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FUZZY NODE RUPTURE DEGREE OF SOME FUZZY GRAPH FAMILIES

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Abstract. In the event of the malfunction of nodes and/or links connecting nodes, the vulnerability parameters defined in graph theory may be employed as a metric of the quality of service received *via* a network. In graph theory, numerous different vulnerability parameters have been defined, including toughness, rupture degree, tenacity, integrity, connectivity and others. The modelling of real-world problems is made more effective by the use of fuzzy graphs as a specific type of graph. These issues can be more accurately addressed through the use of membership values as they better represent the inherent uncertainties involved. However, despite this advantage, there has been limited research conducted on the vulnerability parameters in fuzzy graphs. In this paper, we consider the importance of fuzzy graphs and the limitations of the existing literature on vulnerability parameters for these graphs. We propose a definition of the fuzzy node rupture degree, a commonly used concept in graph theory, for application in fuzzy graph theory. Furthermore, this study examines the theoretical aspects of fuzzy wheel graphs, fuzzy cycle graphs and fuzzy star graphs, with the objective of providing general formulas that can be applied in practice, and also an algorithm for calculating the fuzzy node rupture degree of a given fuzzy graph is presented.

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1. Introduction

A network is a collection up of nodes and links, just like communication networks, transportation networks, and water distribution networks, among others. Inefficiencies in links or nodes can have an influence on overall network service quality. Network vulnerability is the term used to describe this issue. Vulnerability evaluates the network's ability to withstand disturbances caused by the breakdown of certain nodes or links. A network should be designed to withstand initial disruptions and allow for possible reconstruction attempts [1].

The vulnerability of a network can be quantified by utilising graph parameters such as connectivity [2], integrity [1], tenacity [3], toughness [4], rupture degree [5], among others. It is notable that the connectivity of nodes and links has been a commonly used parameter. The connectivity of a graph only deals with the number of elements that are not functioning. However, this parameter is typically insufficient, as it does not provide insight into the remaining components of the graph after a disruption. Consequently, another vulnerability parameter, rupture degree, was introduced. This parameter considers the number of connected components and

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the number of elements in the largest remaining graph within which mutual communication can still occur besides the number of elements that are not functioning [2].

Fuzzy sets were defined by Zadeh in 1965 as a natural extension of fuzzy logic. An object may be a partial element of a fuzzy set, while it may or may not be an element of a set [6]. Fuzzy graphs are a specialisation of graphs which are defined by Rosenfeld [7] and Yeh *et al.* [8], independently in 1975. One of the important differences between fuzzy graphs and graphs is that in a graph, the connection strength between any two nodes is 0 or 1, while in fuzzy graphs it can be any real number between $[0, 1]$. Although fuzzy graphs and graphs are structurally similar to each other, fuzzy graphs gain a special importance when there are an uncertainty in the nodes and/or the links. Modelling by grading the relations is closer to reality in a world where everything is not black and white. For instance, when modelling the traffic flow of a city with a graph, membership values can be assigned to the links of the graph in response to the number of vehicles passing on a road. With the vulnerability of a fuzzy graph modelled in this way indicates the effect of damage to crossroads or roads on the entire traffic flow, or which crossroads or roads are most critical and what measures can be taken in advance in case of damage. Although fuzzy graphs offer a more realistic modelling opportunity, there are only a few studies in the literature when examined in terms of vulnerability. This study aims to fill this major gap in the literature. Although many vulnerability parameters have been defined in graph theory and the vulnerability of various graphs has been analysed, only connectivity [9] and integrity [10] parameters have been defined for fuzzy graphs.

There is no other parameter available to determine the number of sub-networks where communication continues or the number of centres of the largest sub-network where communication continues.

The aim of this study is to establish a new vulnerability parameter for fuzzy graphs that considers both the quantity of damaged nodes and the number of sub-networks that remain connected, as well as the number of nodes within the largest sub-network that remain connected. Fuzzy graphs are preferred as they allow for modelling that is closer to reality. This study defines the rupture degree for fuzzy graphs by considering the reduction of flow and the strength of connectedness based on the study of Mathew *et al.* [11], which are important in fuzzy graphs, instead of the disconnectedness of the graph that is the basis of vulnerability parameters in graph theory.

In this study, the required definitions and theorems are given in Section 2. In Section 3, the new vulnerability parameter fuzzy node rupture degree is defined and also some families of graphs are examined and the general formulas for the fuzzy node rupture degree of these fuzzy graphs are obtained. Additionally, an algorithm to evaluate the fuzzy node rupture degree of a given graph is given in Section 4.

The fuzzy node rupture degree is a vulnerability measure applicable in many different domains. For telecommunication networks, for instance, it can be utilized in determining the most vulnerable nodes, *i.e.* those whose failure would result in serious communication disruption. This measure is useful in estimating the contribution of certain nodes to the network reliability when such nodes become inoperative. In social network analysis, it can determine the effect of central individuals (*e.g.* influencers) on the structural reliability and information dissemination within a community. This is highly relevant in research on online platforms' susceptibility to targeted attacks or the spread of misinformation. In bioinformatics, the fuzzy node rupture degree can be used to identify key proteins in protein-protein interaction networks, and hence potential biological drug targets. In the case of transportation and logistics networks, this measure can evaluate the effects of road closures or major junction failures on traffic flow and connection. To drive this point home, consider the effects of a highway closure in an urban setting; such analysis can guide the development of more resilient infrastructure. These uses show that the fuzzy node rupture degree is not only a theoretical measure but has immense practical importance in several fields.

2. Preliminaries

The definition of fuzzy graph, some basic concept definitions and and theorems used in this study are given in this section. The references [2, 9, 12] can be used for he definitions not given.

Definition 2.1 ([12]). A fuzzy graph $G = (V, \sigma, \mu)$, also called f-graph, is a triple where V represents the set of nodes, $\sigma : V \to [0, 1]$ is the fuzzy subset of V and $\mu : VxV \to [0, 1]$ is the fuzzy relation on σ where $\mu(v_1, v_2) \leq \sigma(v_1) \wedge \sigma(v_2)$, $\forall v_1, v_2 \in V$.

Definition 2.2 ([13]). An arc which is an edge of a fuzzy graph is the weakest arc if it has the smallest membership value in G, denoted by $d(\mu)$ and the strongest arc if it has the largest membership value in G, denoted by $h(\mu)$.

Definition 2.3 ([7]). The strength of connectedness between two nodes v_1 and v_2 is equal to the largest value of the strengths of all paths between v_1 and v_2 in a fuzzy graph $G = (V, \sigma, \mu)$, represented by $\mathrm{CONN}_G(v_1, v_2)$.

Definition 2.4 ([11]). If $\mu(x, y) > \mathrm{CONN}_{G-(x,y)}(x, y)$, then the arc (x, y) is called α-strong.

If $\mu(x, y) = \mathrm{CONN}_{G-(x,y)}(x, y)$, then the arc (x, y) is called β-strong.

