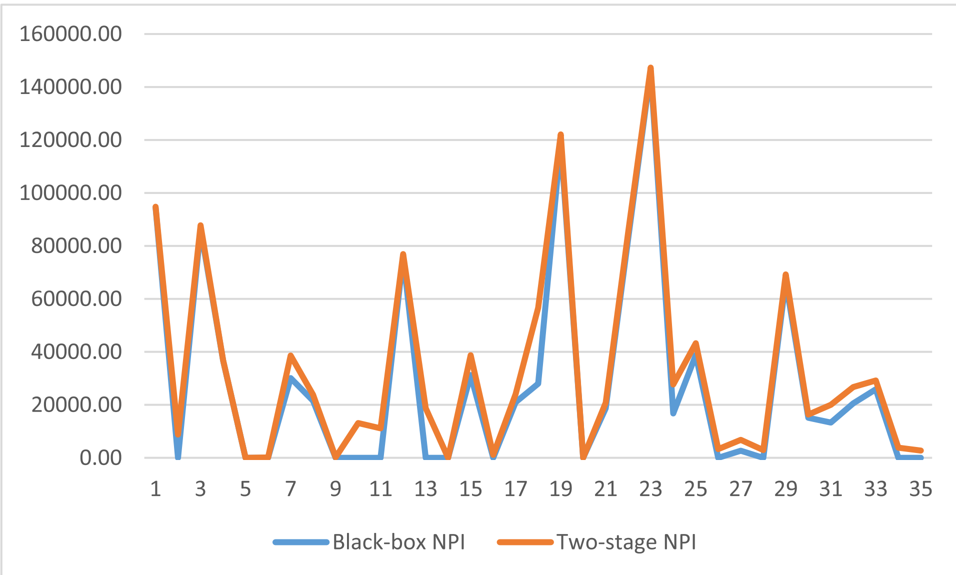

FIGURE 5. Black-box and Two-stage NPIs.

underscores the importance of adopting advanced DEA models for a more nuanced and accurate assessment and ranking of operational efficiency in the banking sector.

5.3. Managerial interpretations

Based on these findings and considering the specific variables of fund from customers, number of accounts, deposits, number of transactions, loans, and profits, Guilan bank managers can consider the following practical implications:

- *Targeted Resource Allocation*: utilize the insights and ranking order from the two-stage NPI approach to allocate resources more effectively. Addressing inefficiencies identified in specific stages related to fund from customers and number of accounts can lead to optimized input resource utilization and cost management strategies. For instance, reallocating funds from less productive areas to those generating higher transaction volumes or loan profits can enhance overall operational efficiency.
- *Strategic Planning*: use the accurate benchmarks provided by the two-stage NPI approach to inform strategic planning efforts. Identify best practices from branches with similar benchmarks in deposits, transactions, loans, and profits and implement targeted improvement initiatives. This strategic approach can involve optimizing deposit-to-loan ratios and profitability metrics across branches to improve overall financial performance.
- *Operational Adjustments*: focus on improving input inefficiencies in both stages as highlighted by the two-stage NPI results, particularly in managing fund from customers and account resources. This may involve revising operational processes, enhancing customer engagement strategies, or leveraging technology to streamline transaction processes.
- *Long-term Sustainability*: continuous monitoring and adjustment based on the insights provided by the more powerful two-stage NPI approach, compared to traditional black-box approaches, can contribute significantly to long-term sustainability. This advanced method allows banks to adapt to evolving customer needs, regulatory changes, and market dynamics while maintaining financial stability. Such a proactive approach ensures that banks remain resilient and capable of achieving sustainable growth over the long term.

In conclusion, leveraging the insights from the two-stage potential improvement approach empowers Guilan bank managers to make informed decisions, optimize resource allocation, and improve overall operational efficiency across fund management, transaction processing, loan administration, and profitability management. By focusing on these areas, Guilan banks can enhance performance, customer satisfaction, and competitive positioning in the banking industry.

6. Conclusions and future research

This paper introduces novel methods for assessing improvement potential within network data envelopment analysis (DEA), focusing on a two-stage network model evaluated through both radial and non-radial approaches. The findings underscore that the non-radial approach surpasses the radial approach in assessing efficiency and identifying improvement potential at various stages of the production process.

Applied within the Iranian banking sector in Guilan Province, this study highlights the practical significance of these methodologies. The results demonstrate their effectiveness in pinpointing inefficiencies and proposing targeted strategies to enhance operational efficiency in banking branches.

Looking ahead, it's important to note that the developed methodology is specifically tailored for two-stage series production systems. Future research can enhance these models by extending them to more complex network settings, such as multi-component and multi-function production cases, and addressing special data scenarios like negative output values and probabilistic data. Addressing these complexities will broaden the applicability of DEA methodologies in real-world settings characterized by data uncertainties. Moreover, refining directional efficiency scores and applying the models across diverse industries and organizational contexts offer promising avenues for further investigation.

In conclusion, the methodologies presented here provide a robust framework for improving decision-making processes aimed at enhancing efficiency and productivity within intricate operational environments.

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Data availability statement

The data supporting the findings of this study are available upon request from the corresponding author, R. Kazemi Matin.

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