ON SOME STRUCTURAL PROPERTIES OF A DIVISOR GRAPH

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Abstract. A divisor graph G_n of a positive integer n is a simple graph with vertices as proper divisors of n, in which two distinct divisors are adjacent if and only if they are relatively prime. Also G'_n is a graph with vertices as divisors of n except n, such that the two distinct vertices are adjacent if and only if their greatest common divisor is one. These graph play a fundamental role in the study of the comaximal graphs associated to commutative rings. We study the graph theoretic properties of both G_n and G'_n. Formally, we determine the clique number, the independence number, the chromatic number and the automorphism group of the divisor graphs G_n and G'_n. We also find the bounds for their metric dimension along with the characterization of graphs attaining them. Further, we discuss their resolving polynomial.

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1. INTRODUCTION

Let G be a finite simple undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G)$ consisting of unordered pairs of vertices. The numbers $n = |V(G)|$ is order and $m = |E(G)|$ is size of G. Two adjacent vertices u and v are represented by $u \sim v$. A vertex of degree 0 is said to be isolated vertex and a vertex of degree one is a pendent vertex. The set of vertices $\{x \in V \mid x \sim y\}$, denoted by $N_G(v)$, is called the *neighbourhood* of v (sometimes known as an open neighbourhood). The closed neighbourhood of a vertex $v \in V$ is given as $N[v] = \{v\} \cup N(v)$. The *degree* $d_{v_i}(G)$ (or simply d_i, if G is clear) of v_i is the number of elements in the set $N_G(v_i)$. The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \cup G_2$, is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The *join* of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2, denoted by $G_1 \vee G_2$, is defined as the graph $G_1 \cup G_2$ together with all the edges joining each vertex of V_1 to every vertex of V_2. For arbitrary $v_1 \neq v_2 \in V$, if $N[v_1] = N[v_2]$ (or $N(v_1) = N(v_2)$), then we say v_1 and v_2 are true twins (false twins), also we note that $N[v_1] = N[v_2]$ if $v_1 \sim v_2$, otherwise $N(v_1) = N(v_2)$. They are called twins if they are either true or false twins. If $v_1 \sim v_2$, then they are known as adjacency twins, otherwise non adjacent twins. For two different vertices $u, v \in V$, the distance between them is denoted by $d(u, v)$, and is defined as the length of a shortest path connecting them. The eccentricity $e(v)$ of a vertex v is defined as $e(v) = \max_{u \in V} d(v, u)$. The diameter $\mathrm{diam}(G)$ of G is the maximum eccentricity of any vertex in G.

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For a commutative ring R with non-zero identity, Shamra and Bhatwadekar [25] introduced the concept of comaximal graph $\Gamma(R)$ of R with vertex set as the elements of R, and $a \neq b \in \Gamma(R)$ are adjacent if and only if $aR + bR = R$, where aR is the ideal generated by $a \in R$. Interesting results related to rings and comaximal graphs exists in literature, like: ring R is a finite ring if and only if chromatic number of $\Gamma(R)$ is finite, chromatic number of $\Gamma(R)$ is same as number of maximal ideals and number of units of R (see, [25]). In [18], the authors showed that for some rings the fact that $\Gamma(R_1) \cong \Gamma(R_2)$ implies that $R_1 \cong R_2$. For some recent development about comaximal graphs of R, see [11,24,28] and the references cited therein. For studying the spectral analysis of graphs defined on groups, rings, vector spaces and other algebraic structures, the structure of these graphs are written in terms of certain graph operations (joined union), see [10,19,30]. Like wise the structure of comaximal graph of commutative rings can be written in terms of graph operation so that their graph theoretic properties and spectral theory can be easily investigated [3,4,6–8,23]. For the graph operation joined union, we need an underlying graphs like divisor graphs for zero divisor graphs [10,17], a divisor type graph for power graphs of groups [19,20] and a relatively prime divisor type graph for comaximal graphs of commutative rings [3,7].

In Section 2, we give the basic properties of divisor type graphs, like number of pendents, independence number, clique number, cut vertices, chromatic number and automorphism group. Section 3 gives the results related to metric dimension and resolving polynomial of these divisor type graphs.

2. STRUCTURAL PROPERTIES OF G_n AND G_n'

For a positive integer n, let $\tau(n)$ (see [16]) be the number of positive factors of n and let $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$ be its *canonical decomposition*, where $r, n_1, n_2, \ldots, n_t$ are positive integers and $p_1, p_2, \ldots, p_t$ are distinct primes. Then it is well known that $\tau(n) = (n_1 + 1)(n_2 + 1) \ldots (n_t + 1)$.

The *Euler's totient function* $\phi(n)$ denotes the number of positive integers less or equal to n and relatively prime to n. An integer d is a proper divisor of n if d divides n and $1 < d < n$. Let $d_1, d_2, \ldots, d_t$ be the proper divisors of n. Let G_n be a simple graph with vertex set $\{d_1, d_2, \ldots, d_t\}$ in which two distinct vertices d_i and d_j are adjacent if and only if they are relatively prime, that is, $(d_i, d_j) = 1$, for $1 \leq i < j \leq t$, where (x, y) is the greatest common factor of x and y. For a positive integer n in canonical decomposition, it follows that the size of G_n is $|V(G_n)| = \prod_{i=1}^{t}(n_i + 1) - 2$. Next, we consider a new graph $G_n' = K_1 \vee G_n$, where $V(K_1) = \{1\}$. Thus, G_n' is a simple graph with vertices as $d_i \neq n$ and two distinct vertices d_i and d_j are connected if and only if $(d_i, d_j) = 1$. We observe that $(1, d_i) = 1$ is always true, so the vertex 1 is adjacent to all other vertices of G_n'. We call the divisors $p_1, p_2, \ldots, p_t$ as the primary divisors of $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$, like $p_i^{n_i}$ has only one primary divisor, p_1 and p_2 are primary divisors of $p_i^{n_i} p_j^{n_j}$ and so on.

