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TWO-ECHELON SUPPLY MODEL WITH CREDIT FINANCING IN A FINITE HORIZON

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Abstract. An inventory model is developed for multiple items in which items can be both perishable and non-perishable. The demand is assumed as a linear function in time, and shortages are considered negligible. In this model, a two-echelon supply chain network is considered along with credit financing. Multiple suppliers and multiple retailers are also considered in this article such that each supplier sells one type of item but the retailers can buy one or more than one type of item. To minimize the total transportation cost, an intermediate is introduced between suppliers and retailers, and the maximum number of possible shortest route delivery patterns between suppliers and intermediate and the intermediate and retailers are also calculated in this paper. The planning horizon is taken as finite. A MINLP is formulated and the optimum ordering time, optimum ordering quantity, and optimum route is calculated. A numerical example is solved at the end.

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1. Introduction

Many times, the buyer does not have the full amount to pay thus the supplier may offer the retailer a trade credit in the form of permissible payment delay [8]. Since, some items like food, and fashion goods deteriorate over time, considering this scenario, Liao *et al.* [14] assumed a model of deteriorating items in which payment delay is considered permissible. Also, in real-world cases, considering credit financing increases demand and helps in growing business. Motivated by this, Taleizadeh *et al.* [26] introduced an EPQ model considering credit financing in supply chain management. Additionally, for many items, the demand rate either remains constant or increases linearly despite price changes. Assuming this, Singh and Rana [24] gave an inventory model considering linear demand and deterioration, emphasizing the use of linear demand models in real-life scenarios such as the growing items inventory model. Authors like Jayanthi [11], Choudhury and Mahata [5] also proposed innovative approaches to solve inventory models, assuming the circumstances of permissible delays in payments for deteriorating items. Sometimes the retailer wants more than one type of product, assuming this, Rezaei and Davoodi [21] considered a supply chain framework with multiple products and multiple suppliers. However, in some cases, such as the clothing distribution network, a buyer may purchase clothes of different brands from various companies in one city and then sell them to multiple clothing shop owners in another city.

Keywords. MINLP, credit financing, multiple items, two-echelon, deterioration, finite planning horizon, route delivery pattern.

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Thus, in these kinds of situations, it is better to consider multiple retailers along with considering multiple buyers. Since, in minimizing aggregate supply cost in a supply chain network, routing also plays a vital role, accepting it [27] gave an optimum supply chain network design to optimize the total cost. However, the optimum route for deteriorating items may not always be the optimum route for non deteriorating items. Thus, finding the shortest, and cheapest possible route considering both deteriorating and non-deteriorating items, along with minimizing overall cost is a significant challenge. Pourhassan *et al.* [20] also formulated a multi-echelon location routing inventory model to reduce overall driving hours using Multi-Objective Whale Optimization approach. However, in many real-life situations, the locations of suppliers and buyers are fixed and optimization of their locations is not possible. In such cases, the only feasible approach is to find shortest route covering maximum number of locations along with minimizing overall costs including the inventory perishability cost. Additionally, an integrated supply chain management plan permits companies to increase overall efficiency, taking it into account, Mousavi *et al.* [18] considered a two-echelon supply chain model with several vendors and several buyers. Sana [23] also presented a twoechelon supply chain network model to increase the overall profit. Furthermore, Abdul-Jalbar *et al.* [1] formulated a two-echelon inventory model considering a single vendor supplying two buyers, who can delay payment until the end of the credit period, and compared the integrated and the decentralized policies. Herbon and David [9] also explored a two-echelon integrated supply chain network under the consideration of credit financing.

This study mainly aims to answer the following question: How can a warehouse owner reduce the total cost of buying, transporting, storing, and financing different products when getting them from many suppliers and sending them to many customers, especially when some products spoil over time and rented vehicles are used for delivery?

2. Literature review

Several researchers have contributed to the field of inventory and supply chain management by exploring different aspects such as deterioration, credit financing, multiple-item management, and supply chain network complexity. Table 1 provides a comparative overview of key studies, highlighting the factors considered in each.

Markov *et al.* [16] and Sakulsom and Tharmmaphornphilas [22] focus primarily on multi-echelon supply chain networks, but do not address deterioration, credit financing, or multiple items under a finite planning horizon. Ben Ahmed *et al.* [2] and Chiu *et al.* [4] extended this work to a finite planning horizon. However, they do not consider deterioration, credit financing, and a multi-item inventory and supply chain model, which is crucial for real-world applications involving perishable goods.

Deterioration has been a significant area of study in supply chain models. Considering it, Padiyar *et al.* [19] gave a multi-echelon inventory and supply chain model under an infinite planning horizon. Ghiami and Beullens [7] introduced a two-echelon supply chain model for deteriorating items and presented a continuous resupply policy. Mondal *et al.* [17] incorporate deterioration in his model, including credit financing under an infinite planning horizon. Dye [6], Taleizadeh and Nematollahi [25] and Budania and Mishra [3] extended this work to a finite planning horizon. Li *et al.* [13] also developed an inventory model for deteriorating items with linear demand under trade credit within a finite planning horizon. Their focus on permissible payment delays contributes to the literature, yet, these studies focus on single-item, single-echelon inventory and supply chain systems, limiting their applicability to real-world multi-item, multi-echelon inventory scenarios.

In contrast, Rezaei and Davoodi [21] and Iraj *et al.* [10] address multi-item, multi-echelon inventory and supply chain systems. However, their models lack credit financing, which plays a crucial role in decision making for companies operating under financial constraints. Similarly, Manna *et al.* [15] incorporate multiple items but do not consider deterioration, credit financing, or multiechelon supply chain network.

