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HIGHER-ORDER DUALITY RESULTS FOR MULTIOBJECTIVE FRACTIONAL VARIATIONAL PROBLEMS INVOLVING SUPPORT FUNCTIONS AND ITS APPLICATION

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Abstract. This study investigates a class of multiobjective fractional variational problems that incorporate support functions. Utilizing the concept of higher-order pseudoinvexity, we establish duality results for these problems. We present an example of a non-trivial functional that demonstrates higher-order pseudoinvexity, distinguishing it from first-order pseudoinvexity. Our theoretical analysis elucidates the relationship between the primal and dual values within the proposed model. To validate the weak duality theorem, we also provide a practical example.

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1. Introduction

The calculus of variations is a fundamental tool for addressing a wide range of real-world problems, such as orbit optimization, rigid-body dynamics, and variational theory. In science and engineering, the primary goal often involves determining the optimal value of a definite integral that includes a specific function under fixed boundary conditions. Hanson's [10] pioneering work in 1964 revealed connections between duality results in mathematical programming and the calculus of variations. Following this, Mond and Hanson [22] extended these duality concepts to scalar-valued optimization problems. A dual pair of problems is termed symmetric when the original problem serves as the dual of its own dual. This concept of symmetric dual programs was developed and refined by researchers like Dorn [6] and Dantzig *et al.* [4]. Mond and Hanson [23] focused on the definition of symmetric duality in relation to the variational problems.

In the context of multiobjective variational programming, the focus shifts to optimizing multiple objective functions, which may involve either minimization or maximization. The earliest dual models for multiobjective variational problems, developed by Bector and Hussain [2], were based on convexity assumptions. Mishra and Mukherjee [19] used the parametric approach to study the usual duality results for a class of multiobjective fractional variational programs. Gulati *et al.* [9] studied the Mond-Weir type symmetric dual for a multiobjective

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variational problem and proved the weak and strong duality theorems under generalized convexity assumptions. Mishra *et al.* [20] studied the symmetric dual pair for nondifferentiable multiobjective fractional variational problems and discussed the usual duality theorems. The efficient solutions of multiobjective fractional variational problems and the non-fractional problem were related by Mitilelu and Stancu-Minasian [21] using a parametric approach. Further, Kailey and Gupta [15] achieved appropriate duality results for a pair of nondifferentiable multiobjective symmetric fractional variational dual programs over arbitrary cones under generalized convexity assumptions. More recently, researchers such as Upadhyay *et al.* [34] and Pokharna and Tripathi [28] have contributed to the study of multiobjective problems, highlighting their growing relevance in optimization theory.

Second-order dual models for variational programming were introduced by Chen [3]. Motivated by Chen [3], many authors have studied second-order duality for variational programming problems [8, 11, 13, 25, 27]. Various duality results for Mond-Weir type second-order variational dual programs were explored by Husain *et al.* [11]. Later, Gulati and Mehndiratta [8] provided a modified version of the converse duality theorem for the second-order dual model of a variational problem, which was studied by Husain *et al.* [11]. Since then, many authors [1,12,30] have worked on variational problems. For the interval-valued variational problems, the duality results were discussed by Dhingra and Kailey [5]. The study of these problems becomes challenging when the objective functions involve support functions, making it nondifferentiable in nature. Singh *et al.* [31] addressed this challenge by studying duality for multiobjective variational problems under second-order (Φ, ρ)-invexity. Additionally, Prasad *et al.* [29] studied symmetric dual problems for second-order nondifferentiable fractional variational problems under cone constraints using support functions and established the standard duality results under second-order F-convexity assumptions.

Building on this foundation, we investigate higher-order duality results for nondifferentiable multiobjective fractional variational problems under the framework of higher-order pseudoinvexity. The remainder of the paper is organized as follows: Section 2 introduces key definitions and the concept of higher-order pseudoinvexity, highlighting a functional that belongs exclusively to this class. Section 3 discusses the duality results for the proposed primal-dual pair under higher-order pseudoinvexity assumptions. A practical application validating the weak duality theorem is presented in Section 4. Finally, Section 5 offers concluding remarks on the work.

2. Notations and preliminaries

In this section, we introduce the necessary notations, definitions, and preliminary results that will be utilized in the subsequent analysis. Throughout the paper, we denote by $\mathcal{I} \subseteq \mathbb{R}$ the closed real interval $[a_0, a_1]$. The notation $\mathbb{R}^n$ represents an n-tuple of real numbers $(a_1, a_2, \ldots, a_n)$, and $\mathbb{R}_+$ refers to the set of non-negative real numbers, including zero. Additionally, x^T denotes the transpose of the vector x.

Suppose that the collection of piecewise smooth functions $x\colon \mathcal{I} \to \mathbb{R}^l$, equipped with the norm

$$\|x\| = \|x\|_\infty + \|Dx\|_\infty,$$

where the differentiation operator D is defined as

$$\mu = Dx \ \text{ if and only if } \ x(t) = \nu_1 + \int_{a_0}^{t} \mu(g)\,\mathrm{d}g,$$

for a given boundary value ν_1, is denoted by $F(\mathcal{I}, \mathbb{R}^l)$. Therefore, the usual differential operator $\frac{\mathrm{d}}{\mathrm{d}t} \equiv D$, except at discontinuities. Note that the spaces $F(\mathcal{I}, \mathbb{R}^n)$ and $F(\mathcal{I}, \mathbb{R}^m)$ are symbolized by $\mathcal{U}$ and $\mathcal{V}$, respectively.

Also, we suppose that for $x \in \mathcal{U}$ and $y \in \mathcal{V}$, with derivatives $\dot{x}$ and $\dot{y}$, the function $f^j\colon \mathcal{I} \times \mathcal{U} \times \mathcal{U} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}_+$ is twice continuously differentiable for each $j \in \mathcal{L} = \{1, 2, \ldots, k\}$. Furthermore, $f^j_x, f^j_{\dot{x}}, f^j_y$, and $f^j_{\dot{y}}$ represent the gradient vectors of the scalar-valued function $f^j(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ with respect to $x, \dot{x}, y$, and $\dot{y}$, respectively.

More precisely, we have

$$f_x^j = \begin{pmatrix} \dfrac{\partial f^j}{\partial x_1} \\ \vdots \\ \dfrac{\partial f^j}{\partial x_n} \end{pmatrix} \quad \text{and} \quad f_{\dot{x}}^j = \begin{pmatrix} \dfrac{\partial f^j}{\partial \dot{x}_1} \\ \vdots \\ \dfrac{\partial f^j}{\partial \dot{x}_n} \end{pmatrix}.$$

We consider the multiobjective variational problem subject to certain constraints as follows:

$$\begin{aligned} \text{(P) Minimize } & \int_{a_0}^{a_1} \kappa(t, x(t), \dot{x}(t))\,\mathrm{d}t = \left(\int_{a_0}^{a_1} \kappa^1(t, x(t), \dot{x}(t))\,\mathrm{d}t, \ldots, \int_{a_0}^{a_1} \kappa^k(t, x(t), \dot{x}(t))\,\mathrm{d}t \right) \\ \text{subject to } & x(a_0) = 0 = x(a_1), \\ & \dot{x}(a_0) = 0 = \dot{x}(a_1), \\ & h(t, x(t), \dot{x}(t)) \leqq 0, \qquad t \in \mathcal{I}, \end{aligned}$$

where $\kappa\colon \mathcal{I} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ and $h\colon \mathcal{I} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^l$ are continuously differentiable functions. Let $\mathfrak{S}$ be the set of all feasible solutions of (P) and is defined as:

$$\mathfrak{S} = \{x \in \mathcal{V} \mid x(a_0) = 0 = x(a_1),\ \dot{x}(a_0) = 0 = \dot{x}(a_1),\ h(t, x(t), \dot{x}(t)) \leqq 0\}.$$

