RAIRO-Oper. Res. 59 (2025) 3069–3077
https://doi.org/10.1051/ro/2025112

RAIRO Operations Research
www.rairo-ro.org

ON THE SECOND LARGEST LAPLACIAN EIGENVALUE OF TREES

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Abstract. We determine all trees for which the second largest Laplacian eigenvalue is at most $2+\sqrt{3}$ as well as all trees with exactly one Laplacian eigenvalue exceeding $2+\sqrt{3}$.

Mathematics Subject Classification. 05C50, 15A18.

Received July 22, 2024. Accepted July 30, 2025.

1. Introduction

We consider simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, if $uv \in E(G)$, we say u is adjacent to v in G, denoted by $u \sim v$, otherwise we say u is not adjacent to v. For $v \in V(G)$, let $d_G(v)$ be the degree of v in G, *i.e.*, the number of edges incident with v in G. Let $A(G) = (a_{uv})_{u,v\in V(G)}$ be the adjacency matrix, where $a_{uv} = 1$ if $u \sim v$ and $a_{uv} = 0$ otherwise. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G. The Laplacian characteristic polynomial $\phi(G)$ of G is just the characteristic polynomial of $L(G)$, that is, $\phi(G) = \det(xI_n - L(G))$, where $n = |V(G)|$ and I_n is the identity matrix of order n. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues of G. Evidently, $L(G)$ is a real symmetric matrix with row sums zero, and we know from Gershgorin's theorem that the Laplacian eigenvalues of G are nonnegative real numbers and the smallest one is zero. We can assume that the Laplacian eigenvalues of G are

$$\lambda_1(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0.$$

That is, $\lambda_i(G)$ is the ith largest Laplacian eigenvalue of G and $\lambda_{n-i+1}(G)$ is the ith smallest Laplacian eigenvalue for $i = 1, \ldots, n$. Let $\overline{G}$ denote the complement of a graph G of order $n \geq 2$. Let J_n denote the matrix with all entries equal to 1. Clearly, $L(G) + L(\overline{G}) = nI_n - J_n$, so $\lambda_i(G) + \lambda_{n-(i+1)+1}(\overline{G}) = n$ for $i = 1, \ldots, n-1$. Thus, studying the ith largest Laplacian eigenvalue of a graph is equivalent to studying the $(i+1)$th smallest Laplacian eigenvalue of its complement for $i = 1, \ldots, n-1$.

Investigating the properties of the Laplacian spectrum is an important field not only for matrix theory and related applications in chemistry and physics, but for graph theory as well [5, 6, 9, 13, 15]. In particular, it is of interest to determine the graphs with constraints on the Laplacian eigenvalues. For example, Guo [7] determined the trees for which the second largest Laplacian eigenvalues achieve the first three smallest values. Petrović *et al.* [14] determined all connected bipartite graphs with one Laplacian eigenvalue exceeding three. Li *et al.* [11] determined all connected graphs with the second largest Laplacian eigenvalue at most three, and

Keywords. Laplacian spectrum, second largest Laplacian eigenvalue, trees.

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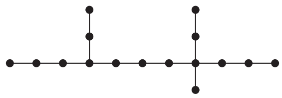

FIGURE 1. Graph $D^4_{3,2;3,2,1}$.

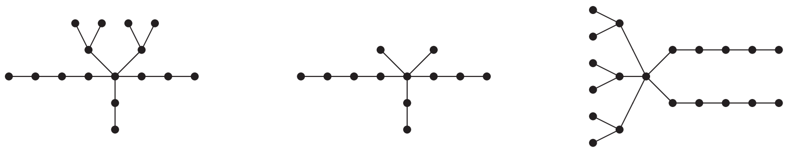

FIGURE 2. Graphs $F_{2,4,3,2}$, $S_{4,3,2,1,1}$, and $F_{3,5,5}$.

hence all trees with the second largest Laplacian eigenvalue at most three. More results along this line may be found in the literature, see, *e.g.*, [1, 8, 12, 13]. Motivated by these work, in this paper, we characterize all trees with one Laplacian eigenvalue exceeding $2+\sqrt{3}$ (*i.e.*, trees with the second largest Laplacian eigenvalue at most $2+\sqrt{3}$). One reason we consider only $2+\sqrt{3}$ is that there are infinite trees with $2+\sqrt{3}$ as the second largest Laplacian eigenvalue.

To state our result, we introduce the notations used.

Let P_n and $K_{1,n-1}$ be the path and the star on n vertices, respectively.

For positive integers k, $p_1 \geq \cdots \geq p_s$ and $q_1 \geq \cdots \geq q_t$ with $s \geq 2, t \geq 2$, let $D^k_{p_1,\ldots,p_s;q_1,\ldots,q_t}$ be the tree obtained from a path of length k with end vertices u and v by attaching s pendant paths of lengths $p_1,\ldots,p_s$ at u and attaching t pendant paths of lengths $q_1,\ldots,q_t$ at v. For example, the tree $D^4_{3,2;3,2,1}$ is shown in Figure 1.

