RAIRO-Oper. Res. 59 (2025) 2993–3014
https://doi.org/10.1051/ro/2025096

RAIRO Operations Research
www.rairo-ro.org

JOINT CHANCE-CONSTRAINED SURVIVABLE CAPACITATED MULTI-COMMODITY NETWORK DESIGN PROBLEM WITH DEMAND UNCERTAINTY

Salman Khodayifar* and Mohammadreza Farjaie

Abstract. In this paper, we present a cost-capacity efficient model for the survivable capacitated network design problem with demand uncertainty. To deal with uncertainty, various methods were presented in the optimization theory. We focus on the method of joint chance-constrained programming and formulate the corresponding deterministic model taking into account the dependence between the random variables. In this case, we use the copula theory, and the dependency is driven by an appropriate Archimedean copula. We propose a customized approximation optimization approach and use piecewise tangent approximation (PTA) and piecewise linear approximation (PLA) methods to obtain its lower and upper bounds, respectively. Finally, computational results show the efficiency of the proposed model and approximation methods for relatively large-scale networks.

Mathematics Subject Classification. 90B06, 93B70.

Received November 13, 2023. Accepted July 6, 2025.

1. Introduction

The Survivable Capacitated Networks Design Problem (SCNDP for short) is one of the most essential issues in telecommunication and transportation networks. A survivable network is a network that is designed to remain operational in the event of a component(s) failure(s) (cable cuts, electronic failures on switching centers, and so forth). In other words, it can still satisfy some of the network requirements after a single failure, that is a single link or node failure. To emphasize the importance of this issue in telecommunication and transportation networks, respectively, it is enough to recall that data loss incurred due to a line break could be catastrophic to network users, and also the destruction of some important highways, such as bridges, can cause disconnectedness of the communication network between cities. The objective is to select a suitable capacity of arcs in such a way that network requirements can be met with the total minimum cost. In a capacitated network design problem, capacity may be selected from a specific range, or only finitely many choices may be available; these cases are referred to as continuous capacities and discrete capacities.

In SCNDP, researchers usually consider survivability in two different ways: diversification and reservation. Diversification consists of dividing the demand of source-sink node pairs into two or more node-disjoint (or arc-disjoint) paths, and in reservation after the failure of a component(s) of the network, part of the demand

Keywords. Network design problem, survivability, joint chance-constrained programming, copula theory.

Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran.
*Corresponding author: s.khodayifar@iasbs.ac.ir

for node pairs can still be satisfied by rerouting. To implement a reservation, the links on the network ought to have enough capacity to support the new flows after the rerouting has been made. For more studies about the SCNDP as a survey, see [2, 22, 33].

For the first time, Minoux [29] considered survivability only with the reservation method in a general multi-commodity flow model with continuous capacities. Dahl and Stoer [11] applied discrete capacity structure and survivability issues (diversification and reservation) to a telecommunication model in Norway. Then, Rios *et al.* [31] studied the survivable capacitated network design problem with discrete capacity levels and a quadratics cost function.

While in some problems, only the failure of an arc (a node) is considered, the state of general failure scenarios allows us to consider the arcs (or nodes) that can fail simultaneously. In this regard, Manish *et al.* [27] in the field of flow network optimization proposed models and algorithms for designing survivable multi-commodity flow networks with general failure scenarios. Recently, Salehi and Anisi [32] also considered the design of a survivable multi-commodity network with general failure scenarios in which each failure scenario can include multiple arcs, so that if the arc fails, that arc loses only part of its capacity or is wholly failed. They set a limit on the total input flow to each node and considered the capacity for each node.

In many real-world applications, the observed values are often uncertain, for example random, fuzzy, etc. In summary, three different types of approaches can be considered to model with uncertainty data in real-world problems including, fuzzy techniques, stochastic optimization, and robust optimization-based techniques. Regarding the stochastic variants, there are notably fewer results published about the stochastic SCNDP so far. To the best of our knowledge, the only results about the stochastic SCNDP with diversification/reservation requirements are contained in Terblanche *et al.* [35], Koster *et al.* [23], and Ljubic *et al.* [25]. Terblanche *et al.* [35] stated the survivable network design problem taking into account a set of non-simultaneous demand vectors. It is worth noting that, in this work, each vector in the set of non-simultaneous demand vectors is a realization of a random event with the probability of one and so the way of dealing with demand uncertainty within this work is not in the stochastic programming framework. Koster *et al.* [23] considered the network design problem under demand uncertainty and stated a combined model incorporating both survivability and robustness. Ljubic *et al.* [25] considered two-stage stochastic versions of survivable network design problems with edge-connectivity requirements and solved the proposed models by a two-stage branch and cut algorithm based on L-shaped optimality cuts. Table 1 reviews some survivable network design problems with different aspects.

Contributions of this paper. Given the importance and wide application of the SCNDP in the case of data uncertainty, the main contributions of this paper are as follows:

(1) This paper develops a cost-capacity efficient model for the SCNDP with different capacity levels and a piecewise linear convex objective function based on the reservation strategy.
(2) We suppose that the demand vector can be reported as the standard normal distributed random variables, in the case of dependent, and to deal with uncertainty we focus on the method of joint chance-constrained programming. It is assumed that the dependence is driven by an Archimedean copula.
(3) This paper develops a mixed integer non-linear programming (MINLP) model and finds the lower and upper bounds by using the piecewise tangent approximation (PTA) and piecewise linear approximation (PLA) methods, respectively.

The concept of chance-constrained programming was first introduced by Charnes *et al.* [6]. For an introduction to stochastic programming (especially chance-constrained programming) see, *e.g.*, Birge *et al.* [5]. Van de Panne and Popp [36] considered the problem of general linear individual chance constraints with a normal distribution, and at the same time, Kataoka [18] explored the same problem with a random right-hand side. Miller and Wagner [28] generalized the problem of general linear individual chance constraints to a problem with general linear joint chance constraints, and considered the members of the right-hand side vector as independent random variables. Then, Jagannathan [17] generalized the result to the dependent case and also applied the assumption of uncertainty to the coefficient matrix. Recently, Cheng *et al.* [7] have expressed the problem of joint chance

TABLE 1. Review of some existing survivable network design models.

Year	Reference	Model features	Type of mathematical programming to deal with the uncertainty	Component(s) of survivability	Solution method
1981	Minoux [29]	Certain	–	Reservation	Simplex method
1997	Alevras *et al.* [1]	Certain	–	Diversification and reservation	Cutting-plane method
1998	Alevras *et al.* [3]	Certain	–	Diversification and reservation	Cutting-plane method
1998	Dahl and Stoer [11]	Certain	–	Diversification and reservation	Cutting-plane method
2000	Rios *et al.* [31]	Certain	–	Diversification and reservation	Lagrangian relaxation method
2007	Cole *et al.* [10]	Certain	–	Reservation	Cutting-plane method
2008	Manish and Cole [27]	Certain	–	Reservation	Heuristic algorithm
2011	Koster and Kutschka [23]	Uncertain	Robust optimization	Diversification	Branch-and-cut method
2011	Terblanche *et al.* [35]	Uncertain	Robust optimization	Diversification	Branch-and-cut method
2016	Ljubic *et al.* [25]	Uncertain	Stochastic optimization	Arc-connectivity	Branch-and-cut method
2019	Salehi and Anisi [32]	Certain	–	Diversification and reservation	Benders decomposition approach
	Current-study	Uncertain	Joint chance-constrained programming based on copula theory	Reservation	Approximation algorithm

constraint with dependent rows and a dependent random right-hand side vector that has an elliptical distribution for the general uncertainty linear optimization problem. They demonstrated the convexity of the feasible solution set for both problems and, utilizing copula theory, derived deterministic models corresponding to each. Recently, Hosseini *et al.* [14], and Khanjani *et al.* [20], considered the stochastic constrained shortest path problem and the stochastic geometric programming based on chance-constrained programming, respectively.

Also, the application of the chance-constrained programming is not limited to optimization and has been used in other sciences such as management. Recently, Khanjani *et al.* [19] and Xiao *et al.* [37] applied chance-constrained programming in data envelopment analysis and portfolio efficiency estimation, respectively.

The rest of the paper is organized as follows. Section 2, contains some of the required definitions of statistics and copula theory in the body of the paper. In Section 3, we develop a cost-capacity efficient model for the SCNDP with different capacity levels and a piecewise linear convex objective function based on the reservation strategy. In Section 4, first, SCNDP with joint chance constraints and considering the dependence between random variables is presented, and then the corresponding deterministic model is obtained with copula theory. In Section 5, we propose an iterative algorithm based on PLA and PTA methods to solve the proposed deterministic model. In Section 6, we present the computational results for proposed approximation methods with several instances. Finally, the paper ends in Section 7 with some conclusions and some possible future works.

2. PRELIMINARIES AND DEFINITIONS

Copula theory was first used in mathematics in 1959 by Sklar [34]. Generally, a copula is a function that joins a cumulative multivariate distribution function to its univariate marginal distribution functions. Recently, copulas have been used in the various fields of applied mathematics such as quantitative finance to model and minimize risk [24,26], network optimization [7,9,14,21], and geometric programming [20]. Nevertheless, an understanding of copulas is important in mathematical optimization such as joint chance-constrained programming. In this section, we will state some of the concepts about copula theory taken from [30]. The notion of copula in probability theory is presented to show the dependence of random variables.

First, let us recall the definition of the one-dimensional quantile function.