Definition 2.1 ([15]). A point $x \in \mathfrak{S}$ is an efficient solution (or Pareto optimal) of (P) if there exists no other $\bar{x} \in \mathfrak{S}$ such that

$$\int_{a_0}^{a_1} \kappa(t, \bar{x}(t), \dot{\bar{x}}(t))\,\mathrm{d}t \leq \int_{a_0}^{a_1} \kappa(t, x(t), \dot{x}(t))\,\mathrm{d}t.$$

Definition 2.2 ([15]). A point $x \in \mathfrak{S}$ is a weak efficient solution of (P) if there exists no other $\bar{x} \in \mathfrak{S}$ such that

$$\int_{a_0}^{a_1} \kappa(t, \bar{x}(t), \dot{\bar{x}}(t))\,\mathrm{d}t < \int_{a_0}^{a_1} \kappa(t, x(t), \dot{x}(t))\,\mathrm{d}t.$$

Definition 2.3 ([15]). The support function of a compact convex set $\mathcal{D} \subset \mathbb{R}^n$ is defined as

$$\mathcal{S}(\tilde{x}|\mathcal{D}) = \max\big\{\tilde{x}^T r\colon r \in \mathcal{D}\big\}.$$

Since the support function is convex and finite everywhere, it admits a subdifferential. That is, there exists $z \in \mathbb{R}^n$ such that

$$\mathcal{S}(r|\mathcal{D}) \geqq \mathcal{S}(\tilde{x}|\mathcal{D}) + z^T(r - \tilde{x}), \quad \text{for all } r \in \mathcal{D}.$$

The subdifferential of $\mathcal{S}(\tilde{x}|\mathcal{D})$ is given by

$$\partial\mathcal{S}(\tilde{x}|\mathcal{D}) = \big\{z \in \mathcal{D}\colon z^T\tilde{x} = \mathcal{S}(\tilde{x}|\mathcal{D})\big\}.$$

For any convex set $H \subset \mathbb{R}^n$, the normal cone to H at a point $\tilde{x} \in H$ is defined by

$$N_H(\tilde{x}) = \big\{r \in \mathbb{R}^n : r^T(z - \tilde{x}) \leqq 0,\ \text{for all } z \in H\big\}.$$

It follows that for a compact convex set $\mathcal{D}$, a vector r belongs to the normal cone $N_{\mathcal{D}}(\tilde{x})$ if and only if

$$\mathcal{S}(r|\mathcal{D}) = \tilde{x}^T r,$$

or equivalently,

$$\tilde{x} \in \partial\mathcal{S}(r|\mathcal{D}).$$

Example 2.4. The support function of the Euclidean unit ball

$$\mathcal{D} = \{r \in \mathbb{R}^n : \|r\|_2 \le 1\}$$

is given by

$$\mathcal{S}(\tilde{x}|\mathcal{D}) = \|\tilde{x}\|_2,$$

where $\|\cdot\|_2$ denotes the Euclidean (2-norm) norm.

As discussed in problem (P), our goal is to minimize the objective function. Convex functions play an important role in this context, as they possess the property that any line segment connecting two points on the graph lies entirely above or on the graph itself. This property guarantees that every local minimum is also a global minimum, which simplifies the solution process considerably. Nonetheless, many real-world problems involve non-convex functions, highlighting the need for broader generalizations of convexity.

Mond and Weir [24] relaxed the convexity requirement to the weaker assumption of pseudo-convexity. For variational problems, symmetric duality under invexity assumptions was investigated by Smart and Mond [32]. The concept of invexity was further generalized to pseudo-invexity by Kim and Lee [17]. Second-order duality results under the assumption of second-order invexity were studied in [3,11,25], and this notion was subsequently extended to higher-order invexity by Padhan and Nahak [26]. For the class of (F, α, ρ, d)-convexity assumptions, duality results for variational problems were established by Kailey and Gupta [15], and these results were extended to the second-order case in [13].

In the present work, we further extend these results by establishing higher-order duality results under the framework of higher-order pseudo-invexity assumptions.

Definition 2.5. The functional $\int_{a_0}^{a_1} f(t, x(t), \dot{x}(t), y(t), \dot{y}(t))\,\mathrm{d}t$, $f\colon \mathcal{I}\times\mathcal{U}\times\mathcal{U}\times\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ is defined as follows:

(i) **First-order pseudoinvex** [17] at u, $\dot{u}$ with respect to $\eta\colon \mathcal{I}\times\mathcal{U}\times\mathcal{U}\times\mathcal{U}\times\mathcal{U}\to\mathbb{R}^n$, for fixed y, $\dot{y}$, if it satisfies for all $x, \dot{x} \in \mathcal{U}$,

$$\int_{a_0}^{a_1} \eta(t, x, \dot{x}, u, \dot{u})\{(f)_x(t, u, \dot{u}, y, \dot{y}) - D(f)_{\dot{x}}(t, u, \dot{u}, y, \dot{y})\}\,\mathrm{d}t \geqq 0$$

implies

$$\int_{a_0}^{a_1} (f(t, x, \dot{x}, y, \dot{y}) - f(t, u, \dot{u}, y, \dot{y}))\,\mathrm{d}t \geqq 0.$$

(ii) **Higher-order pseudoinvex** at u, $\dot{u}$ with respect to $\eta\colon \mathcal{I}\times\mathcal{U}\times\mathcal{U}\times\mathcal{U}\times\mathcal{U}\to\mathbb{R}^n$ and $\mathfrak{H}\colon \mathcal{I}\times\mathcal{U}\times\mathcal{U}\times\mathcal{V}\times\mathcal{V}\times\mathbb{R}^n\to\mathbb{R}$, for fixed y, $\dot{y}$, if it satisfies for all $(x, \dot{x}, p) \in \mathcal{U}\times\mathcal{U}\times\mathbb{R}^n$,

$$\int_{a_0}^{a_1} \eta(t, x, \dot{x}, u, \dot{u})\big\{f_x(t, u, \dot{u}, y, \dot{y}) - Df_{\dot{x}}(t, u, \dot{u}, y, \dot{y}) + \nabla_p\mathfrak{H}\big(t, u, \dot{u}, y, \dot{y}, p^j\big)\big\}\,\mathrm{d}t \geqq 0$$

implies

$$\int_{a_0}^{a_1} f(t, x, \dot{x}, y, \dot{y})\,\mathrm{d}t \geqq \int_{a_0}^{a_1} \big(f(t, u, \dot{u}, y, \dot{y}) + \mathfrak{H}(t, u, \dot{u}, y, \dot{y}, p) - (p)^T\nabla_p\mathfrak{H}(t, u, \dot{u}, y, \dot{y}, p)\big)\,\mathrm{d}t.$$

Remark 2.6. If $\mathfrak{H} = 0$, the Definition 2.5(ii) simplifies to the definition of a first-order pseudoinvex functional. Consequently, every first-order pseudoinvex functional is also higher-order pseudoinvex; however, the converse does not hold. To illustrate this, we provide the following example.