For nonnegative integers a, b and positive integers $q_1 \geq \cdots \geq q_b$, denote $F_{a,q_1,\ldots,q_b}$ the tree obtained from $aK_{1,3} \cup P_{q_1+1} \cup \cdots \cup P_{q_b+1}$ by identifying a pendant vertex from each $K_{1,3}$ and an end vertex of each P_{q_i+1} for $i = 1,\ldots,b$. If $a = 0$ and $b \geq 3$, we write $S_{q_1,\ldots,q_b}$ instead of $F_{a,q_1,\ldots,q_b}$. For example, $F_{2,4,3,2}$, $S_{4,3,2,1,1}$ and $F_{3,5,5}$ are displayed in Figure 2.

Let

$$\mathcal{T}_1 = \{S_{6,1,1}, S_{6,2,1}, S_{6,3,1}, S_{6,2,2}, S_{6,3,2}, S_{6,4,1}\},$$

$$\begin{aligned}\mathcal{T}_2 = &\left\{D^1_{2,1;2,1}, D^1_{2,2;2,1}, D^1_{2,2;2,2}\right\} \cup \left\{D^1_{2,1;2,p,q} : p = 1,2\right\} \cup \left\{D^1_{2,1;3,p,q} : p = 1,2\right\} \\ &\cup \left\{D^1_{2,1;4,1,1}, D^1_{2,1;4,2,1}\right\} \cup \left\{D^1_{2,1;1,1,1}, D^1_{2,1;1,1,1,1}, D^1_{2,2;1,1,1}, D^1_{2,1;3,3,1}\right\},\end{aligned}$$

$$\begin{aligned}\mathcal{T}_3 = &\left\{D^1_{3,q;2,t} : q = 1,2,3 \text{ and } t = 1,2\right\} \cup \left\{D^1_{3,q;3,t} : q = 1,2 \text{ and } t = 1,2\right\} \\ &\cup \left\{D^1_{3,1;3,3}, D^1_{3,1;1,1,1}\right\},\end{aligned}$$

$$\begin{aligned}\mathcal{T}_4 = &\left\{D^1_{4,q;2,1} : q = 1,2,3,4\right\} \cup \left\{D^1_{4,q;2,2} : q = 1,2\right\} \cup \left\{D^1_{4,q;3,1} : q = 1,2\right\} \\ &\cup \left\{D^1_{4,1;3,2}, D^1_{4,1;4,1}\right\},\end{aligned}$$

$$\mathcal{T}_5 = \left\{D^1_{5,1;2,1}, D^1_{5,2;2,1}\right\}.$$

By Observation 3.3 and Theorem 3.4 of [11], trees T with $\lambda_2(T) \leq 3$ are $K_{1,n}$ with $n \geq 1$, P_n with $n = 4,5,6$, $F_{1,1,1}$, or $S_{\ell_1,\ldots,\ell_s}$ with $s \geq 3$ and $\ell_1 = 2$.

The main result of this paper is as follows.

Theorem 1.1. *Let T be a nontrivial tree. Then $\lambda_2(T) \leq 2+\sqrt{3}$ if and only if T is one of the following trees:*

(i) $K_{1,n}$ *with* $n \geq 1$*;*
(ii) P_n *with* $n = 4, \dots, 12$*;*
(iii) *a tree in* $\cup_{i=1}^{5} \mathcal{T}_i$*;*
(iv) $S_{\ell_1,\dots,\ell_s}$ *with* $s \geq 3$ *and* $2 \leq \ell_1 \leq 5$*;*
(v) F_a *for* $a \geq 2$*;*
(vi) $F_{a,q_1,\dots,q_b}$ *for* $a \geq 1, b \geq 1$ *and* $1 \leq q_1 \leq 5$.

As immediate consequences, we determine completely all trees with exactly one Laplacian eigenvalue exceeding $2+\sqrt{3}$ as well as all trees for which the second Laplacian value is equal to $2+\sqrt{3}$.

2. Preliminaries

For a subset V_0 of vertices of a graph G, $G[V_0]$ denotes the subgraph induced by V_0. A path $u_1 \dots u_r$ (with $r \geq 2$) in a graph G is called a pendant path (of length $r-1$) at u_1 if $d_G(u_1) \geq 2$, the degrees of $u_2, \dots, u_{r-1}$ (if any exists) are all equal to two in G, and $d_G(u_r) = 1$. If P is a pendant path of G at u with length $r \geq 1$, we say G is obtained from H by attaching a pendant path of length r at u with $H = G[V(G) \setminus (V(P) \setminus \{u\})]$.

The union of two disjoint graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. We write kG_1 instead of the union of k copies of G_1.

The following lemma is obtained by the well-known Courant-Weyl inequalities, see [13].

Lemma 2.1. *Let G be a graph with $e \in E(\overline{G})$. Let $G' = G + e$. Then*

$$\lambda_1(G') \geq \lambda_1(G) \geq \lambda_2(G') \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G') \geq \lambda_n(G) = 0.$$

Lemma 2.2 ([10])**.** *For a nontrivial connected graph G with at least three vertices, $\lambda_2(G) \geq d_2$, where d_2 is the second largest degree of G.*

For nonnegative integers a, b, let $T_{a,b} = F_{a,q_1,\dots,q_b}$, where $q_1 = \cdots = q_b = 5$.

