

RESULTS ON THE ELLIPTIC SOMBOR INDEX

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Abstract. The newly introduced elliptic Sombor index of a graph G is defined as follows

$$\text{ESO}(G) = \sum_{uv \in E} (d(u) + d(v)) \sqrt{d(u)^2 + d(v)^2},$$

in which E and $d(u)$ are the edge set of G and the degree of the vertex u in G , respectively. In this paper, we compute the elliptic Sombor index for certain graphs. Furthermore, we obtain new results and bounds for the elliptic Sombor index in a graph.

Mathematics Subject Classification. 05C40, 05C90.

Received May 7, 2024. Accepted June 20, 2025.

1. INTRODUCTION

Let G be a simple graph with vertex set V and edge set E such that $|V|$ and $|E|$ denote the order and size of G , respectively. The degree of a vertex $u \in V$ is the number of vertices adjacent to u . For two adjacent u and v connected by an edge in G , we denote it as $uv \in E$. The complement of G , denoted by \bar{G} , is defined such that $V(\bar{G}) = V(G)$ and an edge exists between two vertices in \bar{G} if and only if there is no edge between them in G . As usual, C_n , P_n , and K_n denote a cycle graph, path graph, and complete graph of order n . A complete bipartite graph, denoted $K_{p,q}$, is a bipartite graph where every vertex in one set is adjacent to every vertex in the other set. We denote the star graph $K_{1,n-1}$, a complete bipartite graph with one vertex connected to $n - 1$ other vertices, by S_n [1].

The Cartesian product $G_1 \square G_2$ of graphs G_1 and G_2 is a graph such that its vertex set is the Cartesian product $V(G_1) \times V(G_2)$, and two vertices (t, w) and (t', w') are adjacent in $G_1 \square G_2$ if and only if either $t = t'$ and $ww' \in E(G_2)$ or $w = w'$ and $tt' \in E(G_1)$. A connected graph G is called a biregular graph if G is a bipartite graph with two partite sets V_1 and V_2 such that each vertex in V_1 has degree Δ and each vertex in V_2 has degree δ .

The generalized Dutch windmill graph, denoted $D_n^{(m)}$, is constructed by taking n cycles of length m (where $n \geq 2$ and $m \geq 3$) and connecting them all at a single shared vertex. Specifically, when the cycle length m is 3 (forming triangles), this graph is known as the friendship graph and is written as F_n [1, 2].

Topological indices, numerical descriptors derived from the structural graph of a molecule, have emerged as powerful tools in mathematical chemistry. They provide quantitative descriptors of molecular structures,

Keywords. Elliptic Sombor index, degree, topological index, Sombor index.

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enabling the establishment of correlations between structural features and various physicochemical properties, such as boiling point, solubility, stability, and reactivity [3]. The ability of these indices to predict molecular properties without the need for physical synthesis makes them invaluable in virtual screening processes for identifying potential drug candidates or materials with desired characteristics. For more details on the degree-based topological indices, one can explore the survey [4].

The first Zagreb index ($M_1(G)$) of a graph G was introduced by Gutman and Trinajstić [5], and the second Zagreb index ($M_2(G)$) was proposed by Gutman *et al.* [6]. They are defined as follows

$$M_1(G) = \sum_{uv \in E} (d(u) + d(v)) = \sum_{u \in V} d(u)^2,$$

and

$$M_2(G) = \sum_{uv \in E} d(u)d(v).$$

Furtula and Gutman [7] introduced the following forgotten index and obtained some interesting results. This index is defined as follows

$$F(G) = \sum_{uv \in E} (d(u)^2 + d(v)^2) = \sum_{u \in V} d(u)^3.$$

For more details on the Zagreb indices and the forgotten index, we refer to [8–12].