Example 2.7. Let $\mathcal{I} = [0.2, 3]$, $n = 1$, $m = 1$, and let $\mathcal{U}$ and $\mathcal{V}$ be the spaces of piecewise smooth functions $x\colon \mathcal{I}\to[0,\infty)$ and $y\colon \mathcal{I}\to[0,\infty)$, respectively. Define the following functions:

$$f\colon \mathcal{I}\times\mathcal{U}\times\mathcal{U}\times\mathcal{V}\times\mathcal{V}\to\mathbb{R} \text{ as } f(t, x, \dot{x}, y, \dot{y}) = (x+1)^2 + \sin^2(x+1),$$

$$\eta\colon \mathcal{I}\times\mathcal{U}\times\mathcal{U}\times\mathcal{U}\times\mathcal{U}\to\mathbb{R} \text{ as } \eta(t,x,\dot{x},u,\dot{u}) = u + x^2 + 1,$$
$$\mathfrak{H}^1\colon \mathcal{I}\times\mathcal{U}\times\mathcal{U}\times\mathcal{V}\times\mathcal{V}\times\mathbb{R}\to\mathbb{R} \text{ as } \mathfrak{H}^1(t,u,\dot{u},y,\dot{y},p) = u - 2.$$

Then the inequality of Definition 2.5(ii) can be viewed as

$$\begin{aligned}&\int_{0.2}^{3}\eta(t,x,\dot{x},u,\dot{u})\big\{f_x(t,u,\dot{u},y,\dot{y}) - Df^1_{\dot{x}}(t,u,\dot{u},y,\dot{y}) + \nabla_p\mathfrak{H}^1(t,u,\dot{u},y,\dot{y},p)\big\}\,\mathrm{d}t\\ &\quad = \int_{0.2}^{3}\big(u+x^2+1\big)\{2(u+1)+\sin(2(u+1))\}\,\mathrm{d}t\\ &\quad \geqq 0, \qquad \text{for } x(t)\in\mathcal{U},\ u(t) = \frac{1}{4} \text{ and } p\in\mathbb{R}.\end{aligned}$$

Next, evaluate:

$$\begin{aligned}&\int_{0.2}^{3}\big(f(t,x,\dot{x},y,\dot{y}) - f(t,u,\dot{u},y,\dot{y}) - \mathfrak{H}^1(t,u,\dot{u},y,\dot{y},p) + (p)^T\nabla_p\mathfrak{H}^1(t,u,\dot{u},y,\dot{y},p)\big)\,\mathrm{d}t\\ &\quad = \int_{0.2}^{3}\big((x+1)^2 + \sin^2(x+1) - (u+1)^2 - \sin^2(u+1) - (u-2)\big)\,\mathrm{d}t\\ &\quad \geqq 0, \qquad \text{for } x(t)\in\mathcal{U},\ u(t) = \frac{1}{4} \text{ and } p\in\mathbb{R}.\end{aligned}$$

Thus, the functional $\int_{0.2}^{3} f(t,x,\dot{x},y,\dot{y})\,\mathrm{d}t$ is higher-order pseudoinvex at $u(t) = \frac{1}{4}$. However, verifying first-order pseudoinvexity:

$$\begin{aligned}\int_{0.2}^{3}\eta(t,x,\dot{x},u,\dot{u})\big\{f_x(t,u,\dot{u},y,\dot{y}) - Df^1_{\dot{x}}(t,u,\dot{u},y,\dot{y})\big\}\,\mathrm{d}t &= \int_{0.2}^{3}\big(u+x^2+1\big)\{2(u+1)+\sin(2(u+1))\}\,\mathrm{d}t\\ &\geqq 0, \qquad \text{for } x(t)\in\mathcal{U},\ u(t) = \frac{1}{4} \text{ and } p\in\mathbb{R}.\end{aligned}$$

but

$$\begin{aligned}\int_{0.2}^{3}(f(t,x,\dot{x},y,\dot{y}) - f(t,u,\dot{u},y,\dot{y}))\,\mathrm{d}t &= \int_{0.2}^{3}\big((x+1)^2 + \sin^2(x+1) - (u+1)^2 - \sin^2(u+1)\big)\,\mathrm{d}t\\ &\ngeqq 0, \qquad \text{for } x(t)\in\mathcal{U},\ u(t) = \frac{1}{4} \text{ and } p\in\mathbb{R}.\end{aligned}$$

Hence, the above functional is higher-order pseudoinvex but not first-order pseudoinvex.

This example demonstrates that there exist functionals that are higher-order pseudoinvex without being first-order pseudoinvex.

3. Higher-order dual formulation

Researchers have been exploring the concept of duality for many years. Mond and Hanson [22] introduced a dual model for variational programming problems, along with related results. Later, Husain *et al.* [11] extended these ideas to second-order duality for variational problems. Gulati and Mehndiratta [8] expanded this further to cover second-order multiobjective variational problems, focusing on both optimal solutions and duality results. Recently, Singh *et al.* [31] and Dubey *et al.* [7] studied duality in multiobjective problems, while Kumar *et al.* [18] worked on nondifferentiable second-order symmetric multiobjective fractional variational programming with cone constraints. In this section, we present the higher-order multiobjective fractional variational primal and dual formulations.

For each $j\in\mathcal{L}$, consider the following:

(i) $f^j : \mathcal{I} \times \mathcal{U} \times \mathcal{U} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}_+$, and $g^j : \mathcal{I} \times \mathcal{U} \times \mathcal{U} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}_+/\{0\}$ are the continuously differentiable functions;

(ii) $\mathfrak{H}_1^j, \mathfrak{K}_1^j : \mathcal{I} \times \mathcal{U} \times \mathcal{U} \times \mathcal{V} \times \mathcal{V} \times \mathbb{R}^m \to \mathbb{R}$ and $\mathfrak{H}_2^j, \mathfrak{K}_2^j : \mathcal{I} \times \mathcal{U} \times \mathcal{U} \times \mathcal{V} \times \mathcal{V} \times \mathbb{R}^n \to \mathbb{R}$ are the higher-order approximation differentiable functions;

(iii) $p^j : \mathcal{I} \to \mathbb{R}^m$ and $q^j : \mathcal{I} \to \mathbb{R}^n$ for $j \in \mathcal{L}$;

(iv) $\hat{B}^j, \hat{E}^j, \hat{D}^j$, and $\hat{H}^j$, are compact convex sets in $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^m$, and $\mathbb{R}^m$, respectively;