Lemma 2.3. *For $a, b \geq 0$ with $3a + 5b > 5$,*

$$\lambda_2(T_{a,b}) \leq 2 + \sqrt{3}$$

with equality if and only if $a \geq 2$.

Proof. Note that $T_{a,b}$ is symmetric. By the graphic approach in [2], the spectrum of $L(T_{a,b})$ is the union of the spectra of the following divisor matrices:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & a+b & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \underbrace{\begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}}_{(a-1)-\text{times}},$$

$$\underbrace{\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}}_{(b-1)-\text{times}} \text{ and } (1).$$

By a direct calculation and the method to describing the spectra of a tridiagonal matrix in [3], the corresponding characteristic polynomials are respectively $xh(x), ((x-1)(x^2-4x+1))^{a-1}, g(x)^{b-1}$ and $x-1$, where

$$g(x) = x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1,$$

and

$$h(x) = x^7 - (13+a+b)x^6 + (65+12a+12b)x^5 - (156+55a+54b)x^4 \\ + (183+119a+112b)x^3 - (96+120a+106b)x^2 + (19+46a+40b)x - 3a - 5b - 1.$$

Thus

$$\phi(T_{a,b}) = x(x-1)^a(x^2-4x+1)^{a-1}g(x)^{b-1}h(x). \tag{2.1}$$

Suppose first that $b=0$. Then $a \geq 2$ as $3a+5b > 5$. Note that $h(x) = g(x)(x-1)(x^2-(a+4)x+3a+1)$. By (2.1),

$$\phi(T_{a,0}) = x(x-1)^a(x^2-4x+1)^{a-1}(x^2-(a+4)x+3a+1).$$

So the Laplacian eigenvalues of $T_{a,0}$ are 0 with multiplicity 1, 1 with multiplicity a, $2 \pm \sqrt{3}$ with multiplicity $a-1$, and $\frac{a+4\pm\sqrt{a^2-4a+12}}{2}$ with multiplicity 1. Note that $\frac{a+4-\sqrt{a^2-4a+12}}{2} < 3$ and $\frac{a+4+\sqrt{a^2-4a+12}}{2} > 4$. Thus $\lambda_2(T_{a,0}) = 2+\sqrt{3}$, as desired.

Suppose next that $a=0$. Then $b \geq 2$. By (2.1), we have

$$\phi(T_{0,b}) = xg(x)^{b-1}q(x),$$

where

$$q(x) = x^5 - (9+b)x^4 + (8b+28)x^3 - (21b+35)x^2 + (20b+15)x - 5b - 1.$$

Note that

$$\begin{aligned} g(0) &= -1 < 0, \\ g(2-\sqrt{3}) &= 1 > 0, \\ g(1) &= -1 < 0, \\ g(2) &= 1 > 0, \\ g(3) &= -1 < 0, \\ g(2+\sqrt{3}) &= 1 > 0. \end{aligned}$$

So all roots of $g(x)=0$ are less than $2+\sqrt{3}$. Note also that

$$\begin{aligned} q(2-\sqrt{3}) &= -b+1 < 0, \\ q(1) &= b-1 > 0, \\ q(2) &= -b+1 < 0, \\ q(3) &= b-1 > 0, \\ q(2+\sqrt{3}) &= -b+1 < 0. \end{aligned}$$

Then the roots of $q(x)=0$ lie in the intervals $(2-\sqrt{3},1)$, $(1,2)$, $(2,3)$, $(3,2+\sqrt{3})$ and $(2+\sqrt{3},+\infty)$, respectively. As $\lambda_2(T_{0,b})$ is the second largest root of $g(x)q(x)=0$, we have $\lambda_2(T_{0,b}) < 2+\sqrt{3}$, as desired.

TABLE 1. Forbidden subgraphs and their second largest Laplacian eigenvalues.

G	$\lambda_2(G)$	G	$\lambda_2(G)$	G	$\lambda_2(G)$
$S_{7,1,1}$	3.7678	$S_{6,1,1,1}$	3.7387	$S_{6,5,1}$	3.7453
$S_{6,4,2}$	3.7382	$S_{6,3,3}$	3.7369	$D^5_{1,1;1,1}$	4.1701
$D^4_{1,1;1,1}$	4.1149	$D^3_{1,1;1,1}$	4	$D^2_{1,1;2,1}$	3.8408
$D^2_{1,1;1,1,1}$	4	$D^1_{6,1;1,1}$	3.7679	$D^1_{5,1;3,1}$	3.7519
$D^1_{5,1;2,2}$	3.7624	$D^1_{5,3;2,1}$	3.7369	$D^1_{4,2;4,1}$	3.7618
$D^1_{4,1;3,3}$	3.7577	$D^1_{4,2;3,2}$	3.7719	$D^1_{4,3;3,1}$	3.7360
$D^1_{4,3;2,2}$	3.7557	$D^1_{3,2;3,3}$	3.7679	$D^1_{2,1;5,1,1}$	3.7537
$D^1_{2,1;4,3,1}$	3.7448	$D^1_{2,1;4,2,2}$	3.7497	$D^1_{2,1;3,3,2}$	3.7465
$D^1_{3,1;2,1,1}$	3.7491	$D^1_{2,2;2,1,1}$	3.7871	$D^1_{2,1;2,1,1,1}$	3.7543
$D^1_{4,1;1,1,1}$	3.7773	$D^1_{3,2;1,1,1}$	3.8027	$D^1_{3,1;1,1,1,1}$	3.8315
$D^1_{2,2;1,1,1,1}$	3.8920	$D^1_{2,1;1,1,1,1,1}$	3.7996		