If $\mu(x, y) < \mathrm{CONN}_{G-(x,y)}(x, y)$, then the arc (x, y) is called δ-arc. An arc is called strong either it is α-strong or β-strong.

Definition 2.5 ([7]). A node v in a fuzzy graph $G = (V, \sigma, \mu)$ is called a fuzzy cut node if the deletion of this node reduces the strength of connectedness between any two nodes of G. An arc is called a fuzzy bridge if the deletion of this arc reduces the strength of connectedness between any two nodes of G.

Definition 2.6 ([9]). Fuzzy node cut (FNC) of a fuzzy graph $G = (V, \sigma, \mu)$ is the set of nodes $S = \{u_1, u_2, \ldots, u_n\}$ where $\mathrm{CONN}_{G-S}(x, y) < \mathrm{CONN}_G(x, y)$ or $G{-}S$ is trivial for every node x and y of G.

Definition 2.7 ([14]). A fuzzy graph $G = (V, \sigma, \mu)$ is called a fuzzy cycle if it contains more than one weakest arc where G^* is a cycle.

Theorem 2.8. *If a fuzzy graph $G = (V, \sigma, \mu)$ has at most one α-strong arc where G^* is a cycle, then G does not contain a fuzzy cut node [11].*

Definition 2.9 ([14]). A fuzzy cycle $G = (V, \sigma, \mu)$ is called a locamin cycle if every node of G is incident to the weakest arc of G and a multimin cycle is the fuzzy cycle containing more than one weakest arc.

Definition 2.10 ([15]). The sequence of strengths arcs in a fuzzy graph $G = (V, \sigma, \mu)$ given in a nonincreasing order as $\{q_1, q_2, \ldots q_m\}$ is called the arc strength sequence where q_1 is depth of μ and q_m is height of μ for $|\mu^*| = m$.

Definition 2.11 ([9]). The strong weight $s(S)$ of a fuzzy node cut S of a fuzzy graph $G = (V, \sigma, \mu)$ is the sum of the smallest membership values among the strong arcs incident to the nodes in S. That is $s(S) = \sum_{v_1 \in S} \mu(v_1, v_2)$, where $\mu(v_1, v_2)$ is the minimum of the weights of strong arcs incident on v_1.

3. Fuzzy node rupture degree of fuzzy graphs

This section presents a new vulnerability parameter for fuzzy graphs called fuzzy node rupture degree. This parameter takes into account the number of damaged nodes, the number of remaining connected sub-networks, and the number of nodes within the largest connected sub-network. This definition is based on the reduction of flow and strength of connectedness, rather than the disconnection of the fuzzy graphs.

The following two definitions contain the variables $m_f(G{-}S)$ and $\omega_f(G{-}S)$ that are used in the definition of fuzzy node rupture degree.

Definition 3.1 ([15]). Let $G = (V, \sigma, \mu)$ be a connected fuzzy graph and S be a fuzzy node cut in G. Maximum weight is the maximum of the degree of membership values of a pair of nodes whose strength of connectedness is reduced after the fuzzy node cut is removed from G, denoted by $m_f(G{-}S)$. If $G{-}S$ is trivial, then $m_f(G{-}S) = 0$, otherwise

$$m_f(G{-}S) = \max_{S \subset \sigma*} \{\mathrm{CONN}_{G-S}(v_1, v_2) : \mathrm{CONN}_{G-S}(v_1, v_2) < \mathrm{CONN}_G(v_1, v_2)\}$$

where $u, v \in \sigma^* - S$.

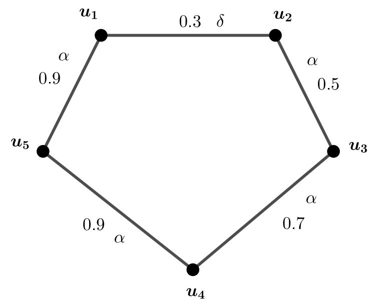

FIGURE 1. An example of a fuzzy graph G.

Definition 3.2 ([15])**.** Let $G = (V, \sigma, \mu)$ be a connected fuzzy graph and S be a fuzzy node cut in G. The reduced connectedness strength is the maximum of the differences in connectedness between pair of nodes whose strength of connectedness is reduced after the fuzzy node cut is removed from G, denoted by $\omega_f(G{-}S)$.

$$\omega_f(G{-}S) = \max\{\mathrm{CONN}_G(v_1, v_2) - \mathrm{CONN}_{G-S}(v_1, v_2) : v_1, v_2 \in \sigma^*\}.$$

The parameter of the fuzzy node rupture degree has been defined and examined in detail on an example.

Definition 3.3. Let $G = (V, \sigma, \mu)$ be a connected fuzzy graph. The fuzzy node rupture degree of G is defined as

$$r_f(G) = \max_{S \subseteq \sigma^*}\{\omega_f(G{-}S) - s(S) - m_f(G{-}S) : \omega(G{-}S) > 1\}$$

where S is a fuzzy node cut, $s(S)$ is the strong weight of S, $m_f(G{-}S)$ is the maximum weight of $G{-}S$ and $\omega_f(G{-}S)$ is the reduced connectedness strength of $G{-}S$.

Example 3.4. According to Figure 1, u_3, u_4 and u_5 nodes are 1-FNC.

For $S_1 = \{u_3\}$, $s(S_1) = 0.5$, $\mathrm{CONN}_{G-S_1}(u_1, u_2) = 0.3 < \mathrm{CONN}_G(u_1, u_2) = 0.5$ and $\mathrm{CONN}_{G-S_1}(u_2, u_4) = 0.3 < \mathrm{CONN}_G(u_2, u_4) = 0.5$. We obtain $m_f(G{-}S_1) = 0.3$ and $\omega_f(G{-}S_1) = \mathrm{CONN}_G(u_1, u_2) - \mathrm{CONN}_{G-S_1}(u_1, u_2) = 0.5 - 0.3 = 0.2$. Therefore; $r_f(G) = \omega_f(G{-}S_1) - s(S_1) - m_f(G{-}S_1) = 0.2 - 0.5 - 0.3 = -0.6$.

For $S_2\{u_4\}$, $s(S_2) = 0.7$ with $\mathrm{CONN}_{G-S_2}(u_1, u_2) = 0.3 < \mathrm{CONN}_G(u_1, u_2) = 0.5$ and $\mathrm{CONN}_{G-S_2}(u_3, u_5) = 0.3 < \mathrm{CONN}_G(u_3, u_5) = 0.7$. We get $m_f(G{-}S_2) = 0.3$ and $\omega_f(G{-}S_2) = \mathrm{CONN}_G(u_3, u_5) - \mathrm{CONN}_{G-S_2}(u_3, u_5) = 0.7 - 0.3 = 0.4$. So, $r_f(G) = \omega_f(G{-}S_2) - s(S_2) - m_f(G{-}S_2) = 0.4 - 0.7 - 0.3 = -0.6$.