Despite these contributions, existing models lack an integrated approach that simultaneously addresses multiple deteriorating items, credit financing, and a two echelon supply chain network under a finite planning horizon. This research aims to bridge this gap by developing a comprehensive model that incorporates these critical factors, hence improving supply chain efficiency and financial viability for perishable goods.

TABLE 1. A brief literature review of some relevant articles.

Article	Multiple items	Deterioration	Credit financing	Supply Chain Network	Finite planning horizon
Rezaei & Davoodi [21]	✓	✗	✗	✓	✓
Dye [6]	✗	✓	✓	✗	✓
Taleizadeh & Nematollahi [25]	✗	✓	✓	✗	✓
Chiu *et al.* [4]	✗	✗	✗	✓	✓
Manna *et al.* [15]	✓	✗	✗	✗	✓
Sakulsom & Tharmmaphornphilas [22]	✗	✗	✗	✓	✗
Markov *et al.* [16]	✗	✗	✗	✓	✗
Padiyar *et al.* [19]	✗	✓	✗	✓	✗
Ben Ahmed *et al.* [2]	✗	✗	✗	✓	✓
Iraj *et al.* [10]	✓	✓	✗	✓	✗
Mondal *et al.* [17]	✗	✓	✓	✗	✗
Kumar *et al.* [12]	✗	✓	✓	✗	✓
Budania & Mishra [3]	✗	✓	✓	✗	✓

2.1. Model formulation

In this chapter, a two-echelon system is considered such that the warehouse owner or intermediate (say ϵ) imports inventory from "τ" number of suppliers and exports it to "Φ" number of customers, where τ and Φ are natural numbers. It is assumed that exactly one type of product is imported from each supplier. We want to minimize the total cost of ϵ. In this model, deteriorating and non-deteriorating items are considered such that each type of product can have different deterioration rates. It is assumed that for each type of product, the sum of all customer's demands and the deteriorated inventory is equal to the imported inventory. It is also assumed that ϵ can sell the items to the customer at at-most the maximum retail price of the product.

In certain situations, the intermediate does not have the vehicles so he may rent the vehicles. Keeping this in mind, it is assumed that ϵ rents vehicles with capacity as per the amount of inventory he wants to carry. And since ϵ can rent the vehicle that has capacity as much as he wants, thus, to decrease the total transportation cost, it is considered that each supplier is visited exactly once in each cycle to carry the required amount of inventory, and each customer is visited exactly once in each cycle to send the required amount of inventory. Suppliers, customers, and ϵ are considered as vertices, and the shortest direct routes from any supplier to supplier, any customer to customer, any supplier to ϵ, or ϵ to any customer as an edge (say e).

It is assumed that ϵ sent the rented vehicles to carry the inventory from suppliers at the start of each cycle. Then, after some fixed time when every type of product is available at ϵ, he sends items to each customer as per the demand. It is also assumed that each customer may want one or more than one type of product. It is also considered that ϵ finances money in each cycle as per his requirement.

Assumptions

- Planning horizon is finite.
- Demand for each product is linear in time for each customer.
- ϵ can sell each product at at-most the maximum retail price (MRP) rate of the product.
- Shortages are considered negligible.
- The cost of renting vehicle "m" is constant in each cycle.
- ϵ borrows money at the beginning of the cycle and gives it back on the completion of the cycle.
- The holding cost of the warehouse is fixed and independent of the quantity.

Notations

$I_j^{i+1}(t)$ = Inventory level of the jth item in units in the $(i+1)$th cycle at any time t.
$Q_j^{i+1}(t)$ = Initial inventory (order quantity) of jth item in units in the $(i+1)$th cycle.
X_j = The initial demand rate for the jth item per unit of time.
Y_j = The change in demand rate for the jth item per unit time.
d_j = Deterioration rate of the jth item per unit time.
C_j = Buying cost of the jth item.
x_e = Number of times an edge is travelled in a cycle.
y_e = Number of times an edge is travelled in a cycle.
h = Holding cost of inventory per unit time.
P_1 = Set of route delivery patterns (RDP) between intermediate and customers.
P_2 = Set of route delivery patterns (RDP) between intermediate and suppliers.

$$\Delta_{i+1}^{p} = \begin{cases} 0, & \text{if RDP } p \text{ is not selected in the } (i+1)\text{th cycle,} \\ 1, & \text{if RDP } p \text{ is selected in the } (i+1)\text{th cycle.} \end{cases}$$

$$\delta_j = \begin{cases} 0, & \text{if the } j\text{th item does not deteriorate with time,} \\ 1, & \text{if the } j\text{th item deteriorates with time.} \end{cases}$$

E_1 = Set of edges between intermediate and customers.
E_2 = Set of edges between intermediate and suppliers.
l = Interest rate per unit time.
A_{i+1} = Amount of money ϵ borrow in $(i+1)$th cycle.
M_j = Maximum retail price (MRP) of jth item.
β = Number of days the warehouse is occupied in a cycle.
V_m = Cost of renting vehicle "m".
μ = Number of rented vehicles in a cycle.
F_O = Number of types of products that the customer "O" wants.
k_q = Number of times supplier "q" is visited in a cycle.
k_r = Number of times customer "r" is visited in a cycle.
H = Finite planning horizon.

3. Mathematical model

The change in inventory for product j in $(i+1)$th cycle is given by

$$\frac{\mathrm{d}I_{i+1}^{j}(t)}{\mathrm{d}t} = -\alpha_j \delta_j I_{i+1}^{j}(t) - X_j - Y_j t; \qquad t_i \le t \le t_{i+1}. \tag{1}$$

Boundary condition: $I_{i+1}^{j}(t_{i+1}) = 0$.