Suppose finally that $a > 0$ and $b > 0$. As above, all roots of $g(x) = 0$ are less than $2 + \sqrt{3}$. As

$$\begin{aligned} h(0) &= -3a - 5b - 1 < 0, \\ h(2-\sqrt{3}) &= (1+\sqrt{3})a > 0, \\ h(1) &= -2(a+b-1) < 0, \\ h(2) &= a + 3b - 3 > 0, \\ h(3) &= -2(b-1) < 0, \\ h\left(\frac{73}{20}\right) &= \frac{214}{605}a + \frac{363}{5350}b + \frac{844}{5589} > 0, \\ h(2+\sqrt{3}) &= (1-\sqrt{3})a < 0, \end{aligned}$$

the roots of $h(x) = 0$ are in the intervals $(0, 2-\sqrt{3})$, $(2-\sqrt{3}, 1)$, $(1, 2)$, $(2, 3)$, $(3, \frac{73}{20})$, $(\frac{73}{20}, 2+\sqrt{3})$ and $(2+\sqrt{3}, +\infty)$, respectively. Thus only one root of $h(x) = 0$ exceeds $2 + \sqrt{3}$. If $a \geq 2$, then as a root of $x^2 - 4x + 1 = 0$, $2 + \sqrt{3}$ is also a Laplacian eigenvalue of $T_{a,b}$. Therefore $\lambda_2(T_{a,b}) \leq 2 + \sqrt{3}$ with equality if and only if $a \geq 2$.

Combining above three cases, $\lambda_2(T_{a,b}) \leq 2 + \sqrt{3}$ with equality if and only if $a \geq 2$. □

By Lemmas 2.1 and 2.3, we have

Corollary 2.1. *Let $T \cong F_{a,q_1,\ldots,q_b}$ for $a \geq 2, b \geq 0$ and $1 \leq q_1 \leq 5$. Then $\lambda_2(T) = 2 + \sqrt{3}$.*

3. Proof of Theorem 1.1

Let H be a graph with $\lambda_2(H) > 2 + \sqrt{3}$. Let G be a connected graph with $\lambda_2(G) \leq 2 + \sqrt{3}$. By Lemma 2.1, H can not be a subgraph of G, that is, H is forbidden in G. In this paper, we are concerned with a connected graph G with $\lambda_2(G) \leq 2 + \sqrt{3}$, so any connected graph H with $\lambda_2(H) > 2 + \sqrt{3}$ is called a forbidden subgraph. It is not difficult to check that the following 32 trees in Table 1 are forbidden subgraphs.

For simplicity, we compute the second largest Laplacian eigenvalues of some graphs, which are listed in Table 2. We refer to Table 2 frequently in our proof.

TABLE 2. $\lambda_2(G)$ of graphs G.

G	$\lambda_2(G)$	G	$\lambda_2(G)$	G	$\lambda_2(G)$
$S_{6,4,1}$	$2+\sqrt{3}$	$S_{6,3,2}$	$2+\sqrt{3}$	$D^1_{3,2;3,2}$	$2+\sqrt{3}$
$D^1_{3,3;2,2}$	3.7202	$D^1_{3,1;3,3}$	3.7030	$D^1_{4,4;2,1}$	3.7014
$D^1_{4,2;2,2}$	$2+\sqrt{3}$	$D^1_{4,2;3,1}$	3.7163	$D^1_{4,1;3,2}$	$2+\sqrt{3}$
$D^1_{4,1;4,1}$	$2+\sqrt{3}$	$D^1_{5,2;2,1}$	$2+\sqrt{3}$	$D^1_{2,1;1,1,1,1}$	$2+\sqrt{3}$
$D^1_{2,2;1,1,1}$	$2+\sqrt{3}$	$D^1_{3,1;1,1,1}$	3.7046	$D^1_{2,1;3,2,2}$	$2+\sqrt{3}$
$D^1_{2,1;3,3,1}$	3.7263	$D^1_{2,1;4,2,1}$	$2+\sqrt{3}$		