For $S_3 = \{u_5\}$, $s(S_3) = 0.9$, $\mathrm{CONN}_{G-S_3}(u_1, u_2) = 0.3 < \mathrm{CONN}_G(u_1, u_2) = 0.5$ and $\mathrm{CONN}_{G-S_3}(u_1, u_4) = 0.3 < \mathrm{CONN}_G(u_1, u_4) = 0.9$. We obtain $m_f(G{-}S_3) = 0.3$ and $\omega_f(G{-}S_3) = 0.6$. So, $r_f(G) = 0.6 - 0.9 - 0.3 = -0.6$.

For $S_4 = \{u_1, u_3\}$, $s(S_4) = 0.9 + 0.5 = 1.4$ with $\mathrm{CONN}_{G-S_4}(u_2, u_4) = 0 < \mathrm{CONN}_G(u_2, u_4) = 0.5$ and $\mathrm{CONN}_{G-S_4}(u_2, u_5) = 0 < \mathrm{CONN}_G(u_2, u_5) = 0.5$. We get $m_f(G{-}S_4) = 0$ and $\omega_f(G{-}S_4) = 0.5$. Hence, $r_f(G) = 0.5 - 1.4 - 0 = -0.9$.

For $S_5 = \{u_1, u_4\}$, $s(S_5) = 0.9 + 0.7 = 1.6$ with $\mathrm{CONN}_{G-S_5}(u_2, u_5) = 0 < \mathrm{CONN}_G(u_2, u_5) = 0.5$ and $\mathrm{CONN}_{G-S_5}(u_3, u_5) = 0 < \mathrm{CONN}_G(u_3, u_5) = 0.7$. We obtain $m_f(G{-}S_5) = 0$ and $\omega_f(G{-}S_5) = 0.7$. Therefore, $r_f(G) = 0.7 - 1.6 - 0 = -0.9$.

For $S_6 = \{u_2, u_4\}$, $s(S_6) = 0.5 + 0.7 = 1.2$ with $\mathrm{CONN}_{G-S_6}(u_3, u_5) = 0 < \mathrm{CONN}_G(u_3, u_5) = 0.7$ and $\mathrm{CONN}_{G-S_6}(u_1, u_3) = 0 < \mathrm{CONN}_G(u_1, u_3) = 0.7$. We get $m_f(G{-}S_6) = 0$ and $\omega_f(G{-}S_6) = 0.7$. So, $r_f(G) = 0.7 - 1.2 - 0 = -0.5$.

For $S_7 = \{u_2, u_5\}$, $s(S_7) = 0.5 + 0.9 = 1.4$ with $\mathrm{CONN}_{G-S_7}(u_1, u_4) = 0 < \mathrm{CONN}_G(u_1, u_4) = 0.9$ and $\mathrm{CONN}_{G-S_7}(u_1, u_3) = 0 < \mathrm{CONN}_G(u_1, u_3) = 0.7$. We obtain $m_f(G{-}S_7) = 0$ and $\omega_f(G{-}S_7) = \max\{0.9, 0.7\} = 0.9$. And then, $r_f(G) = 0.9 - 1.4 - 0 = -0.5$.

For $S_8 = \{u_3, u_5\}$, $s(S_8) = 0.5 + 0.9 = 1.4$ with $\mathrm{CONN}_{G-S_8}(u_2, u_4) = 0 < \mathrm{CONN}_G(u_2, u_4) = 0.5$ and $\mathrm{CONN}_{G-S_8}(u_1, u_4) = 0 < \mathrm{CONN}_G(u_1, u_4) = 0.9$. We get $m_f(G{-}S_8) = 0$ and $\omega_f(G{-}S_8) = \max\{0.5, 0.9\} = 0.9$. Therefore, $r_f(G) = 0.9 - 1.4 - 0 = -0.5$.

For $S_9 = \{u_3, u_4\}$, $s(S_9) = 0.5 + 0.7 = 1.2$ with $\mathrm{CONN}_{G-S_9}(u_1, u_2) = 0.3 < \mathrm{CONN}_G(u_1, u_2) = 0.5$ and $\mathrm{CONN}_{G-S_9}(u_2, u_5) = 0.3 < \mathrm{CONN}_G(u_2, u_5) = 0.5$. We obtain $m_f(G{-}S_9) = 0.3$ and $\omega_f(G{-}S_9) = 0.2$. So, $r_f(G) = 0.2 - 1.2 - 0.3 = -1.3$.

For $S_{10} = \{u_4, u_5\}$, $s(S_{10}) = 0.7 + 0.9 = 1.6$ with $\mathrm{CONN}_{G-S_{10}}(u_1, u_2) = 0.3 < \mathrm{CONN}_G(u_1, u_2) = 0.5$ and $\mathrm{CONN}_{G-S_{10}}(u_1, u_3) = 0.3 < \mathrm{CONN}_G(u_1, u_3) = 0.7$. We get $m_f(G{-}S_{10}) = 0.3$ and $\omega_f(G{-}S_{10}) = \max\{0.2, 0.4\} = 0.4$. Therefore, $r_f(G) = 0.4 - 1.6 - 0.3 = -1.5$.

If the number of elements of S, which is a fuzzy node cut, increases the strong weight of S also increases. In this case the fuzzy node rupture degree decreases. But by the definition of the fuzzy node rupture degree must be maximum.

By the definition;

$$r_f(G) = \max\{-0.6, -0.6, -0.6, -0.9, -0.9, -0.5, -0.5, -0.5, -1.3, -1.5\} = -0.5.$$

The parameter has been applied to various types of graphs and the general results for the fuzzy node rupture degree of some types of fuzzy graphs are obtained and given below.

Theorem 3.5. *Let $G = (V, \sigma, \mu)$ be a fuzzy cycle graph with $n \geq 4$. Then the fuzzy node rupture degree is*

$$r_f(G) = -d(\mu).$$

Proof. Let S be a fuzzy node cut of a fuzzy cycle G. There are two cases according to the existence of a fuzzy cut node of G.

Case 1. Let G contain a fuzzy cut node, say a, and let $S = \{a\}$. So $|S| = 1$ and for $x, y \,\epsilon\, \sigma^*$, $\mathrm{CONN}_G(x, y) = s(S)$. If the set S is removed from the graph then $\mathrm{CONN}_{G-S}(x, y) = d(\mu) = q_1$. Hence $m_f(G{-}S) = q_1$. Largest difference of strength of connectedness in $G{-}S$ is $\omega_f(G{-}S) = s(S) - q_1$. Thus, fuzzy node rupture degree is

$$r_f(G) = \max_{S \subseteq \sigma^*}\{\omega_f(G{-}S) - s(S) - m_f(G{-}S) : \omega(G{-}S) > 1\}$$
$$r_f(G) = s(S) - q_1 - s(S) - q_1 = -2q_1. \tag{1}$$