Where, $Y_j > 0$ if demand for jth product is increasing with time, and $Y_j = 0$ if demand for jth product is constant with time.

Solving equation (1), we get

$$I_{i+1}^{j}(t) = e^{\alpha_j (t_{i+1}-t)}\left(\frac{X_j}{\alpha_j} + \frac{Y_j}{\alpha_j} t_{i+1} - \frac{{Y_j}^2}{\alpha_j}\right) - \left(\frac{X_j}{\alpha_j} + \frac{Y_j}{\alpha_j} t - \frac{{Y_j}^2}{\alpha_j}\right); \quad t_i \le t \le t_{i+1} \tag{2}$$

if $\delta_j = 1$ with $t_0 = 0$ and $t_k = H$.

And

$$I_{i+1}^{j}(t) = X_j(t_{i+1} - t) + \frac{Y_j}{2}\left(t_{i+1}^2 - t^2\right); \quad t_i \le t \le t_{i+1} \tag{3}$$

if $\delta_j = 0$ with $t_0 = 0$ and $t_k = H$.

Thus,

$$Q_{i+1}^{j} = e^{\alpha_j(t_{i+1}-t_i)}\left(\frac{X_j}{\alpha_j} + \frac{Y_j}{\alpha_j}t_{i+1} - \frac{Y_j}{\alpha_j}^2\right) - \left(\frac{X_j}{\alpha_j} + \frac{Y_j}{\alpha_j}t_i - \frac{Y_j}{\alpha_j}^2\right) \quad \text{if} \quad \delta_j = 1. \tag{4}$$

And

$$Q_{i+1}^{j} = X_j(t_{i+1} - t_i) + \frac{Y_j}{2}\left(t_{i+1}^2 - t_i^2\right) \quad \text{if} \quad \delta_j = 0. \tag{5}$$

Now, the cost equation can be written as

$$\min \sum_{i=0}^{k-1}\left(\sum_{j=1}^{\tau} C_j Q_{i+1}^{j} + h\beta + A^{i+1}l(t_{i+1} - t_i) + \sum_{p\in P_1}\sum_{e\in E_1} x_e y_e \Delta p^{i+1} + \sum_{p\in P_2}\sum_{e\in E_2} x_e y_e \Delta p^{i+1} + \sum_{m=1}^{\mu} V_m\right). \tag{6}$$

Such that

$$\sum_{i=0}^{k-1}\sum_{j=1}^{\tau}\left(X_j(t_{i+1} - t_i) + \frac{Y_j}{2}\left(t_{i+1}^2 - t_i^2\right) + d_j\delta_j\int_{t_i}^{t_{i+1}} I_{i+1}^{j}(t)\,\mathrm{d}t\right) = \sum_{i=0}^{k-1}\sum_{j=1}^{\tau} I_{i+1}^{j}(t_i) \tag{7a}$$

$$\sum_{i=0}^{k-1}\left(\sum_{j=1}^{\tau} C_j I_j^{i+1}(t_i) + A^{i+1}l(t_{i+1} - t_i) + \sum_{p\in P_1}\sum_{e\in E_1} x_e y_e \Delta p^{i+1} + \sum_{p\in P_2}\sum_{e\in E_2} x_e y_e \Delta p^{i+1}\right) < \sum_{j=1}^{\tau} M_j I_j^{i+1}(t_i) \tag{7b}$$

$$0 < F_0 \le \tau \tag{7c}$$

$$\Delta_{i+1}^{p} \in \{0, 1\} \tag{7d}$$

$$\delta_j \in \{0, 1\} \tag{7e}$$

$$x_e \in \{0, 1, 2\} \tag{7f}$$

$$F_0 \in \mathbb{N} \tag{7g}$$

$$k_q = 1 \tag{7h}$$

$$k_r = 1. \tag{7i}$$

Since, constraint (7c), (7g), (7h), (7i) do not affect the objective function (Eq. (6)) and hence can be omitted.

Now, if we consider only suppliers and ϵ and if we can find a shortest route between each supplier to supplier and between each supplier to ϵ, then by considering this path as an edge and the suppliers and ϵ as vertices, we can construct a complete graph.

Similarly, if we consider only customers and ϵ and if we can find a shortest route between each customer to customer and between each customer to ϵ, then by considering this path as an edge and the customers and ϵ as vertices, we can construct a complete graph.

Now, following τ cases arise for the suppliers:

Case 1. When ϵ sent 1 vehicle to each supplier, then total number of possible routes are τ.

Case 2. When ϵ sent 1 vehicle to two suppliers, say τ_1 and τ_2 then for minimum supplying cost, the items will be supplied *via* any of the following roots:

(1) $\varepsilon \to \tau_1 \to \tau_2 \to \varepsilon$
(2) $\varepsilon \to \tau_1 \to \tau_2 \to \tau_1 \to \varepsilon$

(3) $\varepsilon \to \tau_2 \to \tau_1 \to \varepsilon$
(4) $\varepsilon \to \tau_2 \to \tau_1 \to \tau_2 \to \varepsilon$.

Thus, total number of different possible routes for Case 2 are $2\tau(\tau-1)$.

Observe that when no item deteriorate with time, in that case, travelling through route (1) is same as travelling through route (3) because in this case the travelling time does not matter. (For example: if the travelling time between ϵ and τ_1 is greater than that between ϵ and τ_2, then in case of deteriorable items with the same deterioration, and demand rate, it is better to travel *via* the route $\varepsilon \to \tau_1 \to \tau_2 \to \varepsilon$ than *via* the route $\varepsilon \to \tau_2 \to \tau_1 \to \varepsilon$ to prevent deterioration for a longer time. But in case of no deterioration, it does not matter.) Thus, total number of different possible routes for Case 2 in absence of deterioration are $2\tau(\tau-1)-1$.