Proof of Theorem 1.1. Since $\lambda_2(P_n) = 2 - 2\cos\frac{(n-2)\pi}{n}$, we have $\lambda_2(P_n) \le 2+\sqrt{3}$ for $n \le 12$ with equality if and only if $n = 12$. Any graph in $\cup_{i=1}^5 \mathcal{T}_i$ is a subgraph of some graph in Table 2 or some graph $F_{a,q_1,\dots,q_b}$ (and hence $T_{a,b}$) with $1 \le q_1 \le 5$. Note that the second largest Laplacian eigenvalue of any graph in Table 2 is at most $2+\sqrt{3}$. By Lemma 2.3, $\lambda_2(T_{a,b}) \le 2+\sqrt{3}$ for $a, b \ge 0$ with $3a+5b > 5$. By Lemma 2.1, the second largest Laplacian eigenvalue of any graph in $\cup_{i=1}^5 \mathcal{T}_i$ and $F_{a,q_1,\dots,q_b}$ also does not exceed $2+\sqrt{3}$.

Let T be a tree on $n \ge 3$ vertices with second largest Laplacian eigenvalue at most $2+\sqrt{3}$. It suffices to show that T is one tree in (i)–(vi) in Theorem 1.1.

Let Δ be the maximum degree and d_2 the second largest degree of T. By Lemma 2.2, $d_2 \le \lambda_2(T) \le 2+\sqrt{3}$, so $d_2 \le 3$.

If $d_2 = 1$, then $T \cong K_{1,n}$ for some $n \ge 1$.

Suppose that $d_2 = 2$. If $\Delta = 2$, then $T \cong P_n$ with $n \ge 3$. Since $\lambda_2(P_n) = 2 - 2\cos\frac{(n-2)\pi}{n}$, we have $n \le 12$.

Suppose that $\Delta \ge 3$. Then there is exactly one vertex of degree at least 3 in T, so $n \ge 5$ and it is easily seen that $T \cong S_{\ell_1,\dots,\ell_\Delta}$ for $\ell_1 \ge \cdots \ge \ell_\Delta \ge 1$ and $\ell_1 \ge 2$. From Table 1, $S_{7,1,1}$ is a forbidden subgraph, so $\ell_1 \le 6$. It follows that $2 \le \ell_1 \le 6$. Suppose that $\ell_1 = 6$. From Table 1, $S_{6,1,1,1}$ is a forbidden subgraph, so $\Delta = 3$. Similarly, as $S_{6,5,1}$ is a forbidden subgraph, we have $\ell_2 \le 4$. If $\ell_2 = 4$, then, as $S_{6,4,2}$ is a forbidden subgraph, we have $\ell_3 = 1$, so $T \cong S_{6,4,1}$. If $\ell_2 = 3$, then, as $S_{6,3,3}$ is a forbidden subgraph, we have $\ell_3 \le 2$, so $T \cong S_{6,3,1}$, $S_{6,3,2}$. If $\ell_2 \le 2$, then $T \cong S_{6,1,1}$, $S_{6,2,1}$, $S_{6,2,2}$. Thus $T \in \mathcal{T}_1$.

If $\ell_1 \le 5$, then $T \cong S_{\ell_1,\dots,\ell_\Delta}$ with $2 \le \ell_1 \le 5$ and $\Delta \ge 3$.

Suppose next that $d_2 = 3$.

Let u be a vertex with degree 3 in T and v a vertex with maximum degree in T such that $d_T(u,v)$ is as large as possible. It follows that T contains a subgraph $T' := D^k_{p_1,p_2;q_1,q_2}$, where $k = d_T(u,v)$. If $k \ge 6$, then T' contains $S_{7,1,1}$, which is a forbidden subgraph from Table 1, a contradiction. So $k \le 5$. Since $D^5_{1,1;1,1}$, $D^4_{1,1;1,1}$ and $D^3_{1,1;1,1}$ are all forbidden subgraphs, we have $k \le 2$.

Case 1. $\Delta = 3$.

Note that there are exactly two pendant paths of lengths p_1 and p_2 at u and two pendant paths of lengths q_1, q_2 at v.

Suppose that $k = 2$. If $p_1 \ge 2$ or $q_1 \ge 2$, then T' contains $D^2_{2,1;1,1}$, which is a forbidden subgraph according to Table 1, a contradiction. So $p_1 = q_1 = 1$. Let w be the common neighbor of u and v in T. Evidently, $d_T(w) = 2, 3$.

If $d_T(w) = 2$, then $T \cong D^2_{1,1;1,1} = F_2$.

Suppose that $d_T(w) = 3$. Then either there is a pendant path at w in T or there is exactly one vertex, say z of degree 3 different from u and v that is adjacent with w in T. In former case, as $S_{7,1,1}$ is a forbidden graph, we have $T \cong F_{2,q}$ with $1 \le q \le 5$. In later case, as $D^2_{2,1;1,1}$ is a forbidden graph, the two pendant paths at z in T are of length 1, so $T \cong F_3$.

Suppose next that $k = 1$. As $D^1_{6,1;1,1}$ is a forbidden subgraph of T, we get $p_1, q_1 \leq 5$. Suppose without loss of generality that $p_1 \geq q_1$.