If $|S| = 2$ for $S = \{a, b\}$ where $a, b \,\epsilon\, \sigma^*$ so that S consist of adjacent elements then $s(S) = s(a) + s(b)$ when the set S is removed from the fuzzy graph G, $\mathrm{CONN}_{G-S}(x, y) = d(\mu) = q_1$ for $x, y \epsilon \sigma^*$ is obtained. Hence, $m_f(G{-}S) = q_1$. Since largest difference of strength of connectedness in $G{-}S$ is $\omega_f(G{-}S) = \mathrm{CONN}_G(x, y) - \mathrm{CONN}_{G-S}(x, y)$ we have $\omega_f(G{-}S) = s(a) - q_1$ for $\mathrm{CONN}_G(x, y) = \min\{s(a) - s(b)\} = s(a)$. So, fuzzy node rupture degree is

$$r_f(G) = \max_{S \subseteq \sigma^*}\{\omega_f(G{-}S) - s(S) - m_f(G{-}S) : \omega(G{-}S) > 1\}$$
$$r_f(G) = s(S) - q_1 - s(a) - s(b) - q_1 = -s(b) - 2q_1. \tag{2}$$

Let $S = \{a, b\}$ for $a, b \,\epsilon\, \sigma^*$ such that S consist of non-adjacent elements. Hence $s(S) = s(a) + s(b)$. If the set S is removed from the fuzzy cycle graph G then $\mathrm{CONN}_{G-S}(x, y) = 0$ for $x, y \epsilon \sigma^*$. Thus $\mu_f(G{-}S) = 0$. The largest difference of strength of connectedness in $G{-}S$ is $\omega_f(G{-}S) = s(b)$ for $\mathrm{CONN}_G(x, y) = \max\{s(a) - s(b)\} = s(b)$. From the definition of fuzzy node rupture degree;

$$r_f(G) = \max_{S \subseteq \sigma^*}\{\omega_f(G{-}S) - s(S) - m_f(G{-}S) : \omega(G{-}S) > 1\}$$
$$r_f(G) = s(b) - s(a) - s(b) = -s(a). \tag{3}$$

Since $q_1 \leq s(S) \leq q_m$, by the definition of fuzzy node rupture degree with (2) and (3) it is obtained that

$$r_f(G) = -d(\mu) = -q_1. \tag{4}$$

If $\mid S \mid \geq 3$ such that S consist of adjacent elements then $s(S) \geq q_m + (\mid S \mid -1)q_1$. If the set S is removed from the fuzzy cycle graph G then $\mathrm{CONN}_{G-S}(x, y) = q_1$ for $x, y \epsilon \sigma^*$. Hence $m_f(G-S) = q_1$. Since $\mathrm{CONN}_G(x, y) \leq q_m$ and $\mathrm{CONN}_{G-S}(x, y) = q_1$. The largest difference of strength of connectedness in $G-S$ is $\omega_f(G-S) \leq q_m - q_1$. Thus the fuzzy node rupture degree is

$$r_f(G) = \max_{S \subseteq \sigma^*} \{\omega_f(G-S) - s(S) - m_f(G-S) : \omega(G-S) > 1\}$$
$$r_f(G) \leq q_m - q_1 - q_m + (1+ \mid S \mid)q_1 - q_1$$
$$r_f(G) \leq -q_1(\mid S \mid +1).$$

By $\mid S \mid \geq 3$,

$$r_f(G) \leq -4q_1. \tag{5}$$

If $\mid S \mid \geq 3$ such that S consist of non-adjacent elements then $s(S) \geq q_m + (\mid S \mid -1)q_1$. When the set S is removed from the fuzzy cycle graph G, $\mathrm{CONN}_{G-S}(x, y) = 0$ for $x, y \epsilon \sigma^*$. Thus $m_f(G-S) = 0$. So $\omega_f(G-S) \leq q_m$ from $\mathrm{CONN}_G(x, y) \leq q_m$ and $\mathrm{CONN}_{G-S}(x, y) = 0$. The fuzzy node rupture degree is $r_f(G) \leq -q_1(\mid S \mid -1)$. By $\mid S \mid \geq 3$,

$$r_f(G) \leq -2q_1. \tag{6}$$

From (1), (4), (5), (6) and by the definition of fuzzy node rupture degree, we obtain

$$r_f(G) = -d(\mu) = -q_1.$$

Case 2. Let S be a fuzzy node cut of a fuzzy cycle graph G. Since there is no fuzzy cut node, $\mid S \mid \geq 2$. The weakest arcs in the fuzzy cycle graph are β-strong the remaining arcs are α-strong arcs. Since there is no fuzzy cut node in the fuzzy cycle graph G, each node is incident to the weakest arc. Thus $s(v_i) = d(\mu) = q_1$ where $i = 1, \ldots, n$. When the set S that consist of adjacent elements is removed from the fuzzy cycle graph G, the strength of connectedness between the node pairs doesn't change. Thus the set S should consist of non-adjacent elements and $s(S) = 2d(\mu) = 2q_1$ for $\mid S \mid = 2$. If the set S is removed from the fuzzy cycle graph G then $\mathrm{CONN}_G(x, y) = 0$ for $x, y \epsilon \sigma^*$. Thus $m_f(G-S) = 0$. In the fuzzy cycle graph G, the strength of connectedness between the remaining pair of nodes is $\mathrm{CONN}_G(x, y) = d(\mu) = q_1$ with $x, y \epsilon \sigma^*$ except for the pairs where α-strong arcs are incident. Thus $\omega_f(G-S) = \mathrm{CONN}_G(x, y) - \mathrm{CONN}_{G-S}(x, y) = d(\mu) = q_1$. So, the fuzzy node rupture degree is

$$r_f(G) = \max_{S \subseteq \sigma^*} \{\omega_f(G-S) - s(S) - m_f(G-S) : \omega(G-S) > 1\}$$
$$r_f(G) = -q_1 = -d(\mu). \tag{7}$$

Let $\mid S \mid \geq 3$, so $s(S) \geq 3q_1$. When the set S is removed from the fuzzy cycle graph G, $\mathrm{CONN}_{G-S}(x, y) = 0$ for $x, y \epsilon \sigma^*$. Thus $m_f(G-S) = 0$. Therefore $\omega_f(G-S) = q_1$ from $\mathrm{CONN}_G(x, y) = d(\mu) = q_1$. The fuzzy node rupture degree is

$$r_f(G) = \max_{S \subseteq \sigma^*} \{\omega_f(G-S) - s(S) - m_f(G-S) : \omega(G-S) > 1\}$$
$$r_f(G) \leq -2q_1 = -2d(\mu). \tag{8}$$

By (7), (8) and by the definition of fuzzy node rupture degree, we get

$$r_f(G) = -d(\mu) = -q_1.$$

Thus we obtain the result by Case 1 and Case 2.

$$r_f(G) = -d(\mu) = -q_1.$$

□

Theorem 3.6. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with only one weakest arc (v_1, v_2) with $v_1, v_2 \epsilon \sigma^*$ where G^* is a cycle graph. Then the fuzzy node rupture degree is*

$$r_f(G) = \max\{-2q_1, -s(v_1), -s(v_2)\}.$$

Proof. Let (v_1, v_2) be the weakest arc of a cycle graph such that $v_1, v_2, \ldots, v_n$ are the nodes of the cycle graph. This arc is δ-arc. The others are α-strong arcs. Therefore the nodes v_1 and v_2 are fuzzy end nodes and the nodes v_i for $i = 3, \ldots, n$ are fuzzy cut nodes.