Case 3. When ϵ sent 1 vehicle to three suppliers say τ_1, τ_2, and τ_3 then the items will be supplied *via* any of the following roots:

(1) $\varepsilon \to \tau_1 \to \tau_2 \to \tau_3 \to \varepsilon$
(2) $\varepsilon \to \tau_2 \to \tau_3 \to \tau_1 \to \varepsilon$
(3) $\varepsilon \to \tau_3 \to \tau_1 \to \tau_2 \to \varepsilon$
(4) $\varepsilon \to \tau_3 \to \tau_2 \to \tau_1 \to \varepsilon$
(5) $\varepsilon \to \tau_1 \to \tau_3 \to \tau_2 \to \varepsilon$
(6) $\varepsilon \to \tau_2 \to \tau_1 \to \tau_3 \to \varepsilon$
(7) $\varepsilon \to \tau_1 \to \tau_2 \to \tau_3 \to \tau_2 \to \varepsilon$
(8) $\varepsilon \to \tau_1 \to \tau_3 \to \tau_2 \to \tau_3 \to \varepsilon$
(9) $\varepsilon \to \tau_2 \to \tau_1 \to \tau_3 \to \tau_1 \to \varepsilon$
(10) $\varepsilon \to \tau_2 \to \tau_3 \to \tau_1 \to \tau_3 \to \varepsilon$
(11) $\varepsilon \to \tau_3 \to \tau_1 \to \tau_2 \to \tau_1 \to \varepsilon$
(12) $\varepsilon \to \tau_3 \to \tau_2 \to \tau_1 \to \tau_2 \to \varepsilon$
(13) $\varepsilon \to \tau_1 \to \tau_2 \to \tau_3 \to \tau_1 \to \varepsilon$
(14) $\varepsilon \to \tau_1 \to \tau_2 \to \tau_3 \to \tau_1 \to \varepsilon$
(15) $\varepsilon \to \tau_1 \to \tau_3 \to \tau_2 \to \tau_1 \to \varepsilon$
(16) $\varepsilon \to \tau_2 \to \tau_3 \to \tau_1 \to \tau_2 \to \varepsilon$
(17) $\varepsilon \to \tau_3 \to \tau_1 \to \tau_2 \to \tau_3 \to \varepsilon$
(18) $\varepsilon \to \tau_3 \to \tau_2 \to \tau_1 \to \tau_3 \to \varepsilon$
(19) $\varepsilon \to \tau_1 \to \tau_2 \to \tau_3 \to \tau_2 \to \tau_1 \to \varepsilon$
(20) $\varepsilon \to \tau_1 \to \tau_3 \to \tau_2 \to \tau_3 \to \tau_1 \to \varepsilon$
(21) $\varepsilon \to \tau_2 \to \tau_1 \to \tau_3 \to \tau_1 \to \tau_2 \to \varepsilon$
(22) $\varepsilon \to \tau_2 \to \tau_3 \to \tau_1 \to \tau_3 \to \tau_2 \to \varepsilon$
(23) $\varepsilon \to \tau_3 \to \tau_1 \to \tau_2 \to \tau_1 \to \tau_3 \to \varepsilon$
(24) $\varepsilon \to \tau_3 \to \tau_2 \to \tau_1 \to \tau_2 \to \tau_3 \to \varepsilon$.

Thus, total number of different possible shortest routes for Case 3 are $4\tau(\tau-1)(\tau-2)$.

Since, when no item deteriorate with time, travelling *via* routes (1), (2), (3) are same as travelling *via* routes (4), (5), (6), respectively.

Thus, total number of different possible routes for Case 3 in absence of deterioration are $4\tau(\tau-1)(\tau-2)-3$.

Similarly, when ϵ sent 1 vehicle to four suppliers, then total number of different possible routes are $8\tau(\tau-1)(\tau-2)(\tau-3)$ in presence of deterioration and $8\tau(\tau-1)(\tau-2)(\tau-3)-12$ when there is no deterioration.

Thus, maximum number of shortest possible routes between τ suppliers and ϵ are $\tau+2\tau(\tau-1)+4\tau(\tau-1)(\tau-2)+8\tau(\tau-1)(\tau-2)(\tau-3)+\ldots+2^{\tau-1}\tau(\tau-1)(\tau-2)\ldots 2.1$ for deteriorating items, and $\tau+2\tau(\tau-1)+4\tau(\tau-1)(\tau-2)+8\tau(\tau-1)(\tau-2)(\tau-3)+\ldots+2^{\tau-1}\tau(\tau-1)(\tau-2)\ldots 2.1-\sum_{n=1}^{\tau} S_n$ where $S_n = S_{n-1}^2+n-1$ such that $S_0=0$ for non-deteriorating items.

Similarly, maximum number of shortest possible routes between Φ customers and ϵ are $\Phi+2\Phi(\Phi-1)+4\Phi(\Phi-1)(\Phi-2)+8\Phi(\Phi-1)(\Phi-2)(\Phi-3)+\ldots+2^{\Phi-1}\Phi(\Phi-1)(\Phi-2)\ldots 2.1$ for deteriorating items, and

$\Phi + 2\Phi(\Phi-1) + 4\Phi(\Phi-1)(\Phi-2) + 8\Phi(\Phi-1)(\Phi-2)(\Phi-3) + \ldots + 2^{\Phi-1}\Phi(\Phi-1)(\Phi-2)\ldots 2.1 - \sum_{n=1}^{\Phi} S_n$ where $S_n = S_{n-1}^2 + n - 1$ such that $S_0 = 0$ for non-deteriorating items.