Among the plethora of topological indices, the Sombor index has garnered significant attention in recent years due to its unique definition based on the Euclidean distance between degrees of adjacent vertices [13]. Its sensitivity to the degree distribution within a graph has led to its successful application in various chemical and material science contexts [13–18]. The Sombor index is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2}.$$

The recently introduced elliptic Sombor index by Gutman in 2024 [19] represents a novel approach to quantifying molecular structure within the framework of topological indices, building upon the established utility of the Sombor index in correlating structure with physicochemical properties. The elliptic Sombor index is defined as follows

$$ESO(G) = \sum_{uv \in E(G)} (d(u) + d(v))\sqrt{d(u)^2 + d(v)^2}.$$

Some mathematical properties and applications of the elliptic Sombor index are analyzed in [19]. Espinal *et al.* [20] investigated the extremal values of the elliptic Sombor index of chemical graphs and chemical trees. Rada *et al.* [21] analyzed the Sombor index and the elliptic Sombor index of benzenoid systems, molecular graphs of benzenoid hydrocarbons. Kulli in [22] introduced and studied the modified elliptic Sombor index in a graph. Qi and Lin [23] determined the bicyclic graph with the maximum ESO index among all such graphs of a fixed order. In [24], the authors investigated the elliptic Sombor index for polymer-like graphs constructed by point-attaching primary subgraphs. They derived explicit formulas and lower bounds for the ESO index in various graph structures with applications to chemically significant systems like polyphenylenes, triangulanes, and nanostar dendrimers. The authors in [25] investigated optimization problems for the general ESO index. The work combined analytical techniques, graph transformations, and extremal graph theory to characterize optimal structures, extending prior results on the ESO index and offering unified insights into degree-based topological indices. In [26], the authors evaluated the predictive potential of six topological indices, including the ESO index and the reverse elliptic Sombor index, for 38 polycyclic aromatic hydrocarbons using regression models. In addition to the theoretical results presented here, a comprehensive comparative analysis between the elliptic Sombor index and several classical degree-based indices including the Zagreb indices, the Forgotten index, the Randić index, and the Sombor index has been conducted in a separate study [27]. This study establishes

sharp mathematical relationships and inequalities among these indices and provides numerical comparisons for irregular graph families such as star graphs, highlighting the limitations of some ESO bounds in such cases.

This study is motivated by the opportunity to delve into the fundamental mathematical characteristics of this newly defined index. By deriving a series of mathematical results for the elliptic Sombor index, this work aims to provide an initial yet crucial understanding of its behavior and its relationship to graph structural parameters. These findings will contribute to establishing the theoretical foundation of this index and pave the way for its subsequent exploration in chemical applications and comparative studies with existing molecular descriptors. In this paper, we first compute the elliptic Sombor index for certain families of graphs. Furthermore, we report some new results for the elliptic Sombor index of a graph.

2. PRELIMINARIES

In this section, some inequalities and known results that will be used in the main results are stated.

Lemma 2.1 ([28]). *Let n be a positive integer, and let a, b, A , and B be real constants. Assume that for each $i = 1, 2, \dots, n$, the values a_i and b_i are positive real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$. Then,*

$$\left| n \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i \sum_{i=1}^n b_i \right| \leq \beta(n)(A - a)(B - b),$$

where $\beta(n) = n \lfloor \frac{n}{2} \rfloor (1 - \frac{1}{n} \lfloor \frac{n}{2} \rfloor)$.

Lemma 2.2 ([29]). *Let $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ be non-negative real numbers, and r and R be real constants. Then*

$$\sum_{i=1}^n b_i^2 + rR \sum_{i=1}^n a_i^2 \leq (r + R) \left(\sum_{i=1}^n a_i b_i \right),$$

where $ra_i \leq b_i \leq Ra_i$ for $1 \leq i \leq n$.

Lemma 2.3 ([30]). *Let $a = (a_i)_{i=1}^n$ and $b = (b_i)_{i=1}^n$ be two decreasing sequences of real numbers. Assume that for each integer k , $1 \leq k \leq n - 1$, we have $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$. Then for any convex function f , $\sum_{i=1}^k f(a_i) \leq \sum_{i=1}^k f(b_i)$.*

Lemma 2.4 ([31]). *Let G be a connected graph with n vertices and m edges. Then*

$$M_1(G) \leq m(m + 1),$$

with equality for $n > 3$ if and only if $G \simeq K_3$ or $G \simeq S_n$.