(iv) $\mathcal{F}^j(x,y) = \int_{a_0}^{a_1} \{f^j(t,x,\dot{x},y,\dot{y}) + \mathcal{S}(x|\hat{B}^j) - y^T z^j + \mathfrak{H}_1^j(t,x,\dot{x},y,\dot{y},p^j) - (p^j)^T \nabla_{p^j} \mathfrak{H}_1^j(t,x,\dot{x},y,\dot{y},p^j)\}\,\mathrm{d}t$;
$\mathcal{G}^j(x,y) = \int_{a_0}^{a_1} \{g^j(t,x,\dot{x},y,\dot{y}) - \mathcal{S}(x|\hat{E}^j) + y^T r^j + \mathfrak{K}_1^j(t,x,\dot{x},y,\dot{y},p^j) - (p^j)^T \nabla_{p^j} \mathfrak{K}_1^j(t,x,\dot{x},y,\dot{y},p^j)\}\,\mathrm{d}t$;
$\mathcal{M}^j(u,v) = \int_{a_0}^{a_1} \{f^j(t,u,\dot{u},v,\dot{v}) - \mathcal{S}(v|\hat{D}^j) + u^T w^j + \mathfrak{H}_2^j(t,u,\dot{u},v,\dot{v},q^j) - (q^j)^T \nabla_{q^j} \mathfrak{H}_2^j(t,u,\dot{u},v,\dot{v},q^j)\}\,\mathrm{d}t$;
$\mathcal{N}^j(u,v) = \int_{a_0}^{a_1} \{g^j(t,u,\dot{u},v,\dot{v}) + \mathcal{S}(v|\hat{H}^j) - u^T s^j + \mathfrak{K}_2^j(t,u,\dot{u},v,\dot{v},q^j) - (q^j)^T \nabla_{q^j} \mathfrak{K}_2^j(t,u,\dot{u},v,\dot{v},q^j)\}\,\mathrm{d}t$, provided that $\mathcal{G}^j(x,y) > 0,\ \mathcal{N}^j(u,v) > 0,\ \mathcal{F}^j(x,y) \geqq 0,\ \mathcal{M}^j(u,v) \geqq 0,\ \forall j \in \mathcal{L}$.

Consider the following symmetric higher-order fractional variational primal problem, referred to as (SHFVP):

$$\text{Minimize } \left(\frac{\mathcal{F}^1(x,y)}{\mathcal{G}^1(x,y)}, \ldots, \frac{\mathcal{F}^k(x,y)}{\mathcal{G}^k(x,y)}\right)$$
$$\text{Subject to } x(a_0) = 0 = x(a_1), \quad y(a_0) = 0 = y(a_1),$$
$$\dot{x}(a_0) = 0 = \dot{x}(a_1), \quad \dot{y}(a_0) = 0 = \dot{y}(a_1),$$
$$\sum_{j=1}^{k} \lambda^j \left\{ \left[f_y^j - Df_{\dot{y}}^j - z^j + \nabla_{p^j} \mathfrak{H}_1^j\right] - \left[g_y^j - Dg_{\dot{y}}^j + r^j + \nabla_{p^j} \mathfrak{K}_1^j\right] \frac{\mathcal{F}^j(x,y)}{\mathcal{G}^j(x,y)} \right\} \leqq 0,$$
$$y^T \sum_{j=1}^{k} \lambda^j \left\{ \left[f_y^j - Df_{\dot{y}}^j - z^j + \nabla_{p^j} \mathfrak{H}_1^j\right] - \left[g_y^j - Dg_{\dot{y}}^j + r^j + \nabla_{p^j} \mathfrak{K}_1^j\right] \frac{\mathcal{F}^j(x,y)}{\mathcal{G}^j(x,y)} \right\} \geqq 0,$$
$$\lambda > 0, \quad x(t) \geq 0, \quad t \in \mathcal{I},$$
$$z^j \in \hat{D}^j, \quad r^j \in \hat{H}^j, \quad j \in \mathcal{L} = \{1, 2, \ldots, k\}.$$

The symmetric higher-order fractional variational dual formulation corresponding to the above primal problem is presented below and is denoted by (SHFVD):

$$\text{Maximize } \left(\frac{\mathcal{M}^1(u,v)}{\mathcal{N}^1(u,v)}, \ldots, \frac{\mathcal{M}^k(u,v)}{\mathcal{N}^k(u,v)}\right)$$
$$\text{Subject to } u(a_0) = 0 = u(a_1), \quad v(a_0) = 0 = v(a_1),$$
$$\dot{u}(a_0) = 0 = \dot{u}(a_1), \quad \dot{v}(a_0) = 0 = \dot{v}(a_1),$$
$$\sum_{j=1}^{k} \lambda^j \left\{ \left[f_x^j - Df_{\dot{x}}^j + w^j + \nabla_{q^j} \mathfrak{H}_2^j\right] - \left[g_x^j - Dg_{\dot{x}}^j - s^j + \nabla_{q^j} \mathfrak{K}_2^j\right] \frac{\mathcal{M}^j(u,v)}{\mathcal{N}^j(u,v)} \right\} \geqq 0,$$
$$u^T \sum_{j=1}^{k} \lambda^j \left\{ \left[f_x^j - Df_{\dot{x}}^j + w^j + \nabla_{q^j} \mathfrak{H}_2^j\right] - \left[g_x^j - Dg_{\dot{x}}^j - s^j + \nabla_{q^j} \mathfrak{K}_2^j\right] \frac{\mathcal{M}^j(u,v)}{\mathcal{N}^j(u,v)} \right\} \leqq 0,$$
$$\lambda > 0, \quad v(t) \geq 0, \quad t \in \mathcal{I},$$
$$w^j \in \hat{B}^j, \quad s^j \in \hat{E}^j, \quad j \in \mathcal{L}\{1, 2, \ldots, k\}.$$

Parametric prospective formulation

For sake of simplicity, we consider

$$\hat{m}^j = \frac{\mathcal{F}^j(x,y)}{\mathcal{G}^j(x,y)}$$

$$= \frac{\int_{a_0}^{a_1}\Big\{f^j(t,x,\dot{x},y,\dot{y})+\mathcal{S}\Big(x|\hat{B}^j\Big)-y^Tz^j+\mathfrak{H}_1^j\big(t,x,\dot{x},y,\dot{y},p^j\big)-\big(p^j\big)^T\nabla_{p^j}\mathfrak{H}_1^j\big(t,x,\dot{x},y,\dot{y},p^j\big)\Big\}\,\mathrm{d}t}{\int_{a_0}^{a_1}\Big\{g^j(t,x,\dot{x},y,\dot{y})-\mathcal{S}\Big(x|\hat{E}^j\Big)+y^Tr^j+\mathfrak{K}_1^j(t,x,\dot{x},y,\dot{y},p^j)-(p^j)^T\nabla_{p^j}\mathfrak{K}_1^j(t,x,\dot{x},y,\dot{y},p^j)\Big\}\,\mathrm{d}t},$$

$$\hat{n}^j = \frac{\mathcal{M}^j(u,v)}{\mathcal{N}^j(u,v)}$$

$$= \frac{\int_{a_0}^{a_1}\Big\{f^j(t,u,\dot{u},v,\dot{v})-\mathcal{S}\Big(v|\hat{D}^j\Big)+u^Tw^j+\mathfrak{H}_2^j\big(t,u,\dot{u},v,\dot{v},q^j\big)-\big(q^j\big)^T\nabla_{q^j}\mathfrak{H}_2^j\big(t,u,\dot{u},v,\dot{v},q^j\big)\Big\}\,\mathrm{d}t}{\int_{a_0}^{a_1}\Big\{g^j(t,u,\dot{u},v,\dot{v})+\mathcal{S}\Big(v|\hat{H}^j\Big)-u^Ts^j+\mathfrak{K}_2^j(t,u,\dot{u},v,\dot{v},q^j)-(q^j)^T\nabla_{q^j}\mathfrak{K}_2^j(t,u,\dot{u},v,\dot{v},q^j)\Big\}\,\mathrm{d}t}.$$

The parametric perspective of the aforementioned problems (SHFVP) and (SHFVD) is defined as follows:

(SSHFVP) Minimize $\hat{m} = \big(\hat{m}^1,\hat{m}^2,\dots,\hat{m}^k\big)$

subject to

$$x(a_0)=0=x(a_1),\quad y(a_0)=0=y(a_1),$$

$$\dot{x}(a_0)=0=\dot{x}(a_1),\quad \dot{y}(a_0)=0=\dot{y}(a_1),$$

$$\mathcal{F}^j(x,y)-\hat{m}^j\mathcal{G}^j(x,y)=0,\qquad \forall j\in\mathcal{L} \tag{1}$$

$$\sum_{j=1}^k\lambda^j\Big\{\Big[f_y^j-Df_{\dot{y}}^j-z^j+\nabla_{p^j}\mathfrak{H}_1^j\Big]-\hat{m}^j\Big[g_y^j-Dg_{\dot{y}}^j+r^j+\nabla_{p^j}\mathfrak{K}_1^j\Big]\Big\}\leqq 0, \tag{2}$$

$$y^T\sum_{j=1}^k\lambda^j\Big\{\Big[f_y^j-Df_{\dot{y}}^j-z^j+\nabla_{p^j}\mathfrak{H}_1^j\Big]-\hat{m}^j\Big[g_y^j-Dg_{\dot{y}}^j+r^j+\nabla_{p^j}\mathfrak{K}_1^j\Big]\Big\}\geqq 0, \tag{3}$$

$$\lambda>0,\quad x(t)\geq 0,\quad t\in\mathcal{I},$$

$$z^j\in\hat{D}^j,\quad r^j\in\hat{H}^j,\quad j\in\mathcal{L}.$$

(SSHFVD) Maximize $\hat{n} = (\hat{n}^1,\hat{n}^2,\dots,\hat{n}^k)$

subject to

$$u(a_0)=0=u(a_1),\quad v(a_0)=0=v(a_1),$$

$$\dot{u}(a_0)=0=\dot{u}(a_1),\quad \dot{v}(a_0)=0=\dot{v}(a_1),$$

$$\mathcal{M}^j(u,v)-\hat{n}^j\mathcal{N}^j(u,v)=0,\qquad \forall j\in\mathcal{L} \tag{4}$$

$$\sum_{j=1}^k\lambda^j\Big\{\Big[f_x^j-Df_{\dot{x}}^j+w^j+\nabla_{q^j}\mathfrak{H}_2^j\Big]-\hat{n}^j\Big[g_x^j-Dg_{\dot{x}}^j-s^j+\nabla_{q^j}\mathfrak{K}_2^j\Big]\Big\}\geqq 0, \tag{5}$$

$$u^T\sum_{j=1}^k\lambda^j\Big\{\Big[f_x^j-Df_{\dot{x}}^j+w^j+\nabla_{q^j}\mathfrak{H}_2^j\Big]-\hat{n}^j\Big[g_x^j-Dg_{\dot{x}}^j-s^j+\nabla_{q^j}\mathfrak{K}_2^j\Big]\Big\}\leqq 0, \tag{6}$$

$$\lambda>0,\quad v(t)\geq 0,\quad t\in\mathcal{I},$$

$$w^j\in\hat{B}^j,\quad s^j\in\hat{E}^j,\quad j\in\mathcal{L}.$$

The sets of feasible solutions for (SSHFVP) and (SSHFVD) are denoted by $\hat{M}$ and $\hat{N}$, respectively, under the assumption that $\hat{m}$ and $\hat{n}$ are nonnegative.

In the following, we discuss the duality theorems for (SSHFVP) and (SSHFVD), which are also applicable to the problems (SHFVP) and (SHFVD).

Theorem 3.1 (Weak duality)**.** *Let $(x,y,\hat{m},\lambda,z,r,p)\in\hat{M}$ and $(u,v,\hat{n},\lambda,w,s,q)\in\hat{N}$. Suppose that the following conditions hold:*

(i) $\sum_{j=1}^k\lambda^j\int_{a_0}^{a_1}\{(f^j(t,\cdot,\cdot,v,\dot{v})+(\cdot)^Tw^j)-\hat{n}^j(g^j(t,\cdot,\cdot,v,\dot{v})-(\cdot)^Ts^j)\}\,\mathrm{d}t$ *is higher-order pseudoinvex at u and $\dot{u}$ with respect to η_1^T, $\sum_{j=1}^k\lambda^j\{\mathfrak{H}_2^j-\hat{n}^j\mathfrak{K}_2^j\}$ and $q^j\in\mathbb{R}^n$ for fixed v and $\dot{v}$,*

(ii) $\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\{(-f^j(t,x,\dot{x},\cdot,\cdot)+(\cdot)^Tz^j)+\hat{m}^j(g^j(t,x,\dot{x},\cdot,\cdot)+(\cdot)^Tr^j)\}\,\mathrm{d}t$ *is higher-order pseudoinvex at* y *and* $\dot{y}$ *with respect to* η_2^T, $\sum_{j=1}^{k}\lambda^j\{-\mathfrak{H}_1^j+\hat{m}^j\mathfrak{K}_1^j\}$ *and* $p^j\in\mathbb{R}^m$ *for fixed* x *and* $\dot{x}$,

(iii) $\eta_1(t,x,\dot{x},u,\dot{u})+u\geqq 0$, *where* $\eta_1^T:\mathcal{I}\times\mathcal{U}\times\mathcal{U}\times\mathcal{U}\times\mathcal{U}\to\mathbb{R}^n$,

(iv) $\eta_2(t,v,\dot{v},y,\dot{y})+y\geqq 0$, *where* $\eta_2^T:\mathcal{I}\times\mathcal{V}\times\mathcal{V}\times\mathcal{V}\times\mathcal{V}\to\mathbb{R}^m$,

(v) $\int_{a_0}^{a_1}(g^j(t,x,\dot{x},v,\dot{v})-\mathcal{S}(x|\hat{E}^j)+v^Tr^j)\,\mathrm{d}t>0$, $\forall j\in\mathcal{L}$.

Then, it follows that

$$\hat{m}\nleq\hat{n}.$$

Proof. Let $(x,y,\hat{m},\lambda,z,r,p)$ be a feasible solution for the primal problem (SSHFVP) and $(u,v,\hat{n},\lambda,w,s,q)$ be a feasible solution for the dual problem (SSHFVD). Then, from equation (5) and hypothesis (iii), we have

$$\eta_1^T(t,x,\dot{x},u,\dot{u})\sum_{j=1}^{k}\lambda^j\Big\{\Big[f_x^j-Df_{\dot{x}}^j+w^j+\nabla_{q^j}\mathfrak{H}_2^j\Big]-\hat{n}^j\Big[g_x^j-Dg_{\dot{x}}^j-s^j+\nabla_{q^j}\mathfrak{K}_2^j\Big]\Big\}$$
$$\geqq -u^T\sum_{j=1}^{k}\lambda^j\Big\{\Big[f_x^j-Df_{\dot{x}}^j+w^j+\nabla_{q^j}\mathfrak{H}_2^j\Big]-\hat{n}^j\Big[g_x^j-Dg_{\dot{x}}^j-s^j+\nabla_{q^j}\mathfrak{K}_2^j\Big]\Big\}.$$