Definition 2.1. Consider the distribution function $F : \mathbb{R} \to [0,1]$, its quantile function is defined as

$$F^{-1}(p) := \inf \{x \in \mathbb{R} | F(x) \geq p\}.$$

Definition 2.2. An m-dimensional copula (m-copula) is a distribution function $C : [0,1]^m \to [0,1]$,

$$C(u_1, u_2, \ldots, u_m) = \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2, \ldots, U_m \leq u_m),$$

where the random variables U_1, U_2,..., U_m are uniformly distributed on $[0,1]$.

Theorem 2.3 (Sklar's Theorem 1959 [34])**.** *For any m-dimensional distribution function $F : \mathbb{R}^m \to [0,1]$ with marginals $F_1, F_2, \ldots, F_m$, there exists a copula C such that*

$$F(z_1, z_2, \ldots, z_m) = C(F_1(z_1), \ldots, F_m(z_m)), \qquad \forall (z_1, z_2, \ldots, z_m) \in \mathbb{R}^m.$$

If, moreover, F_i is continuous for all $i = 1, 2, \ldots, m$, then C is uniquely given by

$$C(u) = F\left(F_1^{(-1)}(u_1), \ldots, F_m^{(-1)}(u_m)\right),$$

otherwise, C is uniquely determined only on $Ran(F_1) \times Ran(F_2) \times \ldots \times Ran(F_m)$, where $Ran(F_i)$ denotes the range of the cumulative distribution function F_i.

In this paper, we utilize two classes of copulas, which are:

(1) Independent (or product copula), defined as

$$C_{\Pi}(u_1, u_2, \ldots, u_m) = \prod_{i=1}^{m} u_i.$$

(2) Gumbel-Hougaard copula with dependency parameter $\theta \geq 1$, defined as

$$C_{\theta}(u_1, u_2, \ldots, u_m) = \exp\left\{-\left[\sum_{i=1}^{m} (-\ln u_i)^{\theta}\right]^{1/\theta}\right\}.$$

By using Sklar's theorem, it can be proved that random variables are independent if and only if their copula is the independence copula. The family of Gumbel's copulas was first proposed by Gumbel [13], hence they are called Gumbel's family. This family also appears in Hougaard [15], Hutchinson and Lai [16] refer to it as the Gumbel-Hougaard family. It is also clear that, if the dependence parameter θ set to 1 ($\theta = 1$) in the Gumbel-Hougaard copula, the independent copula is obtained.

Definition 2.4. A copula C is called Archimedean if there exists a continuous strictly decreasing function $\psi : [0,1] \to [0, +\infty]$ so that $\psi(1) = 0$ and

$$C(u) = \psi^{(-1)}\left(\sum_{i=1}^{m} \psi(u_i)\right),$$

where, ψ is called generator of C. If $\lim_{t \to 0} \psi(t) = +\infty$, then C is called a strict Archimedean copula and ψ is called a strict generator.

Archimedean copulas are popular due to their ease of construction and the diverse range of families within this classification. The most commonly cited families of Archimedean copulas are listed in Table 2.

TABLE 2. Some Archimedean copulas.

Copula family	Dependency parameter (θ)	Copula generator ($\psi(t)$)	Inverse of generator ($\varphi^{-1}(t)$)
Independent (product)	–	$-\ln t$	e^{-t}
Gumbel-Hougaard	$\theta \geq 1$	$(-\ln t)^{\theta}$	$e^{-t^{1/\theta}}$
Frank	$\theta > 0$	$-\ln\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$	$-\theta^{-1}\ln\left(1+e^{-t}\left(e^{-\theta}-1\right)\right)$
Clayton	$\theta > 0$	$\frac{1}{\theta}\left(t^{-\theta}-1\right)$	$(1+t)^{-1/\theta}$
Joe	$\theta \geq 1$	$-\ln\left[1-(1-t)^{\theta}\right]$	$1-\left(1-e^{-t}\right)^{1/\theta}$

TABLE 3. Definition of the sets.

Sets	Definitions
N	Set of nodes
A	Set of arcs
K	Set of commodities
S	Set of all scenarios
A_s	Set of all operating arcs in A in scenario s
M_k	Set of intermediate nodes corresponding to the commodity k
$R = \{1, 2, ..., r_{\max}\}$	Set of available capacity levels for installation on arcs

3. SURVIVABLE CAPACITATED NETWORK DESIGN PROBLEM (SCNDP)

Let $G = (N, A)$ be a given supply network where N is the set of nodes, and A is the set of arcs in the network. Furthermore, let K be a set of commodities, so that, for each commodity $k \in K$, $O(k)$ is the source node of commodity k and $D(k)$ is the sink node of commodity k and M_k is the set of intermediate nodes corresponding to the commodity k. Here, we define two important parameters corresponding to the commodity $k \in K$ as follows:

d_k: Demand of commodity k.
ρ_k: Reservation parameter, *i.e.*, the minimum amount of commodity demand k that must be satisfied after the failure of a single arc.

The reservation parameter implies that at least $\rho_k d_k$ units of flow must be carried out after a single arc failure. We define S as a set of all scenarios, and each scenario represents an operational state of the network. Scenario $s = 0$ corresponds to a state in which all arcs are operational, and other scenarios represent a state in which only one arc fails. For each scenario s, $G_s = (N, A_s)$ represents the operating network corresponding to that scenario, and A_s is the set of all operating arcs in A. Additionally, the notations used are explained in Tables 3–5.

It is important to note that the variable cost is a convex function, which we will approximate using a piecewise linear convex function. To simplify our notation, we suppose that each piecewise linear convex function contains exactly $r_{\max}$ linear pieces. Let U_{ij}^{r-1} and U_{ij}^{r} denote the lower and upper bounds for arc (i, j), and for each piece $r \in R$. Also, we suppose that c_{ij}^{r} show the variable cost corresponding to capacity level r on the arc (i, j). To guarantee convexity, it is assumed that $c_{ij}^{1} < c_{ij}^{2} < \ldots < c_{ij}^{r_{\max}}$. In other words, the variable cost increases from one capacity level to the next.

So, according to the definitions of Tables 4 and 5, we can write

$$x_{ij}^{ks} = \sum_{r \in R} x_{ij}^{ksr}, c_{ij}(x_{ij}^{ks}) = \sum_{r \in R} c_{ij}^{r} x_{ij}^{ksr}.$$

TABLE 4. Definition of the parameters.

Parameters	Definitions
U_{ij}^{r}	Upper bound of capacity level r on the arc (i,j)
U_{ij}^{r-1}	Lower bound of capacity level r on the arc (i,j), such that, $U_{ij}^{r} > U_{ij}^{r-1}$, $\forall r \in R$
e_{ij}^{kr}	Weight of the commodity flow k corresponding to the capacity level r on the arc (i,j)
f_{ij}^{r}	Fixed cost for installing capacity level r on the arc (i,j)
c_{ij}^{r}	Variable cost corresponding to capacity level r on the arc (i,j)
c_{ij}	Convex cost function of the total flow on the arc (i,j)
$\rho_k^s = \begin{cases} 1, & \text{if } s = 0 \\ \rho_k, & \text{if } s \neq 0 \end{cases}$	Reservation parameter corresponding scenario s

TABLE 5. Definition of decision variables.

Decision variables	Definitions
y_{ij}^{r}	If the capacity level r is installed on the arc (i,j), it is equal to one, otherwise it is equal to zero
x_{ij}^{ksr}	Flow of commodity k in the operating state s on the arc (i,j) for capacity level r
x_{ij}^{ks}	Total flow of commodity k in the operating state s on the arc (i,j)

Additionally, according to the convex cost function $c_{ij}\left(x_{ij}^{ks}\right)$, the segment flows x_{ij}^{ksr} can be obtained according to the total arc flow x_{ij}^{ks} using the following relation:

$$x_{ij}^{ksr} = \begin{cases} 0, & \text{if } x_{ij}^{ks} \leq U_{ij}^{r-1}, \\ x_{ij}^{ks} - U_{ij}^{r-1}, & \text{if } U_{ij}^{r-1} \leq x_{ij}^{ks} \leq U_{ij}^{r}, \\ U_{ij}^{r} - U_{ij}^{r-1}, & \text{if } x_{ij}^{ks} \geq U_{ij}^{r}. \end{cases}$$

Based on the observations and definitions mentioned above, the proposed model of the SCNDP can be expressed as follows:

$$\min \sum_{(i,j)\in A} \sum_{r\in R} f_{ij}^{r} y_{ij}^{r} + \sum_{(i,j)\in A} \sum_{r\in R} \sum_{s\in S} \sum_{k\in K} c_{ij}^{r} x_{ij}^{ksr} \tag{3.1}$$

s.t.

$$\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks} = 0, \qquad \forall i \in M_k, \ \ \forall k \in K, \ \ \forall s \in S, \tag{3.2}$$

$$\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks} \geq \rho_k^s d_k, \qquad i = O(k), \ \ \forall k \in K, \ \ \forall s \in S, \tag{3.3}$$

$$x_{ij}^{ks} = \sum_{r\in R} x_{ij}^{ksr}, \qquad \forall (i,j) \in A_s, \ \ \forall s \in S, \ \ \forall k \in K, \tag{3.4}$$

$$1 = y_{ij}^{0} \geq y_{ij}^{1} \geq \ldots \geq y_{ij}^{|R|}, \qquad \forall (i,j) \in A, \tag{3.5}$$

$$\sum_{k\in K} e_{ij}^{kr} x_{ij}^{ksr} \leq \left(U_{ij}^{r} - U_{ij}^{r-1}\right) y_{ij}^{r}, \qquad \forall (i,j) \in A_s, \ \ \forall s \in S, \ \ \forall r \in R, \tag{3.6}$$

$$x_{ij}^{ksr} \geq 0, \qquad \forall\,(i,j) \in A, \quad \forall k \in K, \quad \forall s \in S, \quad \forall r \in R, \tag{3.7}$$

$$y_{ij}^{r} \in \{0,1\}, \qquad \forall\,(i,j) \in A, \quad \forall r \in R. \tag{3.8}$$

Equation (3.1) shows the objective function of the problem, which is to minimize fixed and variable costs. Constraints set (3.2) and (3.3) show the flow protection constraints or, in other words, the routing constraints for each commodity $k \in K$. Constraints set (3.4) calculates the total flow of commodity k on each operating arc per scenario is equal to the sum of the flows of commodity k on the capacity levels installed on that arc. The set of constraints (3.5) implies that if the capacity level r is installed on an arc, the capacity level $r-1$ must also be installed on that arc. The constraints set in (3.6), referred to as weighted boundary constraints. Constraints (3.7) and (3.8) are the integrality and continuous restrictions on the decision variables.