For $1 \leq i \leq t$, let

$$A_{d_i} = \{x \in \mathbb{Z}_n : (x, n) = d_i\}.$$

Clearly, $A_{d_i} \cap A_{d_j} = \phi$, when $i \neq j$, implying that the sets $A_{d_1}, A_{d_2}, \ldots, A_{d_t}$ are pairwise disjoint and partitions the vertex set of $\Gamma(\mathbb{Z}_n)$ as

$$V(\Gamma(\mathbb{Z}_n)) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_t} \cup \{0\} \cup U(\mathbb{Z}_n),$$

where $U(\mathbb{Z}_n) = \{x \in \mathbb{Z}_n : (x, y) = 1, \text{for each } y \in \mathbb{Z}_n\}$, and there are $\phi(n)$ such elements in $\mathbb{Z}_n$. Also, $|A_{d_i}| = \phi\left(\frac{n}{d_i}\right)$, for $1 \leq i \leq t$, (see, [30]). Furthermore, no two members of the set A_{d_i} are adjacent, for each d_i, if $v_i \in A_{d_i}$ is adjacent to $v_j \in A_{d_j}$ for some $i \neq j$, then v_i is adjacent to every $v_j \in A_{d_j}$ and $x_i \in A_{d_i}$ is adjacent to $x_j \in A_{d_j}$ if and only if $(d_i, d_j) = 1$, (see [6]).

The *joined union* $G[G_1, \ldots, G_n]$ of graphs $G_1, \ldots, G_n$ with the underlying graph G is the union of graphs $G_1 \cup G_2 \cup \cdots \cup G_n$ together with the edges from each vertex of G_i to every vertex of G_j, whenever $v_i \sim v_j$ in G. It follows that the join of two graphs G_1 and G_2 is $G_1 \vee G_2 = K_2[G_1, G_2]$.

The following lemma says that $\Gamma(\mathbb{Z}_n)$ is the joined union of certain complete graphs and null graphs.

Lemma 2.1 ([3])**.** *For the positive integer n and its proper divisor d_i, the following hold.*

(i) *For each $i = 1, 2, \ldots, t$, $\Gamma(A_{d_i}) \cong \overline{K}_{\phi\left(\frac{n}{d_i}\right)}$, where $\Gamma(A_{d_i})$ is induced subgraph of A_{d_i}.*

(ii) *The comaximal graph of $\mathbb{Z}_n$ is*

$$\Gamma(\mathbb{Z}_n) = K_{\phi(n)} \vee \left(K_1 \cup G_n\left[\overline{K}_{\phi\left(\frac{n}{d_1}\right)}, \overline{K}_{\phi\left(\frac{n}{d_2}\right)}, \ldots, \overline{K}_{\phi\left(\frac{n}{d_t}\right)} \right] \right)$$

$$= G'_n\left[K_{\phi(n)}, K_1, \overline{K}_{\phi\left(\frac{n}{d_1}\right)}, \overline{K}_{\phi\left(\frac{n}{d_2}\right)}, \ldots, \overline{K}_{\phi\left(\frac{n}{d_t}\right)} \right].$$

Therefore, the comaximal graph $\Gamma(\mathbb{Z}_n)$ can be written as joined union of graphs, the underlying graph of the joined union is $K_1 \vee (G_n \cup \{0\}) = G'_n$, since $V(K_1) = \{0\}$. Further, from [7], we have

$$\Gamma(\mathbb{Z}_n) = G_1 \vee G'_2 \cong K_{\phi(n)} \vee G'_2,$$

where G'_2 is the graph on $n - \phi(n)$ vertices and is the union of K_1 and the graph G_n. This is the other motivation for studying $G'_n = K_1 \vee G_n$.

Clearly, the diameter of G'_n is 2 and if G_n is connected then $\mathrm{diam}(G_n) = 2$, since p_i's form a clique and any other vertex of G_n is adjacent to some vertex of clique. Thus, it follows that G'_n is of diameter at most two.

The following results concerns about the basic properties of G_n

Lemma 2.2 ([23])**.** *For the graph G_n, the following holds.*

(i) *If n is a prime, then G_n is empty an graph.*
(ii) *If $n = p_1^{n_1}, n_1 \geq 2$, then $G_n \cong \overline{K}_{n_1 - 1}$.*
(iii) *G_n is connected if and only if n is product of distinct primes.*

By Lemma 2.2 with $n = p_1^{n_1}, n_1 \geq 2$, it follows that G'_n is a star graph and all basic properties of star graph are known. So, while discussing any property of G'_n, we take $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$, with $t \geq 2$, unless otherwise stated. Moreover, for $n = p_1 p_2 \ldots p_t, t \geq 2$, we have $1 \leq d_{v_i}(G_n) \leq n - 1$. Also, by Lemma 2.2, it follows that G_n need not be connected, however G'_n is always connected. From now onwards, we will be dealing with only connected graphs.

The very first results gives the number of pendent vertices in G'_n.

Proposition 2.3. *Let n be a positive integer. Then the following hold:*

(i) *If $n = p_1 p_2 \ldots p_t, t \geq 2$, then G'_n has no pendent vertices.*
(ii) *If $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}, t \geq 2$, then there are $\prod_{i=1}^{t} n_i - 1$ pendent vertices in G'_n. Moreover they are of the form $p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}$, where $1 \leq a_i \leq n_i$ for $i = 1, 2, \ldots, t$.*

Proof. (i) By Lemma 2.2, G_n is connected for $n = p_1 p_2 \ldots p_t$ so is G'_n. For the second part. Let $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$. Then the proper divisors of n of the form $p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}$ (excluding n), where $1 \leq a_i \leq n_i$ for $1 \leq i \leq t$ are never relatively prime with any other divisors of n (excluding 1), since their common divisor is at least p_i. Such vertices are adjacent only to the trivial divisor 1 and they are not adjacent to themselves, as for divisors e_1 and e_2 of the form $p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}$, we always have $(e_1, e_2) \neq 1$. Also, it is easy to see that the count of such divisors in G_n are $n_1 n_2 \ldots n_t - 1$. $\square$

Remark 2.4. From Proposition 2.3, it is clear that $2 \leq d_{v_i}(G'_n) \leq n - 1$ if $n = p_1 p_2 \ldots p_t$. Also, for $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$, it follows that $1 \leq d_{v_i}(G'_n) \leq n - 1$.