Let S be a fuzzy node cut. If $\mid S \mid = 1$ then $\mathrm{CONN}_G(x, y) = s(S)$ for $x, y \epsilon \sigma^*$. If the set S is removed from the cycle graph G then $\mathrm{CONN}_{G-S}(x, y) = d(\mu) < \mathrm{CONN}_G(x, y) = s(S)$. $m_f(G{-}S) = d(\mu) = q_1$ and $\omega_f(G{-}S) = s(S) - q_1$. Thus the fuzzy node rupture degree is

$$\begin{aligned} r_f(G) &= \max_{S \subseteq \sigma^*}\{\omega_f(G{-}S) - s(S) - m_f(G{-}S) : \omega(G{-}S) > 1\} \\ r_f(G) &= s(S) - q_1 - s(S) - q_1 \\ r_f(G) &= -2q_1. \end{aligned} \tag{9}$$

Let $\mid S \mid = 2$ where S contains adjacent elements but does not contain v_1 and v_2. Thus $S = \{v_i, v_{i+1}\}$ for $i = 3, \ldots, n-1$; $v_i \epsilon \sigma^*$. Hence $S(S) = s(v_i) + s(v_{i+1})$ when the set S is removed from the cycle graph G. $\mathrm{CONN}_{G-S}(x, y) = q_1 < \mathrm{CONN}_G(x, y) = \min\{s(v_i), s(v_{i+1})\} = s(v_i)$. Therefore $m_f(G{-}S) = q_1$ and $\omega_f(G{-}S) = s(v_i) - q_1$. So, the fuzzy node rupture degree is

$$\begin{aligned} r_f(G) &= \max_{S \subseteq \sigma^*}\{\omega_f(G{-}S) - s(S) - m_f(G{-}S) : \omega(G{-}S) > 1\} \\ r_f(G) &= s(v_i) - q_1 - s(v_i) - s(v_{i+1}) - q_1 \\ r_f(G) &= -2q_1 - s(v_{i+1}). \end{aligned}$$

The fuzzy node rupture degree should be maximum by the definition. So $s(v_{i+1}) \geq q_2$ is accepted. Then

$$r_f(G) \leq -2q_1 - q_2. \tag{10}$$

Let $|S| = 2$ where the set S consists of non-adjacent elements and $\{v_1, v_2\} \notin S$. Thus $s(S) = s(v_i) + s(v_j)$ for $S = \{v_i, v_j\}$ with $i, j = 3, \ldots, n$. If the set S is removed from the cycle graph G, then $\mathrm{CONN}_{G-S}(v_1, v_2) = q_1$ with $\mathrm{CONN}_G(v_1, v_2) \geq q_2$ and $\mathrm{CONN}_{G-S}(x, y) = 0 < \mathrm{CONN}_G(x, y) = \max\{s(v_i), s(v_j)\} = s(v_j)$. Therefore $m_f(G{-}S) = q_1$, $\omega_f(G{-}S) = s(v_j)$. So the fuzzy node rupture degree is

$$\begin{aligned} r_f(G) &= \max_{S \subseteq \sigma^*}\{\omega_f(G{-}S) - s(S) - m_f(G{-}S) : \omega(G{-}S) > 1\} \\ r_f(G) &= s(v_j) - s(v_i) - s(v_j) - q_1 \\ r_f(G) &= -q_1 - s(v_i). \end{aligned}$$

By the definition of fuzzy node rupture degree

$$r_f(G) \leq -q_2 - q_1. \tag{11}$$

Let $\mid S \mid = 2$ where the set S consists of non-adjacent elements and also v_1 and v_2. If $S = \{v_1, v_i\}$ with $i = 3, \ldots, n$; $v_i \in \sigma^*$ then $s(S) = s(v_1) + s(v_i)$, $\mathrm{CONN}_{G-S}(v_n, v_2) = 0 < \mathrm{CONN}_G(v_n, v_2) = q_2$.

Thus $\mathrm{CONN}_{G-S}(v_{i-1}, v_{i+1}) = 0 < \mathrm{CONN}_G(v_{i-1}, v_{i+1}) = s(v_i)$. So, $m_f(G{-}S) = 0$ and $\omega_f(G{-}S) = s(v_i)$. Therefore the fuzzy node rupture degree is

$$\begin{aligned} r_f(G) &= s(v_i) - s(v_1) - s(v_i) \\ r_f(G) &= -s(v_1). \end{aligned}$$

If $S = \{v_2, v_i\}$ with $i = 3, \ldots, n$; $v_i \in \sigma^*$ then

$$r_f(G) = -s(v_2).$$

So, the fuzzy node rupture degree is

$$r_f(G) = \max\{-s(v_1), -s(v_2)\}. \tag{12}$$

Let $\mid S \mid \geq 3$ such that the set S consists of adjacent elements. Hence $s(S) \geq q_m + (\mid S \mid -1)q_2$. If the set S is removed from the fuzzy graph G, then $\text{CONN}_{G-S}(x, y) = q_1$ for $x, y \in \sigma^*$. Thus $m_f(G-S) = q_1$ and $\omega_f(G-S) \leq q_m - q_1$ for $\text{CONN}_G(x, y) \leq q_m$. The fuzzy node rupture degree is

$$\begin{aligned} r_f(G) &= \max_{S \subseteq \sigma^*}\{\omega_f(G-S) - s(S) - m_f(G-S) : \omega(G-S) > 1\} \\ r_f(G) &\leq q_m - q_1 - q_m + (1- \mid S \mid)q_2 - q_1 \\ r_f(G) &\leq (1- \mid S \mid)q_2 - 2q_1. \end{aligned}$$

Since the fuzzy node rupture degree must be maximum by the definition, it must be $\mid S \mid = 3$. So,

$$r_f(G) \leq -2q_2 - 2q_1. \tag{13}$$

Let$\mid S \mid \geq 3$ such that the set S consists of non-adjacent elements. Thus $s(S) \geq q_m + (\mid S \mid -1)q_2$ when the set S is removed from the fuzzy graph G, $\text{CONN}_{G-S}(x, y) = 0$ and $\text{CONN}_{G-S}(x, y) = q_1$ for $x, y \in \sigma^*$. Hence $m_f(G-S) = q_1$ and $\omega_f(G-S) \leq q_m$ for $\text{CONN}_G(x, y) \leq q_m$, $x, y \in \sigma^*$.