4. Stochastic SCNDP with joint chance-constrained programming

In the real-world applications of the SCNDP, the demand of customers is usually uncertain. In this case, the constraints set (3.3) are as follows

$$\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks} \geq \rho_k^s \widetilde{d}_k, \qquad i = O(k), \quad \forall k \in K, \quad \forall s \in S. \tag{4.1}$$

In this section, to deal with uncertainty, we focus on the method of joint chance-constrained programming and formulate the corresponding deterministic model taking into account the dependence between the random variables. Constraints in chance-constrained programming are jointly satisfied with a probability confidence level $0 < p \leq 1$. By entering demand uncertainty into the deterministic SCNDP model, we can reformulate the constraints set (4.1) into the following joint chance constraint with a confidence parameter p

$$\mathbb{P}\left(\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks} \geq \widetilde{d}_k\right], \qquad i = O(k), \quad \forall k \in K, \quad \forall s \in S\right) \geq p. \tag{4.2}$$

To avoid the repetition of random variables $\widetilde{d}_k$ from the random vector $\widetilde{d}$ and to use Sklar's Theorem, the above constraint can be written as follows

$$\mathbb{P}\left(\min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right], i = O(k)\right\} \geq \widetilde{d}_k, \qquad \forall k \in K\right) \geq p. \tag{4.3}$$

Then, the SCNDP with the joint chance-constrained programming can be rewritten as

$$\min \sum_{(i,j)\in A}\sum_{r\in R} f_{ij}^r y_{ij}^r + \sum_{(i,j)\in A}\sum_{r\in R}\sum_{s\in S}\sum_{k\in K} c_{ij}^r x_{ij}^{ksr}$$

s.t.

(3.2), (3.4)–(3.8) and (4.3).

Now suppose that the parameter $\widetilde{d}_k$ has a standard normal distribution with continuous distribution function Φ. We know that the random vector $\widetilde{d} = (\widetilde{d}_1, \widetilde{d}_2, \ldots, \widetilde{d}_{|K|})$ has normal cumulative distribution function, referred to as F. In the following, we assume that the elements of the random vector $\widetilde{d}$ are dependent. In this case, the deterministic model for the problem of SCNDP with a joint chance constraint can be derived from the following theorem.

Theorem 4.1. *Suppose that the parameter $\widetilde{d}_k$ has a standard normal distribution with marginal continuous distribution function Φ and the elements of $\widetilde{d}$ are dependent. Also, suppose that the vector $\widetilde{d}$ has a cumulative*

distribution function F induced by a copula C. If C is Archimedean copula with generator ψ, then joint chance constraint (4.3) is equivalent to the following constraints

$$\begin{aligned}
&q_k \geq \Phi^{-1}(\psi^{-1}(\psi(p)z_k)), && \forall k \in K,\\
&q_k \leq \frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right], && i = O(k), \forall k \in K, \forall s \in S,\\
&\sum_{k\in K} z_k = 1,\\
&z_k \geq 0,\ q_k \geq 0, && \forall k \in K.
\end{aligned}$$

Proof. Assuming the theorem and the definition of the cumulative distribution function F, the constraint (4.3) can be rewritten as follows

$$F\left(\left(\min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right],\ i = O(k)\right\}\right), \forall k \in K\right) \geq p.$$

Now, according to the Sklar's Theorem, we have

$$C\left(\Phi\left(\min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right],\ i = O(k)\right\}\right), \forall k \in K\right) \geq p.$$

Because C is Archimedean, we have

$$\psi^{-1}\left(\sum_{k\in K}\psi\left(\Phi\left(\min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right],\ i = O(k)\right\}\right)\right)\right) \geq p,$$

or equivalently,

$$\sum_{k\in K}\psi\left(\Phi\left(\min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right],\ i = O(k)\right\}\right)\right) \leq \psi(p). \tag{4.4}$$

By definition of auxiliary non-negative vector $z = (z_1, z_2, \ldots, z_{|K|})$ with $\sum_{k\in K} z_k = 1$, the inequality (4.4) is equivalent to

$$\begin{aligned}
&\sum_{k\in K}\psi\left(\Phi\left(\min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right],\ i = O(k)\right\}\right)\right) \leq \psi(p)\sum_{k\in K} z_k,\\
&\sum_{k\in K} z_k = 1,\ z_k \geq 0, \qquad \forall k \in K.
\end{aligned} \tag{4.5}$$

In the following, we prove that the constraints (4.5) are equivalent to

$$\begin{aligned}
&\psi\left(\Phi\left(\min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right],\ i = O(k)\right\}\right)\right) \leq \psi(p)z_k, && \forall k \in K,\\
&\sum_{k\in K} z_k = 1,\ z_k \geq 0, && \forall k \in K.
\end{aligned} \tag{4.6}$$

It is easy to see that the constraints (4.6) imply the constraints (4.5). To demonstrate the converse, we define the following variable

$$\widetilde{z}_k = \frac{\psi\left(\Phi\left(\min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right], i = O(k)\right\}\right)\right)}{\psi(p)}, \qquad \forall k \in K.$$

Then, we have

$$\sum_{k\in K} \widetilde{z}_k \leq 1, \text{and} \sum_{k\in K} \psi\left(\Phi\left(\min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right],\ i = O(k)\right\}\right)\right) \leq \psi(p). \tag{4.7}$$

Now, suppose that $z_k = \frac{\widetilde{z}_k}{\sum_{k\in K}\widetilde{z}_k},\ \forall k \in K$. Inequality (4.4) implies that $\widetilde{z}_k \leq z_k, \forall k \in K$.

Then, we have

$$\psi\left(\Phi\left(\min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right],\ i = O(k)\right\}\right)\right) \leq \psi(p) z_k, \qquad \forall k \in K.$$

Since $\psi(.)$ is non-negative, therefore the variables $\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{|K|}$ are non-negative and implies that the variables $z_1, z_2, \ldots, z_{|K|}$ be also non-negative. Finally, it is easy to see that $\sum_{k\in K} z_k = 1$. Since, the function ψ is decreasing and the function Φ is increasing, we have:

$$\min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right],\quad i = O(k)\right\} \geq \Phi^{-1}\left(\psi^{-1}\left(\psi(p) z_k\right)\right), \qquad \forall k \in K,$$

$$\sum_{k\in K} z_k = 1,\ z_k \geq 0, \qquad \forall k \in K.$$

Now, by defining

$$q_k = \min_{s\in S}\left\{\frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right],\ i = O(k)\right\},$$

we have

$$\begin{aligned}
&q_k \geq \Phi^{-1}(\psi^{-1}(\psi(p) z_k)), && \forall k \in K,\\
&q_k \leq \frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right], && i = O(k),\ \ \forall k \in K,\ \ \forall s \in S,\\
&\sum_{k\in K} z_k = 1,\\
&z_k \geq 0,\ q_k \geq 0, && \forall k \in K.
\end{aligned}$$

□

Now, by using of Gumbel-Hougaard generator $\psi(t) = (-\ln t)^\theta; \theta \geq 1$ and inverse of the generator $\varphi^{-1}(t) = e^{-t^{1/\theta}}$, we have

$$\Phi^{-1}\left(\psi^{-1}(\psi(p) z_k)\right) = \Phi^{-1}\left(p^{{z_k}^{1/\theta}}\right).$$

Therefore, the deterministic model corresponding to the model of stochastic SCNDP with joint chance constraint can be driven by the Gumbell-Hougaard copula with dependency parameter $\theta \geq 1$ and confidence level p, as follows

$$\min \sum_{(i,j)\in A}\sum_{r\in R} f_{ij}^r y_{ij}^r + \sum_{(i,j)\in A}\sum_{r\in R}\sum_{k\in K}\sum_{s\in S} c_{ij}^r x_{ij}^{ksr} \tag{4.8}$$

s.t.

$$\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks} = 0, \qquad \forall i \in M_k, \quad \forall k \in K, \quad \forall s \in S,$$

$$q_k \geq \Phi^{-1}\left(p^{{z_k}^{1/\theta}}\right), \qquad \forall k \in K,$$

$$q_k \leq \frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right], \qquad i = O(k), \quad \forall k \in K, \quad \forall s \in S,$$

$$x_{ij}^{ks} = \sum_{r\in R} x_{ij}^{ksr}, \qquad \forall (i,j) \in A_s, \quad \forall k \in K, \quad \forall s \in S,$$

$$1 = y_{ij}^0 \geq y_{ij}^1 \geq \ldots \geq y_{ij}^{|R|}, \qquad \forall (i,j) \in A,$$

$$\sum_{k\in K} e_{ij}^{kr} x_{ij}^{ksr} \leq \left(U_{ij}^r - U_{ij}^{r-1}\right) y_{ij}^r, \qquad \forall (i,j) \in A_s, \quad \forall r \in R, \quad \forall s \in S,$$

$$\sum_{k\in K} z_k = 1,$$

$$x_{ij}^{ksr} \geq 0, \qquad \forall (i,j) \in A, \quad \forall k \in K, \quad \forall s \in S, \quad \forall r \in R,$$

$$z_k \geq 0, \; q_k \geq 0, \qquad \forall k \in K,$$

$$y_{ij}^r \in \{0,1\}, \qquad \forall (i,j) \in A, \quad \forall r \in R.$$

This problem is an MINLP model which is often known to be NP-hard. So, finding an optimal solution for this problem in polynomial time is challenging. To solve this problem, we use the iterative approximate method based on piecewise lines and piecewise tangent lines (using one-order Taylor series), which we will describe in the next section. In each iteration, an upper bound is obtained from the PLA method, and a lower bound is derived from the PTA method. The iterative process continues until the stopping condition is met.