The following result gives the independence number of G'_n.

Proposition 2.5. *Let n be a positive integer. Then the following hold:*

(i) *If $n = p_1 p_2 \ldots p_t, t \geq 2$, then the independence number of G'_n is t.*
(ii) *If $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}, t \geq 2$, then the independence number of G'_n is at least $\prod_{i=1}^{t} n_i - 1$.*

Proof. (i) For $n = p_1 p_2 \ldots p_t$, the divisors $l_i = \prod_{\substack{j=1 \\ i \neq j}}^{t} p_j$ are adjacent only to p_i and they form pendent vertices in G_n. The number of such divisor l_i in G'_n are $\binom{t}{t-1} = t$. So $\{l_1, l_2, \ldots, l_t\}$ is maximal independent set and such a set is not unique. Thus the independence number of G'_n is t.

(ii) For $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$, the proper divisors of the form $p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}$ (excluding n), where $1 \leq a_i \leq n_i$ for $1 \leq i \leq t$ form an isolated vertices in G_n and hence set of pendent vertices in G'_n. So, this set is independent set of cardinality $\prod_{i=1}^{n} n_i - 1$, but it may not be the maximum independent set, since $d_i = \prod_{\substack{j=1 \\ i \neq j}}^{t} p_j^{n_j}$ are also the part of the maximum independent set. $\qquad \square$

The following result discuss the independence number of G'_n for some special values of n.

Proposition 2.6. *Let n be a positive integer. Then the following hold:*

(i) *For $n = p_1^{n_1} p_2^{n_2}$, where $n_1 \geq n_2$, the independence number of G'_n is $n_1(n_2 + 1) - 1$.*

(ii) *For $n = p_1^{n_1} p_2^{n_2} p_3^{n_3}$, the independence number of G'_n is*

$$
\begin{cases}
n_1^2(3 + n_1) - 1, & \text{if } n_1 = n_2 = n_3, \\
n_3(n_1^2 + 2n_1 + 1) - 1 & \text{if } n_1 = n_2 < n_3 \text{ and } n_1^2 < n_3, \\
n_1(n_1 n_3 + 2n_3 + n_1) - 1 & \text{if } n_1 = n_2 < n_3 \text{ and } n_1^2 \geq n_3, \\
n_2(n_1 n_2 + 2n_1 + 1) - 1 & \text{if } n_1 = n_3 < n_2 \text{ and } n_1^2 < n_2, \\
n_1(n_1 n_2 + 2n_2 + n_1) - 1 & \text{if } n_1 = n_3 < n_2 \text{ and } n_1^2 \geq n_2, \\
n_1 n_2 + n_1 n_3 + n_2 n_3 + n_1 n_2 n_3 - 1 & \text{otherwise.}
\end{cases}
$$

Proof. If $n = p_1^{n_1} p_2^{n_2}$, with $n_1 \geq n_2$, then the divisors $p_1^{e_1} p_2^{e_2}, 1 \leq e_i \leq n_i$ (except n) form an independent set of cardinality $n_1 n_2 - 1$. Also, $p_1^i, i = 1, 2, \ldots, n_1$ and $p_2^j, j = 1, 2, \ldots, n_2$ are other two independent set where each vertex set of the first set is connected to every other vertex of second set in G'_n. Since, $n_1 \geq n_2$, so the divisor set p_1^i together with the divisors $p_1^{e_1} p_2^{e_2}$ form a maximal independent set of cardinality $n_1 n_2 + n_1 - 1$.

(ii) If $n = p_1^{n_1} p_2^{n_2} p_3^{n_3}$, then by Proposition 2.3, there are $n_1 n_2 n_3 - 1$ pendents in G'_n. Further, let A_i be the set consisting of divisors $p_i^{\alpha_i}$ with $i = 1, 2, 3$, A_4 be the set consisting of divisors $p_1^i p_2^j$, A_5 be the set consisting of divisors $p_1^i p_3^k$ and A_6 be the set consisting of divisors $p_2^j p_3^k$, where $i = 1, 2, \ldots, n_1, j = 1, 2, \ldots, n_2$ and $k = 1, 2, \ldots, n_3$. Clearly each such set induces a totally disconnected graph and form independent sets in G'_n of cardinalities $n_1, n_2, n_3, n_1 n_2, n_1 n_3$ and $n_2 n_3$, respectively. If $n_1 = n_2 = n_3$, then the pendents in G'_n together with the vertices $p_1^i p_2^j$, $p_1^i p_3^k$ and $p_2^j p_3^k$ is one such maximal independent set of cardinality $n_1 n_2 + n_1 n_3 + n_2 n_3 + n_1 n_2 n_3 - 1$ in G'_n. If $n_1 = n_2 < n_3$ and $n_1^2 < n_3$, then the pendents together with $A_3 \cup A_5 \cup A_6$ form a maximal independent set of cardinality $2n_1 n_3 + n_3 + n_1^2 n_3 - 1$. Now, if If $n_1 = n_2 < n_3$ and $n_1^2 \geq n_3$, then the pendents together with $A_4 \cup A_5 \cup A_6$ form a maximal independent set of cardinality $n_1^2 + 2n_1 n_3 + n_1^2 n_3 - 1$. If $n_1 = n_3 < n_2$ and $n_1^2 < n_2$, then the pendents together with $A_2 \cup A_4 \cup A_6$ form a maximal independent set of cardinality $2n_1 n_2 + n_2 + n_1^2 n_2 - 1$, otherwise $A_4 \cup A_5 \cup A_6$ is a maximal independent set. Finally, for the remaining cases, it is clear that $A_4 \cup A_5 \cup A_6$ form a maximal independent set along with the pendent vertices of G'_n. $\qquad \square$

In the next result, we give the clique number of G'_n.

