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AUGMENTED PROGRESSIVE HEDGING ALGORITHM FOR A CAPACITATED FIRM SUBJECT TO DEMAND AND SUPPLY UNCERTAINTIES CONSIDERING DISCOUNT

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Abstract. Decision-makers face different uncertainties, and ignoring them leads to negative consequences and significant losses. In the business environment, sourcing is one of the most critical decisions for capacitated firms. So, this article develops a multi-period model to formulate the uncertainties in demand, production line capacity and disruption time. Considering the cited uncertainty helps the decision-makers make the best decisions about production planning, supplier selection, and order amount. It provides a more accurate analysis of the final cost of fulfilling each demand unit. Another contribution of this model is introducing discounts in order, which makes it so complicated to solve. The augmented progressive hedging algorithm is introduced as the solution approach. The original progressive hedging algorithm is augmented by an updating method for the penalty coefficient and clustering the scenarios instead of solving each problem one by one. This contribution leads to reducing the running time with acceptable accuracy.

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1. Introduction

Due to the increasing competition between companies in recent years, attracting and satisfying customers' demand by providing high-quality products at an acceptable price at the proper time is one of the main challenges of supply chain managers. On the other hand, expanding supply chains to international levels to decrease total cost and increase product quality has made firms face a huge amount of uncertainty, which increases the complexity of supply chain management. Deciding supply policies plays a critical role in companies' manufacturing and logistics management, and most experienced companies consider it the most crucial activity of organizations. Thus, addressing uncertainty in sourcing and making decisions to decrease their effects have received a lot of attention since each wrong assessment or decision causes the firms to face negative consequences and numerous losses. The strike of two workers of General Motors in 1998 is an example; their strike in two manufacturing sites stopped 100 other manufacturing and 26 assembly sites from working, which left the stores

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empty for a few months. So, most manufacturers have been encouraged to use multi-sourcing opportunities to decrease the supply chain risk [19].

In the related literature, supply uncertainty is divided into the following categories:

(1) **Disruption:** A random event that makes the supplier or other parts of the supply chain stop their regular activity entirely for a specific period, usually random. In most models, disruption creates a situation in which the supplier cannot provide any product at that time. In general, a system stays in the non-disruption state for a random period and again goes back to the disruption state for another random period.
(2) **Uncertainty in yield:** In this type of uncertainty, the number of items delivered by suppliers or the number of items produced in the production process is a random variable which depends on the order amount.
(3) **Uncertainty in lead-time:** Shows uncertainty in the delivery time.
(4) **Uncertainty in capacity:** This represents uncertainty in the supplier's delivery capacity or the firm's production capacity. It is defined as a random variable independent of the order amount.
(5) **Uncertainty in input cost:** Includes uncertainty in the firm's-imposed procurement cost.

The boundary between these uncertainties is so vague for the firm. For example, if *disruption* is defined as a Bernoulli decision variable, it can be seen as a specific case of uncertainty in yield. However, generally, uncertainty in yield shows a continuous state of uncertainty while it is a discrete disruption. Hence, there are many different approaches for modelling and solving the problems of these two groups. The same is valid for uncertainty in both capacity and lead time. Uncertainty in capacity is discrete, while uncertainty in lead time is considered continuous [20].

The main concern of this article is considering different uncertainties of supply and demand simultaneously. It increases the number of scenarios and complexity in solving the problems, so the progressive hedging algorithm (PHA) decomposition method is used to decrease running time and operational costs. The article has the following structure. Section 2 gives a literature review of the problem. Section 3 defines a problem. Section 4 formulates the problem. Section 5 is parameter tuning. Section 6 presents the solution procedure. Section 7 deals with numerical results. Section 8 conducts a sensitivity analysis of the problem. Finally, Section 9 is devoted to conclusions and recommendations.

2. Literature review

The management of uncertainty in supply chain decision-making has been a key focus in operations research, particularly in areas such as supplier selection, capacity disruptions, and demand fluctuations. Given that incorporating discounts adds an additional layer of complexity, this literature review is structured into three sections: uncertainties without discount, uncertainties with discount, and optimization approaches. The optimization section explores various mathematical and computational techniques used to address uncertainty in supply chains, including exact methods, heuristics, and metaheuristic algorithms.

2.1. Uncertainties without discount

Disruption is the most frequently studied uncertainty factor, appearing in nine papers [2,5,6,11,12,18,21,24, 26]. Among these, six also incorporate demand uncertainty, either as a random variable [2,11,18,26] or through a scenario-based approach [6,12], while the remaining three [5,21,24] assume deterministic demand. Silbermayr and Minner [18] examined disruption and demand uncertainty in a dual-supplier setting, where ordering costs depend on past supplier choices, incentivizing long-term commitments. Freeman *et al.* [6] and Azad and Hassini [2] also studied disruption alongside demand uncertainty. Freeman *et al.* [6] formulated a single-period model with policies such as dual sourcing, downward substitution, and in-house production, incorporating buyer capacity constraints and scenario-based demand uncertainty. Azad and Hassini [2] explored supply chains utilizing both external producers and in-house production while considering recovery scenarios following disruptions. Zheng *et al.* [26] analyzed a two-stage supply chain where a retailer selects between reliable and unreliable suppliers, factoring in demand updates under uncertainty. Joshi and Luong [11] proposed a dual-sourcing model with a primary supplier subject to disruptions and a backup supplier, using capacity reservation contracts to enhance resilience under random demand fluctuations. Taghavi *et al.* [21] addressed supply disruptions using a two-stage risk-averse stochastic programming approach, integrating supply and production scheduling with scenario-based disruptions but assuming deterministic demand. Finally, Han *et al.* [12] examined dual-sourcing strategies under disruption and demand uncertainty, considering varying risk attitudes in decision-making.

Capacity uncertainty is explicitly addressed in two works [8, 10]. Jakšič and Fransoo [10] studied a sourcing problem where a firm relies on two suppliers, one with zero lead time and the other with fixed but non-zero lead time, while facing capacity uncertainty and random demand fluctuations. Gupta *et al.* [8] examined supply chain disruptions using a game-theoretical approach, incorporating capacity uncertainty alongside deterministic demand to analyze its impact on pricing and ordering decisions.

Lead-time uncertainty has been explored in two studies, with [22, 27] directly modeling its impact. In [22], the authors applied robust optimization to address the integrated lot-sizing and supplier selection problem under lead-time uncertainty, considering deterministic demand. In [27], a two-stage stochastic programming model was developed to account for delivery lateness and quality performance uncertainties, where poor-quality components required corrective actions, leading to additional delays. Demand uncertainty was incorporated as a deterministic factor in this study.

While other works considered only one source of supply uncertainty, Gupta *et al.* [8] addressed both disruption and capacity uncertainty together. Sawik [17] studied disruption with deterministic demand, developing an integrated system to determine production quantities in each factory, supplier portfolio, and demand allocation. To address this problem, they defined two approaches: a two-period model distinguishing pre- and post-disruption phases and a multi-period model for extended analysis.

2.2. Uncertainties with discount

Regardless of uncertainty, Alfares and Turnadi [1] proposed a classic supplier selection model considering transportation, inventory, shortage, and ordering costs with discounts. A heuristic algorithm and genetic algorithm (GA) are applied to solve the problem. Meena and Sarmah [14] define the possibility of disruption. They develop a model in which a set of unreliable suppliers differ in capacity and disruption probability. The undisrupted suppliers are employed to compensate for the number of products not received due to supplier disruption. Considering incremental discount, determining the minimum and maximum order quantity, and computing different lot sizes make the model an integer non-linear programming problem, which is solved using GA. The structure of Bohner and Minner's [4] problem is similar to that of the model developed by Meena and Sarmah [14], but instead of GA, they propose an exact solution method. Besides the all-units and incremental discounts, they linearize their model by defining some parameters. Manerba *et al.* [13] developed a two-stage stochastic programming model in which demand and purchase prices from suppliers are divided into two different groups: deterministic and stochastic. In the first stage, the authors decide which suppliers are active and how many products have to be ordered from which supplier. In the second stage, some scenarios are defined due to the stochastic part. Discount is also modelled in this problem. In [24], a multi-period, multi-product mixed integer programming model was developed in which the firm employed various techniques to mitigate supply chain risks. The inclusion of discounts from unreliable suppliers made it impossible to solve the model using exact methods for larger instances.