5. Solution method

5.1. Lower bound: Piecewise tangent approximation (PTA method)

In this section, we use the PTA method to obtain a lower bound for the proposed MINLP model (4.8). For this purpose, for each variable z_k, we consider the partition from the interval $(0,1]$ in the form $z_{k1} < z_{k2} < \ldots < z_{kT}$. In the following, we name the points $z_{k1}, z_{k2}, \ldots, z_{kT}$ as interpolation points. For PTA method, we need the following convexity lemma from Chen *et al.* [9].

Lemma 5.1. *If $p \geq \frac{1}{2}$ and $\theta \geq 1$, then $H(z) := \Phi^{-1}(p^{z^{1/\theta}})$ is convex on $[0,1]$.*

Due to the convexity of the function $\Phi^{-1}(p^{z^{1/\theta}})$, we approximate the function $H(z_k) = \Phi^{-1}(p^{{z_k}^{1/\theta}})$ in interpolation points $z_{k1}, z_{k2}, \ldots, z_{kT}$ with the first-order Taylor expansion as

$$H(z_k) \cong H(z_{kt}) + H'(z_{kt})(z_k - z_{kt}) = a_{kt} + b_{kt} z_k. \tag{5.1}$$

According to Lemma 5.1 and equation (5.1), the following theorem proposes the lower bound for the proposed model (4.8).

Theorem 5.2. *Consider the model*

$$\min\ t_1 = \sum_{(i,j)\in A}\sum_{r\in R} f_{ij}^r y_{ij}^r + \sum_{(i,j)\in A}\sum_{r\in R}\sum_{k\in K}\sum_{s\in S} c_{ij}^r x_{ij}^{ksr} \tag{5.2}$$

s.t.

$$\begin{aligned}
&\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks} = 0, && \forall i \in M_k,\ \ \forall k \in K,\ \ \forall s \in S,\\
&q_k \geq a_{kt} + b_{kt} z_k, && \forall k \in K,\ \ \forall t \in \{1,2,\ldots,T\},\\
&q_k \leq \frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right], && i = O(k),\ \ \forall k \in K,\ \ \forall s \in S,\\
&x_{ij}^{ks} = \sum_{r\in R} x_{ij}^{ksr}, && \forall (i,j) \in A_s,\ \ \forall k \in K,\ \ \forall s \in S,\\
&1 = y_{ij}^0 \geq y_{ij}^1 \geq \cdots \geq y_{ij}^{|R|}, && \forall (i,j) \in A,\\
&\sum_{k\in K} e_{ij}^{kr} x_{ij}^{ksr} \leq \left(U_{ij}^r - U_{ij}^{r-1}\right) y_{ij}^r, && \forall (i,j) \in A_s,\ \ \forall r \in R,\ \ \forall s \in S,\\
&\sum_{k\in K} z_k = 1,\\
&x_{ij}^{ksr} \geq 0, && \forall (i,j) \in A,\ \ \forall k \in K,\ \ \forall s \in S,\ \ r \in R,\\
&z_k \geq 0,\ q_k \geq 0, && \forall k \in K,\\
&y_{ij}^r \in \{0,1\}, && \forall (i,j) \in A,\ \ \forall r \in R,
\end{aligned}$$

where

$$b_{kt} = H'(z_{kt}), a_{kt} = H(z_{kt}) - b_{kt} z_{kt}. \tag{5.3}$$

The value of the optimal objective function of the model (5.2) is a lower bound for the value of the optimal objective function of the proposed MINLP model (4.8).

Proof. Instead of the second constraints set of the model (4.8), we can write

$$q_k \geq \max_{t\in\{1,2,\ldots,T\}} \{a_{kt} + b_{kt} z_k\}, \qquad \forall k \in K.$$

Since the function $\Phi^{-1}(p^{{z_k}^{1/\theta}})$ is convex in the interval $[0,1]$, then for any interpolation point we have

$$\Phi^{-1}\left(p^{{z_k}^{1/\theta}}\right) \geq a_{kt} + b_{kt} z_k, \qquad \forall k \in K,\ \ \forall t \in \{1,2,\ldots,T\},$$

and so

$$\Phi^{-1}\left(p^{{z_k}^{1/\theta}}\right) \geq \max_{t\in\{1,2,\ldots,T\}} \{a_{kt} + b_{kt} z_k\}, \qquad \forall k \in K.$$

And, finally, we have

$$q_k \geq \Phi^{-1}\left(p^{{z_k}^{1/\theta}}\right) \geq \max_{t\in\{1,2,\ldots,T\}} \{a_{kt} + b_{kt} z_k\}, \qquad \forall k \in K, \tag{5.4}$$

or equivalently,

$$\left\{(z_k, q_k) : q_k \geq \Phi^{-1}\left(p^{{z_k}^{1/\theta}}\right), \forall k \in K\right\} \subseteq \left\{(z_k, q_k) : q_k \geq a_{kt} + b_{kt} z_k, \forall k \in K, \forall t \in \{1,2,\ldots,T\}\right\}.$$

Thus, the feasible region of the proposed MINLP model (4.8) is a subset of the feasible region of the problem (5.2). Therefore, the result is valid. □

To obtain an upper bound for the MINLP model (4.8), we will use the PLA method in the next section.

5.2. Upper bound: Piecewise linear approximation (PLA method)

In this case, according to the convexity of the function $\Phi^{-1}(p^{z_k{}^{1/\theta}})$, the value of line passing through two successive interpolation points $z_{kt}, z_{k,t+1}$, greater than or equal of the value of function $H(z_k) = \Phi^{-1}(p^{z_k{}^{1/\theta}})$ in interval $[z_{kt}, z_{k,t+1}]$. By applying the PLA method, we have

$$H(z_k) \simeq H(z_{kt}) + \frac{H(z_{k,t+1}) - H(z_{kt})}{z_{k,t+1} - z_{kt}}(z_k - z_{kt}). \tag{5.5}$$

According to Lemma 5.1 and equation (5.5), the following theorem guarantees the upper bound for the proposed model (4.8).

Theorem 5.3. *Consider the following problem*

$$\min\ t_2 = \sum_{(i,j)\in A}\sum_{r\in R} f_{ij}^r y_{ij}^r + \sum_{(i,j)\in A}\sum_{r\in R}\sum_{k\in K}\sum_{s\in S} c_{ij}^r x_{ij}^{ksr} \tag{5.6}$$

s.t.

$$\begin{aligned}
&\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks} = 0, && \forall i \in M_k,\ \ \forall k \in K,\ \ \forall s \in S,\\
&q_k \geq g_{kt} + w_{kt} z_k, && \forall k \in K,\ \ \forall t \in \{1, 2, \ldots, T-1\},\\
&q_k \leq \frac{1}{\rho_k^s}\left[\sum_{\{j:(i,j)\in A_s\}} x_{ij}^{ks} - \sum_{\{j:(j,i)\in A_s\}} x_{ji}^{ks}\right], && i = O(k),\ \ \forall k \in K,\ \ \forall s \in S,\\
&x_{ij}^{ks} = \sum_{r\in R} x_{ij}^{ksr}, && \forall (i,j) \in A_s,\ \ \forall k \in K,\ \ \forall s \in S,\\
&1 = y_{ij}^0 \geq y_{ij}^1 \geq \ldots \geq y_{ij}^{|R|}, && \forall (i,j) \in A,\\
&\sum_{k\in K} e_{ij}^{kr} x_{ij}^{ksr} \leq \left(U_{ij}^r - U_{ij}^{r-1}\right) y_{ij}^r, && \forall (i,j) \in A_s,\ \ \forall r \in R,\ \ \forall s \in S,\\
&\sum_{k\in K} z_k = 1,\\
&x_{ij}^{ksr} \geq 0, && \forall (i,j) \in A,\ \ \forall k \in K,\ \ \forall s \in S, r \in R,\\
&z_k \geq 0,\ q_k \geq 0, && \forall k \in K,\\
&y_{ij}^r \in \{0,1\}, && \forall (i,j) \in A,\ \ \forall r \in R,
\end{aligned}$$

where

$$w_{kt} = \frac{H(z_{k,t+1}) - H(z_{kt})}{z_{k,t+1} - z_{kt}}, g_{kt} = H(z_{kt}) - b_{kt} z_{kt}. \tag{5.7}$$

Then the value of the optimal objective function of the model (5.6) is an upper bound for the value of the optimal objective function of the proposed MINLP model (4.8).