Proposition 2.7. *Let $n = n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}, t \geq 2$, be a positive integer. Then, the clique number of G'_n is $t + 1$.*

Proof. For $n = p_1 p_2 \ldots p_t$, the divisor p_i is always adjacent to p_j, since $(p_i, p_j) = 1$, for each $i \neq j$. Thus p_i's induce a clique of size t in G_n and thereby along with the divisor 1, they form a clique of size $t + 1$ in G'_n.

For $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$, the proper divisors p_i^j for $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, n_i$ form independent sets of size $\alpha_i = n_i$. Also, each vertex of α_i is adjacent to every vertex of α_j and it follows that $\{p_1, p_2, \ldots p_t\} \cup \{1\}$ form a clique of size $t + 1$ in G'_n. $\qquad \square$

The following is a consequence of Proposition 2.7 and gives the clique of G_n.

Corollary 2.8. *Let $n = p_1 p_2 \ldots p_t, t \geq 3$ be a positive integer. Then, the clique number of G_n is t.*

For a vertex v of G, with two disjoint non-empty subsets of A and B of $V(G) \setminus \{v\}$, we say that (A, B) is a separation of $G(V(G) \setminus \{v\})$ if $A \cup B = V(G) \setminus \{v\}$, and there is no edge of $G(V(G) \setminus \{v\})$ with one end point in A and other in B. For connected graph G, we say v is a cut vertex of G if and only if there exists a separation of $G(V(G) \setminus \{v\})$. Note that $G(X)$, where X is a subset of $V(G)$, is the subgraph of G induced by the vertices in X.

Next result gives the number of cut vertices of G_n.

Proposition 2.9. *Let $n = p_1 p_2 \cdots = p_t, t \geq 3$ be the product of t distinct positive integers. Then p_i, for $i = 1, 2, \ldots, t$ are precisely the cut vertices of G_n.*

Proof. Since each $l_i = \prod_{\substack{j=1 \\ i \neq j}}^{t} p_i$ is a pendent vertex in G_n and l_i is adjacent to p_i in G_n for $i = 1, 2, \ldots, t$. Thus, by the connectedness of G_n, it follows that p_i is a cut vertex of G_n. Suppose that there is a cut vertex v in G_n other than p_i. Let (X, Y) be a separation of $G_n[V(G_n) \setminus \{v\}]$. As $(p_i, p_j) = 1$ for $i \neq j$, so all the vertices $p_i, i = 1, 2, \ldots, t$ are either in X or in Y. We may assume that each p_i is in X, and note that Y is a non empty set, so there is a vertex $v \in Y$. But $(v, p_i) = 1$ for some i, that is v is adjacent to some p_i, which contradicts the assumption that (X, Y) is a separation of $G_n[V(G_n) \setminus \{v\}]$. Therefore, any cut vertex of G_n is of the form p_i. $\square$

A vertex colouring of G is an assignment of colors to vertices such that adjacent vertices receive different colors. The chromatic number of G is the minimum number of colors required for vertex coloring of G. The following results gives the chromatic number of G_n and G'_n.

Proposition 2.10. *Let $n = p_1 p_2 \ldots p_t, t \geq 3$ be a positive integer. Then, the chromatic number of G_n is same as its clique number.*

Proof. Since p_i's form a clique of size t in G_n, and all other vertices are adjacent to some p_i. More precisely, p_i is adjacent to all vertices which are not multiple of p_i, similarly $p_j, j \neq i$ is adjacent to all other vertices except multiples of p_j (including multiples of p_i). Now, there are vertices like $p_i p_j$, that is, multiples of both p_i and p_j which are adjacent to some $p_k, k \notin \{i, j\}$. Lastly, there are vertices like $v_i = \prod_{\substack{j=1 \\ i \neq j}}^{t} p_i$ which are adjacent to p_i. Thus, in this manner all the vertices (other than p_i) of G_n are adjacent to some p_i's. We assign $c_i, i = 1, 2, \ldots, t$ different colors to the vertices of clique K_t in G_n. Since vertex p_i receives color c_i, so we assign color c_j to its neighbours (excluding its neighbours in K_t) which are not adjacent to p_j, since p_j is already colored with c_j. Next, we assign color c_k to those vertices which are adjacent to both p_i and p_j, where $k \notin \{i, j\}$. Proceeding in this manner, at the end we are left with $v_i = \prod_{\substack{j=1 \\ i \neq j}}^{t} p_i$ which are adjacent to only p_i, we assign them any colour from c_i, for $i = 2, 3, \ldots, t$. Thereby, we require only t colors to color the vertices of G_n as each vertex of G_n is adjacent to some p_i, for $i = 1, 2, \ldots, t$. $\square$

With the similar proof as in Proposition 2.10, we have the following result for G'_n.

Corollary 2.11. *Let $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}, t \geq 3$ be a positive integer. Then, the chromatic number of G'_n is same as clique its number.*

An automorphism of a graph G is a permutation in which the vertex set of G is mapped onto itself in such a way that the adjacency and non-adjacency relation of vertices is preserved. The set of all automorphisms of G is a permutation group known as the automorphism group of G, denoted by $Aut(G)$. In the next result, we present the automorphism group of G_n and G'_n.

Proposition 2.12. *Let $n = p_1 p_2 \ldots p_t, t \geq 3$ be a positive integer. Then, the automorphism group of G_n and G'_n is same and equals to S_t, the symmetric group of order $t!$.*

Proof. Since p_i's form a clique K_t and all other vertices of G_n are adjacent to some p_i, though the neighbours of p_i's are not same. Since the vertices of K_t can be permuted in $t!$ ways, so the remaining vertices $V(G_n \setminus K_t)$ can be permuted with such $t!$ permutations. Thus, we arrive at the automorphism group of G_n is isomorphic to the symmetric group S_t on t elements. Also, we note that the vertex 1 of G'_n is adjacent to all other vertices, it follows that any automorphism fixes this vertex and its automorphism group is same as that of G_n.   □

3. METRIC DIMENSION OF G'_n

For a subset $W = \{x_1, x_2, \ldots, x_k\}$ of $V(G)$ and a vertex $v \in V(G)$, the vector (k-vector) with respect to W is written as

$$r(v|W) = (d(v, x_1), d(v, x_2), \ldots, d(z, x_k)),$$

is known as the representation of v with respect to W. A set W is known as the resolving (locating set) for G if any two distinct vertices of G have different representations related to subset W. A resolving set of minimum cardinality is a basis (metric basis) of G and the cardinality of such a set is called metric dimension of G, denoted by $\beta(G)$.