2.3. Optimization approaches

In the pursuit of optimizing supply chain operations under uncertainty, researchers have explored various solution methodologies, ranging from exact algorithms to heuristic and metaheuristic approaches. Some studies, such as Bohner and Minner [4] and Manerba *et al.* [13], have focused on exact methods, leveraging mathematical programming techniques like branch-and-cut and valid inequalities to ensure optimal solutions. Others, like

Freeman *et al.* [6] and Sawik [17], have relied on powerful solvers such as CPLEX and Gurobi to handle complex stochastic mixed-integer programming (MIP) formulations.

For problems where exact methods become computationally prohibitive, researchers have turned to heuristic and metaheuristic approaches. The work of Alfares and Turnadi [1] exemplifies this shift, combining the Modified Silver Meal heuristic with genetic algorithms (GA) to strike a balance between computational efficiency and solution quality. Similarly, Yavari *et al.* [24] decomposed a multi-product model and applied GA, particle swarm optimization (PSO), and a hybrid PSO-GA approach to solve it effectively.

Dynamic programming has emerged as another favored technique, particularly in studies like Silbermayr and Minner [18] and Jakšič and Fransoo [10], where sequential decision-making under uncertainty is a key challenge. Meanwhile, researchers such as Azad and Hassini [2] and Zhou *et al.* [27] have turned to decomposition techniques like Benders decomposition to accelerate the solution process, especially when dealing with large-scale optimization models.

Beyond traditional optimization, some works have integrated game theory principles to model competitive and cooperative interactions among supply chain entities. For instance, Gupta *et al.* [8] applied a game-theoretic approach to analyze strategic supplier decisions under uncertainty. Additionally, hybrid methodologies have gained traction, as demonstrated by Thevenin *et al.* [22], which combined row and column generation with heuristic strategies like fix-and-optimize and genetic algorithms to efficiently navigate complex problem landscapes.

Despite these advancements, there remain challenges in effectively integrating multiple uncertainties, such as demand fluctuations, supply disruptions, and capacity constraints. Some studies, such as Esmaeili-Najafabadi *et al.* [5] and Zheng *et al.* [26], have rigorously analyzed model convexity and developed tailored solution methods to ensure robust decision-making. Others, like Joshi and Luong [11] and Han *et al.* [12], have adopted multi-objective optimization techniques, transforming complex trade-offs into more manageable single-objective formulations.

The current article has the following features which make it different from the others in the literature:

(1) Developing a model which simultaneously considers uncertainty in demand, production capacity, the ratio of the disrupted item, and the time of disruption occurrence.
(2) Considering suppliers with limited capacity.
(3) Considering discount and uncertainties, which are mentioned in the above first phrase, among models in the literature that considered discount, just one of them studied disruption with definite demand.
(4) Proposing the augmented PHA in terms of updating the penalty coefficient and clustering the scenarios instead of solving each problem one by one to improve the algorithm performance.

To better understand the contributions mentioned above, Table 1 summarizes the cited features.

3. Problem definition

The structure of the problem, assumptions, and all the used notations are given in Table 2 to gain a better insight into the model.

In this article, a firm under study meets its raw material requirements through several unreliable suppliers. Each supplier may be disrupted. After the disruption in ts_s, the recovery operation will start in $ts_s + 1$. The time and cost of recovering the suppliers are functions of the number of disrupted products. After receiving the raw materials, the firm must start production of the final product. The production line also has capacity uncertainty and may not be able to produce the final product at full capacity upon the arrival of raw materials. If the firm tends to pay the fixed costs, it will be able to set up recovery production lines, which, of course, have uncertainty in capacity. Like suppliers, the time and cost of recovery of production lines depend on the unavailable capacity. It is possible to recover suppliers, establish recovery production lines, and hold inventory for these lines to address these uncertainties. Figure 1 depicts the timeline of recovery operations for each part of the system.

TABLE 1. Literature review.

Article	Problem's structure		Uncertainty						Discount
			Supply					Demand	
			Disruption	Yield	Lead-time	Capacity	Input Cost		
[18]	Multi product	Multi period	Yes	–	–	–	–	Random Variable	–
[4]	Multi product	Single period	Yes	–	–	–	–	Deterministic	Yes
[1]	Multi product	Multi period	–	–	–	–	–	Deterministic	Yes
[6]	Multi product	Single period	Yes	–	–	–	–	Scenario-based	–
[10]	Single product	Multi period	–	–	–	Yes	–	Random Variable	–
[13]	Multi product	Single period	–	–	–	–	Yes	Scenario-based	Yes
[17]	Single product	Multi period	Yes	–	–	Yes	–	Deterministic	–
[2]	Multi product	Multi period	Yes	–	–	–	–	Random Variable	–
[5]	Multi product	Multi period	Yes	–	–	–	–	Deterministic	–
[8]	Multi product	Multi period	–	–	–	Yes	–	Deterministic	–
[26]	Multi product	Single period	Yes	–	–	–	–	Random Variable	–
[11]	Single product	Single period	Yes	–	–	–	–	Random Variable	–
[21]	Multi product	Multi period	Yes	–	–	–	–	Deterministic	–
[12]	Single product	Multi period	Yes	–	–	–	–	Scenario-based	–
[22]	Single product	Multi period	–	–	Yes	–	–	Deterministic	–
[27]	Single product	Multi period	–	–	Yes	–	–	Deterministic	–
[24]	Multi product	Multi period	Yes	–	–	–	–	Deterministic	Yes
This paper	Single product	Multi period	Yes	–	–	Yes	–	Scenario-based	Yes

Due to the disruption occurrence, the problem is divided into two stages. In the first stage, decisions regarding selecting different suppliers and the order quantity from those who provide discounts are determined. In the second stage, different scenarios are considered, decisions about auxiliary suppliers' selection, their reorder quantity (without any discount), set up of the recovery production lines, and final production scheduling are made. Also, some parts of the ordered initial products may be stored in the first stage, and depending on the conditions, they will be used in the second stage to supply the required items for the recovery production lines.

Table 2. Notations.

Sets		
T		Time period
I		Supplier
J		Production line
S		Scenario
K		Discount layer
Parameters		
ph		Time horizon
$capp_j$		The maximum capacity of production line j
bcu_{ik}		Maximum orderable products from supplier i in the discount layer k
$capr^s_{jt}$		The capacity of production line j at time t under scenario s
de_s		Demand in scenario s
pr_s		Probability of occurrence of scenario s
cb		The cost of each unit of unfulfilled demand in the first stage
csu_{ik}		The cost of ordering one unit of raw material from supplier i in the discount layer k
cp_j		The cost of producing each product in production line j
ch_j		The inventory cost of each unit of raw material for production line $j (j > 1)$
crs^s_i		The cost of recovering supplier i under scenario s
crp^s_j		The cost of recovering production line j under scenario s
fs_i		The fixed ordering cost from supplier i
fl_j		The fixed cost from production line j
fo^s_i		The ratio of the received raw material from supplier i to the firm under scenario s
fp^s_j		A ratio of the capacity of production line j available in scenario s
ts_s		The start time of disruption in scenario s
tl_i		Delivery time of raw material from supplier i to the firm
trs^s_i		The time required to recover supplier i under scenario s
trp^s_j		The time required to recover production line j under scenario s
PS_i		The Probability of disruption does not occur from supplier i
$\mathrm{PD}_i(l, s)$		The Probability of a disruption in the delivered goods from supplier i in scenario s in uncertainty level l
PP_j		The Probability of capacity loss does not occur in production line j
$\mathrm{PC}_j(l, s)$		The Probability of loss of capacity in disruption level l for production line j in scenario s
ND		The number of samples taken from demand distribution
Variables		
First stage	qpo_i	Order quantity from supplier i for production line j in the first stage – $[0, 1]$
	sd_{ik}	1: If the discount layer k is active for supplier i 0: Otherwise
	dd_{ik}	The ratio of ordering from the maximum orderable items from supplier i in the discount layer k – $[0, 1]$
	bpo_i	1: If supplier i is selected as the supplier in the first stage 0: Otherwise
Second stage	qp^s_{jt}	The quantity of final product produced in the production line j at time t under scenario s
	qro^s_{ij}	Order quantity from supplier i for production line j in the second stage under scenario s – $[0, 1]$
	qh^s_j	The amount of raw material stored for the production line j: $j > 1$ – $[0, 1]$
	qb^s_j	The amount of unfulfilled demand in the first stage in the production line j under scenario s – $[0, 1]$
	brp^s_j	1: If the production line j under Scenario s is activated in the second stage 0: Otherwise
	bro^s_i	1: If supplier i is selected as the supplier in the second stage 0: Otherwise
	ba^s_i	An auxiliary variable used in the constraint (11)