If $q_1 = 1$, then $T \cong D^1_{p_1,p_2;1,1} = F_{1,p_1,p_2}$ for $1 \leq p_1 \leq 5$.

Suppose in the following that $q_1 \geq 2$. Then $p_1 = 2, 3, 4, 5$.

Case 1.1. $p_1 = 2$.

In this case, $T \cong D^1_{2,1;2,1}$, $D^1_{2,2;2,1}$, $D^1_{2,2;2,2}$, so $T \in \mathcal{T}_2$.

Case 1.2. $p_1 = 3$.

Then $q_1 = 2, 3$.

If $q_1 = 2$, then $T \cong D^1_{3,p_2;2,q_2}$ with $1 \leq p_2 \leq 3$ and $1 \leq q_2 \leq 2$. It is easily checked that $T \in \mathcal{T}_3$.

Suppose that $q_1 = 3$. Then $q_2 = 2, 3$. Suppose that $q_2 = 2$. Since $D^1_{3,3;3,2}$ is a forbidden subgraph, we have $p_2 \leq 2$, so $T \cong D^1_{3,p_2;3,q_2}$, where $p_2 = 1, 2$, $q_2 = 1, 2$ and $p_2 \leq q_2$. Thus $T \in \mathcal{T}_3$. If $q_2 = 3$, then as $D^1_{3,2;3,3}$ is a forbidden subgraph, we have $p_2 = 1$, *i.e.*, $T \cong D^1_{3,1;3,3} \in \mathcal{T}_3$.

Case 1.3. $p_1 = 4$.

Then $q_1 = 2, 3, 4$. Suppose that $q_1 = 2$. If $q_2 = 1$, then $T \cong D^1_{4,p_2;2,1}$ with $1 \leq p_2 \leq 4$, and so $T \in \mathcal{T}_4$. If $q_2 = 2$, then, as $D^1_{4,3;2,2}$ is a forbidden subgraph, $p_2 \leq 2$, it follows that $T \cong D^1_{4,1;2,2}$ or $D^1_{4,2;2,2}$, and so $T \in \mathcal{T}_4$. Suppose that $q_1 = 3$. As $D^1_{4,1;3,3}$ is a forbidden subgraph, $q_2 \leq 2$. If $q_2 = 1$, then since $D^1_{4,3;3,1}$ is a forbidden subgraph, we have $p_2 \leq 2$, which follows that $T \cong D^1_{4,1;3,1}$ or $D^1_{4,2;3,1}$, and so $T \in \mathcal{T}_4$. If $q_2 = 2$, then as $D^1_{4,2;3,2}$ is a forbidden subgraph, we have $p_2 = 1$, *i.e.*, $T \cong D^1_{4,1;3,2} \in \mathcal{T}_4$. If $q_1 = 4$, then since $D^1_{4,2;4,1}$ is a forbidden subgraph, we have $p_2 = 1$, *i.e.*, $T \cong D^1_{4,1;4,1} \in \mathcal{T}_4$.

Case 1.4. $p_1 = 5$.

Since $D^1_{5,1;3,1}$ is a forbidden subgraph, $q_1 = 2$. As $D^1_{5,1;2,2}$ is a forbidden subgraph, $q_2 = 1$, and as $D^1_{5,3;2,1}$ is a forbidden subgraph, $p_2 \leq 2$. It follows that $T \cong D^1_{5,1;2,1}$ or $D^1_{5,2;2,1}$. Thus $T \in \mathcal{T}_5$.

Case 2. $\Delta \geq 4$.

As $D^2_{1,1;1,1,1}$ is a forbidden subgraph, we have $d_T(u, v) = 1$. As $d_T(u, v)$ is as large as possible, each vertex of degree 3 is adjacent to v in T, and there are two pendant paths at any neighbor of v with degree 3 in T. Let a be the number of vertices of degree 3 in T, and let $t = \Delta - a$. Evidently $a \leq \Delta$. Then there are exactly two pendant paths of lengths, say p_1 and p_2, at u and t pendant paths of lengths, say $q_1, \ldots, q_t$, at v, where $p_1 \geq p_2 \geq 1$, $q_1 \geq \cdots \geq q_t \geq 1$. It follows that T contains a subgraph isomorphic to $F_{a,q_1,\ldots,q_t}$.

Case 2.1. $a \geq 2$.

As $S_{6,1,1,1}$ is a forbidden subgraph, any pendant path at v has length at most 5. As $D^2_{1,1;2,1}$ is a forbidden subgraph, each pendant path at any neighbor of v with degree 3 in T is of length 1. Thus $T \cong F_{a,q_1,\ldots,q_t}$ for $a \geq 2$, $1 \leq q_1 \leq 5$.

Case 2.2. $a = 1$.

In this case, $T \cong D^1_{p_1,p_2;q_1,\ldots,q_t}$, where $p_1 \geq p_2 \geq 1$ and $q_1 \geq \cdots \geq q_t \geq 1$. As $S_{6,1,1,1}$ is a forbidden subgraph, we have $q_1 \leq 5$.