Proof. The second constraints set of the model (5.6) is

$$q_k \geq g_{kt} + w_{kt} z_k, \qquad \forall k \in K,\ \ \forall t \in \{1, 2, \ldots, T-1\}.$$

So, instead of this constraint, we can write

$$q_k \geq \max_{t\in\{1,2,\ldots,T-1\}} \{g_{kt} + w_{kt}z_k\}, \qquad \forall k \in K.$$

By the property of the convexity $\Phi^{-1}(p^{z_k{}^{1/\theta}})$, the inequalities

$$g_{kt} + w_{kt}z_k \geq \Phi^{-1}\left(p^{z_k{}^{1/\theta}}\right), \qquad \forall k \in K, \quad \forall t \in \{1, 2, \ldots, T-1\},$$

are hold. Then we have

$$\max_{t\in\{1,2,\ldots,T-1\}} \{g_{kt} + w_{kt}z_k\} \geq \Phi^{-1}\left(p^{z_k{}^{1/\theta}}\right), \quad \forall k \in K.$$

And, finally,

$$q_k \geq \max_{t\in\{1,2,\ldots,T-1\}} \{g_{kt} + w_{kt}z_k\} \geq \Phi^{-1}\left(p^{z_k{}^{1/\theta}}\right), \quad \forall k \in K,$$

or equivalently,

$$\left\{(z_k, q_k) : q_k \geq g_{kt} + w_{kt}z_k, \forall k \in K, \forall t \in \{1, 2, \ldots, T\}\right\} \subseteq \left\{(z_k, q_k) : q_k \geq \Phi^{-1}\left(p^{z_k{}^{1/\theta}}\right), \forall k \in K\right\}.$$

This expression means that the feasible region of the model (5.6) is a subset of the feasible region of the MINLP model (4.8). Therefore, the result is valid. □

The pseudo-code of the methods of calculating the lower and upper bounds is driven by Algorithm 1.

Algorithm 1.

Initialization. Set $t := 1$, and go to step 1.

Step 1. After getting the interpolation points through

$$z_{k1} = 0.01, z_{kt} = 0.1 \times (t-1), \quad t = 2, 3, 4, \ldots, 11, \quad k \in K,$$

determine a_{kt} and b_{kt} by equation (5.3) for a given θ, and also determine g_{kt} and w_{kt} by equation (5.7) for a given θ.

Step 2. Solve the model (5.2) to determine a lower bound t_1^* for the proposed MINLP model (4.8).

Step 3. Solve the model (5.6) to determine an upper bound t_2^* for the proposed MINLP model (4.8).

Step 4. If the Gap (Gap $= \frac{t_2^* - t_1^*}{t_2^*} \cdot 100\%$) of between two bounds t_1^* and t_2^* is smaller than a predefined small value γ, stop, otherwise $t := t + 1$ and return to Step 1.

6. Computational results

6.1. Experimental setup

In this section, we consider the computational results to capture the efficiency of the proposed model and the solution methods. So, we considered two case studies: (i) a general illustrative network and (ii) the different classes of randomly generated instances by the Barabasi method [4]. All the test instances are carried out on a core i5-5200 and 2.20 GHz computer with 8.00 GB RAM. The proposed models are coded in GAMS 25.1.2, and the CPLEX solver is used as the optimization solver for solving MILP models.

– We set the confidence parameter $p = 0.6$.

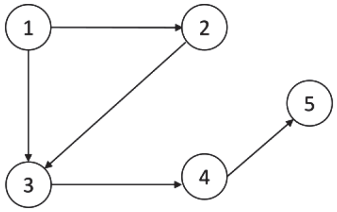

FIGURE 1. Illustrative general network.

– From Algorithm 1, the interpolation points, corresponding to commodity $k \in K$, are $z_{k1} = 0.01, z_{kt} = 0.1 \times (t-1),\ t = 2, 3, 4, \ldots, 11$.
– We set the maximum capacity level as $r_{\max} = 3$.
– Demand $\widetilde{d}_k$ for every commodity $k \in K$ has standard normal distribution.

6.2. The general illustrative network

We consider a network with five nodes and five arcs, shown in Figure 1.

– We set the dependency parameter in the Gumbel-Hougaard copula as $\theta = 1, 2, 5, 7$.
– Fixed cost f_{ij}^r for all $(i, j) \in A$ and $r = 1, 2, \ldots, |R|$, is generated by uniform distribution as $(1, 6)$.
– Variable cost c_{ij}^r for all $(i, j) \in A$ and $r = 1, 2, \ldots, |R|$, is generated by uniform distribution as $(3r, 3(r+1))$.
– Capacity U_{ij}^r, for all $(i, j) \in A$ and $r = 1, 2, \ldots, |R|$, is generated by uniform distribution as $(2r, 2(r+1))$.

Suppose that, there are two commodities, and each commodity has a single source node and a single sink node. Also, assume that nodes 1 and 4 are the source and sink nodes for commodity 1, and nodes 2 and 5 are the source and sink nodes for commodity 2, respectively. Consider two scenarios for this network. In the first scenario, no arc failure occurs, and in the second scenario, only arc (1, 3) is failed. Also, consider three capacity levels, according to the assumptions of the problem, the first capacity level is permanently installed, and the other capacity levels will be installed as necessary. Also, according to the assumptions of the problem, the value of the reservation parameter in the first scenario ($s = 0$) is equal to one and in the second scenario for both commodities is equal to 0.5. The weight value of each commodity corresponding to each capacity level on each arc is considered equivalent to one for convenience.

We present the results for this example in Table 6.

As we see in Table 6, the second column shows the number of interpolation points, and the third and fourth columns show the values of the optimal objective functions of model (5.2) and model (5.6), respectively. Also, in this table, the fifth column shows the gap between the values of these two approximations. The gap is calculated as follows

$$\text{Gap} = \frac{t_2^* - t_1^*}{t_2^*} \cdot 100\%.$$

In this table, the average gap indicates the average of the gaps derived from the lower and upper bound approximations for a fixed dependency parameter θ, considering different numbers of interpolation points.

Figure 2 shows that the lower approximation increases and the upper approximation decreases when the number of interpolation points increases. In this figure, (a)–(d) show the lower and upper bound approximations for $\theta = 1, 2, 5$, and 7, respectively.

Figure 3 shows the changes in the average gap to the dependency parameter. According to Figure 3 and Table 6, the average gap increases when the dependency parameter θ increases. Also, according to Table 6, the gap between the values of lower and upper bound approximations decreases when the number of interpolation points increases, so that if $\theta = 1, 2, 5, 7$ and $T = 7$, the gap is zero, which shows that in this case, the optimal

TABLE 6. Computational results.

θ	Number of interpolation points	t_1^* (Lower bound)	t_2^* (Upper bound)	Gap (%)
1	$T = 5$	52.42	53.89	2.72
	$T = 6$	53.33	53.48	0.28
	$T = 7$	53.48	53.48	0.00
Average gap (%)				1.00
2	$T = 5$	41.25	42.34	2.57
	$T = 6$	42.12	42.33	0.49
	$T = 7$	42.33	42.33	0.00
Average gap (%)				1.02
5	$T = 5$	34.18	35.21	2.93
	$T = 6$	34.92	35.18	0.74
	$T = 7$	35.18	35.18	0.00
Average gap (%)				1.22
7	$T = 5$	33.20	34.21	2.95
	$T = 6$	33.94	34.19	0.73
	$T = 7$	34.17	34.17	0.00
Average gap (%)				1.22