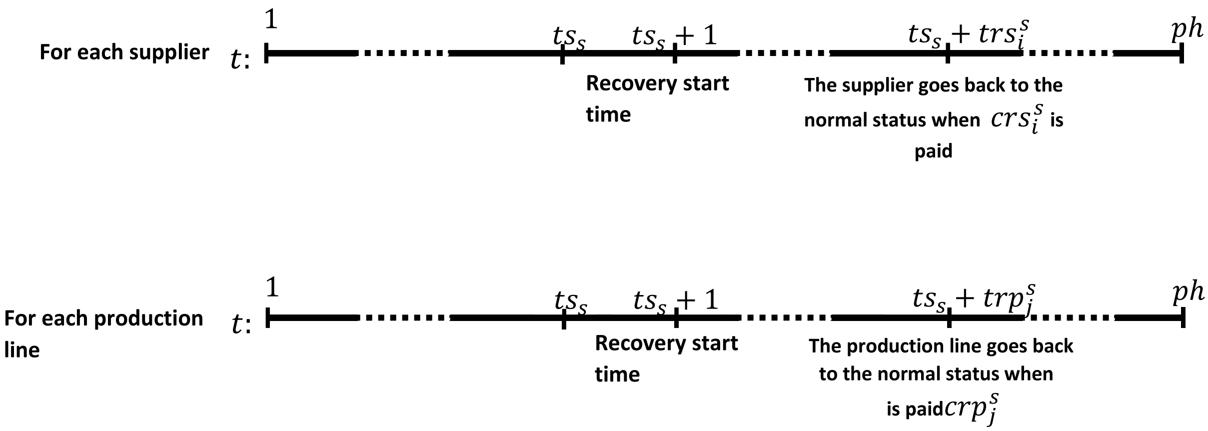

Figure 1. Recovery timeline.

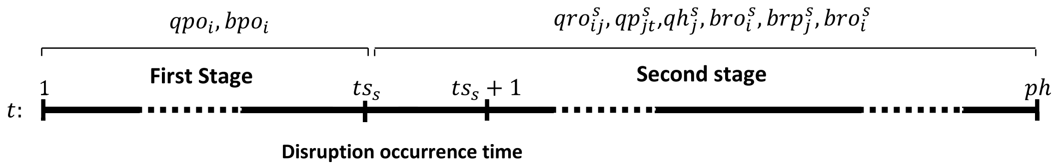

Figure 2. Decision-making timeline.

Figures 2 and 3 illustrate the decision-making timeline and the general structure of supplier selection policies, respectively.

3.1. Assumptions

(1) Demand for the raw material and final products are the same, represented by a probability function with a known mean and standard division.
(2) The final product demand must be completely satisfied at the end of the planning horizon. A penalty should be paid for each unit of unfulfilled demand after the disruption.
(3) The ordered production from suppliers may be disrupted; some percentages of ordered items do not reach the production lines. On the other hand, the start time of disruption is also uncertain.
(4) In a time horizon, disruption occurs just once.
(5) There is also an uncertainty in the non-efficiency of production lines.
(6) Transporting the raw material from the main production lines to the auxiliary ones can start after the recovery phase.

4. Problem formulation

Objective function:

$$Z = \sum_{s=1}^{S} pr_s/de_s \left(\sum_{i=1}^{I} (fs_i(bpo_i + bro_i^s - ba_i^s) + crs_i^s \cdot bro_i^s) + de_s \left(\sum_{k=1}^{K} csu_{ik} \cdot dd_{ik} \cdot bcu_{ik} + \sum_{j=1}^{J} csu_{i1} \cdot qro_{ij}^s \right) \right.$$

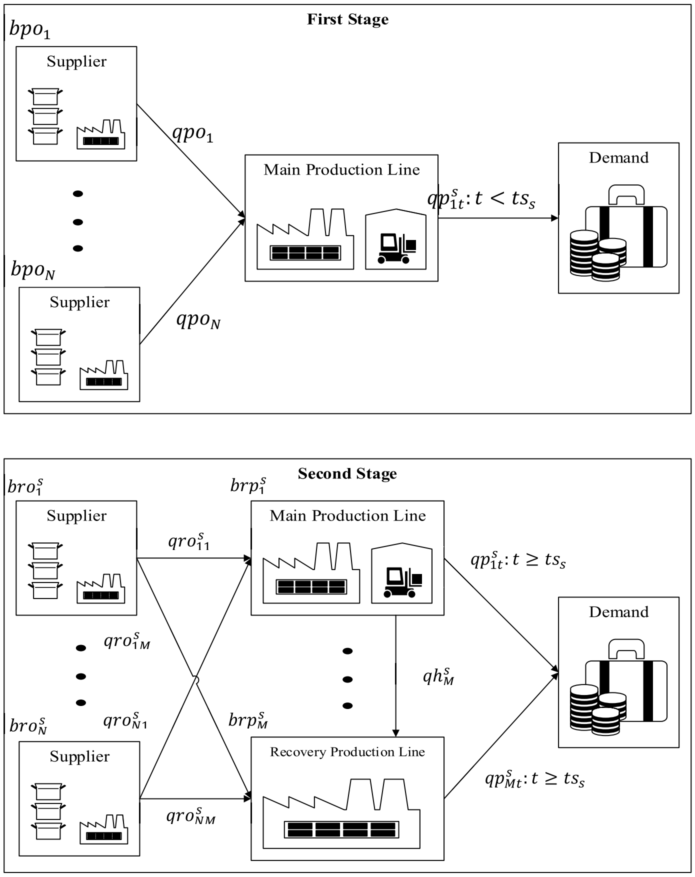

FIGURE 3. Sourcing structure.

$$+\sum_{j=1}^{J}\left(\left(ch_j \cdot qh_j^s + \left(crp_j^s + fl_j\right)brp_j^s\right) + \sum_{t=1}^{T} cp_j \cdot qp_{jt}^s\right) + cb \cdot \left(de_s - \sum_{j=1}^{J}\sum_{t=1}^{t} qp_{jt}^s\right)\Bigg) \tag{1}$$

Constraints:

$$qpo_i \leq bpo_i \qquad \forall i \epsilon I \tag{2}$$

$$\sum_{i \in I} qpo_i = 1 \tag{3}$$

$$qpo_i = \sum_{k \in K} dd_{ik} \cdot bcu_{ik} \qquad \forall i \epsilon I \tag{4}$$

$$dd_{ik} \leq sd_{ik} + sd_{ik-1} \qquad \forall i \epsilon I \quad \forall k \epsilon K : k \geq 2, k \leq |K| - 1 \tag{5}$$

$$dd_{ik} \leq sd_{ik} \qquad \forall i \epsilon I \quad \forall k \epsilon K : k = |K|, 1 \tag{6}$$

$$\sum_{k \in K : k \leq |K|-1} sd_{ik} = 1 \qquad \forall i \epsilon I \tag{7}$$

$$\sum_{k \in K} dd_{ik} = 1 \qquad \forall i \epsilon I \tag{8}$$

$$qro_{ij}^s \leq brp_j^s \qquad \forall i \epsilon I \quad \forall j \epsilon J \quad \forall s \epsilon S \tag{9}$$

$$qro_{ij}^s \leq bro_i^s \qquad \forall i\epsilon I \ \ \forall j\epsilon J \ \ \forall s\epsilon S \tag{10}$$

$$2ba_i^s \leq bro_i^s + bpo_i \qquad \forall i\epsilon I \ \ \forall s\epsilon S \tag{11}$$

$$\sum_{i\in I} qro_{ij}^s + qh_j^s = qb_j^s \qquad \forall j\epsilon J \ \ \forall s\epsilon S \tag{12}$$

$$\sum_{tt\epsilon T:tt\leq t} qp_{1tt}^s \leq de_s\left(\sum_{i\in I:tl_i\leq t-1} fo_i^s \cdot qpo_i + \sum_{i\in I:tl+ts_s+trs_{i\,i}^s\leq t-1} qro_{i1}^s\right) \qquad \forall t\epsilon T: t\leq \max\Big\{ts_s, \max_i tl_i\Big\} \ \ \forall s\epsilon S \tag{13}$$

$$\sum_{tt\epsilon T:tt\leq t} qp_{1tt}^s \leq de_s \times \left(\sum_{i\in I:tl_i\leq t-1} fo_i^s \cdot qpo_i + \sum_{i\in I:tl+ts_s+trs_{i\,i}^s\leq t-1} qro_{i1}^s - \sum_{j\in J} qh_j^s\right) \qquad \forall t\epsilon T: t> \max\Big\{ts_s, \max_i tl_i\Big\} \ \ \forall s\epsilon S \tag{14}$$