Since, the items may be both perishable and non-perishable, so let τ' items are non-deteriorating and $(\tau - \tau')$ items are deteriorating, thus two cases arises:

Case 1. When items are deteriorating:

For fixed k, to calculate optimum t_i, derivative of $\sum_{i=0}^{k-1}\sum_{j=1}^{\tau-\tau'} C_j I_{i+1}^j(t_i)$ with respect to t_i must be $0 \quad \forall \quad i = 0,, k-1$ with $t_0 = 0$ and $t_k = H$. (Notice that the term $A_{i+1}l(t_{i+1} - t_i)$ is not taken in calculating the optimum t_i because A_{i+1} is not known initially and will determined after finding optimum time interval . Also, in case when A_{i+1} is already known, then we can include this term in calculating optimum t_i and this will also give us optima but the only difference is that a term $l(A_{i+1} - A_i)$ will be added in Eq. (8)).

Which implies

$$\sum_{j=1}^{(\tau-\tau')} C_j\left(-e^{d_j(t_{i+1}-t_i)}\left(X_j + Y_j t_{i+1} - \frac{Y_j}{d_j}\right) + e^{d_j(t_i - t_{i-1})}(X_j + Y_j t_i) - \frac{Y_j}{d_j}\right) = Z(\text{say}) = 0 \tag{8}$$

for each $i = 1, \ldots, k-1 \quad$ with $\quad t_0 = 0, \quad t_k = H$.

Now

$$\frac{\partial Z}{\partial t_i} = \sum_{j=1}^{(\tau-\tau')} C_j\left(d_j e^{d_j(t_{i+1}-t_i)}\left(X_j + Y_j t_{i+1} - \frac{Y_j}{d_j}\right) + d_j e^{d_j(t_i - t_{i-1})}(X_j + Y_j t_i) + Y_j e^{d_j(t_i - t_{i-1})}\right). \tag{9}$$

Using equation (8), we get

$$\frac{\partial Z}{\partial t_i} = \sum_{j=1}^{(\tau-\tau')}\left(C_j\left(2d_j e^{d_j(t_i - t_{i-1})}(X_j + Y_j t_i) + Y_j\left(e^{d_j(t_i - t_{i-1})} - 1\right)\right) + (A_i - A_{i+1})ld_l\right) > 0. \tag{10}$$

Since, $\frac{\partial Z}{\partial t_i}$ for each $i = 1, \ldots, k-1$, thus Z is strictly increasing in t_i, and hence it will be zero for at-most one value of t_i for each $i = 1, \ldots, k-1$.

Furthermore,

$$\frac{\partial Z}{\partial t_{i-1}} = -\sum_{j=1}^{(\tau-\tau')}\left(C_j d_j e^{d_j(t_i - t_{i-1})}(X_j + Y_j t_i)\right) < 0. \tag{11}$$

And

$$\frac{\partial Z}{\partial t_{i+1}} = -\sum_{j=1}^{(\tau-\tau')}\left(C_j d_j e^{d_j(t_{i+1} - t_i)}(X_j + Y_j t_{i+1})\right) < 0. \tag{12}$$

Which implies that the hessian matrices are positive definite. Thus, the solution if exists will give global minima.

Case 2. When items are non-deteriorating:

For fixed k, to calculate optimum t_i, derivative of $\sum_{i=0}^{k-1}(C_j I_i^j(t_i))$ with respect to t_i must is zero be 0 for each $i = 1, \ldots, k-1$ with $t_0 = 0$ and $t_k = H$. But the derivative of $\sum_{i=0}^{k-1}(C_j I_i^j(t_i))$ in whole interval $(0, H)$. Thus, it is a constant function with respect to t_i thus no fixed optimum time interval exists for the non-deteriorating items. Hence, it is better to consider the optimum interval obtained by Case 1 as global optima when both types of items are supplied.

Theorem 1. *The sum of first three terms of the total cost is convex in k.*

Proof. Since, constant function and linear functions are both concave and convex, and if the jth item is non-deteriorating, then the term $\sum_{i=0}^{k-1}(C_jQ_{i+1}^j)$ is a quadratic function having positive coefficient of highest degree and thus it is convex, and since finite sum of convex functions is convex. Thus, it is sufficient to prove that TC convex in k for non-deteriorating items.

Thus, it is sufficient to prove that $\sum_{i=0}^{k-1}(\sum_{j=1}^{\tau-\tau'}(C_jQ_{i+1}^j))$ is convex in k.

For fixed k, the interval $(0,H)$ is divided into k parts. Now, suppose $k=k+1$, the interval $(0,H)$ is divided into $k+1$ parts *i.e.* the interval is arbitrary inserted at one point based on the former.