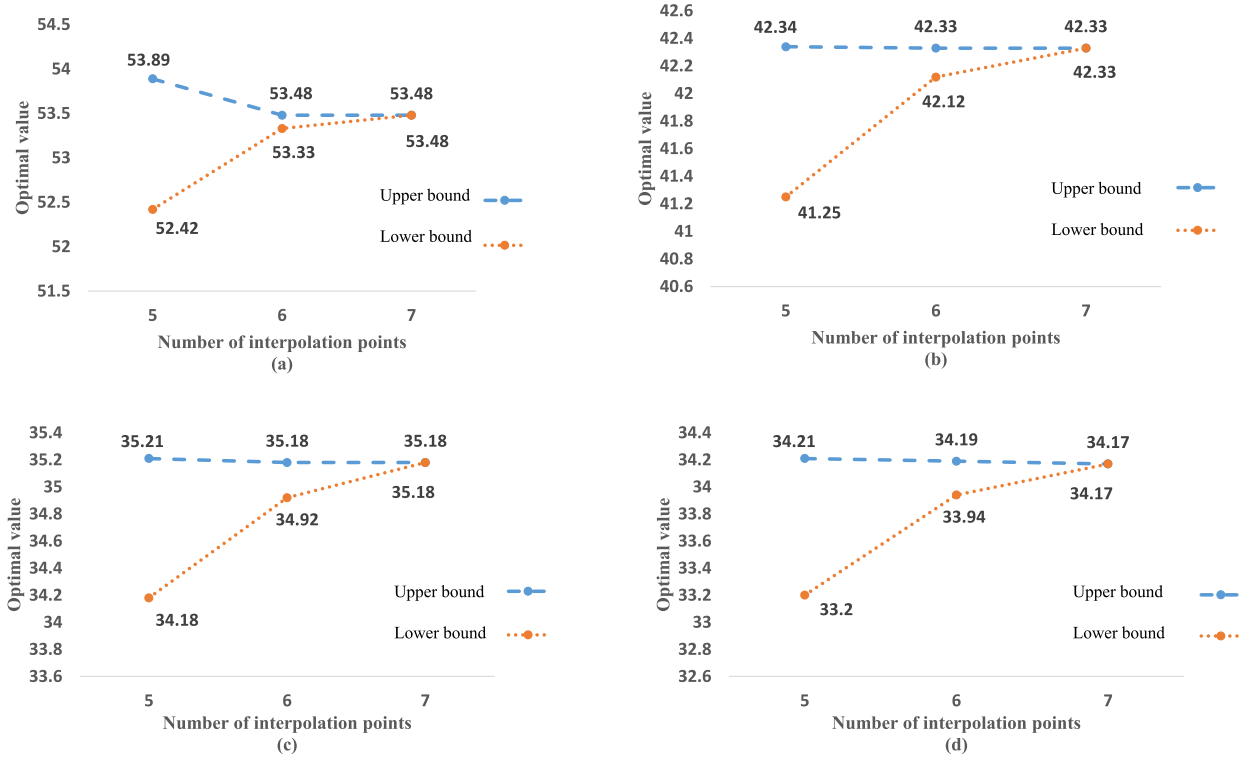

FIGURE 2. Lower and upper bound approximations for the different number of interpolation points, (a) $\theta = 1$, (b) $\theta = 2$, (c) $\theta = 5$, and (d) $\theta = 7$.

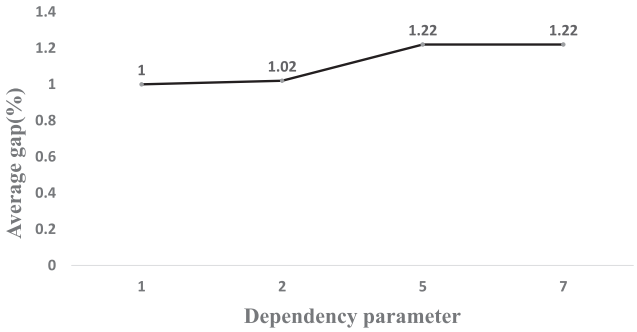

FIGURE 3. The changes in the average gaps.

solution of the model (4.8) has been found. For example, for the case of $\theta = 5$, and $T = 7$, the optimal solution obtained from Algorithm 1 is as follows:

$$\begin{aligned}
&x_{12}^{122} = 0.180,\ x_{13}^{112} = 0.360,\ x_{23}^{122} = 0.180,\ x_{23}^{212} = 0.360,\ x_{23}^{222} = 0.180,\ x_{34}^{112} = 0.360,\\
&x_{34}^{122} = 0.180,\ x_{34}^{212} = 0.360,\ x_{34}^{222} = 0.180,\ x_{45}^{212} = 0.360\ , x_{45}^{222} = 0.180,\\
&x_{12}^{12} = 0.180,\ x_{13}^{11} = 0.360,\ x_{23}^{12} = 0.180,\ x_{23}^{21} = 0.360,\ x_{23}^{22} = 0.180,\ x_{34}^{11} = 0.360,\\
&x_{34}^{12} = 0.180,\ x_{34}^{21} = 0.360,\ x_{34}^{22} = 0.180,\ x_{45}^{21} = 0.360,\ x_{45}^{22} = 0.180,
\end{aligned}$$

and,

$$\begin{aligned}
&y_{12}^1 = y_{12}^2 = y_{13}^1 = y_{13}^2 = y_{23}^1 = y_{23}^2 = y_{34}^1 = y_{34}^2 = y_{45}^1 = y_{45}^2 = 1,\\
&z_1^* = 0.500,\ z_2^* = 0.500,\quad q_1^* = 0.360,\ q_2^* = 0.360,\quad t_1^* = t_2^* = 35.18.
\end{aligned}$$

6.3. Large-scale networks

- We set the dependency parameter in Gumbel-Hougaard copula as $\theta = 1, 2, 5$.
- Fixed cost f_{ij}^r for all $(i, j) \in A$ and $r = 1, 2, \ldots, |R|$, is generated by uniform distribution $(1, 10)$.
- Variable cost c_{ij}^r for all $(i, j) \in A$ and $r = 1, 2, \ldots, |R|$, is generated by uniform distribution $(3r, 3(r + 1))$.
- Capacity(i, j), U_{ij}^r, for all $(i, j) \in A$ and $r = 1, 2, \ldots, |R|$, is generated by uniform distribution $(2r, 2(r + 1))$.

In this section, three classes of instances with 10, 20, and 30 nodes were generated using the Barabasi method [4]. For each class, 200 instances were created, resulting in a total of 600 instances.

In the network with 10 nodes, we consider two commodities and two scenarios so that each commodity has a single source node and a single sink node. In this network, assume that nodes 2 and 9 are the source and sink nodes for commodity 1, and nodes 1 and 6 are the source and sink nodes for commodity 2, respectively. In the first scenario, no arc failure occurs, and in the second scenario, arc (5, 6) fails. This network is shown in Figure 4.

Also, in the networks with 20 and 30 nodes, we consider three commodities and three scenarios. In both networks, assume that nodes 2 and 9 are the source and sink nodes for commodity 1, and nodes 1 and 16 are the source and sink nodes for commodity 2, and nodes 3 and 15 are the source and sink nodes for commodity 3, respectively. Networks with 20 and 30 nodes are shown in Figures 5 and 6, respectively.

In the network with 20 nodes, the first scenario has no arc failures. In the second scenario, arc (4, 7) fails, and in the third scenario, arc (7, 9) fails. In the network with 30 nodes, the first scenario has no arc failures. In the second scenario, arc (2, 7) fails, and in the third scenario, arc (2, 12) fails. In addition, according to the assumption of the problem, for the network with 10 nodes, the value of the reservation parameter in the first scenario ($s = 0$) is equal to one, and in the second scenario for both commodities, it is equal to 0.5. For the

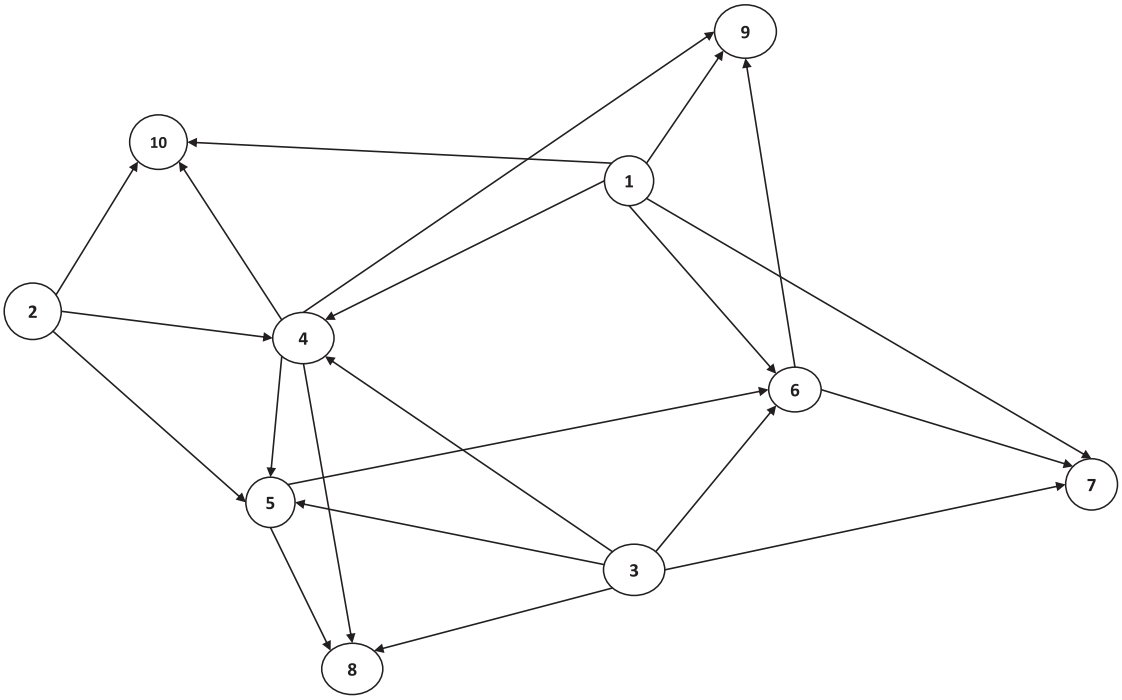

FIGURE 4. A network with 10 nodes has been generated using the Barabasi method [4].