Lemma 2.5 ([19]). *For any connected graph G of order n ,*

$$\text{ESO}(P_n) \leq \text{ESO}(G) \leq \text{ESO}(K_n).$$

Equality holds if and only if $G \simeq P_n$ or $G \simeq K_n$.

3. MAIN RESULTS

In this section, we obtain several bounds for the elliptic Sombor index in terms of some graph parameters.

First, we compute the elliptic Sombor index for certain graphs. Let G be a graph of order n and size m with the maximum and minimum degrees Δ and δ , respectively.

(1) k -regular graph

$$\begin{aligned} \text{ESO}(G) &= \sum_{uv \in E} (d(u) + d(v)) \sqrt{d(u)^2 + d(v)^2} \\ &= \sum_{uv \in E} (2k) \sqrt{2k^2} \\ &= 2\sqrt{2}mk^2 = n\sqrt{2}k^3. \end{aligned}$$

(2) Cycle graph C_n

$$\text{ESO}(C_n) = n\sqrt{2}(2)^3 = 8n\sqrt{2}.$$

(3) Biregular graph

$$\text{ESO}(G) = m(\delta + \Delta) \sqrt{\Delta^2 + \delta^2}.$$

(4) k -dimensional cube graph Q_k

The number of edges in Q_k is $2^{k-1}k$ and the degree of any vertex is k . Since the number of vertices in Q_k is $n = \frac{2m}{k} = 2^k$, using item (1), we have

$$\text{ESO}(Q_k) = n\sqrt{2}k^3 = 2^k k^3 \sqrt{2}.$$

(5) Complete bipartite graph $K_{p,q}$

The result comes directly from item (3). Therefore

$$\text{ESO}(K_{p,q}) = pq(p+q) \sqrt{p^2 + q^2}.$$

(6) Star graph $S_n = K_{1,n-1}$

The result comes directly from item (3). Hence

$$\text{ESO}(S_n) = n(n-1) \sqrt{(n-1)^2 + 1}.$$

(7) Wheel graph W_n

This graph contains $n-1$ edges $e = uv$ with $d(u) = d(v) = 3$ and $n-1$ edges $e = uv$ with $\{d(u), d(v)\} = \{3, n-1\}$. Therefore,

$$\begin{aligned} \text{ESO}(W_n) &= (n-1)(3+3) \sqrt{9+9} + (n-1)(3+n-1) \sqrt{(n-1)^2 + 9} \\ &= (n-1) \left[18\sqrt{2} + (n+2) \sqrt{(n-1)^2 + 9} \right]. \end{aligned}$$

(8) Ladder graph $L_n = P_n \square K_2$ for $n \geq 3$

This graph contains $2n$ vertices and $3n-2$ edges. Based on the structure of the graph L_n , there are 2 edges $e = uv$ where $d(u) = d(v) = 2$, 4 edges $e = uv$ with $\{d(u), d(v)\} = \{2, 3\}$, and the remaining $3n-8$ edges with end vertices of degree 3.

$$\begin{aligned} \text{ESO}(L_n) &= 2(2+2) \sqrt{4+4} + 4(2+3) \sqrt{4+9} + (3n-8)(3+3) \sqrt{9+9} \\ &= 20\sqrt{13} + (54n-128)\sqrt{2}. \end{aligned}$$

(9) **Book graph** $B_n = S_{n+1} \square K_3$

$$\begin{aligned} \text{ESO}(B_n) &= 4n\sqrt{8} + 2(n+1)\sqrt{2(n+1)^2} + 2n(n+3)\sqrt{(n+1)^2 + 4} \\ &= 8\sqrt{2}n + 2\sqrt{2}(n+1)^2 + 2n(n+3)\sqrt{(n+1)^2 + 4} \\ &= 2\sqrt{2}(n^2 + 6n + 1) + 2n(n+3)\sqrt{(n+1)^2 + 4}. \end{aligned}$$