Now let,

$$f(k,0,H)=\sum_{i=0}^{k-1}\left(\sum_{j=1}^{\tau-\tau'}\left(C_jQ_{i+1}^j\right)\right). \tag{13}$$

Which implies

$$f(k,0,H)=\sum_{j=1}^{\tau-\tau'}\left(C_j\sum_{i=0}^{k-1}\left(\int_{t_i}^{t_{i+1}}(X_j+Y_jt)e^{d_j(t-t_i)}\,\mathrm{d}t\right)\right). \tag{14}$$

Then,

$$\begin{aligned} f(k+1,0,H)-f(k,0,H)=\sum_{j=1}^{\tau-\tau'}\Bigg(C_j\Bigg(&\int_{t_{k-1}}^{t_k}(X_j+Y_jt)e^{d_j(t-t_{k-1})}\,\mathrm{d}t\\ &+\int_{t_k}^{t_H}(X_j+Y_jt)e^{d_j(t-t_k)}\,\mathrm{d}t\\ &-\int_{t_{k-1}}^{H}(X_j+Y_jt)e^{d_j(t-t_{k-1})}\,\mathrm{d}t\Bigg)\Bigg) \end{aligned} \tag{15}$$

$$\begin{aligned} \Rightarrow f(k+1,0,H)-f(k,0,H)=\sum_{j=1}^{\tau-\tau'}C_j\Bigg(&\int_{t_{k-1}}^{t_k}(X_j+Y_jt)e^{d_j(t-t_{k-1})}\,\mathrm{d}t\\ &+\int_{t_k}^{H}(X_j+Y_jt)e^{d_j(t-t_k)}\,\mathrm{d}t\\ &-\int_{t_{k-1}}^{t_k}(X_j+Y_jt)e^{d_j(t-t_{k-1})}\,\mathrm{d}t\\ &+\int_{t_k}^{H}(X_j+Y_jt)e^{d_j(t-t_{k-1})}\,\mathrm{d}t\Bigg) \end{aligned} \tag{16}$$

$$\Rightarrow f(k+1,0,H)-f(k,0,H)=\sum_{j=1}^{\tau-\tau'}C_j\int_{t_k}^{H}(X_j+Y_jt)(e^{d_j(t-t_k)}-e^{d_j(t-t_{k-1})})\,\mathrm{d}t. \tag{17}$$

Since, e^t is an increasing function in t. Thus,

$$\implies f(k+1,0,H)-f(k,0,H)<0. \tag{18}$$

Thus, $f(k,0,H)$ is strictly decreasing in k.

Now, it is sufficient to prove that

$$2f(k,0,H) \leq f(k+1,0,H) + f(k-1,0,H). \tag{19}$$

Now, let

$$f(k,0,H) - f(k-1,0,H) - (f(k+1,0,H) - f(k,0,H)) = A. \tag{20}$$

Then,

$$\begin{aligned} A = \sum_{j=1}^{\tau-\tau'} C_j \Bigg(& \int_{t_{k-1}}^{H} (X_j + Y_j t)\Big(e^{d_j(t-t_{k-1})} - e^{d_j(t-t_{k-2})}\Big)\,\mathrm{d}t \\ & - \int_{t_k}^{H} (X_j + Y_j t)\Big(e^{d_j(t-t_k)} - e^{d_j(t-t_{k-1})}\Big)\,\mathrm{d}t \Bigg) \end{aligned} \tag{21}$$

$$\begin{aligned} \Rightarrow A = \sum_{j=1}^{\tau-\tau'} C_j \Bigg(& \int_{t_{k-1}}^{t_k} (X_j + Y_j t)\Big(e^{d_j(t-t_{k-1})} - e^{d_j(t-t_{k-2})}\Big)\,\mathrm{d}t \\ & + \int_{t_k}^{H} (X_j + Y_j t)\Big(e^{d_j(t-t_{k-1})} - e^{d_j(t-t_{k-2})}\Big)\,\mathrm{d}t \\ & - \int_{t_k}^{H} (X_j + Y_j t)\Big(e^{d_j(t-t_k)} - e^{d_j(t-t_{k-1})}\Big)\,\mathrm{d}t \Bigg) \end{aligned} \tag{22}$$

$$\begin{aligned} \Rightarrow A = \sum_{j=1}^{\tau-\tau'} C_j \Bigg(& \int_{t_{k-1}}^{t_k} (X_j + Y_j t)\Big(e^{d_j(t-t_{k-1})} - e^{d_j(t-t_{k-2})}\Big)\,\mathrm{d}t \\ & + \int_{t_k}^{H} (X_j + Y_j t)\Big(2e^{d_j(t-t_{k-1})} - e^{d_j(t-t_{k-2})} - e^{d_j(t-t_k)}\Big)\,\mathrm{d}t \Bigg). \end{aligned} \tag{23}$$

Since, e^t is a convex function. Hence, $A < 0$, which implies, $f(k,0,H)$ is convex in k.

And thus, the sum of first three terms of the total cost is convex in k.

Note: Although the last three terms of total cost are constants for each cycle but these constants may vary in each cycle thus their convexity cannot be determined for complete horizon but if transportation cost is very small as compare to the other costs than the total cost is assumed to be convex. □

Theorem 2. *Length of previous cycle is greater than the next cycle i.e.* $t_{i+1} - t_i < t_i - t_{i-1}$.

Proof. Equation (8) can be written as

$$\sum_{j=1}^{\tau-\tau'} C_j \Bigg((X_j + Y_j t_i)\Big(e^{d_j(t_i - t_{i-1})} - 1\Big) - d_j \int_{t_i}^{t_{i+1}} (X_j + Y_j t) e^{d_j(t-t_i)}\,\mathrm{d}t \Bigg) = 0 \tag{24}$$

$$\implies \sum_{j=1}^{\tau-\tau'} C_j \Bigg((X_j + Y_j t_i)\Big(e^{d_j(t_i - t_{i-1})} - 1\Big) - d_j (X_j + Y_j t_i) \int_{t_i}^{t_{i+1}} e^{d_j(t-t_i)}\,\mathrm{d}t \Bigg) > 0 \tag{25}$$

$$\implies (X_j + Y_j t_i) \sum_{j=1}^{\tau-\tau'} \Big(e^{d_j(t_i - t_{i-1})} - e^{d_j(t_{i+1} - t_i)}\Big) > 0. \tag{26}$$