$$\sum_{tt\epsilon T:tt\leq t} qp_{jtt}^s \leq de_s\left(\sum_{i\in I:tl+ts_s+trs_{i\,i}^s\leq t-1} qro_{ij}^s\right) \qquad \forall j\epsilon J: j>1 \quad \forall t\epsilon T: t\leq \max\Big\{ts_s, \max_i tl_i\Big\} \ \ \forall s\epsilon S \tag{15}$$

$$\sum_{tt\epsilon T:tt\leq t} qp_{jtt}^s \leq de_s\left(\left(\sum_{i\in I:tl+ts_s+trs_{i\,i}^s\leq t-1} qro_{ij}^s\right) + qh_j^s\right) \qquad \forall j\epsilon J: j>1 \quad \forall t\epsilon T: t> \max\Big\{ts_s, \max_i tl_i\Big\} \ \ \forall s\epsilon S \tag{16}$$

$$qp_{1t}^s \leq capp_1 \qquad \forall t\epsilon T: t< ts_s \ \ \forall s\epsilon S \tag{17}$$

$$qp_{jt}^s \leq capr_{jt}^s \cdot brp_j^s \qquad \forall j\epsilon J \ \ \forall t\epsilon T: t\geq ts_s \ \ \forall s\epsilon S \tag{18}$$

$$qp_{jt}^s \leq qb_j^s \cdot de_s \qquad \forall j\epsilon J \ \ \forall t\epsilon T: t\geq ts_s \ \ \forall s\epsilon S \tag{19}$$

$$qb_j^s \leq brp_j^s \qquad \forall j\epsilon J \ \ \forall s\epsilon S \tag{20}$$

$$qh_j^s \leq brp_j^s \qquad \forall j\epsilon J \ \ \forall s\epsilon S. \tag{21}$$

4.1. Explanation about formulation

Table 3 explains equations of the model.

5. Parameters tuning

5.1. Demand

There are two approaches in the literature to handle uncertainty: the probability and scenario-based approaches. In the first approach, probability distributions are considered, but in the second approach, a set of probable values (which will probably occur in the future) is developed. Mont Carlo sampling is one of the conventional methods in scenario generation, so we use this method to generate the demand scenarios (which follow the Normal distribution function). The main challenge of this method is estimating the sample size. A statistical method is applied in which the number of scenarios is defined by reaching a certain level of accuracy in the answers. This accuracy is defined as the confidence interval of the expected cost.

The $1-\alpha$ percent confidence interval is calculated as follows:

$$\left[E(X) - \varphi_{\frac{\alpha}{2}}\frac{S(n)}{\sqrt{n}}, E(X) + \varphi_{\frac{\alpha}{2}}\frac{S(n)}{\sqrt{n}}\right] \tag{22}$$

TABLE 3. Description of equations.

Equation No.	Description
(1)	The objective function represents the cost of fulfilling one unit of demand, guiding the firm in sourcing, production planning, and cost estimation for each produced item. It integrates multiple cost components: $(de_s(\sum_{k=1}^{K} csu_{ik} \cdot dd_{ik} \cdot bcu_{ik} + \sum_{j=1}^{J} csu_{i1} \cdot qro_{ij}^s))$ indicates ordering cost of raw material from suppliers. $(\sum_{t=1}^{T}\sum_{j=1}^{J} cp_j \cdot qp_{jt}^s)$ represents the production cost, capturing manufacturing expenses. $(\sum_{j=1}^{J}(ch_j \cdot qh_j^s)$ shows the inventory holding cost, reflecting storage and stock management. $(\sum_{i=1}^{I}(fs_i(bpo_i+bro_i^s-ba_i^s)+crs_i^s\cdot bro_i^s)$ indicates the fixed ordering cost for suppliers and the recovery cost for suppliers. $(\sum_{j=1}^{J}(crp_j^s+fl_j)brp_j^s)$ represents the fixed cost of launching and recovery cost for production lines. $(cb\cdot(de_s-\sum_{j=1}^{J}\sum_{t=1}^{t} qp_{jt}^s))$ shows the penalty cost for unfulfilled demand. These costs are aggregated and divided by the total demand, providing a comprehensive measure of the per-unit fulfillment cost.
(2)	Before an order is made to a supplier, the same supplier has to be selected in the first stage.
(3)	All demand should be allocated to the suppliers in the first stage.
(4)	The order quantity from a supplier in the first stage equals a certain percentage of the maximum orderable items from that supplier in the discount layer.
(5) and (6)	If the discount layer is activated, it is possible to determine the ratio of the maximum orderable items from a supplier.
(7)	Only one discount layer can be selected.
(8)	The sum of the order ratios of the maximum quantity of items in the discount layer should equal one.
(9)	In the second stage, determination of the order quantity from a supplier for the production line is possible only when it is activated.
(10)	In the second stage, determination of the order quantity from a supplier for the production line is possible only when the supplier is selected.
(11)	The fixed cost should be calculated only once if a supplier is selected in the first and second stages.
(12)	The order quantity to the suppliers in the second stage and the amount of raw material held for use in the recovery lines of the second stage should be equal to the unfulfilled demand of the first stage.
(13)	Before the recovery phase is completed, the production quantity of the main production line should be less than the order quantity of the raw material from the suppliers.
(14)	After the recovery phase is completed, the production quantity of the main production line should be less than the order quantity of the raw material from the suppliers minus the quantity transferred to recovery production lines.
(15)	Before the recovery phase is completed, the production quantity of recovery lines should be less than the order quantity of the raw material from suppliers.
(16)	After the recovery phase is completed, the production quantity of recovery lines should be less than the order quantity of the raw material from suppliers, plus the inventory transferred to recovery lines.
(17)	Before the disruption, the production capacity of the main production line should be less than the maximum capacity.
(18)	If the production line is activated and after the disruption occurs, the production capacity of the lines should be less than the available capacity.
(19)	The production quantity should be less than the unfulfilled demand for the first stage.
(20)	Not fulfilling demand in the production line is only considered if that line has been activated before.
(21)	Transferring the inventory to the recovery production line is only considered if that line has been activated before.

where n is the sample size, $Z \approx N(0,1)$, and $\varphi_{\frac{\alpha}{2}}$ equals to the minimum value that satisfies the following inequality:

$$Pr\big(Z \leq \varphi_{\frac{\alpha}{2}}\big) \leq 1-\alpha. \tag{23}$$

The calculation of $S(n)$ is also presented hereunder:

$$S(n) = \sqrt{\frac{\sum_{i=1}^{n} \left(E(X) - X_s\right)^2}{n-1}} \tag{24}$$

where X_s equals to the expected cost in scenario s.

On the other hand, if there is sample standard deviation ($S(n)$) and maximum possible error ($E(r)$) for $1-\alpha$ percent confidence interval, the total number of scenarios for a specific confidence interval equal to the minimum integer variable (ND) that holds the following inequality [15]:

$$\text{ND} \geq \left[\frac{\varphi_{\frac{\alpha}{2}} \times S(n)}{Er}\right]^2. \tag{25}$$

To consider uncertainty in demand, we solved the problems with a small number of demand scenarios $n(10)$ and a 95% confidence interval and calculated the desired number of scenarios. The sampling was done from $de \approx N(13\,000, 40\,000)$. It is worth mentioning that the maximum possible error is $0.05 \times E(X)$.

5.2. Disruption for suppliers and loss in production lines capacity

The uncertainly level is a crucial parameter influencing the generation of scenarios, impacting two key parameters: the ratio of ordered items received (fo_i^s) and the available capacity of the production line (fp_j^s). Uncertainly level is defined as the number of possible values for fo_i^s and fp_j^s. For example, in a model with two uncertainly levels ($l = 2$), each supplier either delivers the entire order or nothing. This implies that fo_i^s can be either 1 or 0. Two-uncertainly level model ($l = 2$), fo_i^s or fp_j^s can take 0 and 1 values. In three-uncertainly level model ($l = 3$), the values of 0, 0.5, 1 are acceptable. 0, 0.35, 0.7, 1 are allowable for four-uncertainly level model ($l = 4$), and 0, 0.25, 0.5, 0.75, 1 are applied in five-uncertainly level one ($l = 5$).

The probability of a shortage in the delivered goods from suppliers 1, 2, and 3, respectively, is $\text{PS}_1 = 0.9$, $\text{PS}_2 = 0.8$, $\text{PS}_3 = 0.7$. The Probability of reduction of capacity for production line 1 and 2 is also $\text{PP}_1 = 0.95$ and $\text{PP}_2 = 0.85$.