The concept of resolving set and the metric dimension of a graph was first introduced by Slater [29] and was used to uniquely identifying the location of a vertex in a graph. Harary and Melter [13] independently studied metric dimension and resolving sets of G. The metric dimension and resolving set of a graph are important invariants and have applications in coin-weighing problems [27], pharmaceutical chemistry, robot navigation, diverse as combinatorial optimization, and sonar [5], digital images [21]. The problem of finding the metric dimension of a graph is NP-complete and many authors have studied and investigated it among several classes of graphs. Results related to metric dimension of Cayley digraphs can be seen in [12], wheels [26], unicyclic graphs [22], trees [29].

We have the following known results about resolving set of G.

Lemma 3.1 ([14]). *Suppose G is a connected graph and v_1, v_2 are twin vertices in G. Then $d(v_1, v_3) = d(v_2, v_3)$ for any vertex $v_3 \in V(G) \setminus \{v_1, v_2\}$.*

Corollary 3.2 ([14]). *Suppose G is a connected graph and v_1, v_2 are twins in G and let A be its resolving set. Then $v_1 \in A$ or $v_2 \in A$. Furthermore, if $v_1 \in A$ and $v_2 \notin A$, then $(A \setminus \{v_1\}) \cup \{v_2\}$ is also resolving set G.*

From Corollary 3.2, we have the following observation.

Remark 3.3. Let G be a connected graph of order n and let A be a twin set in G with $|A| = k \geq 2$. Then, each resolving set of A contains at least $k - 1$ vertices of A.

For a graph G of order n and diameter d, the following hold (see, [9, 15])

$$\dim(G) \leq N - d.$$

With $\operatorname{diam}(G'_n) = 2$, we have the following result for G'_n.

Theorem 3.4. *Let G'_n be the divisor graph of order N. Then*

$$\dim(G'_n) \leq N - 2.$$

By Lemma 3.1, Corollary 3.2 and Remark 3.3, we have the following result whose proof follows trivially.

Proposition 3.5. *Let G'_n be a graph of order N. Then, the following hold.*

(i) *Let η be the number of twin sets of cardinality at least two. Then*

$$\dim(G'_n) \geq N - \eta.$$

(ii) *Let θ be the number of twin sets of cardinality one. Then*

$$\dim(G'_n) \leq N - \eta + \theta.$$

As the divisor 1 is the only twin of size 1, so we have the following result for G'_n.

Corollary 3.6. *Let G'_n be a graph of order N and let η be the number of twin sets of cardinality at least two. Then*

$$\dim(G'_n) \leq N - \eta + 1.$$

Next, we find the value of η and the upper bound for metric dimension of G'_n is given in the following consequence.

Corollary 3.7. *Let G'_n be a graph of order N and let $t \geq 2$ be number of primary divisors of n. Then,*

$$\dim(G'_n) \leq N - \left\{ \binom{t}{1} + \binom{t}{2} + \cdots + \binom{t}{t-1} + \binom{t}{t} \right\} + 1.$$

Proof. Let $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}, t \geq 2$ be the canonical decomposition of n. Clearly, there are t twin sets in G'_n of the form $p_i^{\alpha_i}$, for $1 \leq \alpha_i \leq n_i$ of cardinality n_i with only one primary divisor p_i. Also, there are twin sets with primary factors

$$p_1 p_2, p_1 p_3, \ldots, p_1 p_t, p_2 p_3, p_2 p_4, \ldots, p_2 p_t, \ldots, p_{t-2} p_{t-1}, p_{t-2} p_t, p_{t-1} p_t$$

and the number of such twin sets are

$$(t-1) + (t-2) + (t-3) + \cdots + 2 + 1 = \binom{t}{2}.$$

Similarly, the number of twin sets with exactly 3 primary divisors $p_1 p_2 p_3 \ldots p_{t-2} p_{t-1} p_t$ are $\binom{t}{3}$. Continuing in this manner, the number of twin sets with $k \geq 4$ primary factors are $\binom{n}{k}$. Now, by Proposition 3.5 and Corollary 3.6, the result follows. $\qquad\square$

In the next two results we obtain sharp lower bounds for the metric dimension of G'_n.

Theorem 3.8. *Let $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}, t \geq 2$ be a positive integer and at least one $n_i \geq 2$. Then, the metric dimension of G'_n is*

$$\dim(G'_n) \geq \prod_{i=1}^{t} n_i + \sum_{i=1}^{t} n_i - t - 2,$$

with equality if and only if $t = 2$.

Proof. For $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$, it is clear that G'_n have $\prod_{i=1}^{t} n_i - 1$ pendent vertices (Prop. 2.3, part (ii)) and there are twins of cardinality n_i for divisors $p_i^{n_i}, i = 1, 2, \ldots, t$. Thus, by Lemma 3.1, Corollary 3.2 and Remark 3.3, it follows that the metric dimension of G'_n satisfies

$$\mathrm{diam}(G'_n) \geq \prod_{i=1}^{t} n_i + \sum_{i=1}^{t} n_i - t - 2, \tag{3.1}$$

since there may exist more twins corresponding to the other divisors of n.