From the dual constraint (6), we obtain the following inequality

$$\eta_1^T(t,x,\dot{x},u,\dot{u})\sum_{j=1}^{k}\lambda^j\Big\{\Big[f_x^j-Df_{\dot{x}}^j+w^j+\nabla_{q^j}\mathfrak{H}_2^j\Big]-\hat{n}^j\Big[g_x^j-Dg_{\dot{x}}^j-s^j+\nabla_{q^j}\mathfrak{K}_2^j\Big]\Big\}\geqq 0$$
$$\Longrightarrow\sum_{j=1}^{k}\lambda^j\eta_1^T(t,x,\dot{x},u,\dot{u})\Big\{\big[f_x^j-Df_{\dot{x}}^j+w^j+\nabla_{q^j}\mathfrak{H}_2^j\big]-\hat{n}^j\Big[g_x^j-Dg_{\dot{x}}^j-s^j+\nabla_{q^j}\mathfrak{K}_2^j\Big]\Big\}\geqq 0,$$

which leads to the integral form

$$\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\eta_1^T(t,x,\dot{x},u,\dot{u})\Big\{\Big[f_x^j-Df_{\dot{x}}^j+w^j+\nabla_{q^j}\mathfrak{H}_2^j\Big]-\hat{n}^j\Big[g_x^j-Dg_{\dot{x}}^j-s^j+\nabla_{q^j}\mathfrak{K}_2^j\Big]\Big\}\,\mathrm{d}t\geqq 0. \tag{7}$$

Using equation (7) and hypothesis (i), we obtain the following expression

$$\begin{aligned}&\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\big\{\big(f^j(t,x,\dot{x},v,\dot{v})+x^Tw^j\big)-\hat{n}^j\big(g^j(t,x,\dot{x},v,\dot{v})-x^Ts^j\big)\big\}\,\mathrm{d}t\\&-\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\big\{\big(f^j(t,u,\dot{u},v,\dot{v})+u^Tw^j\big)-\hat{n}^j\big(g^j(t,u,\dot{u},v,\dot{v})-u^Ts^j\big)\big\}\,\mathrm{d}t\\&-\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\Big\{\mathfrak{H}_2^j\big(t,u,\dot{u},v,\dot{v},q^j\big)-\hat{n}^j\mathfrak{K}_2^j\big(t,u,\dot{u},v,\dot{v},q^j\big)\Big\}\,\mathrm{d}t\\&+\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\Big\{\big(q^j\big)^T\nabla_{q^j}\mathfrak{H}_2^j\big(t,u,\dot{u},v,\dot{v},q^j\big)-\hat{n}^j\big(q^j\big)^T\nabla_{q^j}\mathfrak{K}_2^j\big(t,u,\dot{u},v,\dot{v},q^j\big)\Big\}\,\mathrm{d}t\geqq 0.\end{aligned} \tag{8}$$

Using $x^Tw^j\leqq\mathcal{S}(x|\hat{B}^j)$, where $w^j\in\hat{B}^j$ and $x^Ts^j\leqq\mathcal{S}(x|\hat{E}^j)$, where $s^j\in\hat{E}^j$, $j\in\mathcal{L}$, we substitute these into equation (8) to obtain

$$\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\Big\{\Big(f^j(t,x,\dot{x},v,\dot{v})+\mathcal{S}\Big(x|\hat{B}^j\Big)\Big)-\hat{n}^j\Big(g^j(t,x,\dot{x},v,\dot{v})-\mathcal{S}\Big(x|\hat{E}^j\Big)\Big)\Big\}\,\mathrm{d}t$$

$$-\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\left\{\left(f^j(t,u,\dot{u},v,\dot{v})+u^Tw^j\right)-\hat{n}^j\left(g^j(t,u,\dot{u},v,\dot{v})-u^Ts^j\right)\right\}\mathrm{d}t$$
$$-\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\Big\{\mathfrak{H}_2^j\big(t,u,\dot{u},v,\dot{v},q^j\big)-\hat{n}^j\mathfrak{K}_2^j\big(t,u,\dot{u},v,\dot{v},q^j\big)\Big\}\mathrm{d}t$$
$$+\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\Big\{\big(q^j\big)^T\nabla_{q^j}\mathfrak{H}_2^j\big(t,u,\dot{u},v,\dot{v},q^j\big)-\hat{n}^j\big(q^j\big)^T\nabla_{q^j}\mathfrak{K}_2^j\big(t,u,\dot{u},v,\dot{v},q^j\big)\Big\}\mathrm{d}t\geqq 0.$$

Using equation (4), given by $\int_{a_0}^{a_1}\{f^j(t,u,\dot{u},v,\dot{v})-\mathcal{S}(v|\hat{D}^j)+u^Tw^j+\mathfrak{H}_2^j(t,u,\dot{u},v,\dot{v},q^j)-(q^j)^T\nabla_{q^j}\mathfrak{H}_2^j(t,u,\dot{u},v,\dot{v},q^j)\}\,\mathrm{d}t-\hat{n}^j(\int_{a_0}^{a_1}\{g^j(t,u,\dot{u},v,\dot{v})+\mathcal{S}(v|\hat{H}^j)-u^Ts^j+\mathfrak{K}_2^j(t,u,\dot{u},v,\dot{v},q^j)-(q^j)^T\nabla_{q^j}\mathfrak{K}_2^j(t,u,\dot{u},v,\dot{v},q^j)\}\,\mathrm{d}t)=0$ in the above equation, then the resultant equation

$$\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\Big\{\Big(f^j(t,x,\dot{x},v,\dot{v})+\mathcal{S}\Big(x|\hat{B}^j\Big)-\mathcal{S}\Big(v|\hat{D}^j\Big)\Big)-\hat{n}^j\Big(g^j(t,x,\dot{x},v,\dot{v})-\mathcal{S}\Big(x|\hat{E}^j\Big)+\mathcal{S}\Big(v|\hat{H}^j\Big)\Big)\Big\}\mathrm{d}t\geqq 0.$$

Next, applying the inequality $v^Tr^j\leqq\mathcal{S}(v|\hat{H}^j)$ for $r^j\in\hat{H}^j$, with $\hat{n}^j\geqq 0$ and $\lambda^j>0$ for all j, we derive the following inequality:

$$\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\Big\{\Big(f^j(t,x,\dot{x},v,\dot{v})+\mathcal{S}\Big(x|\hat{B}^j\Big)-\mathcal{S}\Big(v|\hat{D}^j\Big)\Big)-\hat{n}^j\Big(g^j(t,x,\dot{x},v,\dot{v})-\mathcal{S}\Big(x|\hat{E}^j\Big)+v^Tr^j\Big)\Big\}\mathrm{d}t\geqq 0. \quad (9)$$

Using the constraint (2) with hypothesis (iv) yields

$$\eta_2^T(t,v,\dot{v},y,\dot{y})\sum_{j=1}^{k}\lambda^j\Big\{-\Big[f_y^j-Df_{\dot{y}}^j-z^j+\nabla_{p^j}\mathfrak{H}_1^j\Big]+\hat{m}^j\Big[g_y^j-Dg_{\dot{y}}^j+r^j+\nabla_{p^j}\mathfrak{K}_1^j\Big]\Big\}$$
$$\geqq -y^T\sum_{j=1}^{k}\lambda^j\Big\{-\Big[f_y^j-Df_{\dot{y}}^j-z^j+\nabla_{p^j}\mathfrak{H}_1^j\Big]+\hat{m}^j\Big[g_y^j-Dg_{\dot{y}}^j+r^j+\nabla_{p^j}\mathfrak{K}_1^j\Big]\Big\}.$$