Equations (26)–(29) calculate the probability of disruption in the delivered goods from suppliers i in uncertainly level l in scenario s ($\text{PD}_i(l,s)$).

$$\text{PD}_i(2,s) = \begin{cases} \text{PS}_i & fo_i^s = 1 \\ (1-\text{PS}_i) & fo_i^s = 0 \end{cases} \tag{26}$$

$$\text{PD}_i(3,s) = \begin{cases} \text{PS}_i & fo_i^s = 1 \\ 0.7 \times (1-\text{PS}_i) & fo_i^s = 0.5 \\ 0.3 \times (1-\text{PS}_i) & fo_i^s = 0 \end{cases} \tag{27}$$

$$\text{PD}_i(4,s) = \begin{cases} \text{PS}_i & fo_i^s = 1 \\ 0.3 \times (1-\text{PS}_i) & fo_i^s = 0.7 \\ 0.7 \times (1-\text{PS}_i) & fo_i^s = 0.35 \\ 0.1 \times (1-\text{PS}_i) & fo_i^s = 0 \end{cases} \tag{28}$$

$$\mathrm{PD}_i(5,s)=\begin{cases}\mathrm{PS}_i & fo_i^s=1\\ 0.4\times(1-\mathrm{PS}_i) & fo_i^s=0.75\\ 0.3\times(1-\mathrm{PS}_i) & fo_i^s=0.5\\ 0.2\times(1-\mathrm{PS}_i) & fo_i^s=0.25\\ 0.1\times(1-\mathrm{PS}_i) & fo_i^s=0.\end{cases} \tag{29}$$

The probability of loss of capacity in uncertainly level l for production line j in scenario s ($\mathrm{PC}_j(l,s)$) is calculated by equations (30)–(33).

$$\mathrm{PC}_j(2,s)=\begin{cases}\mathrm{PP}_j & fp_j^s=1\\ (1-\mathrm{PP}_j) & fp_j^s=0\end{cases} \tag{30}$$

$$\mathrm{PC}_j(3,s)=\begin{cases}\mathrm{PP}_j & fp_j^s=1\\ 0.7\times(1-\mathrm{PP}_j) & fp_j^s=0.5\\ 0.3\times(1-\mathrm{PP}_j) & fp_j^s=0\end{cases} \tag{31}$$

$$\mathrm{PC}_j(4,s)=\begin{cases}\mathrm{PP}_j & fp_j^s=1\\ 0.3\times(1-\mathrm{PP}_j) & fp_j^s=0.7\\ 0.7\times(1-\mathrm{PP}_j) & fp_j^s=0.35\\ 0.1\times(1-\mathrm{PP}_j) & fp_j^s=0\end{cases} \tag{32}$$

$$\mathrm{PC}_j(5,s)=\begin{cases}\mathrm{PP}_j & fp_j^s=1\\ 0.4\times(1-\mathrm{PP}_j) & fp_j^s=0.75\\ 0.3\times(1-\mathrm{PP}_j) & fp_j^s=0.5\\ 0.2\times(1-\mathrm{PP}_j) & fp_j^s=0.25\\ 0.1\times(1-\mathrm{PP}_j) & fp_j^s=0.\end{cases} \tag{33}$$

Capacity of production line j in period t and scenario s is computed by equation (34).

$$capr_{jt}^s=\begin{cases}fp_j^s\cdot capp_j, & ts_s\le t\le ts_s+trp_j^s-1\\ capp_j, & \text{otherwise.}\end{cases} \tag{34}$$

Disruption or loss can start in the first four periods ($ts_s\approx[1,2,3,4]$) with a probability as indicated in equation (35).

$$\mathrm{PT}(s)=\begin{cases}0.1, & ts_s=1\\ 0.2, & ts_s=2\\ 0.3, & ts_s=3\\ 0.4, & ts_s=4.\end{cases} \tag{35}$$

Recovery time and cost for suppliers are function of fo_i^s, while Recovery time and cost for production line are function fp_j^s. These relationships are illustrated in Tables 4 and 5.

5.3. Suppliers and production characteristics

Time and cost parameters of suppliers and production lines are randomly generated from Uniform distribution in the ranges brought in Table 6.

Table 6 presents the parameters for modeling the discount. In this article, the objective function has been defined as the cost of satisfying one unit of demand and the decision is about the ordering quantity from the supplier with the discount in the first stage decision. Hence, the maximum ordering quantity in the relevant

TABLE 4. Parameters of supplies' recovery.

fo_i^s	trs_i^s	crs_i^s
0	12	$100.fs_i$
0.35	10	$10.fs_i$
0.5	6	$10.fs_i$
0.7	8	fs_i
1	0	0

TABLE 5. Parameters of production lines' recovery.

fp_j^s	trp_1^s	trp_2^s	crp_j^s
0	10	6	10 000
0.25	8	4	5000
0.35	8	4	1000
0.5	6	2	1000
0.7	4	2	100
0.75	4	1	100
1	0	0	0

TABLE 6. Time and cost parameters of suppliers and production lines.

tl_1	Uniform $(1, 3)$	ph	Uniform $(20, 40)$
tl_2	Uniform $(3, 5)$	$capp_1$	Uniform $(900, 1100)$
tl_3	Uniform $(5, 3)$	$capp_2$	Uniform $(4000, 6000)$
fs_1	Uniform $(7000, 9000)$	cb	Uniform $(10, 20)$
fs_2	Uniform $(5000, 7000)$	cp_j	Uniform $(0.5, 1.5)$
fs_3	Uniform $(4000, 6000)$	cp_1	Uniform $(0.5, 1.5)$
fl_2	Uniform $(900, 1100)$	ch_2	Uniform $(0.5, 1.5)$

TABLE 7. Discount parameters.

	1	2	3	4	5
bcu_{1k}	0	130 000	520 000	650 000	1 040 000
bcu_{2k}	0	130 000	390 000	650 000	910 000
bcu_{3k}	0	130 000	260 000	520 000	1 040 000
csu_{1k}	12	11	10	9	8
csu_{2k}	16	15	14	13	12
csu_{3k}	14	13	12	11	10

layer shown in Table 7 was divided by the average demand to prevent the objective function to be deviated from the unit value.

In conclusion, the probability of any scenario occurrence is calculated as follows:

$$pr_s = \prod_{j\epsilon J} \mathrm{PC}_j(l,s) \times \prod_{i\epsilon I} \mathrm{PD}_i(l,s) \times \mathrm{PT}(s) \times \frac{1}{\mathrm{ND}}. \tag{36}$$

6. Solution procedure

In stochastic programming problems, the rapid growth in the number of scenarios increases the problem size and computational complexity. As a result, exact methods often cannot solve these problems. Instead, decomposition methods such as the PHA and Benders decomposition can be applied. In the following sections, the defined problem will be solved using PHA and Benders decomposition, and the results will be compared with those obtained from the exact method.

6.1. Benders decomposition

Benders decomposition is a classical partitioning method for MIP problems that relies on a problem manipulation using projection, followed by solution strategies of dualization, outer linearization and relaxation. In this section, a basic development of the method is presented for formulation (1)–(21). The original problem is divided into two smaller ones: a SP and a MP, subjects of Sections 6.1.1 and 6.1.2, respectively. A Benders algorithm is illustrated in Algorithm 1, where Z and φ are the values of the objective functions of the MP and the dual SP, respectively [3].

Algorithm 1. Benders Algorithm.