(10) **Dutch windmill graph** $D_n^{(m)}, n \geq 3, m \geq 2$

The graph $D_n^{(m)}$ contains $n(m-1) + 1$ vertices and mn edges. Based on the structure of this graph, $2n$ edges have end vertices of degrees $2n$ and 2 , and the remaining $n(m-2)$ edges have end vertices of degree 2 . Therefore,

$$\begin{aligned} \text{ESO}(D_n^{(m)}) &= 2n(2n+2)\sqrt{4n^2+4} + n(m-2)4\sqrt{8} \\ &= 8n(n+1)\sqrt{n^2+1} + 8\sqrt{2}n(m-2) \\ &= 8n\left[(n+1)\sqrt{n^2+1} + \sqrt{2}(m-2)\right]. \end{aligned}$$

(11) **Friendship graph** F_n

Since $F_n = D_n^{(3)}$, from item (10) we get

$$\text{ESO}(F_n) = 8n\sqrt{2} + 8n(n+1)\sqrt{n^2+1}.$$

Next, we obtain bounds for the elliptic Sombor index in the following results.

Theorem 3.1. *Let G be a graph of size m with the minimum degree δ and the maximum degree Δ . Then*

$$\text{ESO}(G) \leq \frac{1}{m} \left(M_1(G)\text{SO}(G) + 2\sqrt{2}\beta(m)(\Delta - \delta)^2 \right),$$

where $\beta(m) = m \lfloor \frac{m}{2} \rfloor \left(1 - \frac{1}{m} \lfloor \frac{m}{2} \rfloor \right)$.

Equality holds if and only if G is regular graph.

Proof. Applying Lemma 2.1 with $a_i = (d(u) + d(v))$, $b_i = \sqrt{d(u)^2 + d(v)^2}$, $A = 2\Delta$, $B = \sqrt{2}\Delta$, $a = 2\delta$, and $b = \sqrt{2}\delta$, we get

$$\left| m \sum_{uv \in E} (d(u) + d(v))\sqrt{d(u)^2 + d(v)^2} - \sum_{uv \in E} (d(u) + d(v)) \sum_{uv \in E} \sqrt{d(u)^2 + d(v)^2} \right| \leq 2\sqrt{2}\beta(m)(\Delta - \delta)^2,$$

in which $\beta(m) = m \lfloor \frac{m}{2} \rfloor \left(1 - \frac{1}{m} \lfloor \frac{m}{2} \rfloor \right)$.

Therefore, we get

$$m \text{ESO}(G) - M_1(G)\text{SO}(G) \leq 2\sqrt{2}\beta(m)(\Delta - \delta)^2.$$

Consequently,

$$\text{ESO}(G) \leq \frac{1}{m} \left(M_1(G)\text{SO}(G) + 2\sqrt{2}\beta(m)(\Delta - \delta)^2 \right).$$

The equality in the above relations holds if and only if $\Delta = \delta$. Therefore, we have $m\text{ESO}(G) - M_1(G)\text{SO}(G) = 0$. The equality holds if and only if for any two vertices u and v in G , $d(u) = d(v)$, that is, G is a regular graph. Conversely, if G is the k -regular graph, then using item (1) we have

$$m \text{ESO}(G) = m \left(n\sqrt{2}k^3 \right) = (nk^2) \times \left(\sqrt{2}km \right) = M_1(G)\text{SO}(G).$$

□

Theorem 3.2. *Let G be a graph of size m with the minimum degree δ and the maximum degree Δ . Then*

$$\text{ESO}(G) \geq \left(\frac{\sqrt{2}(\Delta + \delta)}{(\Delta + \delta) + \sqrt{\delta^2 + \Delta^2}} \right) \left(F(G) + \left(\frac{\sqrt{\delta^2 + \Delta^2}}{\sqrt{2}(\delta + \Delta)} \right) (F(G) + 2M_2(G)) \right),$$

with equality holds if and only if G is regular graph.