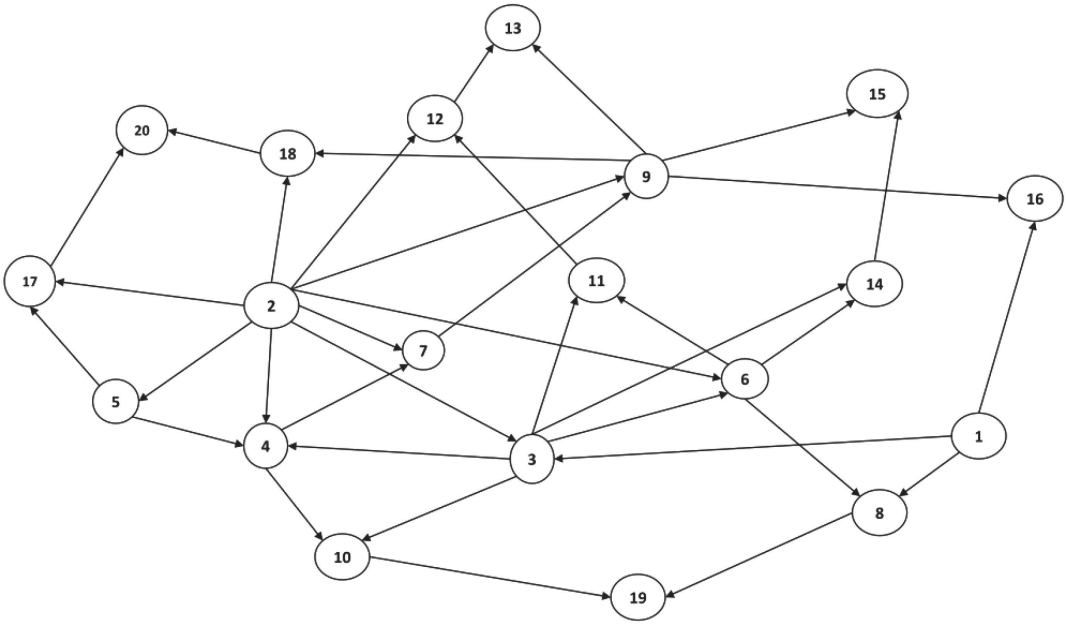

FIGURE 5. A network with 20 nodes has been generated using the Barabasi method [4].

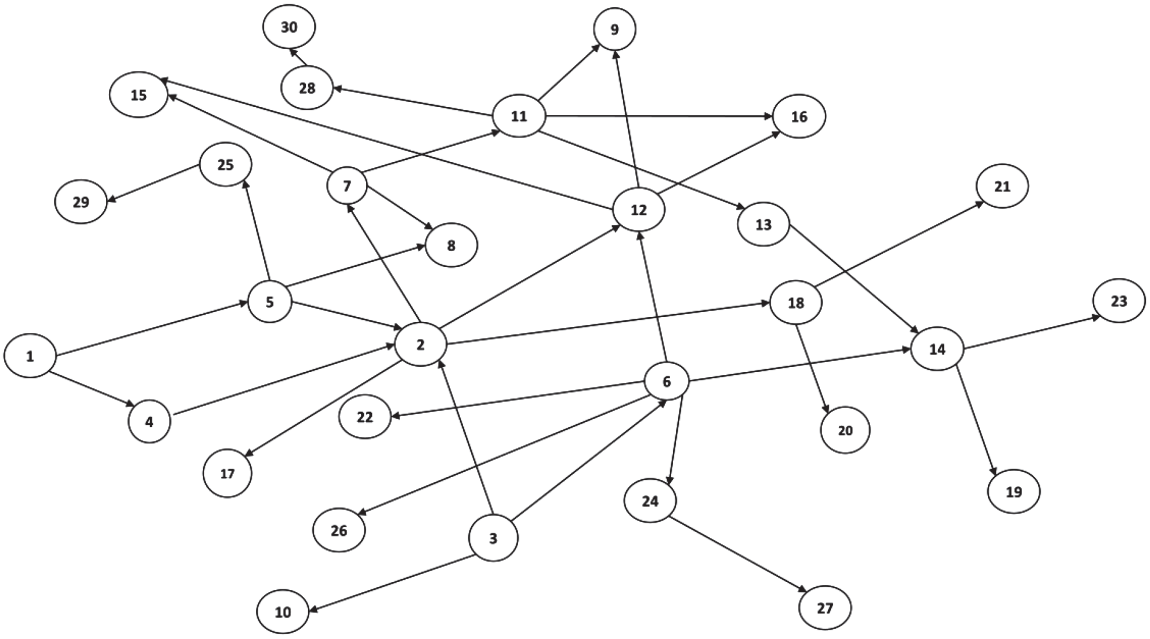

FIGURE 6. A network with 30 nodes has been generated using the Barabasi method [4].

TABLE 7. Computational results for a class of instances (the class with 10 nodes).

θ	Number of interpolation points	t_1^*	t_2^*	Gap (%)
1	$T = 6$	42.93	44.65	4.00
	$T = 7$	43.78	44.62	1.91
	$T = 8$	44.56	44.63	0.15
	$T = 9$	44.58	44.58	0.00
2	$T = 6$	35.06	35.49	1.22
	$T = 7$	35.14	35.45	0.88
	$T = 8$	35.38	35.47	0.25
	$T = 9$	35.45	35.45	0.00
5	$T = 6$	32.30	33.31	3.12
	$T = 7$	32.51	33.27	2.34
	$T = 8$	33.05	33.27	0.67
	$T = 9$	35.25	35.25	0.00

network with 20 nodes, the value of the reservation parameter in the first scenario ($s = 0$) is equal to one, and in the second scenario for both commodities, it is equal to 0.5, and in the third scenario for both commodities, it is equal to 0.6. The weight assigned to each commodity at every capacity level on each arc is simplified to one. We implement Algorithm 1 for each class of instances for the different dependence parameters $\theta = 1, 2$, and 5 and for the interpolation points $T = 6, 7, 8$, and 9. We report the average of the optimal values t_1^* and t_2^*, along with the gap for instances of each class. We report the computational results in Tables 7–9.

In Tables 7–9, the first column shows the value of the dependence parameter, the second column shows the number of interpolation points. The third and fourth columns show the values of the optimal objective functions of model (5.2) and model (5.6), respectively. Also, the fifth column shows the gap between the approximation

TABLE 8. Computational results for the class with 20 nodes.

θ	Number of interpolation points	t_1^*	t_2^*	Gap (%)
1	$T = 6$	106.56	109.90	3.13
	$T = 7$	108.28	109.86	1.45
	$T = 8$	109.70	109.83	0.11
	$T = 9$	109.79	109.79	0.00
2	$T = 6$	76.90	79.24	3.04
	$T = 7$	78.55	79.21	0.84
	$T = 8$	78.92	79.02	0.12
	$T = 9$	79.01	79.01	0.00
5	$T = 6$	58.88	59.81	1.57
	$T = 7$	59.58	59.77	0.31
	$T = 8$	59.70	59.76	0.10
	$T = 9$	59.74	59.74	0.00

TABLE 9. Computational results for the class with 30 nodes.

θ	Number of interpolation points	t_1^*	t_2^*	Gap (%)
1	$T = 6$	204.41	212.92	4.16
	$T = 7$	209.50	212.87	1.61
	$T = 8$	211.65	212.77	0.53
	$T = 9$	212.73	212.73	0.00
2	$T = 6$	154.93	156.96	1.31
	$T = 7$	156.38	156.89	0.33
	$T = 8$	156.78	156.87	0.06
	$T = 9$	156.78	156.78	0.00
5	$T = 6$	117.86	119.39	1.29
	$T = 7$	119.05	119.37	0.27
	$T = 8$	119.21	119.28	0.06
	$T = 9$	119.22	119.22	0.00

values. As observed in the last column of Tables 7–9, the gap between the lower and upper approximations decreases as the number of interpolation points increases, ultimately reaching zero at $T = 9$. The diagram illustrating upper and lower bounds for various numbers of interpolation points in a 30-node network is presented in Figure 7.

Also, according to Tables 7–9, we observe that the lower and upper bound approximations decrease when the dependency parameter is increased. This seems logical, because if the dependency parameter is increased, the joint chance constraints become less restrictive. For a 30-node network with $T = 6$, Figure 8 illustrates how the upper and lower bounds change as θ increases.

7. Conclusions and future works

In this paper, we presented a cost-capacity efficient model for the survivable capacitated network design problem with demand uncertainty. We assumed that the random variables are dependent and follow a standard normal distribution. To address the uncertainty, we used the joint chance-constrained programming, and then, using the suitable Archimedean copula, we obtained the deterministic model. The resulting model is mixed integer non-linear programming and the PLA and PTA approaches were applied to obtain the proposed model's lower and upper bound approximations. Finally, the computational results demonstrated the efficiency of the

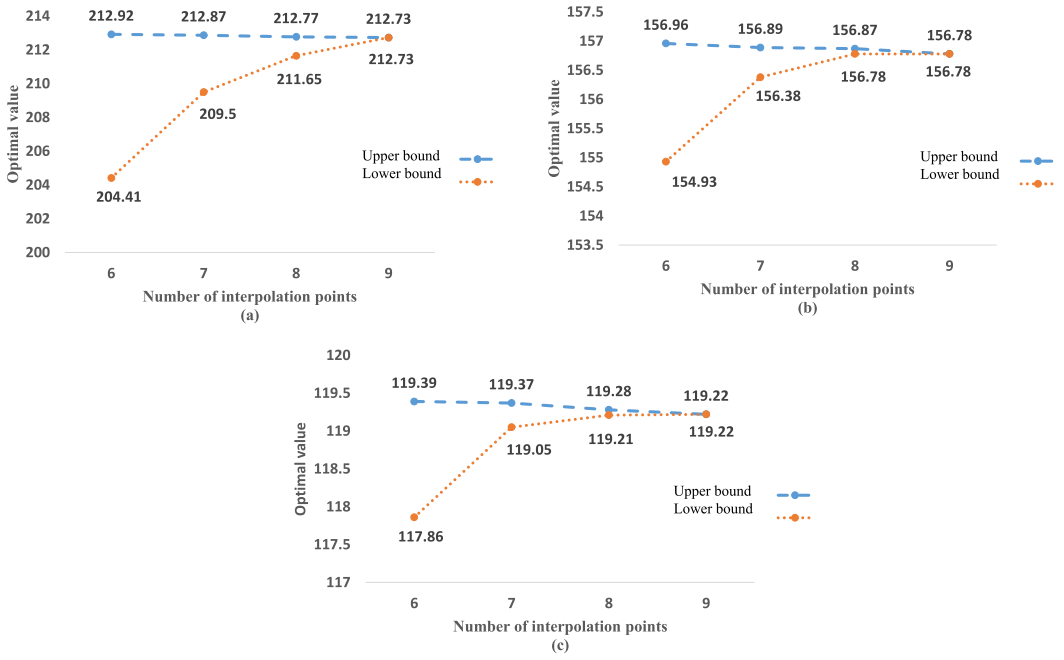

FIGURE 7. Lower and upper approximations for the different number of interpolation points, (a) $\theta = 1$, (b) $\theta = 2$, (c) $\theta = 5$.