(1) $\text{UB} = \infty, \text{LB} = 0, h = 0$
(2) **While** $\text{UB} - \text{LB} > 0.01$ **do**
(3) solve MP
(4) $\text{LB} = Z$
(5) Add Cut to MP
(6) Update UB, if necessary
$h = h + 1$
(7) **End**

6.1.1. *Benders sub-problem*

The primal linear SP includes (1)–(6), (8)–(10), and (12)–(21) where for fixed values of the integer (complicating) variables, $bpo_i = \overline{bpo_i}$, $bro_i^s = \overline{bro_i^s}$, $ba_i^s = \overline{ba_i^s}$, $bro_i^s = \overline{bro_i^s}$, $brp_j^s = \overline{brp_j^s}$, and $sd_{ik} = \overline{sd_{ik}}$. The dual form of primal SP can be attained after associating the dual variables $\alpha\sigma$ to the constraint (σ). For example, $\alpha 2$ associates with the constraint (2). Hence, a dual SP can be given as:

$$\begin{aligned}\text{Max } \varphi = \sum_{i \epsilon I} \Bigg(& \overline{bpo_i} \cdot \alpha 2_i + \sum_{k \epsilon K: k \geq 2, k \leq |K|-1} \left(\overline{sd_{ik}} + \overline{sd_{ik-1}}\right) \alpha 5_{ik} + \sum_{k \epsilon K: k = |K|, 1} \left(\overline{sd_{ik}}\right) \alpha 6_{ik} + \alpha 8_i \\ & + \sum_{j \epsilon J} \sum_{s \epsilon S} \left(\overline{brp_j^s} \cdot \alpha 9_{ijs} + \overline{bro_i^s} \cdot \alpha 10_{ijs}\right) \Bigg) \\ & + \sum_{j \epsilon J} \sum_{s \epsilon S} \left(\sum_{t \epsilon T: t \geq ts_s} \left(capr_{jt}^s \cdot \overline{brp_j^s} \cdot \alpha 18_{jst} \right) + \overline{brp_j^s} \cdot \alpha 20_{js} + \overline{brp_j^s} \cdot \alpha 21_{js} \right) \\ & + \sum_{s \epsilon S} \sum_{t \epsilon T: t < ts_s} capp_1 \cdot \alpha 17_{st} + \alpha 3 \end{aligned} \tag{37}$$

$$\alpha 2_i + \alpha 3 + \alpha 4_i - \sum_{s \epsilon S} \left(\sum_{t \epsilon T: t \leq \max\{ts_s, \max_i tl_i\} \text{ and } tl_i \leq t-1} de_s \cdot fo_i^s \cdot \alpha 13_{ts} \right.$$

$$- \sum_{t\epsilon T: t > \max\{ts_s, \max_i tl_i\} \text{ and } tl_i \le t-1} de_s \cdot fo_i^s \cdot \alpha 14_{ts} \Bigg) \le 0 \qquad \forall i \in I \tag{38}$$

$$-bcu_{ik} \cdot \alpha 4_i + \alpha 5_{ik} + \alpha 8_i \le csu_{ik} \cdot bcu_{ik} \qquad \forall i\epsilon I, \quad \forall k\epsilon K : k \ge 2, \quad k \le |K| - 1 \tag{39}$$

$$-bcu_{ik} \cdot \alpha 4_i + \alpha 6_{ik} + \alpha 8_i \le csu_{ik} \cdot bcu_{ik} \qquad \forall i\epsilon I \quad \forall k\epsilon K : k = |K|, 1 \tag{40}$$

$$\alpha 9_{ijs} + \alpha 10_{ijs} + \alpha 12_{js} - \sum_{t\epsilon T: t \le \max\{ts_s, \max_i tl_i\} \text{ and } tl + ts_s + trs_{i\,i}^s \le t-1} de_s \cdot \alpha 13_{ts}$$
$$- \sum_{t\epsilon T: t > \max\{ts_s, \max_i tl_i\} \text{ and } tl + ts_s + trs_{i\,i}^s \le t-1} de_s \cdot \alpha 14_{ts} \le pr_s \cdot csu_{ij} \qquad \forall s\epsilon S, i \in I, \quad \forall j\epsilon J : j = 1 \tag{41}$$

$$\alpha 9_{ijs} + \alpha 10_{ijs} + \alpha 12_{js} - \sum_{t\epsilon T: t \le \max\{ts_s, \max_i tl_i\} \text{ and } tl + ts_s + trs_{i\,i}^s \le t-1} de_s \cdot \alpha 15_{jts}$$
$$- \sum_{t\epsilon T: t > \max\{ts_s, \max_i tl_i\} \text{ and } tl + ts_s + trs_{i\,i}^s \le t-1} de_s \cdot \alpha 16_{jts} \le pr_s \cdot csu_{ij} \qquad \forall s\epsilon S, i \in I, \quad \forall j\epsilon J : j > 1 \tag{42}$$

$$\sum_{tt\epsilon T: tt \ge t \text{ and } t \le \max\{ts_s, \max_i tl_i\}} \alpha 13_{s,tt}$$
$$+ \sum_{tt\epsilon T: tt \ge t \text{ and } t \le \max\{ts_s, \max_i tl_i\}} \alpha 14_{s,tt} + \alpha 17_{st} \le pr_s(cp_j - cb/de_s) \qquad \forall t\epsilon T : t < ts_s, \quad \forall s\epsilon S, \quad \forall j\epsilon J : j = 1 \tag{43}$$

$$\sum_{tt\epsilon T: tt \ge t \text{ and } t \le \max\{ts_s, \max_i tl_i\}} \alpha 13_{s,tt} + \sum_{tt\epsilon T: tt \ge t \text{ and } t \le \max\{ts_s, \max_i tl_i\}} \alpha 14_{s,tt}$$
$$+ \alpha 18_{jst} + \alpha 19_{jst} \le pr_s(cp_j - cb/de_s) \qquad \forall t\epsilon T : t \ge ts_s, \quad \forall s\epsilon S, \quad \forall j\epsilon J : j = 1 \tag{44}$$

$$\sum_{tt\epsilon T: tt \ge t \text{ and } t \le \max\{ts_s, \max_i tl_i\}} \alpha 15_{js,tt} + \sum_{tt\epsilon T: tt \ge t \text{ and } t \le \max\{ts_s, \max_i tl_i\}} \alpha 16_{js,tt}$$
$$+ \alpha 18_{jst} + \alpha 19_{jst} \le pr_s(cp_j - cb/de_s) \qquad \forall t\epsilon T : t \ge ts_s, \quad \forall s\epsilon S, \quad \forall j\epsilon J : j > 1 \tag{45}$$

$$\sum_{tt\epsilon T: tt \ge t \text{ and } t \le \max\{ts_s, \max_i tl_i\}} \alpha 15_{js,tt}$$
$$+ \sum_{tt\epsilon T: tt \ge t \text{ and } t \le \max\{ts_s, \max_i tl_i\}} \alpha 16_{js,tt} \le pr_s(cp_j - cb/de_s) \qquad \forall t\epsilon T : t < ts_s, \quad \forall s\epsilon S, \quad \forall j\epsilon J : j > 1 \tag{46}$$

$$\alpha 12_{js} - \sum_{t\epsilon T: t > \max\{ts_s, \max_i tl_i\}} de_s \alpha 14_{st} + \alpha 21_{js} \le pr_s \cdot ch_j \qquad \forall s\epsilon S, \quad \forall j\epsilon J : j = 1 \tag{47}$$

$$\alpha 12_{js} - \sum_{t\epsilon T: t > \max\{ts_s, \max_i tl_i\}} de_s \alpha 16_{jst} + \alpha 21_{js} \le pr_s \cdot ch_j \qquad \forall s\epsilon S, \quad \forall j\epsilon J : j > 1 \tag{48}$$

$$-\alpha 12_{js} - \sum_{t\epsilon T: t \ge ts_s} de_s \alpha 19_{js} + \alpha 20_{js} \le -pr_s \cdot cb/de_s \qquad \forall j\epsilon J, \quad \forall s\epsilon S \tag{49}$$

$\alpha 2_i \le 0, \alpha 3$ free, $\alpha 4_i$ free, $\alpha 5_{ik} \le 0, \alpha 6_{ik} \le 0, \alpha 8_i$ free, $\alpha 9_{ijs} \le 0, \alpha 10_{ijs} \le 0, \alpha 12_{js}$ free, $\alpha 13_{st} \le 0, \alpha 14_{st} \le 0,$ $\alpha 15_{jst} \le 0, \alpha 16_{jst} \le 0, \alpha 17_{st} \le 0, \alpha 18_{jst} \le 0, \alpha 19_{jst} \le 0, \alpha 20_{js} \le 0, \alpha 21_{js} \le 0.$

6.1.2. Benders master problem

The original problem is then equivalent to the following Benders MP:

$$\text{Min } Z = \sum_{s=1}^{S} pr_s \cdot \left(\sum_{i=1}^{I} (fs_i(bpo_i + bro_i^s - ba_i^s) + crs_i^s \cdot bro_i^s)/de_s + \sum_{j=1}^{J} (crp_j^s + fl_j) brp_j^s / de_s \right) + \varphi. \tag{50}$$