Proof. Let $h(x) = \frac{\sqrt{1+x^2}}{1+x}$. We know that $h'(x) = \frac{x-1}{(1+x^2)\sqrt{1+x^2}}$ is non-negative for any $x \geq 1$. Thus $h(x)$ is an increasing function on $x \geq 1$.

Since for any $u \in V(G)$, $0 < \delta \leq d(u) \leq \Delta$, we have $\frac{\delta}{\Delta} \leq \frac{d(u)}{d(v)} \leq \frac{\Delta}{\delta}$.

Using the above results, we obtain

$$\frac{\sqrt{d(u)^2 + d(v)^2}}{d(u) + d(v)} = \frac{\sqrt{1 + \frac{d(u)^2}{d(v)^2}}}{1 + \frac{d(u)}{d(v)}} \leq \frac{\sqrt{1 + \frac{\Delta^2}{\delta^2}}}{1 + \frac{\Delta}{\delta}} = \frac{\sqrt{\delta^2 + \Delta^2}}{\delta + \Delta}.$$

On the other hand, since $(d(u) - d(v))^2 \geq 0$, we have

$$\frac{1}{\sqrt{2}} \leq \frac{\sqrt{d(u)^2 + d(v)^2}}{d(u) + d(v)}.$$

By substituting $a_i = (d(u) + d(v))$, $b_i = \sqrt{d(u)^2 + d(v)^2}$, $r = \frac{1}{\sqrt{2}}$, and $R = \frac{\sqrt{\delta^2 + \Delta^2}}{\delta + \Delta}$ into Lemma 2.2, we get

$$\begin{aligned} \sum_{uv \in E} \left(\sqrt{d(u)^2 + d(v)^2} \right)^2 + \left(\frac{\sqrt{\delta^2 + \Delta^2}}{\sqrt{2}(\delta + \Delta)} \right) \sum_{uv \in E} (d(u) + d(v))^2 \\ \leq \left(\frac{1}{\sqrt{2}} + \frac{\sqrt{\delta^2 + \Delta^2}}{\delta + \Delta} \right) \left(\sum_{uv \in E} (d(u) + d(v)) \sqrt{d(u)^2 + d(v)^2} \right). \end{aligned} \tag{3.1}$$

Since $\sum_{uv \in E} (d(u) + d(v))^2 = F(G) + 2M_2(G)$, from inequality (3.1), we have

$$F(G) + \left(\frac{\sqrt{\delta^2 + \Delta^2}}{\sqrt{2}(\delta + \Delta)} \right) (F(G) + 2M_2(G)) \leq \left(\frac{1}{\sqrt{2}} + \frac{\sqrt{\delta^2 + \Delta^2}}{\delta + \Delta} \right) \text{ESO}(G).$$

Therefore, we get

$$\left(\frac{\sqrt{2}(\Delta + \delta)}{(\Delta + \delta) + \sqrt{2}(\delta^2 + \Delta^2)} \right) \left(F(G) + \left(\frac{\sqrt{\delta^2 + \Delta^2}}{\sqrt{2}(\delta + \Delta)} \right) (F(G) + 2M_2(G)) \right) \leq \text{ESO}(G).$$

Suppose that all the above inequalities are equalities. Therefore, the equality holds if and only if $\delta = \Delta$ and for any $u, v \in V(G)$, $d(u) = d(v)$. That is, the equality holds if and only if G is a regular graph. \square

Theorem 3.3. *Let G be a graph of order n and size m . Then*

$$\text{ESO}(G) \leq M_1(G) \sqrt{m^2 + 1}.$$

Equality holds if and only if $m \leq n - 1$ and $G \simeq S_{m+1} \cup (n - m - 1)K_1$.