If $q_1 = 5$, then as $D^1_{2,1;5,1,1}$ is a forbidden subgraph, we have $p_1 = 1$. So, if $p_1 = 1$, then $T \cong D^1_{1,1;q_1,\ldots,q_t} = F_{1,q_1,\ldots,q_t}$ with $q_1 \leq 5$.

Suppose that $q_1 \leq 4$ and $p_1 \geq 2$.

Case 2.2.1. $q_1 = 1$.

As $D^1_{4,1;1,1,1}$ is a forbidden subgraph, we have $p_1 \leq 3$, *i.e.*, $p_1 = 2, 3$.

Suppose that $p_1 = 2$. Then $p_2 = 1, 2$. If $p_2 = 1$, then as $D^1_{2,1;1,1,1,1,1,1}$ is a forbidden subgraph, we have $\Delta = 4, 5$, so $T \cong D^1_{2,1;1,1,1}$ or $D^1_{2,1;1,1,1,1}$, implying that $T \in \mathcal{T}_2$. If $p_2 = 2$, then as $D^1_{2,2;1,1,1,1}$ is a forbidden subgraph, we have $\Delta = 4$, so $T \cong D^1_{2,2;1,1,1} \in \mathcal{T}_2$. Suppose that $p_1 = 3$. As $D^1_{3,1;1,1,1,1}$ is a forbidden subgraph, we have $\Delta = 4$. As $D^1_{3,2;1,1,1}$ is a forbidden subgraph, we have $q_2 = 1$. Thus $T \cong D^1_{3,1;1,1,1} \in \mathcal{T}_3$.

Case 2.2.2. $q_1 = 2$.

As $D^1_{3,1;2,1,1}$ is a forbidden subgraph, we have $p_1 = 2$. As $D^1_{2,2;2,1,1}$ is a forbidden subgraph, we have $p_2 = 1$. If $\Delta \geq 5$, then T contains $D^1_{2,1;2,1,1,1}$ as a subgraph, which is a contradiction by Lemma 2.1 and Table 1. So $\Delta = 4$, implying that $T \cong D^1_{2,1;2,1,1}, D^1_{2,1;2,2,1}$ or $D^1_{2,1;2,2,2}$. Thus $T \in \mathcal{T}_2$.

Case 2.2.3. $q_1 = 3$.

As $D^1_{3,1;2,1,1}$ and $D^1_{2,2;2,1,1}$ are forbidden subgraphs, we have $p_1 = 2$ and $p_2 = 1$. Since T can not contain $D^1_{2,1;2,1,1,1}$ as a subgraph, we have $\Delta = 4$. Note that

$$\lambda_2(D^1_{2,1;3,3,2}) > 2 + \sqrt{3},$$
$$\lambda_2(D^1_{2,1;3,2,2}) = 2 + \sqrt{3},$$
$$\lambda_2(D^1_{2,1;3,3,1}) < 2 + \sqrt{3}.$$

So we have $T \cong D^1_{2,1;3,1,1}, D^1_{2,1;3,2,1}, D^1_{2,1;3,3,1}$ or $D^1_{2,1;3,2,2}$. That is, either $T \subseteq D^1_{2,1;3,3,1}$ or $T \cong D^1_{2,1;3,2,2}$. Thus $T \in \mathcal{T}_2$.

Case 2.2.4. $q_1 = 4$.

As $D^1_{3,1;2,1,1}$ and $D^1_{2,2;2,1,1}$ are forbidden subgraphs, we have $p_1 = 2$ and $p_2 = 1$. Since $D^1_{2,1;2,1,1,1}$ can not be a subgraph in T, it has $\Delta = 4$. As $D^1_{2,1;4,3,1}$ and $D^1_{2,1;4,2,2}$ are forbidden subgraphs, it has $q_2 \le 2$ and $q_3 = 1$, which follows that $T \cong D^1_{2,1;4,1,1}$ or $D^1_{2,1;4,2,1}$, and so $T \in \mathcal{T}_2$.

Combining all above cases, the proof is completed. □

Corollary 3.1. *Let T be a tree with exactly one Laplacian eigenvalue exceeding $2 + \sqrt{3}$. Then T is one of the following trees:*

(i) $K_{1,n}$ *with* $n \ge 3$;
(ii) P_n *with* $n = 7, \ldots, 12$;
(iii) *a tree in* $\cup_{i=1}^{5} \mathcal{T}_i$;
(iv) $S_{\ell_1,\ldots,\ell_s}$ *with* $s \ge 3$ *and* $2 \le \ell_1 \le 5$;
(v) F_a *for* $a \ge 2$;
(vi) $F_{a,q_1,\ldots,q_b}$ *for* $a \ge 2, b \ge 1$ *and* $1 \le q_1 \le 5$.