(7) and (11)

$$\begin{aligned}\varphi \geq \sum_{i\epsilon I} \Bigg(& bpo_i \cdot \alpha 2_i^h + \sum_{k\epsilon K:k\geq 2,k\leq |K|-1} (sd_{ik} + sd_{ik-1})\alpha 5_{ik}^h + \sum_{k\epsilon K:k=|K|,1} (sd_{ik})\alpha 6_{ik}^h + \alpha 8_i^h \\ & + \sum_{j\epsilon J}\sum_{s\epsilon S}\left(brp_j^s \cdot \alpha 9_{ijs}^h + bro_i^s \cdot \alpha 10_{ijs}^h \right)\Bigg) + \sum_{j\epsilon J}\sum_{s\epsilon S}\left(\sum_{t\epsilon T:t\geq ts_s} \left(capr_{jt}^s \cdot brp_j^s \cdot \alpha 18_{jst}^h \right) + brp_j^s \cdot \alpha 20_{js}^h + brp_j^s \cdot \alpha 21_{js}^h \right) \\ & + \sum_{s\epsilon S}\sum_{t\epsilon T:t<ts_s} capp_1 \cdot \alpha 17_{st}^h + \alpha 3^h \qquad \forall h \in H \end{aligned} \tag{51}$$

$$\begin{aligned}\sum_{i\epsilon I} \Bigg(& bpo_i \cdot \alpha 2_i^h + \sum_{k\epsilon K:k\geq 2,k\leq |K|-1} (sd_{ik} + sd_{ik-1})\alpha 5_{ik}^h + \sum_{k\epsilon K:k=|K|,1} (sd_{ik})\alpha 6_{ik}^h + \alpha 8_i^h \\ & + \sum_{j\epsilon J}\sum_{s\epsilon S}\left(brp_j^s \cdot \alpha 9_{ijs}^h + bro_i^s \cdot \alpha 10_{ijs}^h \right)\Bigg) + \sum_{j\epsilon J}\sum_{s\epsilon S}\left(\sum_{t\epsilon T:t\geq ts_s} \left(capr_{jt}^s \cdot brp_j^s \cdot \alpha 18_{jst}^h \right) + brp_j^s \cdot \alpha 20_{js}^h + brp_j^s \cdot \alpha 21_{js}^h \right) \\ & + \sum_{s\epsilon S}\sum_{t\epsilon T:t<ts_s} capp_1 \cdot \alpha 17_{st}^h + \alpha 3^h \leq 0 \qquad \forall h \in G \end{aligned} \tag{52}$$

equations (51) and (52) are cuts that add to MP. H and G are the sets of iterations having the dual SP bounded and unbounded, respectively.

6.2. PHA

In stochastic programming, if non-anticipatively constraints are released, the problem is decomposable for each scenario, and its solution time decreases. PHA, which is one of the most famous scenario-based decomposition methods for stochastic models. In this algorithm, using the extension of Lagrangian relaxation, non-anticipatively constraints are released, and considering a penalty coefficient in the objective function, it gets closer to the feasible answer in each repetition [25].

PHA is developed to solve mixed-integer linear stochastic programming models. Theoretically, stochastic programming problems with continuous convex decision variables are also convergent to the optimal solution, which satisfies all the non-anticipatively constraints. In some cases that most or all decision variables are integer, PHA works heuristically and with a high precision converges to a bound [19].

Expression (53) introduces a two-stage model

$$\begin{aligned} \min_{x,y} Z &= A^T x + \sum_{s\epsilon S} pr_s \left(B^T y_s \right) \\ Cx &\leq d \\ E_s y_s + F_s x &\leq g \qquad \forall s\epsilon S. \end{aligned} \tag{53}$$

The PHA puts $x = x_s$ to can decomposite the model to each scnario and solve them individually. It is obvious that $x_s = \overline{x}\ \forall s\epsilon S$. So $\sum_{s\epsilon S} \lambda_s (x_s - \overline{x}) + \sum_{s\epsilon S} \rho (x_s - \overline{x})^2$ is added to the objective fuction. $\overline{x}$ and λ_s are estimated by an iterative algorithm. Algorithm 2 briefly demonstrates how PHA works. For more information, see [25].

We improve the original PHA From two perspectives:

First, in [25] the penalty coefficient (ρ) is constant in all iterations, but due to the high sensitivity of efficiency of this method to the penalty coefficient, various articles studied the methods of updating ρ (see [7, 16] as two examples). In the current article, combining two methods of Gul [7] and Mulvey and Vladimirou [16] and considering the model structure, the following formula is taken to update ρ.

$$\rho^{it} = \begin{cases} a\rho^{it-1}, & \Delta_D^{it} \geq \varepsilon \\ b\rho^{it-1} & \text{o.w.} \end{cases} \tag{54}$$

$$\Delta_D^{it} = \sum_{s \epsilon S} pr_s \left(x_s^{it} - \overline{x}^{it}\right)^2 \tag{55}$$

where $a = 1.02$, $b = 0.98$, and $\rho = 1$. They are experimentally selected to achieve better performance.

Algorithm 2. PHA Algorithm.

$it = 0, \rho = \rho_0, \varepsilon = \varepsilon_0, \lambda_s^1 = 0$
While Stopping criteria **do**
  For $s \epsilon S'$
   Solve this model for s and calculate x_s^* and y_s^*
   $\min_{x,y} Z_s\left(\lambda_s^{it-1}, \rho\right) = \left(A^T x_s + B^T y_s\right) + \lambda_s^{it-1}\left(x_s - \overline{x}^{it-1}\right) + \frac{\rho}{2}\left(x_s - \overline{x}^{it-1}\right)^2$
   $Cx_s \leq d$
   $E_s y_s + F_s x_s \leq g$
  $x_s^{it} = x_s^*$
  End
$\overline{x}^{it} = \sum_{s \epsilon S} pr_s x_s^{it}$
$\lambda_s^{it} = \rho\left(x_s^{it} - \overline{x}^{it}\right) + \lambda_s^{it-1}$
$it = it + 1$
End

Second, in the existing literature, the original PHA involves performing a separate PHA for each scenario. For instance, a problem with 2 suppliers, 2 production lines, 2 loss capacity levels, 2 disruption levels, and ND = 27 has 1728 scenarios. Therefore, 1728 problems need to be solved independently, each taking about 0.06 s, and if iterated twice at least, the original PHA takes approximately 207 s at the minimum. This extended time frame is impractical, and the convergence rate is hindered due to the extensive number of scenarios. To address this, we introduce an augmented PHA in this study. The augmented PHA groups scenarios into clusters and solves the clusters collectively instead of addressing them individually. The clustering is based on separating demand and supply scenarios, resulting in ND clusters for the problems defined in the next section. In the given problem, scenarios are clustered into 27 groups, each containing 64 members, and then 27 problems with 64 scenarios are solved. This algorithm optimizes the problem in 38 s, showcasing a significant improvement as the number of scenarios increases.

The augmented PHA stands out as a critical contribution of this article, as outlined in Algorithm 3.

7. Numerical results

Ten different-size problems are defined in Table 8 to evaluate the performance of the proposed algorithms. 5 samples are generated from the uniform distributions in Table 4 to consider parameter changes. Based on sampling, 5 states are defined. For each state, ND samples are drawn from a normal distribution with a mean of 13 000 and a standard deviation of 20 000 for demand. Each of the 10 different-size problems is solved for each state. Finally, 50 instances are solved by exact method, Benders decomposition and augmented PHA. The

Algorithm 3. Augmented PHA.

(1) $it = 0, \rho^0 = \rho_0, \varepsilon = \varepsilon_0, \lambda_{s'}^1 = 0$
(2) Cluster all $s\epsilon S$ to $|S'|$ Clusters $(s'\epsilon S')$
(3) $pr_{s'} = \sum_{s\epsilon Clusters'} pr_s \quad \forall s \,\epsilon\, Cluster\, s'$
(4) **While** Stopping criteria **do**
(5) **For** $s'\epsilon S'$
(6) Solve this model for s' and calculate $x_{s'}^*$ and y_s^*
$$\min_{x,y} Z_{s'}\left(\lambda_{s'}^{it-1}, \rho^{it-1}\right) = \sum_{s\epsilon Clusters'} \left(A^T x_{s'} + B^T y_s\right) + \sum_{s\epsilon Clusters'} \lambda_{s'}^{it-1}\left(x_{s'} - \overline{x}^{it-1}\right)$$
$$+ \sum_{s\epsilon Clusters'} \frac{\rho^{it-1}}{2}\left(x_{s'} - \overline{x}^{it-1}\right)^2$$
$$Cx_s \leq d \quad \forall s \,\epsilon\, Cluster\, s'$$
$$E_s y_s + F_s x_s \leq g \qquad \forall s \,\epsilon\, Cluster\, s'$$
(7) $x_{s'}^{it} = x_{s'}^*$
(8) **End**
(9) $\overline{x}^{it} = \sum_{s'\epsilon S'} pr_{s'} x_{s'}^{it}$
(10) $\Delta_D^{it} = \sum_{s'\epsilon S'} pr_{s'} (x_{s'}^{it} - \overline{x}^{it})^2$
(11) $\rho^{it} = \begin{cases} a\rho^{it-1}, & \Delta_D^{it} \geq \varepsilon \\ b\rho^{it-1} & \text{o.w.} \end{cases}$
(12) $\lambda_{s'}^{it} = \rho^{it}(x_{s'}^{it} - \overline{x}^{it}) + \lambda_{s'}^{it-1}$
(13) $it = it + 1$
(14) **End**

TABLE 8. Problem definition.