Therefore the fuzzy node rupture degree is

$$\begin{aligned} r_f(G) &= \max_{S \subseteq \sigma^*}\{\omega_f(G-S) - s(S) - m_f(G-S) : \omega(G-S) > 1\} \\ r_f(G) &\leq q_m - q_m + (1- \mid S \mid)q_2 - q_1 \\ r_f(G) &\leq (1- \mid S \mid)q_2 - q_1. \end{aligned}$$

Because the fuzzy node rupture degree must be maximum from the definition, $\mid S \mid = 3$. So,

$$r_f(G) \leq -2q_2 - q_1. \tag{14}$$

By (9)–(14) and the definition of the fuzzy node rupture degree, the result is

$$r_f(G) = \max\{-2q_1, -s(v_1), -s(v_2)\}.$$

□

Theorem 3.7. *Let $G = (V, \sigma, \mu)$ be a fuzzy cycle graph with one α-strong arc and $n \geq 4$. Then the fuzzy node rupture degree is*

$$r_f(G) = -d(\mu).$$

Proof. The proof is similar to the proof of Theorem 3.5. □

Theorem 3.8. *Let $G = (V, \sigma, \mu)$ be a self-centered fuzzy cycle graph with n nodes and let the arcs be represented by $e_i = (v_i, v_{i+1})$ for $i = 1, \ldots, n-1$ and $e_n = (v_n, v_1)$. If $0 < t < s \leq 1$ then;*

(i) *$\mu(e) = t$ for $i = 1, \ldots, n$ and $n \geq 4$;*
(ii) *$\mu(e_{2i-1}) = t$, $\mu(e_{2i}) = s$ for $n = 2k$, $k \in \mathbb{Z}$ and $i = 1, \ldots, \frac{n}{2}$;*
(iii) *$\mu(e_{2i-1}) = t$, $\mu(e_{2i}) = s$ and $\mu(e_n) = t$ for $n = 2k+1$ $k \in \mathbb{Z}^+ - \{1\}$ and $i = 1, \ldots, \frac{n-1}{2}$;*

(iv) $\mu(e_{2i-1}) = s$, $\mu(e_{2i}) = t$ *and* $\mu(e_n) = s$ *for* $n = 2k - 1$ $k \in \mathbb{Z}^+ - \{1\}$ *and* $i = 1, \dots, \frac{n-1}{2}$

$$r_f(G) = -d(\mu).$$

Proof. The proof is similar to the proof of Theorem 3.5. □

Theorem 3.9. *Let G be a fuzzy wheel graph where the node v_n is a fuzzy hub and the nodes $v_1, \dots, v_{n-1}$ are on the fuzzy cycle C_{n-1} with $n \geq 5$. Assume that $\mu(v_n, v_i) \leq \mu(e)$ for all $\mu(e)$ values, the membership values of the arcs lying on the cycle, are different for all $e \in E(C_{n-1})$ and for $i = 1, \dots, n-1$ all $\mu(v_n, v_i)$ are equal to the weakest arc on the cycle. Then the fuzzy node rupture degree is*

$$r_f(G) = \begin{cases} \max\{\mathrm{CONN}_G(v_i, v_j) \mid \mu(v_i, v_j) = 0\} - 3d(\mu), & \text{if FCN exists} \\ -2d(\mu), & \text{otherwise.} \end{cases}$$

Proof. All the paths on the fuzzy wheel graph G are strong path. Therefore each arc of G is either α-strong or β-strong. So the weakest arcs of G are β-strong arcs and the rest are α-strong. All the nodes v_i for $i = 1, \dots, n-1$ are adjacent to the hub. Then we obtain $s(v_i) = d(\mu)$. Hence there are two cases according to the existence of fuzzy cut node of G since $\mu(v_n, v_i) \leq \mu(e)$.

Case 1. Let fuzzy wheel graph G contain a fuzzy cut node. A node is a fuzzy cut node if and only if it is a common node of at least two α-strong arcs. In this case, the fuzzy cut node is incident to the α-strong arcs. Since all $\mu(v_n, v_i)$ for $i = 1, \dots, n-1$ are equal to the weakest arc on the cycle C_{n-1}, $q_1 < \mathrm{CONN}_G(v_i, v_j) \leq q_m$ where $i \neq j$ and $i, j = 1, \dots, n-1$.

Let S be a fuzzy node cut and $|S| = 1$. Hence $s(S) = d(\mu) = q_1$. If the set S is removed from the graph, then $\mathrm{CONN}_{G-S}(v_i, v_j) = q_1$ for $i \neq j$ $i, j = 1, \dots, n-1$. Thus $m_f(G-S) = q_1$. The largest difference of strength of connectedness in $G-S$ is $0 \leq \mathrm{CONN}_G(v_i, v_j) - \mathrm{CONN}_{G-S}(v_i, v_j) \leq q_m - q_1$ then $0 \leq \omega_f(G-S) \leq q_m - q_1$. Therefore the fuzzy node rupture degree is

$$-2q_1 \leq r_f(G) \leq q_m - 3q_1. \tag{15}$$

Let $|S| = 2$. So $s(S) = 2d(\mu) = 2q_1$ when the set S is removed from the fuzzy wheel graph $\mathrm{CONN}_{G-S}(v_i, v_j) = q_1$ for $i \neq j$, $i, j = 1, \dots, n-1$. Thus $m_f(G-S) = q_1$. Hence $0 \leq \omega_f(G-S) \leq q_m - q_1$ since $q_1 \leq \mathrm{CONN}_G(v_i, v_j) \leq q_m$ where $i \neq j$ $i, j = 1, \dots, n-1$. The fuzzy node rupture degree is

$$-3q_1 \leq r_f(G) \leq q_m - 4q_1. \tag{16}$$

Let $|S| \geq 3$. So $s(S) \geq 3d(\mu) - 3q_1$. If the set S is removed from the fuzzy graph G then $\mathrm{CONN}_{G-S}(v_i, v_j) \geq 0$ for $i \neq j$, $i, j = 1, \dots, n-1$. Hence $m_f(G-S) \geq 0$ and $\omega_f(G-S) \leq q_m$. Therefore the fuzzy node rupture degree is

$$r_f(G) \leq q_m - 3q_1. \tag{17}$$

So, we get the result $r_f(G) \leq q_m - 3q_1$ by taking the maximum of equations (15)–(17). By the definition of the fuzzy node rupture degree we obtain

$$r_f(G) = \max\{\mathrm{CONN}_G(v_i, v_j) \mid \mu(v_i, v_j) = 0\} - 3q_1. \tag{18}$$