Suppose that equality holds in (3.1), then $\operatorname{diam}(G'_n) = \prod_{i=1}^{t} n_i + \sum_{i=1}^{t} n_i - t - 2$ and it implies that G'_n have only twin sets corresponding to divisors $p_i^{n_i}, i = 1, 2, \ldots, t$ and the twin pendents. In case $t \geq 3$, then there are twins other than pendents and those of $p_i^{n_i}$ like the twins corresponding to $\pi_i = p_1^{n_1} p_2^{n_2} \ldots p_{i-1}^{n_{i-1}} p_{i+1}^{n_{i+1}} \ldots p_t^{n_t}$. In such case equality cannot hold, since the other twins contribute to the metric dimension of G'_n. Thus, we must have $p = 2$ and in this case there are twins corresponding to $p_1^{\alpha_1}, 1 \leq \alpha_1 \leq n_1, p_2^{\alpha_2}, 1 \leq \alpha_2 \leq n_2$ and the pendent twins $p_1^{\alpha_1} p_2^{\alpha_2}, 1 \leq \alpha_i \leq n_i$, for $i = 1, 2$ except the divisor n. Thus, for $t = 2$ equality holds in (3.1) and we actually we need to prove that the metric dimension of G'_n is $n_1 n_2 + n_1 + n_2 - 4$ for $n = p_1^{n_1} p_2^{n_2}$. Since $\dim(G_n) \geq n_1 n_2 + n_1 + n_2 - 4$, we only need to show that $\dim(G_n) \leq n_1 n_2 + n_1 + n_2 - 4$, for that we show that

$$W = \left\{ p_1, \ldots, p_1^{n_1-1}, p_2, \ldots, p_2^{n_2-1}, p_1 p_2, p_1 p_2^2, \ldots, p_1 p_2^{n_2}, p_1^2 p_2, p_1^2 p_2^2, \ldots, p_1^2 p_2^{n_2}, \right.$$
$$\left. \ldots, p_1^{n_1} p_2, p_1^{n_1} p_2^2, \ldots, p_1^{n_1} p_2^{n_2-2} \right\}$$

is the resolving set of G'_n. We prove that W is resolving set of G'_n. Further, for each $v_i \in W$, we see that $r(v_i|w)$ is distinct, since all such $|W|$-vectors are different as the i-th coordinate of $r(v_i|w)$ is zero. Next, we find the representation of the remaining vertices $\{1, p_1^{n_1}, p_2^{n_2}, p_1^{n_1} p_2^{n_2-1}\}$ with respect to W. The $|W|$-vector of vertices $\{1, p_1^{n_1}, p_2^{n_2}, p_1^{n_1} p_2^{n_2-1}\}$ with W are

$$r(1|W) = (1, 1, \ldots, 1), r(p_1^{n_1}|W) = \left(\underbrace{2, 2, \ldots, 2}_{n_1-1}, \underbrace{1, 1, \ldots, 1}_{n_2-1}, \underbrace{2, 2, \ldots, 2}_{n_1 n_2-2} \right)$$

$$r(p_2^{n_2}|W) = \left(\underbrace{1, 1, \ldots, 1}_{n_1-1}, \underbrace{2, 2, \ldots, 2}_{n_2-1}, \underbrace{2, 2, \ldots, 2}_{n_1 n_2-2} \right), r(p_1^{n_1} p_2^{n_2-1}|W) = (2, 2, \ldots, 2).$$

Clearly, the above vectors are different and it follows that W is the resolving set of G'_n. Thus, $\dim(G_n) \leq n_1 n_2 + n_1 + n_2 - 4$ and result follows. This proves the equality case. $\square$

Theorem 3.9. *Let $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}, t \geq 3$ be a positive integer and at least one $n_i \geq 2$. Then the metric dimension of G'_n is*

$$\dim(G'_n) \geq \prod_{i=1}^{t} n_i + \sum_{1 \leq i < j \leq n} n_i n_j + \sum_{i=1}^{t} n_i - \frac{t(t+1)}{2} - 2,$$

with equality if and only if $t = 3$.

Proof. By Proposition 2.3, we see that G'_n has $\prod_{i=1}^{t} n_i - 1$ pendent twin vertices. Also divisors of the form $p_i^{n_i}, i = 1, 2, \ldots, t$ form twins of cardinality n_i. Furthermore, there are twins sets of the form $p_i^{\alpha_i} p_j^{\alpha_j}, i < j, 1 \leq \alpha_i \leq n_i$ with cardinalities $n_i n_j$. Also, there are $t - 1 + t - 2 + \ldots + 2 + 1 = \frac{t(t-1)}{2}$ such twin sets. Besides, there are more twin sets of the form $p_i^{\alpha_i} p_j^{\alpha_j} p_k^{\alpha_k}, i < j < k, 1 \leq \alpha_i \leq n_i$ with cardinalities $n_i n_j n_k$ and other twin sets. Thus, by Lemma 3.1, Corollary 3.2 and Remark 3.3, it follows that the metric dimension of G'_n satisfies

$$\dim(G'_n) \geq \prod_{i=1}^{t} n_i + \sum_{1 \leq i < j \leq n} n_i n_j + \sum_{i=1}^{t} n_i - \frac{t(t+1)}{2} - 2. \tag{3.2}$$

Suppose that equality holds in (3.2), then G'_n has twin sets precisely of the form $p_i^{n_i}, i = 1, 2, \ldots, t$, $p_i^{\alpha_i} p_j^{\alpha_j}, i < j, 1 \leq \alpha_i \leq n_i$ and the pendent twins. If $t \geq 4$, then there are twins sets of the form $l_i = p_1^{n_1} p_2^{n_2} \ldots p_{i-1}^{n_{i-1}} p_{i+1}^{n_{i+1}} \ldots p_t^{n_t}$, which contribute a non zero quantity to the metric of G'_n and in this case equality cannot hold in (3.2). Thus, the equality holds if G'_n has twin sets corresponding to divisors $p_i^{n_i}, i = 1, 2, \ldots, t$, $p_i^{\alpha_i} p_j^{\alpha_j}, i < j, 1 \leq \alpha_i \leq n_i$ and the pendent twins. It follows that G'_n has no non pendent twins with more that

two primary divisors like $\prod_{i=1}^{t} p_i, t \geq 3$. Therefore, we are left with pendent twins, twin sets with two primary divisors $p_i p_j, i < j$ and the twins sets of form p_i. It is clear that this requirement is satisfied only for $n = p_1^{n_1} p_2^{n_2} p_3^{n_3}$ and we prove the equality case. By Remark 3.3,

$$\dim(G_n') \geq n_1 n_2 n_3 + n_1 n_2 + n_1 n_3 + n_2 n_3 + n_1 + n_2 + n_3 - 8.$$