Problem	Number of suppliers	Number of production lines	Number of disruption levels	Number of loss capacity levels	ND
1	2	2	2	2	27
2	3	2	2	2	17
3	2	2	3	2	25
4	2	2	2	4	18
5	2	2	2	5	20
6	3	2	3	2	12
7	2	2	4	2	25
8	3	2	2	4	18
9	3	2	4	2	15

average objective function and solution time over defined states for different-size problems are presented in Table 9.

Table 9 shows that the solution time of the first and second problems (with the smaller sizes) is more with the augmented PHA, but as the problem size increases, the number of scenarios increases. The proposed algorithm causes a considerable decrease in the run time. In problems 9 and 10, where the memory error stopped the exact method, the augmented PHA could find the solution in an acceptable run time. Benders decomposition, on the other hand, requires more time to find a solution compared to the other methods. Even for small instances, its solution time exceeds that of the augmented PHA. However, for problems 9 and 10, Benders decomposition still managed to provide a solution. The results also show that the augmented PHA delivers solutions with acceptable accuracy. It is noticeable that the exact method and the proposed algorithms are implemented using the CPLEX solver of GAMS software.

TABLE 9. Comparison between exact, Benders decomposition, and augmented PHA.

Problem number	Exact		Benders Decomposition			Augmented PHA			
	Objective function	Solution time (second)	Objective function (lower bound)	Solution time (second)	Gap-E(%)	Objective function	Solution time (second)	Gap-E(%)	Gap-B(%)
1	12.815	29	12.819	50	0.03%	12.832	38	0.13%	0.10%
2	12.191	83	12.21	236	0.16%	12.396	100	1.68%	1.52%
3	12.101	196	12.109	465	0.07%	12.118	71	0.14%	0.07%
4	12.69	159	12.724	357	0.27%	12.69	89	0.00%	−0.27%
5	12.612	312	12.635	787	0.18%	12.638	177	0.21%	0.02%
6	11.433	456	11.523	1045	0.79%	11.524	165	0.80%	0.01%
7	11.824	496	11.884	1162	0.51%	12.064	126	2.03%	1.51%
8	12.185	934	12.225	3142	0.33%	12.204	485	0.16%	−0.17%
9	Out of memory		11.58	4191	–	11.68	465	–	0.86%
10	Out of memory		13.920	5405	–	13.959	1127	–	0.41%

Notes. Gap-E(%): The difference in objective function compared with Benders decomposition; Gap-B(%): The difference in objective function compared with exact method.

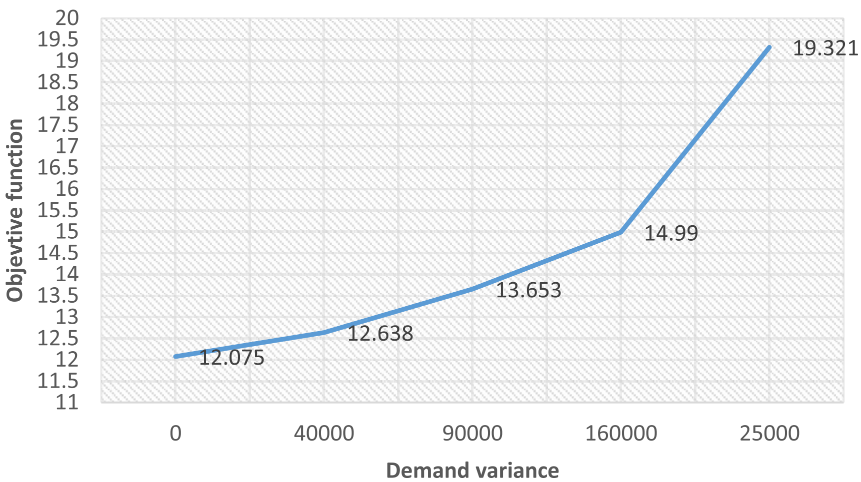

FIGURE 4. Objective function values *versus* variance demand.

8. SENSITIVITY ANALYSIS

According to the related literature and due to the less attention paid to demand uncertainty in the similar-to-this-article studies (specifically those considered discount), this section emphasizes the importance of demand fluctuations.

The problem is solved with different variances in demand. The results can be seen in Figure 4. An exponential increase in the objective function is observed by increasing the variance. This increase indicates the importance of simultaneous consideration of supply and demand uncertainties.

In Figure 5, the treatment of mean demand varies for a constant variance. The rise in demand results in a corresponding increase in the objective function. This increase is explained by the fact that the objective function represents the cost associated with meeting each unit of demand and does not consider income (as

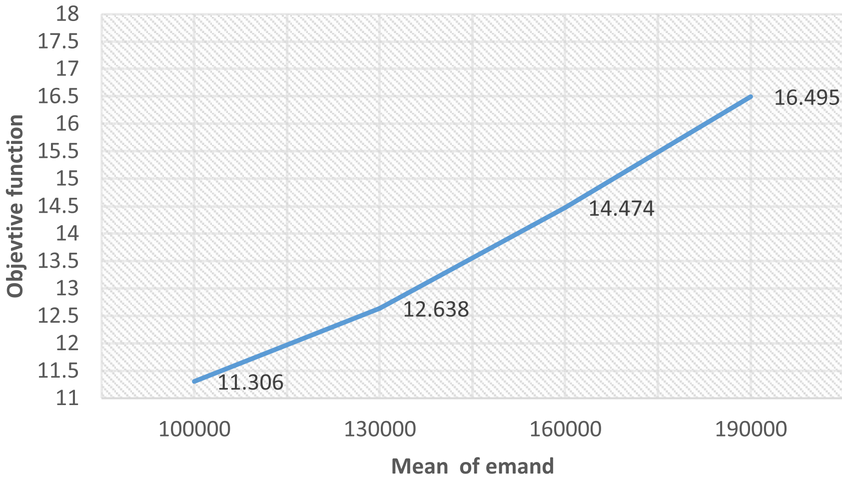

FIGURE 5. Objective function values *versus* mean of demand.

any increase in income also escalates costs). A more critical factor is that the company must fulfil all units of demand. Due to constraints in supplier capacity and potential disruptions, the company may have to procure from more expensive suppliers or meet additional demand in the second stage, incurring shortage costs.

9. Conclusions and recommendations

This article dealt with capacitated firm subject to demand and supply uncertainties. A combination of uncertainties that have not been addressed in the literature was concerned with a multi-period model with an all-unite discount. The closest study to our article in terms of considering supply uncertainty and disruption occurrence is [17], which assumed a definite demand in his article. Also, most articles that considered discounts did not regard uncertainties significantly. Meena and Sarmah [14] and Alfares and Turnadi [1] modelled the discount in their studies but ignored consideration of demand uncertainty. Thus, the model developed in this article comprehensively covered the demand and supply uncertainties with discount. Paying attention to these uncertainties increases the size and complexity of the problem. So, exact solutions are not responsive anymore. Hence, this article initially applied PHA to solve the problem. However, because of the large number of scenarios (which have to be solved independently by the algorithm) and the low convergence rate, we proposed an augmented PHA to accelerate the process. First, the penalty coefficient was updated concerning concepts pertinent to the problem by combining different methods existing in the literature. Different scenarios were then clustered based on the demand. Instead of solving them individually, the groups of scenarios were solved, which led to a considerable decrease in the running time with acceptable accuracy. Finally, because less attention was paid to the uncertainty of demand, a sensitivity analysis was made for the fluctuation in the objective function based on the changes in demand level. The analysis showed that any increase in variance and the mean of demand increases the objective function. This article studied all-unite discounts. Developing scenarios for recovery time such that the supplier or production line cannot be recovered at once and does not return to the normal status gradually can be recommended as a topic for future studies. Another area can be the consideration of incremental discounts instead of all-unit discounts.

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