Case 2. Let fuzzy wheel graph G not contain a fuzzy cut node. Hence every node is incident to a weakest arc. So fuzzy cycle C_{n-1} is a locamin fuzzy cycle and $\mathrm{CONN}_G(x, y) = q_1 = d(\mu)$ for $x, y \in \sigma^*$. $|S| \geq 2$ since G does not have a fuzzy cut node. If $|S| = 2$ then $\mathrm{CONN}_G(x, y) = \mathrm{CONN}_{G-S}(x, y)$ for $x, y \in \sigma^*$. Thus $|S| \geq 3$. Let $|S| \geq 3$. So $s(S) \geq 3d(\mu) \geq 3q_1$. If the set S is removed from the fuzzy wheel graph then $\mathrm{CONN}_{G-S}(x, y) = 0$ for $x, y \in \sigma^*$. Thus $m_f(G-S) = 0$ and $\omega_f(G-S) = d(\mu) = q_1$. The fuzzy node rupture degree is

$$r_f(G) = \max_{S \subseteq \sigma^*}\{\omega_f(G-S) - s(S) - m_f(G-S) : \omega(G-S) > 1\}$$

$$r_f(G) \leq -2q_1 = -2d(\mu).$$

By the definition of fuzzy node rupture degree and $|S|= 3$

$$r_f(G) = -2q_1 = -2d(\mu). \tag{19}$$

The proof is completed from (18) and (19). □

Theorem 3.10. *Let G be a fuzzy wheel graph where the node v_n is a fuzzy hub and the nodes $v_1, \ldots, v_{n-1}$ are on the fuzzy cycle C_{n-1}. If $\mu(v_n, v_i) < \mu(e)$ for $e \in E(C_{n-1})$ then the fuzzy node rupture degree is*

$$r_f(G) = \max\{\mathrm{CONN}_G(v_i, v_j) \mid \mu(v_i v_j) = 0\} - 3d(\mu).$$

Proof. Let G be a fuzzy wheel graph and $\mu(v_n, v_i)$ values are constant for $i = 1, \ldots, n-1$ and $\mu(v_n, v_i) < \mu(e)$ for all $e \in E(C_{n-1})$. Thus all the paths on the wheel G are strong paths. Since a strong path only contains α-strong and β-strong arcs, G does not contain any δ-arc. So v_n is adjacent to all the nodes v_i with $\mu(v_n, v_i) < \mu(e)$ for $i = 1, \ldots, n-1$ and $e \in E(C_{n-1})$. All the arcs (v_n, v_i) are β-strong arcs and $\mu(v_n, v_i) = d(\mu)$. Therefore fuzzy hub v_n is not a fuzzy cut node. Thus $s(v_i) = d(\mu)$ for $i = 1, \ldots, n$ and $s(S) =|S| \cdot d(\mu)$ where S is a fuzzy node cut of G. There are two cases according to the existence of a fuzzy node cut of G.

Case 1. Let fuzzy wheel graph G have a fuzzy cut node. Hence the node is incident to the α-strong arcs. So it has membership value among from $\{q_3, q_4, \ldots, q_m\}$. Let S be a fuzzy node cut and $|S|= 1$. Thus $s(S) = d(\mu) = q_1$. If the set S is removed from G then $\mathrm{CONN}_G(v_i, v_j) = q_2$ where $i \neq j$ $i, j = 1, \ldots, n-1$. Therefore $m_f(G{-}S) = q_2$ and $q_3 - q_2 \leq \omega_f(G{-}S) \leq q_m - q_2$. The fuzzy node rupture degree is

$$r_f(G) = \max_{S \subseteq \sigma^*}\{\omega_f(G{-}S) - s(S) - m_f(G{-}S) : \omega(G{-}S) > 1\}$$
$$q_3 - 2q_2 - q_1 \leq r_f(G) \leq q_m - 2q_2 - q_1. \tag{20}$$

Let $|S|= 2$. So $s(S) = 2q_1 = 2d(\mu)$. Then $q_1 \leq \mathrm{CONN}_G(v_i, v_j) \leq q_m$ for $i \neq j$; $i, j = 1, \ldots, n$. If the set S is removed from the fuzzy wheel graph G then $q_1 \leq \mathrm{CONN}_{G-S}(v_i, v_j) \leq q_2$. Hence $q_1 \leq m_f(G{-}S) \leq q_2$ and $q_1 - q_2 \leq \omega_f(G{-}S) \leq q_m - q_1$. The fuzzy node rupture degree is

$$-2q_2 - q_1 \leq r_f(G) \leq q_m - 4q_1. \tag{21}$$

Let $|S|\geq 3$. Thus $s(S) \geq 3d(\mu)$ and $q_1 \leq \mathrm{CONN}_G(v_i, v_j) \leq q_m$ for $i \neq j$; $i, j = 1, \ldots, n$ when the set S is removed from G, $0 \leq \mathrm{CONN}_{G-S}(v_i, v_j) \leq q_m$. So $0 \leq m_f(G{-}S) \leq q_2$ and $q_1 - q_2 \leq \omega_f(G{-}S) \leq q_m$. The fuzzy node rupture degree is

$$-2q_2 - 2q_1 \leq r_f(G) \leq q_m - 3q_1. \tag{22}$$

By the taking maximum of equations (20)–(22)

$$r_f(G) \leq q_m - 3q_1.$$

From the definition of the fuzzy node rupture degree

$$r_f(G) = \max\{\mathrm{CONN}_G(v_i v_j) \mid \mu(v_i v_j) = 0\} - 3d(\mu). \tag{23}$$

Case 2. Let G not contain a fuzzy cut node in this case. So the node v_i for $i = 1, \ldots, n-1$ is incident to the weakest arc. Thus the fuzzy cycle C_{n-1} is a locamin cycle. Then $\mathrm{CONN}_G(v_i, v_j) = q_2$ for the nodes those are non-adjacent. Since the set S does not contain a fuzzy cut node, $|S|\geq 2$.
If $|S|= 2$, then $s(S) = 2q_1 = 2d(\mu)$. If S contains adjacent elements then $\mathrm{CONN}_G(v_i, v_j) = \mathrm{CONN}_{G-S}(v_i, v_j)$. Hence the set S consists of non-adjacent elements where $\mathrm{CONN}_{G-S}(v_i, v_j) = q_1 < \mathrm{CONN}_G(v_i, v_j) = q_2$, $m_f(G{-}S) = q_1$ and $\omega_f(G{-}S) = q_2 - q_1$. So the fuzzy node rupture degree is

$$r_f(G) = q_2 - 4q_1. \tag{24}$$

If $|S| \geq 3$, then $s(S) \geq 3d(\mu)$. When the set S is removed from G, $\mathrm{CONN}_{G-S}(v_i, v_j) \geq 0$. Hence $m_f(G-S) \geq 0$ and $\omega_f(G-S) \leq q_2$. So the fuzzy node rupture degree is

$$r_f(G) \leq q_2 - 3q_1. \tag{25}$$

When we get the maximum values from (23) to (25), by the definition of fuzzy node rupture degree and $|S| = 3$ we obtain

$$r_f(G) = \max\{\mathrm{CONN}_G(v_i, v_j) \mid \mu(v_i, v_j) = 0\} - 3d(\mu). \tag{26}$$

The proof is completed from (23) and (26). □

Theorem 3.11. *Let G be a fuzzy wheel graph where the node v_n is a fuzzy hub and the nodes $v_1, \ldots, v_{n-1}$ are on the fuzzy cycle C_{n-1}. Assume that $\mu(v_n, v_i) > \mu(e)$ where $e \in E(C_{n-1})$ and $\mu(v_n, v_i)$ are constant for $i = 1, \ldots, n-1$. Then the fuzzy node rupture degree is*

$$r_f(G) = -\max\{\mu(e) \mid e \in E(C_{n-1})\} - d(\mu).$$

Proof. Since $\mu(v_n, v_i) > \mu(e)$ for $i = 1, \ldots, n-1$ and $e \in E(C_{n-1})$ in a fuzzy wheel graph, (v_i, v_n) arcs are α-strong arcs and the rest are δ-arcs. Thus v_n is a fuzzy cut node. So $s(v_i) = q_m = h(\mu)$ for $i = 1, \ldots, n$ and $\mathrm{CONN}_G(v_i, v_j) = q_m$ where $i \neq j$; $i, j = 1, \ldots, n$.