With the use of primal constraint (3), the above inequality concludes

$$\eta_2^T(t,v,\dot{v},y,\dot{y})\sum_{j=1}^{k}\lambda^j\Big\{-\Big[f_y^j-Df_{\dot{y}}^j-z^j+\nabla_{p^j}\mathfrak{H}_1^j\Big]+\hat{m}^j\Big[g_y^j-Dg_{\dot{y}}^j+r^j+\nabla_{p^j}\mathfrak{K}_1^j\Big]\Big\}\geqq 0$$
$$\Longrightarrow \sum_{j=1}^{k}\lambda^j\eta_2^T(t,v,\dot{v},y,\dot{y})\Big\{-\Big[f_y^j-Df_{\dot{y}}^j-z^j+\nabla_{p^j}\mathfrak{H}_1^j\Big]+\hat{m}^j\Big[g_y^j-Dg_{\dot{y}}^j+r^j+\nabla_{p^j}\mathfrak{K}_1^j\Big]\Big\}\geqq 0,$$

which further implies

$$\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\eta_2^T(t,v,\dot{v},y,\dot{y})\Big\{-\Big[f_y^j-Df_{\dot{y}}^j-z^j+\nabla_{p^j}\mathfrak{H}_1^j\Big]+\hat{m}^j\Big[g_y^j-Dg_{\dot{y}}^j+r^j+\nabla_{p^j}\mathfrak{K}_1^j\Big]\Big\}\mathrm{d}t\geqq 0.$$

From hypothesis (ii) and above inequality, we deduce the following expression

$$\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\left\{\left(-f^j(t,x,\dot{x},v,\dot{v})+v^Tz^j\right)+\hat{m}^j\left(g^j(t,x,\dot{x},v,\dot{v})+v^Tr^j\right)\right\}\mathrm{d}t$$

$$
\begin{aligned}
&-\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\left\{\left(-f^j(t,x,\dot{x},y,\dot{y})+y^Tz^j\right)+\hat{m}^j\left(g^j(t,x,\dot{x},y,\dot{y})+y^Tr^j\right)\right\}\mathrm{d}t\\
&-\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\left\{-\mathfrak{H}_1^j\left(t,x,\dot{x},y,\dot{y},p^j\right)+\hat{m}^j\mathfrak{K}_1^j\left(t,x,\dot{x},y,\dot{y},p^j\right)\right\}\mathrm{d}t\\
&+\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\left\{-\left(p^j\right)^T\nabla_{p^j}\mathfrak{H}_1^j\left(t,x,\dot{x},y,\dot{y},p^j\right)+\hat{m}^j\left(p^j\right)^T\nabla_{p^j}\mathfrak{K}_1^j\left(t,x,\dot{x},y,\dot{y},p^j\right)\right\}\mathrm{d}t\geqq 0. \qquad (10)
\end{aligned}
$$

From equation (1) and $v^Tz^j\leqq\mathcal{S}(v|\hat{D}^j)$, for $z^j\in\hat{D}^j,\ \lambda^j>0,\ \forall j$, equation (10) becomes

$$
\sum_{j=1}^{k}\lambda^j\int_{a_0}^{a_1}\left\{\left(-f^j(t,x,\dot{x},v,\dot{v})-\mathcal{S}\left(x|\hat{B}^j\right)+\mathcal{S}\left(v|\hat{D}^j\right)\right)+\hat{m}^j\left(g^j(t,x,\dot{x},v,\dot{v})-\mathcal{S}\left(x|\hat{E}^j\right)+v^Tr^j\right)\right\}\mathrm{d}t \geqq 0. \qquad (11)
$$

Adding equations (9) and (11), we arrive at the following inequality

$$
\sum_{j=1}^{k}\lambda^j\left(\hat{m}^j-\hat{n}^j\right)\int_{a_0}^{a_1}\left(g^j(t,x,\dot{x},v,\dot{v})-\mathcal{S}\left(x|\hat{E}^j\right)+v^Tr^j\right)\mathrm{d}t\geqq 0. \qquad (12)
$$

If $\hat{m}^j\leqq\hat{n}^j$ for all $j\in\mathcal{L}$, and $\hat{m}^i<\hat{n}^i$ for some $i\in\mathcal{L}$, then, under the assumption (v) and condition $\lambda>0$, we have

$$
\sum_{j=1}^{k}\lambda^j\left(\hat{m}^j-\hat{n}^j\right)\int_{a_0}^{a_1}\left(g^j(t,x,\dot{x},v,\dot{v})-\mathcal{S}\left(x|\hat{E}^j\right)+z^Tu^j\right)\mathrm{d}t<0,
$$

which contradicts the inequality (12). Hence, we conclude that $\hat{m}\nleq\hat{n}$. □

Note that, for the fixed value $\lambda=\bar{\lambda}$, the problem (SSHFVD) is abbreviated as $(\text{SSHFVD})_{\bar{\lambda}}$ in the following studies. Moreover, for better readability, we used the following notations in the next theorems.

$$
\begin{aligned}
&f^j=f^j(t,x,\dot{x},y,\dot{y}), && g^j=g^j(t,x,\dot{x},y,\dot{y}),\\
&\mathfrak{H}_1^j=\mathfrak{H}_1^j\left(t,x,\dot{x},y,\dot{y},p^j\right), && \mathfrak{K}_1^j=\mathfrak{K}_1^j\left(t,x,\dot{x},y,\dot{y},p^j\right), \quad \text{for every } \ j\in L.
\end{aligned}
$$

Additionally, following the approach in [14,16], we make use of the following derivative expansion, which will be employed in the proof of the strong duality result (Thm. 3.2). By the chain rule for total derivatives, we have

$$
Df_{\dot{x}}^j=f_{\dot{x}t}^j+f_{\dot{x}x}^j\dot{x}+f_{\dot{x}\dot{x}}^j\ddot{x}+f_{\dot{x}y}^j\dot{y}+f_{\dot{x}\dot{y}}^j\ddot{y} \qquad (13)
$$

where $D\equiv\frac{\mathrm{d}}{\mathrm{d}t}$, $x=x(t)$, $y=y(t)$. From (13), it follows that the partial derivatives of $Df_{\dot{x}}^j$ with respect to the independent variables $(x,\dot{x},y,\dot{y})$ are given by:

$$
\frac{\partial}{\partial x}Df_{\dot{x}}^j=Df_{\dot{x}x}^j,\quad \frac{\partial}{\partial\dot{x}}Df_{\dot{x}}^j=Df_{\dot{x}\dot{x}}^j+f_{\dot{x}x}^j,\quad \frac{\partial}{\partial\ddot{x}}Df_{\dot{x}}^j=f_{\dot{x}\dot{x}}^j, \qquad (14)
$$

$$
\frac{\partial}{\partial y}Df_{\dot{x}}^j=Df_{\dot{x}y}^j,\quad \frac{\partial}{\partial\dot{y}}Df_{\dot{x}}^j=Df_{\dot{x}\dot{y}}^j+f_{\dot{x}y}^j,\quad \frac{\partial}{\partial\ddot{y}}Df_{\dot{x}}^j=f_{\dot{x}\dot{y}}^j,\quad j\in\mathcal{L}. \qquad (15)
$$

Theorem 3.2 (Strong duality theorem)**.** *Let $(\bar{x},\bar{y},\bar{\hat{m}},\bar{\lambda},\bar{z},\bar{r},\bar{p})$ be a weakly efficient solution for (SSHFVP). Assume the following conditions:*

(i) *The Hessian matrices* $\nabla_{p^j p^j}\mathfrak{H}_1^j - \bar{\bar{m}}^j \nabla_{p^j p^j}\mathfrak{K}_1^j$, *are nonsingular for each* $j \in \mathcal{L}$.