Proof. For the graph G of size m and $uv \in E$, we have $d(u) + d(v) \leq m + 1$. Let $a = (d(u) + \alpha, d(v))$ and $b = (m, 1)$ be two decreasing sequences, where $\alpha = m + 1 - (d(u) + d(v))$. Since $d(u) + \alpha = m + 1 - d(v) \leq m$, then by considering the convex function $f(x) = x^2$ on \mathbb{R} , according to Lemma 2.3, $\sum_{i=1}^2 f(a_i) \leq \sum_{i=1}^2 f(b_i)$. Therefore, we have

$$(d(u) + \alpha)^2 + d(v)^2 \leq m^2 + 1.$$

Using the fact that

$$\sqrt{d(u)^2 + d(v)^2} \leq \sqrt{(d(u) + \alpha)^2 + d(v)^2},$$

we have

$$\begin{aligned} \text{ESO}(G) &= \sum_{uv \in E} (d(u) + d(v)) \sqrt{d(u)^2 + d(v)^2} \\ &\leq \sum_{uv \in E} (d(u) + d(v)) \sqrt{(d(u) + \alpha)^2 + d(v)^2} \\ &\leq \sum_{uv \in E} (d(u) + d(v)) \sqrt{m^2 + 1} \\ &= \sqrt{m^2 + 1} \sum_{uv \in E} (d(u) + d(v)) \\ &= \sqrt{m^2 + 1} M_1(G). \end{aligned}$$

The equality holds if and only if $\alpha = (m + 1) - (d(u) + d(v))$ and $(d(u) + \alpha, d(v)) = (m, 1)$. In this case, equality holds if and only if $m \leq n - 1$ and $G \simeq S_{m+1} \cup (n - m - 1)K_1$. □

From Theorem 3.3 and Lemma 2.4, the following results are obtained.

Corollary 3.4. *Let G be a connected graph of size m . Then*

$$\text{ESO}(G) \leq m(m + 1)\sqrt{m^2 + 1},$$

with equality holds if and only if $G \simeq S_{m+1}$.

Corollary 3.5. *Let T be a tree of order n . Then*

$$\text{ESO}(G) \leq n(n - 1)\sqrt{(n - 1)^2 + 1}.$$

Equality holds if and only if $T \simeq S_n$.

Corollary 3.6. *Let T be a tree of order n . Then*

$$\text{ESO}(G) \leq M_1(G)\sqrt{(n - 1)^2 + 1}.$$

Equality holds if and only if $T \simeq S_n$.

Theorem 3.7. *Let G be a graph of order n with the minimum degree δ and the maximum degree Δ . Then*

$$\sqrt{2n}\delta^3 \leq \text{ESO}(G) \leq \sqrt{2n}\Delta^3.$$

Equality holds if and only if G is a regular graph.

Proof. By using the handshaking lemma, we have

$$n\delta \leq \sum_{u \in V} d(u) = 2m \leq n\Delta.$$

Using the definition of the elliptic Sombor index and since $\delta \leq d(u) \leq \Delta$ for any $u \in V$, we have

$$\begin{aligned} \text{ESO}(G) &= \sum_{uv \in E} (d(u) + d(v)) \sqrt{d(u)^2 + d(v)^2} \\ &\leq \sum_{uv \in E} (2\Delta) \sqrt{2\Delta^2} \\ &= 2\sqrt{2}m\Delta^2 \\ &\leq \sqrt{2}n\Delta^3. \end{aligned}$$

Similarly, the lower bound is obtained and we have $\text{ESO}(G) \geq \sqrt{2}n\delta^3$.

The equality holds if and only if $\delta = d(u) = d(v) = \Delta$ for any $uv \in E$. Therefore, equality holds if and only if G is a regular graph. \square

Theorem 3.8. *Let G be a graph. Then*

- (i) $\text{ESO}(G) > \text{ESO}(G - e) + \frac{|d(u)^2 - d(v)^2|}{\sqrt{2}}$,
- (ii) $\text{ESO}(G + e) > \text{ESO}(G) + \frac{|d(u)^2 - d(v)^2|}{\sqrt{2}}$,

for edge $e = uv \in E$ such that vertices u and v are not connected in G .