Since $(X_j + Y_j t_i) > 0$, thus

$$\sum_{j=1}^{\tau-\tau'} \left(e^{d_j(t_i - t_{i-1})} - e^{d_j(t_{i+1} - t_i)} \right) > 0 \tag{27}$$

$$\implies \left(e^{d_j(t_i - t_{i-1})} - e^{d_j(t_{i+1} - t_i)} \right) > 0 \quad \forall \quad j = 1 \quad \text{to} \quad \tau - \tau'. \tag{28}$$

Because if

$$\left(e^{d_j(t_i - t_{i-1})} - e^{d_j(t_{i+1} - t_i)} \right) < 0 \tag{29}$$

for at-least one j then it will imply

$$t_i - t_{i-1} < t_{i+1} - t_i. \tag{30}$$

Implying,

$$\sum_{j=1}^{\tau-\tau'} \left(e^{d_j(t_i - t_{i-1})} - e^{d_j(t_{i+1} - t_i)} \right) < 0 \tag{31}$$

which contradicts inequality (27).

Thus,

$$\left(e^{d_j(t_i - t_{i-1})} - e^{d_j(t_{i+1} - t_i)} \right) > 0 \quad \forall \quad j = 1 \quad \text{to} \quad \tau - \tau'. \tag{32}$$

Which implies

$$t_i - t_{i-1} > t_{i+1} - t_i. \tag{33}$$

Hence the result. □

3.1. Algorithm

- Calculate optimum cycle length using equation (8).
- Calculate the initial inventory using equations (4) and (5).
- Select the route of minimum transportation cost within the interval (including the vehicle renting cost and deterioration cost of the travelling time).
- Find the optimum cost.

3.2. Example

Consider the intermediate "ϵ" is customer himself and let ϵ needs soap, flour, and edible oil. Let supplier "τ_1" sells Soap, supplier "τ_2" sells flour, and supplier "τ_3" sells edible oil. Let the shortest direct route from ϵ to τ_1, τ_2, and τ_3 is of length 20 Km, 20 Km, and 30 Km, respectively and the shortest direct route from τ_1 to τ_2, and τ_3 is of length 10 Km, and 20 Km, respectively and the shortest direct route between τ_2, and τ_3 is of length 20 Km. Let $H = 10$ Days. Let X_1, X_2, X_3 be 700 Kg/10 days, 1500 Kg/10 days, and 700 Kg/10 days, respectively and Y_1, Y_2, Y_3 be 70 Kg/10 days, 150 Kg/10 days, and 70 Kg/10 days, respectively. Let d_1, d_2, d_3 be 0/day, 0.1/day, and 0.2/day, respectively. Let the price of renting vehicle (say V_1') of capacity 1000–2000 Kg is 10 \$/h, the price of renting vehicle (say V_2') of capacity 2000–4000 Kg is 20 \$/h, the price of renting vehicle (say V_3') of capacity 4000–8000 Kg is 40 \$/h, the price of renting vehicle (say V_4') of capacity 8000–16 000 Kg is 80 \$/h, the price of renting vehicle (say V_5') of capacity 16 000–32 000 Kg is 160 \$/h, the price of renting vehicle (say V_6') of capacity 32 000–64 000 Kg is 320 \$/h, the price of renting vehicle (say V_7') of capacity 64 000–100 000 Kg is

640 \$/h. Let the average speed of each vehicle be 100 Km/h. Let the cost of fuel for V_1', V_2', V_3', V_4', V_5', V_6', and V_7' be 1 \$ per Km, 2 \$ per Km, 3 \$ per Km, 4 \$ per Km, 5 \$ per Km, 6 \$ per Km, 7 \$ per Km, 8 \$ per Km respectively. Let $C_1 = 1$ \$ per item, $C_2 = 1$ \$ per item, $C_3 = 1.5$ \$ per item, h = 1 \$/day, $\mathbf{l} = 0.1$/time. Let X borrow money as much as is required to buy inventory. Find an optimum number of ordering cycles. Also, find the optimum route in each cycle.

Answer:

From Table 2, it is clear that optimum number of ordering cycles = 4.

Optimum routes $k = 1$ are: $\varepsilon \to \tau_1 \to \varepsilon$ and $\varepsilon \to \tau_2 \to \tau_3 \to \tau_2 \to \varepsilon$.

Optimum routes for $k = 2$ are: $\varepsilon \to \tau_1 \to \varepsilon$ and $\varepsilon \to \tau_2 \to \tau_3 \to \tau_2 \to \varepsilon$ in both the cycles.

Optimum routes for $k = 3$ are: $\varepsilon \to \tau_2 \to \tau_1 \to \tau_3 \to \tau_2 \to \varepsilon$.

Optimum routes for $k = 4$ are: $\varepsilon \to \tau_1 \to \varepsilon$, $\varepsilon \to \tau_2 \to \tau_3 \to \tau_2 \to \varepsilon$, for first two cycles, and $\varepsilon \to \tau_2 \to \tau_1 \to \tau_3 \to \tau_2 \to \varepsilon$ in last two cycles.

Optimum route for $k = 5$ is: $\varepsilon \to \tau_2 \to \tau_1 \to \tau_3 \to \tau_2 \to \varepsilon$ in all cycles.

Since, flour is deteriorating item, thus in route $\varepsilon \to \tau_2 \to \tau_1 \to \tau_3 \to \tau_2 \to \varepsilon$, it is profitable for the intermediate to collect inventory from τ_2 while coming instead of leaving.