Let $|S| = 1$ and $S = \{v_n\}$. So $s(S) = q_m$. When the set S is removed from the fuzzy wheel graph, then $q_1 \leq \mathrm{CONN}_{G-S}(v_i, v_j) \leq q_{m-1}$ for $i \neq j$; $i, j = 1, \ldots, n-1$. Thus $m_f(G-S) = q_{m-1}$ and $\omega_f(G-S) = q_m - q_1$. The fuzzy node rupture degree is

$$r_f(G) = -(q_{m-1} + q_1). \tag{27}$$

Let $|S| = 2$. So $s(S) = 2q_m = 2h(\mu)$. If the set S is removed from G, then $q_1 \leq m_f(G-S) \leq q_{m-1}$ and $q_{m-1} - q_1 \leq \omega_f(G-S) \leq q_m - q_1$. The fuzzy node rupture degree is

$$-q_m - q_1 - q_{m-1} \leq r_f(G) \leq -q_m - 2q_1. \tag{28}$$

Let $|S| \geq 3$. So $s(S) \geq 3q_m = 3h(\mu)$. Thus $\mathrm{CONN}_G(x, y) \geq 0$ for $x, y \in \sigma^*$, $m_f(G-S) \geq 0$ with $\omega_f(G-S) \leq q_m$. The fuzzy node rupture degree is

$$r_f(G) \leq -2q_m. \tag{29}$$

From the definition of the fuzzy node rupture degree and the maximum values of equation (27)–(29), we get the result

$$r_f(G) = -[\max\{\mu(e) \mid e \in E(C_{n-1})\} + d(\mu)].$$

□

4. An algorithm to find fuzzy node rupture degree

Algorithm 1

The strength of connectedness algorithm given by Altundag, which calculates the strength of connectedness between any two vertices in a fuzzy graph $G : (V, \sigma, \mu)$, is given below [15].

Step 1. Write the adjacency matrix of the fuzzy graph G as matrix A.
Step 2. Obtain the matrix AA by replacing the diagonal elements with ∞ of the matrix A.
Step 3. If $\min\{AA[i,k], AA[k,j] > AA[i,j]$, then write
$C[i,j] = \min\{AA[i,k], AA[k,j]\}$, otherwise $C[i,j] = AA[i,j]$ as the matrix of strength of connectedness.

If this algorithm is applied to a fuzzy graph G, then the results will be obtained as follows [15]:

(1) The strength of connectedness between each nodes of G are obtained.
(2) Determines the strong arcs where $AA[i,j] \neq 0$. If $AA[i,j] < C[i,j]$, then (i,j) is a δ-arc and if $AA[i,j] = C[i,j]$, then (i,j) is an α-strong arc or a β-strong arc.
(3) The strong weights of each nodes are obtained by $s(i) = \min\{AA[i,j] | AA[i,j] = C[i,j]\}$.

4.1. Algorithm 2

This section gives an algorithm to find the node rupture degree of a fuzzy graph $G : (V, \sigma, \mu)$. This algorithm uses the Algorithm 1 in Section 4 above.

Step 1. Determine the strong weights of each nodes of G by using the Algorithm 1. $s(i) = \min\{AA[i, j]|AA[i, j] = C[i, j]\}$.
Step 2. Find a set $S \subseteq \sigma^*$ which is a fuzzy node cut for G.
Step 3. Evaluate the value of $s(S)$ using the strong weights found in Step 1 of the nodes in S.
Step 4. Find the adjacency matrix of the graph $G{-}S$ by replacing the entries on the columns and the rows, corresponding to the nodes in S, of the adjacency matrix G with zero.
Step 5. Obtain the matrix of strength of connectedness C for the fuzzy graph $G{-}S$ by using the Algorithm 1. For the Step 1 of the Algorithm 1 use the adjacency matrix obtained in Step 4 above.
Step 6. If $q < p$, then $r = p - 2q - s(S)$ where $C[i, j] = p$ for the fuzzy graph G and $C[i, j] = q$ for the fuzzy graph $G{-}S$, otherwise go to Step 2.
Step 7. Repeat the Steps 2 through 6 until there is no other set S exists in Step 2.
Step 8. Find the maximum of all the values of r which gives the node rupture degree value $r_f(G)$ of the fuzzy graph G.

5. Discussion and conclusion

Vulnerability metrics in graphs have been well studied in classical and fuzzy graph theory, and parameters such as connectivity, integrity, toughness, tenacity, and rupture degree have played crucial roles in measuring the resilience of networks. However, such traditional metrics fail to fully capture the impact of node removal in uncertain environments where edges and nodes may possess varying degrees of connectedness. Compared with the previous works, the fuzzy node rupture degree provided in this study is a more comprehensive measure of graph vulnerability for fuzzy networks. Compared with the classical vulnerability measures, which consider only the number of disconnected components, our approach involves:

- The resilience of remaining sub-networks after node removals.
- The size of the largest connected component, indicating the network's capacity for maintaining communication.
- The summation of the effects of multiple node deletions, hence a more realistic metric in scenarios with progressive deterioration.

The proposed fuzzy node rupture degree varies from other fuzzy graph vulnerability metrics, such as fuzzy connectivity and fuzzy integrity, in several ways. While previous fuzzy vulnerability metrics had primarily dealt with binary disconnection status, our approach quantifies the incremental loss of connections by considering membership values and strength of connectedness. This gives a more realistic estimate of network robustness, especially in complex systems such as transportation, bioinformatics, and communication networks, where interconnection is not necessarily binary. Furthermore, the algorithm in this study also gives a viable means to compute the fuzzy node rupture degree that can be utilized in realistic network analysis. The computation results on fuzzy cycle, fuzzy wheel, and fuzzy star graphs indicate our measure can be implemented on a wide range of network topologies, and they convey meaningful information regarding the robustness of fuzzy networks.

5.1. Future work

Despite this paper clearly establishing the basic properties of fuzzy node rupture degree, further work can expand its applicability to more complex real-world networks. Some possible avenues for future work are:

- This metric should be used on large real-world databases such as transportation systems and biological networks.

– Develop heuristic algorithms that would efficiently calculate fuzzy node rupture degree for large graphs.
– Exploring multi-layered fuzzy networks, whose interactions between layers influence vulnerability of networks.
– Comparing this metric with other fuzzy vulnerability measures, such as fuzzy toughness and fuzzy tenacity, for further refinement of its applicability.

By bridging the gap between classical vulnerability analysis and fuzzy network modeling, this study makes a contribution to fuzzy graph theory development and provides a solid foundation for future studies on network resilience.

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Conflicts of interest

The authors declare that they have no conflict of interest.

Data availability statement

The research data associated with this article are included within the article.

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