Proof. It is sufficient to prove the case (i). By removing the edge $e = uv \in E$ from graph G , we consider the elliptic Sombor index $G - e$ as $\text{ESO}(G - e)$. Now we add the edge $e = uv$ to $G - e$. Therefore the term $(d(u) + d(v))\sqrt{d(u)^2 + d(v)^2}$ is added to $\text{ESO}(G - e)$. Since $\sqrt{d(u)^2 + d(v)^2} \geq \frac{|d(u) - d(v)|}{\sqrt{2}}$, consequently

$$\begin{aligned} \text{ESO}(G) &> \text{ESO}(G - e) + (d(u) + d(v))\sqrt{d(u)^2 + d(v)^2} \\ &\geq \text{ESO}(G - e) + \frac{|d(u)^2 - d(v)^2|}{\sqrt{2}}. \end{aligned}$$

The case (ii) is similar to case (i). \square

Theorem 3.9. *For a graph G ,*

- (i) if $v \in V$, then $\text{ESO}(G - v) < \text{ESO}(G) - \sum_{uv \in E} \frac{|d(u)^2 - d(v)^2|}{2}$,
- (ii) if $v \notin V$, then $\text{ESO}(G) < \text{ESO}(G + v)$.

Proof. By removing the vertex $v \in V$ and its connected edges, we compute $\text{ESO}(G - v)$. Now, by adding the vertex v back to $G - v$ and including all the edges that were connected to v in the original graph G , and since $\sqrt{d(u)^2 + d(v)^2} \geq \frac{|d(u) - d(v)|}{\sqrt{2}}$

$$\begin{aligned} \text{ESO}(G) &> \text{ESO}(G - v) + \sum_{uv \in E} (d(u) + d(v))\sqrt{d(u)^2 + d(v)^2} \\ &\geq \text{ESO}(G - v) + \sum_{uv \in E} \frac{|d(u)^2 - d(v)^2|}{\sqrt{2}}. \end{aligned}$$

\square

Theorem 3.10. *Let G be a bipartite graph of order n . Then*

$$6\sqrt{5} + 8(n - 3)\sqrt{2} \leq \text{ESO}(G) \leq \begin{cases} \frac{n^4}{4\sqrt{2}} & \text{if } n \text{ is even,} \\ \frac{n(n^2-1)\sqrt{n^2+1}}{4\sqrt{2}} & \text{if } n \text{ is odd.} \end{cases}$$

The left hand side equality holds if and only if $G \simeq P_n$ and the right hand side equality holds if and only if $G \simeq K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Proof. Let G be a bipartite graph of order n . Since using Lemma 2.5, for any connected graph of order n , the path graph P_n gives the minimum value of elliptic Sombor index, thus for any bipartite graph G with n vertices, $\text{ESO}(G) \geq \text{ESO}(P_n) = 6\sqrt{5} + 8(n - 3)\sqrt{2}$.

For the upper bound, using Theorem 3.8, we have

$$\text{ESO}(G) \leq \text{ESO}(K_{p,q}),$$

where $K_{p,q}$ is a complete bipartite graph with $p + q = n$. The equality holds if and only if $G \simeq K_{p,q}$. Therefore

$$\text{ESO}(G) \leq n(n - p)p\sqrt{p^2 + (n - p)^2}.$$

By considering the function $f(x) = n(n - x)x\sqrt{x^2 + (n - x)^2}$, it is easily seen that $f(x)$ is a decreasing function on $\lfloor \frac{n}{2} \rfloor \leq x \leq n - 1$ and consequently,

$$\text{ESO}(G) \leq n(n - p)p\sqrt{p^2 + (n - p)^2} \leq n \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil \sqrt{\lfloor \frac{n}{2} \rfloor^2 + \lceil \frac{n}{2} \rceil^2}.$$

Therefore, in the above inequality, if n is even, $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil = \frac{n}{2}$, and $\text{ESO}(G) \leq \frac{n^4}{4\sqrt{2}}$. Also, if n is odd, $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ and $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$, where we have

$$\text{ESO}(G) \leq \frac{n(n^2 - 1)}{4} \sqrt{\frac{n^2 + 1}{4}}.$$

The equality holds if and only if $G \simeq K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. □

We obtain the following Nordhaus–Gaddum result for the elliptic Sombor index of a graph G .