Proof. Let n be the order of T. As $\lambda_1(T) \le n$ (see [12]), we have $n \ge 4$. Let Δ be the maximum degree of T. Evidently, $\Delta \ge 2$. If $\Delta = 2$, then $T \cong P_n$. By Theorem 1.1, $n \le 12$. Since $\lambda_1(P_n) = 2 - 2\cos\frac{(n-1)\pi}{n} > 2 + \sqrt{3}$, we have $n \ge 7$. Thus $T \cong P_n$ for $7 \le n \le 12$. Suppose that $\Delta \ge 3$. Note that $\lambda_1(T) \ge \Delta + 1$ (see [12]). So $\lambda_1(T) > 2 + \sqrt{3}$. Thus T may be any tree in By Theorem 1.1 with $\Delta \ge 3$. □

By Theorem 1.1 and Corollary 2.1, we have

Corollary 3.2. *Let T be a tree. Then $\lambda_2(T) = 2 + \sqrt{3}$ if and only if either $T \in \{P_{12}, S_{6,4,1}, S_{6,3,2}, D^1_{3,2;3,2}, D^1_{4,2;2,2}, D^1_{4,1;3,2}, D^1_{4,1;4,1}, D^1_{5,2;2,1}, D^1_{2,1;1,1,1,1}, D^1_{2,2;1,1,1}, D^1_{2,1;3,2,2}, D^1_{2,1;4,2,1}\}$ or $T \cong F_{a,q_1,\ldots,q_b}$ for $a \ge 2$, $b \ge 0$, $1 \le q_1 \le 5$ and $3a + \sum_{i=1}^{b} q_i + 1 \ge 7$.*

Corollary 3.3. *Let G be a graph whose complement is a nontrivial tree. Then $\lambda_2(G) \ge n - 2 - \sqrt{3}$ if and only if $\overline{G}$ is one of the following trees:*

(i) $K_{1,n}$ *with* $n \ge 1$;
(ii) P_n *with* $n = 4, \ldots, 12$;
(iii) *a tree in* $\cup_{i=1}^{5} \mathcal{T}_i$;
(iv) $S_{\ell_1,\ldots,\ell_s}$ *with* $s \ge 3$ *and* $2 \le \ell_1 \le 5$;
(v) F_a *for* $a \ge 2$;
(vi) $F_{a,q_1,\ldots,q_b}$ *for* $a \ge 1, b \ge 1$ *and* $1 \le q_1 \le 5$.

Acknowledgments

We thank the referees for their insightful comments and suggestions.

Funding

This work was supported by the National Natural Science Foundation of China (No. 11801410) and the Guangdong Basic and Applied Basic Research Foundation (Nos. 2024A1515010493 and 2025A1515011917).

Data availability statement

The research data associated with this article are included in the article.

References

[1] S. Akbari, E.R. van Dam and M.H. Fakharan, Trees with a large Laplacian eigenvalue multiplicity. *Linear Algebra Appl.* **586** (2020) 262–273.

[2] M. Andelić and C.M. da Fonseca, Equitable partitions and spectra of symmetric trees: revisiting Heilbronner's composition principle. *Discrete Math. Lett.* **14** (2024) 108–117.

[3] C.M. da Fonseca, On the eigenvalues of some tridiagonal matrices. *Comput. Appl. Math.* **200** (2007) 283–286.

[4] N.M.M. de Abreu, Old and new results on algebraic connectivity of graphs. *Linear Algebra Appl.* **423** (2007) 53–73.

[5] R. Grone, On the geometry and Laplacian of a graph. *Linear Algebra Appl.* **150** (1991) 167–178.

[6] R. Grone, R. Merris and V.S. Sunder, The Laplacian spectrum of a graph. *SIAM J. Matrix Anal. Appl.* **11** (1990) 218–238.

[7] J.-M. Guo, On the second largest Laplacian eigenvalue of trees. *Linear Algebra Appl.* **404** (2005) 251–261.

[8] J.-M. Guo, On the Laplacian spectral radius of trees with fixed diameter. *Linear Algebra Appl.* **419** (2006) 618–629.

[9] D.P. Jacobs, E.R. Oliveira and V. Trevisan, Most Laplacian eigenvalues of a tree are small. *J. Comb. Theory Ser. B* **146** (2021) 1–33.

[10] J. Li and Y. Pan, A note on the second largest eigenvalue of the Laplacian matrix of a graph. *Linear Multilinear Algebra* **48** (2000) 117–121.

[11] J. Li, J.-M. Guo and W.C. Shiu, On the second largest Laplacian eigenvalues of graphs. *Linear Algebra Appl.* **438** (2013) 2438–2446.

[12] R. Merris, Laplacian matrices of a graph: a survey. *Linear Algebra Appl.* **197/198** (1994) 143–176.

[13] B. Mohar, The Laplacian spectrum of graphs, in Graph Theory, Combinatorics, and Applications, edited by Y. Alavi, G. Chartrand, O.R. Oellermann and A.J. Schwenk. Vol. 2. Wiley, New York (1991) 871–898.

[14] M. Petrović, I. Gutman, M. Lepović and B. Milekić, On bipartite graphs with small number of Laplacian eigenvalues greater than two and three. *Linear Multilinear Algebra* **47** (2000) 205–215.

[15] L. Xu and B. Zhou, Proof of a conjecture on distribution of Laplacian eigenvalues and diameter, and beyond. *Linear Algebra Appl.* **678** (2023) 92–106.