We will now prove the above inequality in reverse order. We will prove that W given below is the resolving set for G_n'

$$W = \Big\{ p_1, \ldots, p_1^{n_1-1}, p_2, \ldots, p_2^{n_2-1}, p_3, \ldots, p_3^{n_3-1}, p_1 p_2, p_1 p_2^2, \ldots, p_1^{n_1} p_2^{n_2-1}, p_1 p_3, p_1 p_3^2, \ldots,$$

$$p_1^{n_1} p_3^{n_3-1}, p_2 p_3, p_2 p_3^2, \ldots, p_2^{n_2} p_3^{n_3-1}, p_1 p_2 p_3, \ldots, p_1^{n_1} p_2^{n_2} p_3^{n_3-2} \Big\}.$$

Clearly, for each $v_i \in W$, the $|W|$-vector $r(v_i|w)$ is distinct for each v_i. We consider the representation of the vertices in $V(G_n' \setminus W) = \{1, p_1^{n_1}, p_2^{n_2}, p_3^{n_3}, p_1^{n_1} p_2^{n_2}, p_2^{n_2} p_3^{n_3}, p_1^{n_1} p_3^{n_3}, p_1^{n_1} p_2^{n_2} p_3^{n_3-1}\}$ with respect to W. Their $|W|$-vectors are

$$r(1|W) = (1, 1, \ldots, 1), r(p_1^{n_1}|W) = \Big(\underbrace{2, 2, \ldots, 2}_{n_1-1}, \underbrace{1, 1, \ldots, 1}_{n_2 n_3 + n_2 + n_3 - 3}, \underbrace{2, 2, \ldots, 2, 2}_{n_1 n_2 + n_1 n_3 + n_1 n_2 n_3 - 4} \Big),$$

$$r(p_2^{n_2}|W) = \Big(\underbrace{1, 1, \ldots, 1}_{n_1-1}, \underbrace{2, 2, \ldots, 2}_{n_2-1}, \underbrace{1, 1, \ldots, 1}_{n_3-1}, \underbrace{2, 2, \ldots, 2}_{n_1 n_2 + n_2 n_3 - 2}, \underbrace{1, 1, \ldots, 1}_{n_1 n_3 - 1}, \underbrace{2, 2, \ldots, 2}_{n_1 n_2 n_3 - 2} \Big),$$

$$r(p_3^{n_3}|W) = \Big(\underbrace{1, 1, \ldots, 1}_{n_1+n_2-2}, \underbrace{2, 2, \ldots, 2}_{n_3-1}, \underbrace{1, 1, \ldots, 1}_{n_1 n_2 - 1}, \underbrace{2, 2, \ldots, 2}_{n_2 n_3 + n_1 n_3 + n_1 n_2 n_3 - 4} \Big),$$

$$r(p_1^{n_1} p_2^{n_2}|W) = \Big(\underbrace{2, 2, \ldots, 2}_{n_1+n_2-2}, \underbrace{1, 1, \ldots, 1}_{n_3-1}, \underbrace{2, 2, \ldots, 2}_{n_2 n_3 + n_1 n_3 + n_1 n_2 n_3 - 4} \Big),$$

$$r(p_2^{n_2} p_3^{n_3}|W) = \Big(\underbrace{1, 1, \ldots, 1}_{n_1-1}, \underbrace{2, 2, \ldots, 2}_{n_2 + n_3 + n_1 n_2 + n_2 n_3 + n_1 n_3 + n_1 n_2 n_3 - 7} \Big),$$

$$r(p_1^{n_1} p_3^{n_3}|W) = \Big(\underbrace{2, 2, \ldots, 2}_{n_1-1}, \underbrace{1, 1, \ldots, 1}_{n_2-1}, \underbrace{2, 2, \ldots, 2}_{n_3 + n_1 n_2 + n_2 n_3 + n_1 n_3 + n_1 n_2 n_3 - 7} \Big),$$

$$r(p_1^{n_1} p_2^{n_2} p_3^{n_3-1}|W) = (2, 2, \ldots, 2).$$

All the above vectors are different and it follows that W is the resolving set of G_n'. Thus, the equality case follows. $\square$

Based on the calculations in Theorems 3.8 and 3.9, and the upper bound for the metric dimension in Proposition 3.5 and Corollaries 3.6 and 3.7, we leave the following problem for future research.

Problem 1. Is it true that the metric dimension of G_n' for $n = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}, t \geq 4$ and $n_1 \geq n_2 \geq \cdots \geq n_t \geq 2$ is

$$\dim(G_n') = \prod_{i=1}^{t} n_i + \sum_{i=1}^{t} n_i + \sum_{i<j} n_i n_j + \sum_{i<j<k} n_i n_j n_k + \cdots + \sum_{i<j<k<\cdots<t-1} n_i n_j n_k \ldots n_{t-1} - \sum_{i=1}^{t} \binom{t}{i} - 2.$$

For a connected graph G of order n, the collection of all resolving sets (subsets of $V(G)$) of cardinality k is denoted by s_k, and resolving polynomial of G is defined as

$$\psi(G, x) = \sum_{k=\beta(G)}^{n} s_k x^k,$$

where $\beta(G)$ is the metric dimension of G. We note that $s_k = 0$ for $k < \beta(G)$. The sequence $(s_{\beta(G)}, s_{\beta(G)+1}, \ldots, s_n)$ of $\psi(G, x)$ is known as the resolving sequence of G. The resolving polynomial of commuting graphs of dihedral groups and power graphs of finite graphs can be seen in [1, 2].

The following result gives the coefficients s_{n-1} and s_n of the resolving polynomial. The proof follows trivially and is therefore omitted.