Theorem 3.11. *Let G be a graph of order n . Then*

$$\text{ESO}(G) + \text{ESO}(\bar{G}) \leq \sqrt{2}n(n - 1)^3.$$

Equality holds if and only if $G \simeq K_n$ or $G \simeq \bar{K}_n$.

Proof. Let G be a graph of order n with maximum degree Δ . We have $\Delta \leq n - 1$ and for any $uv \in E$,

$$\sqrt{d(u)^2 + d(v)^2} \leq \sqrt{2}(n - 1),$$

and

$$|E(G)| + |E(\bar{G})| = \frac{n(n - 1)}{2}.$$

Therefore,

$$\text{ESO}(G) + \text{ESO}(\bar{G}) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))\sqrt{d_G(u)^2 + d_G(v)^2}$$

$$\begin{aligned}
& + \sum_{uv \in E(\bar{G})} (d_{\bar{G}}(u) + d_{\bar{G}}(v)) \sqrt{d_{\bar{G}}(u)^2 + d_{\bar{G}}(v)^2} \\
& \leq 2(n-1) \binom{n(n-1)}{2} (\sqrt{2}(n-1)) \\
& = \sqrt{2}n(n-1)^3.
\end{aligned}$$

The equality holds if and only if $d_G(u)^2 + d_G(v)^2 = 2(n-1)^2$ for any $uv \in E(G)$ and $d_{\bar{G}}(u)^2 + d_{\bar{G}}(v)^2 = 2(n-1)^2$ for any $uv \in E(\bar{G})$. Therefore, $G \simeq K_n$ or $G \simeq \bar{K}_n$. \square

4. CONCLUSION

In this study, we investigated the elliptic Sombor index ($ESO(G)$), a relatively new topological index, for various graph structures and established several of its mathematical properties. We began by explicitly computing the elliptic Sombor index for a selection of well-known graph families, including regular graphs, cycle graphs, biregular graphs, k -dimensional cube graphs, complete bipartite graphs, star graphs, wheel graphs, ladder graphs, book graphs, Dutch windmill graphs, and friendship graphs.

Furthermore, we derived several upper and lower bounds for the elliptic Sombor index in terms of key graph parameters such as the minimum degree, the maximum degree, the number of vertices, the number of edges, the first Zagreb index, the forgotten index, and the second Zagreb index. These bounds provide valuable estimates for the ESO index and offer insights into its relationship with other graph invariants. We also presented Nordhaus–Gaddum-type results for the elliptic Sombor index.

While some of the derived bounds rely on general inequalities, potentially leaving room for refinement in future work, the exact computations for specific graph classes and the established relationships with fundamental graph parameters contribute to a deeper understanding of the elliptic Sombor index. This study lays a foundation for further exploration of the elliptic Sombor index. Future research directions may include:

- (1) Identifying tighter bounds for the ESO index, possibly through the application of more specialized inequalities.
- (2) Investigating the chemical and pharmacological significance of the ESO index by examining its correlation with molecular properties.
- (3) Exploring the computational complexity of determining the ESO index for large graphs.
- (4) Characterizing extremal graphs with respect to the ESO index within specific graph families.

The results presented here contribute to the growing body of knowledge on degree-based topological indices and their role in characterizing graph structures.

ACKNOWLEDGMENTS

The author gratefully acknowledges the valuable comments and suggestions of the anonymous reviewers, which significantly contributed to improving the quality of this manuscript.

DATA AVAILABILITY STATEMENT

No new data/codes were created or analyzed in this study.

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