From equation (6), it is evident that the total cost is directly proportional to the rental cost of the vehicle and the purchasing cost. Additionally, the total cost is convex in K (Fig. 1). Furthermore, although the demand for soap and edible oil is the same, the minimum quantity of soap ordering is lower than that of edible oil, as shown in Table 2, due to deterioration. Furthermore, while the demand for flour is greater than that of edible oil, the minimum ordering quantity of edible oil for optimal k ($k = 4$) is greater than that of flour, as shown in Table 2, because the rate of deterioration of edible oil is higher. Thus, we can conclude that the total cost is highly sensitive to the rate of deterioration. Figure 2 represents a pictorial representation of the total number of possible RDPs in the case of three suppliers and an intermediate.

3.3. Managerial insights

This article presents a two-echelon inventory and transportation model for effective supply chain decision-making. In this model, each supplier provides exactly one type of product, while customers may demand multiple types, making it essential to synchronize inventory inflows and outflows to prevent stockouts or overstocking. This synchronization also supports effective inventory tracking and order fulfilment systems. The inclusion of both deteriorating and non-deteriorating items, each with distinct deterioration rates, further highlights the need for product-specific inventory handling strategies. Additionally, the model considers flexible transportation by allowing the rental of vehicles, such that each supplier and customer is visited exactly once per cycle, encouraging batch processing and consolidated shipments. The use of the shortest routes between all nodes in the network enables managers to reduce fuel, time, and labour costs effectively. To support smooth operations, the model also incorporates cycle-wise financial planning, which helps ensure adequate cash flow. The restriction that items can only be sold at or below the maximum retail price assists managers in navigating pricing and regulatory constraints while maintaining profit margins. Overall, instead of focusing solely on individual cost components,

TABLE 2. Minimum total cost and total ordering quantity for different cycles.

k	t_0	t_1	t_2	t_3	t_4	t_5	$\sum_{i=0}^{k-1} Q_{i+1}^1$	$\sum_{i=0}^{k-1} Q_{i+1}^2$	$\sum_{i=0}^{k-1} Q_{i+1}^3$	TC*
1	0	10	–	–	–	–	10 500	40 774.23	40 542.51	225 316
2	0	5.607	10	–	–	–	10 500	29 598.3	25 571.3	118 964
3	0	4.177	7.399	10	–	–	10 500	26 923.61	25 703.14	103 082
4	0	3.479	6.142	8.264	10	–	10 500	25 756.6	28 271.61	99 572
5	0	3.073	5.413	7.264	8.764	10	10 500	25 115.54	30 654.65	100 368

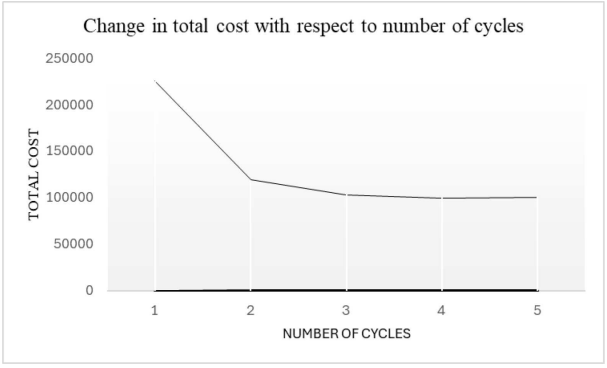

FIGURE 1. The total cost is convex in k.

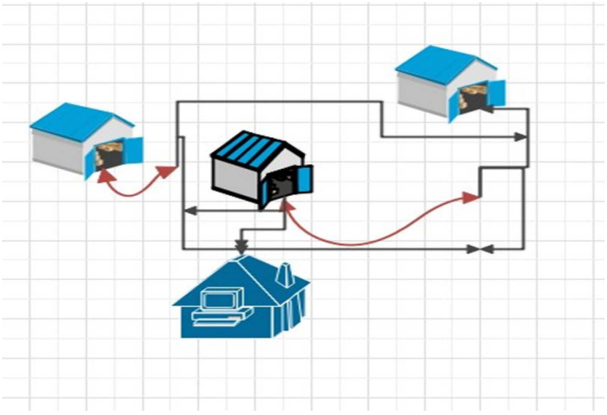

FIGURE 2. Pictorial representation of suppliers and intermediate.

the model guides decision-makers to evaluate trade-offs across the entire supply chain, promoting more efficient and financially sustainable operations.

4. Conclusion

In conclusion, our exploration of the two-echelon supply chain management system for both deteriorating and non-deteriorating items, under credit financing within a finite planning horizon, yields valuable insights for effective managerial decision-making. We observed that the total cost exhibits convexity in k. Furthermore, we found that the total cost is directly proportional to the rate of deterioration. Thus, allocating resources to mitigate deterioration risks may sometime lead to substantial cost savings. It is also noticed that the number shortest route delivery patterns depend upon the number of suppliers and retailers. Also, different possibilities of choosing shortest route delivery patterns gives flexibility. Leveraging this flexibility can lead to improved operational efficiency, as companies can explore alternative routes based on factors such as delivery schedules, traffic conditions, and cost considerations. Furthermore, the possiblity of choosing number of shortest route delivery patterns for non-deteriorating items are excessively lower than that of deteriorating items. In other words, deteriorating items introduce additional complexities in transportation logistics, potentially affecting route planning and overall supply chain efficiency. And for non deteriorating items, choosing shortest route delivery pattern is quite easier. Thus, the contrast in the number of possible shortest route patterns for deteri-

orating and non-deteriorating items implies the importance of tailoring logistics and transportation strategies based on item characteristics.

The main limitation of this article is that it assumes deterministic demand, whereas in real-world scenarios, demand can be stochastic. Future research could extend this work by considering stochastic demand.

Data availability statement

No new data/codes were created or analyzed in this study.

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