Lemma 3.10 ([1]). *Let G be a connected graph of order n. Then, the only resolving set of G of order n is $V(G)$ itself and the resolving set of order $n-1$ can be selected in n possibly distinct ways. Thus, $s_{n-1} = n$ and $s_n = 1$.*

The following result gives the resolving polynomial of G'_n, provided n has at most two primary divisors.

Proposition 3.11. *The resolving polynomial of G'_n for $n = p_1^{n_1} p_2^{n_2}$ is*

$$\psi(G'_n, x) = x^{n-4}\Big(x^4 + nx^3 + (n_1 n_2(n_1 + n_2 + 2) - 1)x^2 + \big(n_1 n_2(n_1 n_2 + n_1 + n_2)$$
$$- n_1 - n_2\big)x + n_1 n_2(n_1 n_2 - 1)\Big).$$

Proof. For $n = p_1^{n_1} p_2^{n_2}$, by Theorem 3.8 the metic dimension of G'_n is $n_1 n_2 + n_1 + n_2 - 4$. So by the definition of resolving set, we need to evaluate the coefficients of resolving sequence $(s_{n-4}, s_{n-3}, s_{n-2}, s_{n-1}, s_n)$, where $n = n_1 n_2 + n_1 + n_2$. Also, we note that there are n_1 twins with primary divisor p_1, n_2 twins with primary divisors p_2 and $n_1 n_2 - 1$ twins with primary divisors p_1, p_2, we denote them by A_1, A_2 and A_3, respectively. For the coefficient s_{n-4}, we have to find all the resolving sets of G'_n of cardinality $n - 4$. By Corollary 3.2 along with combination and multiplications rule, we have

$$s_{n-4} = \binom{n_1}{n_1 - 1}\binom{n_2}{n_2 - 1}\binom{n_1 n_2 - 1}{n_1 n_2 - 2} = n_1 n_2(n_1 n_2 - 1).$$

Next, for s_{n-3}, we are required to find all the resolving sets of G'_n of cardinality $n - 3$. Let S be any resolving set of order $n - 3$ and let $x, y,$ and z be three vertices of G'_n such that $x, y, z \notin S$. In view of Remark 3.3, we have the following possibilities:

$$x = 1, y \in A_1 \text{ and } z \in A_2 \text{ or } x = 1, y \in A_1 \text{ and } z \in A_3$$
$$\text{or } x = 1, y \in A_2 \text{ and } z \in A_3 \text{ or } x \in A_1, y \in A_2 \text{ and } z \in A_3.$$

Thus, with the above possibilities, the number of resolving sets of cardinality $n - 3$ are

$$s_{n-3} = \binom{n_1}{n_1 - 1}\binom{n_2}{n_2 - 1} + \binom{n_1}{n_1 - 1}\binom{n_1 n_2 - 1}{n_1 n_2 - 2} + \binom{n_2}{n_2 - 1}\binom{n_1 n_2 - 1}{n_1 n_2 - 2}$$
$$+ \binom{n_1}{n_1 - 1}\binom{n_2}{n_2 - 1}\binom{n_1 n_2 - 1}{n_1 n_2 - 2} = n_1 n_2 + n_1(n_1 n_2 - 1) + n_2(n_1 n_2 - 1)$$
$$+ n_1 n_2(n_1 n_2 - 1) = n_1 n_2(n_1 n_2 + n_1 + n_2) - n_1 - n_2.$$

For s_{n-2}, consider two vertices x and y of $G_{n'}$ such that $x, y \notin S'$, where S' is any resolving set of order $n - 2$. By Remark 3.2, we have the following possibilities in this case

$$x = 1 \text{ and } y \in A_1, \text{ or } x = 1 \text{ and } y \in A_2, \text{ or } x = 1 \text{ and } y \in A_3,$$

or $x \in A_1$ and $y \in A_2$, or $x \in A_1$ and $y \in A_3$, or $x \in A_2$ and $y \in A_3$.

By combinations and multiplications rule, the value of s_{n-2} is

$$
\begin{aligned}
s_{n-2} = & \binom{n_1}{n_1 - 1} + \binom{n_2}{n_2 - 1} + \binom{n_1 n_2 - 1}{n_1 n_2 - 2} + \binom{n_1}{n_1 - 1}\binom{n_2}{n_2 - 1} \\
& + \binom{n_1}{n_1 - 1}\binom{n_1 n_2 - 1}{n_1 n_2 - 2} + \binom{n_2}{n_2 - 1}\binom{n_1 n_2}{n_1 n_2 - 1} = n_1 + n_2 + n_1 n_2 - 1 \\
& + n_1 n_2 + n_1(n_1 n_2 - 1) + n_2(n_1 n_2 - 1) = n_1 n_2(n_1 + n_2 + 2) - 1.
\end{aligned}
$$

Furthermore by Lemma 3.10, we get $s_{n-1} = n$ and $s_n = 1$. Therefore, the resolving polynomial of G'_n is

$$
\begin{aligned}
\psi(G'_n, x) = \sum_{k=n-4}^{n} s_k x^k = & \, n_1 n_2(n_1 n_2 - 1)x^{n-4} + (n_1 n_2(n_1 n_2 + n_1 + n_2) - n_1 - n_2)x^{n-3} \\
& + (n_1 n_2(n_1 + n_2 + 2) - 1)x^{n-2} + n x^{n-1} + x^n \\
= & \, x^{n-4}\Big(x^4 + n x^3 + (n_1 n_2(n_1 + n_2 + 2) - 1)x^2 + \big(n_1 n_2(n_1 n_2 + n_1 + n_2) \\
& - n_1 - n_2\big)x + n_1 n_2(n_1 n_2 - 1)\Big).
\end{aligned}
$$

$\square$

4. Conclusion

This article gives some graph invariants of G_n and G'_n for some special values of n and in general it remains an open challenge to find the automorphism group, metric dimension, resolving polynomials and other invariants of these graphs.

Conflicts of interest

The authors declare that they have no competing interests.

Data availability statement

No new data/codes were created or analyzed in this study.

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