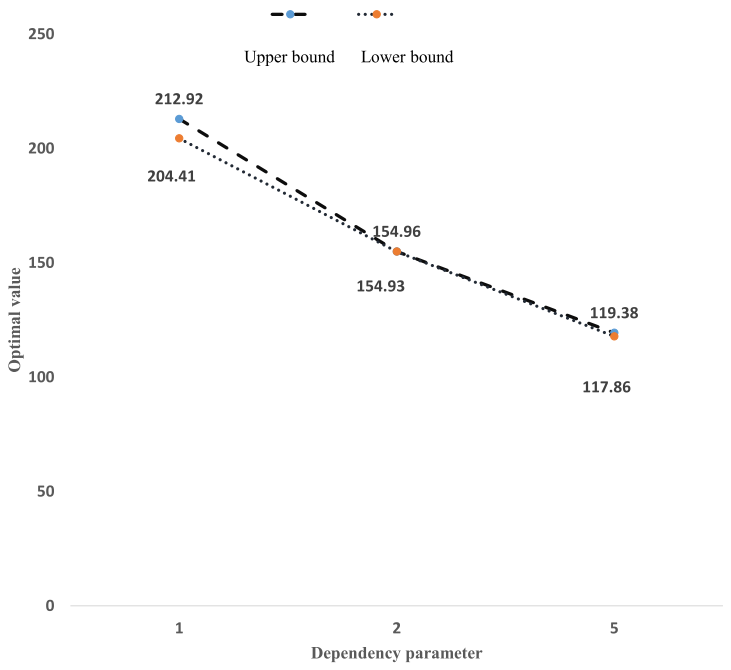

FIGURE 8. The effect of increasing dependency parameter θ on the upper and lower bounds for $T = 6$.

proposed model and solution method. To the best of our knowledge, this paper is one of the primary works applying the joint chance-constrained approach to the survivable capacitated network design problem. Numerous potential future studies can be identified in the field of stochastic survivable network design problems. For instance, we can examine this problem using the distributionally robust approach, particularly with the Wasserstein distance.

Acknowledgments

The authors thank the respected editors and the anonymous referees for their valuable and constructive suggestions on this work, which improved the content substantially.

Data availability statement

No new data/codes were created or analyzed in this study.

References

[1] D. Alevras, M. Grötschel and R. Wessäly, Dimensioning of survivable telecommunications networks. Technical report SC-96-49. Konrad-Zuse-Zentrum, Berlin (1996).

[2] D. Alevras, M. Grötschel and R. Wessäly, Capacity and survivability models for telecommunications networks. Technical report SC-97-24. Konrad-Zuse-Zentrum, Berlin (1997).

[3] D. Alevras, M. Grötschel, P. Jonas, U. Paul and R. Wessäly, Survivable mobile phone network architectures: models and solution methods. *IEEE Commun. Magaz.* **36** (1998) 88–93.

[4] A. Barabasi and R. Albert, Emergence of scaling in random networks. *Science* **286** (2002) 509–512.

[5] J. Birge and F. Louveaux, Introduction to stochastic programming. Springer, New York (2011).

[6] A. Charnes and W.W. Cooper, Chance-constrained programming. *Manage. Sci.* **6** (1959) 73–79.

[7] J. Cheng and A. Lisser, A second-order cone programming approach for linear programs with joint probabilistic constraints. *Oper. Res.* **40** (2012) 325–328.

[8] J. Cheng and A. Lisser, Maximum probability shortest path problem. *Discrete Appl. Math.* **192** (2015) 40–48.

[9] J. Cheng, M. Houda and A. Lisser, Chance-constrained 0–1 quadratic programs using copulas. *Optim. Lett.* **9** (2015) 1283–1295.

[10] J. Cole, C. Lim and F. Sudargho, Survivable network design under optimal and heuristic interdiction scenarios. *J. Global. Optim.* **38** (2007) 181–199.

[11] G. Dahl and M. Stoer, A cutting plane algorithm for multi-commodity survivable network design problems. *Inf. J. Comput.* **10** (1998) 1–11.

[12] K.T. Fang, S. Kotz and K.W. Ng, Symmetric Multivariate and Related Distributions. Chapman and Hall (1990).

[13] E.J. Gumbel, Distributions des valeurs extremes en plusiers dimensions. *Publ. Inst. Statist. Univ. Paris* **9** (1960) 171–173.

[14] Z. Hosseini, A. Babapour, R. Khanjani, S. Khodayifar and P. Pardalos, Joint chance constrained shortest path problem with Copula theory. *J. Comb. Optim.* **40** (2020) 110–140.

[15] P. Hougaard, A class of multivariate failure time distributions. *Biometrika* **73** (1986) 671–678.

[16] T.P. Hutchinson and C.D. Lai, Continuous Bivariate distributions, in Emphasising Applications. Rumsby Scientific Publishing, Adelaide (1990).

[17] R. Jagannathan, Chance-constrained programming with joint constraints. *Oper. Res.* **22** (1974) 358–372.

[18] S. Kataoka, A stochastic programming model. *Econometrica* **31** (1963) 181–196.

[19] R. Khanjani, M. Tavana and D. Caprio, Chance-constrained data envelopment analysis modeling with random-rough data. *RAIRO-Oper. Res.* **52** (2018) 259–284.

[20] R. Khanjani, S. Khodayifar and P. Pardalos, Copula theory approach to stochastic geometric programming. *J. Global. Optim.* **81** (2021) 435–468.

[21] S. Khezri and S. Khodayifar, Joint chance-constrained multi-objective multi-commodity minimum cost network flow problem with copula theory. *Comput. Oper. Res.* **156** (2023) 106–260.

[22] S. Khodayifar and M. Farjaie, Survivable capacitated multicommodity network design problem with independent demand uncertainty. *Math. Res.* **10** (2025) 33–51.

[23] A. Koster and M. Kutschka, An integrated model for survivable network design under demand uncertainty, in 8th International Workshop on the Design of Reliable Communication Networks (DRCN) (2011)
[24] A. Krzemionowski and S. Szymczyk, Portfolio optimization with a copula-based extension of conditional value-at-risk. *Ann. Oper. Res.* **237** (2016) 219–236.
[25] I. Ljubic, P. Mutzel and B. Zey, Stochastic survivable network design problems: theory and practice. *Eur. J. Oper. Res.* **256** (2017) 333–348.
[26] R.K.Y. Low, J. Alcock, R. Faff and T. Brailsford, Canonical vine copulas in the context of modern portfolio management: are they worth it? Asymmetric Dependence in Finance: Diversification, Correlation and Portfolio Management in Market Downturns (2018) 263–289.
[27] G.J. Manish and S. Cole, Models and algorithms for the design of survivable multi-commodity flow networks with general failure scenarios. *Omega* **36** (2008) 1057–1071.
[28] B.L. Miller and H.M. Wagner, Chance constrained programming with joint constraints. *Oper. Res.* **13** (1965) 930–945.
[29] M. Minoux, Optimum synthesis of a network with non-simultaneous multi-commodity flow requirements, in Studies on Graphs and Discrete Programming, edited by P. Hansen (1981) 269–277.
[30] R.B. Nelsen, An Introduction to Copulas, 2nd edition. Springer, New York (2006).
[31] M. Rios, V. Marianov and M. Gutierrez, Survivable capacitated network design problem: new formulation and Lagrangean relaxation. *J. Oper. Res. Soc.* **51** (2000) 574–582.
[32] H. Salehi and M. Anisi, Survivable multi-commodity network flow design: case of node capacities and arc failure. *Int. J. Oper. Res.* **35** (2019) 355–365.
[33] M.B. Salem, R. Taktak and F. Almathkour, The edge-labeled survivable network design problem: formulations and branch-and-cut. *RAIRO-Oper. Res.* **56** (2022) 3393–3404.
[34] M. Sklar, Fonctions de repartition an dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris* **8** (1959) 229–231.
[35] S.E. Terblanche, R. Wessaly and J.M. Hatting, Survivable network design with demand uncertainty. *Eur. J. Oper. Res.* **210** (2011) 10–26.
[36] C. Van de Panne and W. Popp, Minimum-cost cattle feed under probabilistic protein constraints. *Manage. Sci.* **9** (1963) 405–430.
[37] H. Xiao, X. Liu and Z. Zhou, Estimation of portfolio efficiency via stochastic dea. *RAIRO-Oper. Res.* **56** (2022) 2367–2387.