(ii) *The set of vectors* $\{[f_y^j - Df_{\dot{y}}^j - \bar{z}^j + \nabla_{p^j}\mathfrak{H}_1^j] - \bar{\bar{m}}^j[g_y^j - Dg_{\dot{y}}^j + \bar{r}^j + \nabla_{p^j}\mathfrak{K}_1^j];\ j \in \mathcal{L}\}$ *are linearly independent.*

(iii) *The following gradient conditions hold for all* $j \in \mathcal{L}$:

$$\nabla_\theta \mathfrak{H}_1^j(t,\bar{x},\dot{\bar{x}},\bar{y},\dot{\bar{y}},0) = 0, \quad \nabla_\theta \mathfrak{K}_1^j(t,\bar{x},\dot{\bar{x}},\bar{y},\dot{\bar{y}},0) = 0, \qquad \forall \theta \in \{y,\dot{x},\dot{y},\ddot{x},\ddot{y},p^j\}.$$

Additionally,

$$\nabla_x \mathfrak{H}_1^j(t,\bar{x},\dot{\bar{x}},\bar{y},\dot{\bar{y}},0) = \nabla_{q^j}\mathfrak{H}_2^j(t,\bar{x},\dot{\bar{x}},\bar{y},\dot{\bar{y}},0),$$
$$\nabla_x \mathfrak{K}_1^j(t,\bar{x},\dot{\bar{x}},\bar{y},\dot{\bar{y}},0) = \nabla_{q^j}\mathfrak{K}_2^j(t,\bar{x},\dot{\bar{x}},\bar{y},\dot{\bar{y}},0).$$

Then, the following holds:

(i) *There exists* $\bar{w}^j \in \hat{B}^j$ *and* $\bar{s}^j \in \hat{E}^j$ *for each* $j \in \mathcal{L}$, *such that* $(\bar{x},\bar{y},\bar{\bar{m}},\bar{w},\bar{s},\bar{q}=0)$ *is feasible for* $(\mathrm{SSHFVD})_{\bar{\lambda}}$.

(ii) *The objective values of (SSHFVP) and* $(\mathrm{SSHFVD})_{\bar{\lambda}}$ *are equal.*

Furthermore, if the hypotheses of a weak duality theorem are satisfied for all feasible solutions of $(\mathrm{SSHFVP})_{\bar{\lambda}}$ *and* $(\mathrm{SSHFVD})_{\bar{\lambda}}$, *then* $(\bar{x},\bar{y},\bar{\bar{m}},\bar{w},\bar{s},\bar{q}=0)$ *is an efficient solution of* $(\mathrm{SSHFVD})_{\bar{\lambda}}$.

Proof. Since $(\bar{x},\bar{y},\bar{\bar{m}},\bar{\lambda},\bar{z},\bar{r},\bar{p})$ is a weakly efficient solution of (SSHFVP), the Fritz John optimality conditions, along with Lemma 1 of [33] imply the existence of $\rho \in \mathbb{R}_+^k, \chi \in \mathbb{R}^k$, piecewise smooth functions $\omega(t)\colon \mathcal{I} \to \mathbb{R}^m$, $\zeta(t)\colon \mathcal{I} \to \mathbb{R}_+$, and $\theta \in \mathbb{R}_+^k$, such that the following holds:

$$\begin{aligned}\tilde{A} = {} & \sum_{j=1}^k \rho^j \bar{\bar{m}}^j + \sum_{j=1}^k \chi^j \Bigg[\Big(f^j + \mathcal{S}\big(\bar{x}|\hat{B}^j\big) - \bar{y}^T\bar{z}^j + \mathfrak{H}_1^j - \big(\bar{p}^j\big)^T \nabla_{p^j}\mathfrak{H}_1^j\Big) - \bar{\bar{m}}^j\Big(g^j - \mathcal{S}\big(\bar{x}|\hat{E}^j\big) + \bar{y}^T\bar{r}^j \\ & + \mathfrak{K}_1^j - \big(\bar{p}^j\big)^T\nabla_{p^j}\mathfrak{K}_1^j\Big)\Bigg] + (\omega - \zeta\bar{y})^T \Bigg[\sum_{j=1}^k \bar{\lambda}^j \Big\{\Big[f_y^j - Df_{\dot{y}}^j - \bar{z}^j + \nabla_{p^j}\mathfrak{H}_1^j\Big] - \bar{\bar{m}}^j\Big[g_y^j - Dg_{\dot{y}}^j \\ & + \bar{r}^j + \nabla_{p^j}\mathfrak{K}_1^j\Big]\Big\}\Bigg] - \theta^T\bar{\lambda}\end{aligned}$$

satisfies the following conditions at $(\bar{x},\bar{y},\bar{\bar{m}},\bar{\lambda},\bar{z},\bar{r},\bar{p})$:

$$\Big[\tilde{A}_x - D\tilde{A}_{\dot{x}} + D^2\tilde{A}_{\ddot{x}}\Big](x(t) - \bar{x}(t)) \geqq 0 \qquad \forall x(t) \in \mathbb{R}^n, \quad t \in \mathcal{I}, \tag{16}$$
$$\tilde{A}_y - D\tilde{A}_{\dot{y}} + D^2\tilde{A}_{\ddot{y}} = 0, \qquad t \in \mathcal{I}, \tag{17}$$
$$\tilde{A}_\lambda = 0, \qquad t \in \mathcal{I}, \tag{18}$$
$$\tilde{A}_{\hat{m}} = 0, \qquad t \in \mathcal{I}, \tag{19}$$
$$\tilde{A}_p = 0, \qquad t \in \mathcal{I}, \tag{20}$$
$$\begin{aligned}\int_{a_0}^{a_1} \chi^j \Bigg[&\Big(f^j + \mathcal{S}\big(\bar{x}|\hat{B}^j\big) - \bar{y}^T\bar{z}^j + \mathfrak{H}_1^j - \big(\bar{p}^j\big)^T\nabla_{p^j}\mathfrak{H}_1^j\Big) - \bar{\bar{m}}^j\Big(g^j - \mathcal{S}\big(\bar{x}|\hat{E}^j\big) \\ & + \bar{y}^T\bar{r}^j + \mathfrak{K}_1^j - \big(\bar{p}^j\big)^T\nabla_{p^j}\mathfrak{K}_1^j\Big)\Bigg]\mathrm{d}t = 0, \qquad j \in \mathcal{L}, \quad t \in \mathcal{I},\end{aligned}$$
$$\omega^T\Bigg[\sum_{j=1}^k \bar{\lambda}^j\Big\{\Big[f_y^j - Df_{\dot{y}}^j - \bar{z}^j + \nabla_{p^j}\mathfrak{H}_1^j\Big] - \bar{\hat{m}}^j\Big[g_y^j - Dg_{\dot{y}}^j + \bar{r}^j + \nabla_{p^j}\mathfrak{K}_1^j\Big]\Big\}\Bigg] = 0, \qquad t \in \mathcal{I},$$
$$\zeta\bar{y}^T\Bigg[\sum_{j=1}^k \bar{\lambda}^j\Big\{\Big[f_y^j - Df_{\dot{y}}^j - \bar{z}^j + \nabla_{p^j}\mathfrak{H}_1^j\Big] - \bar{\hat{m}}^j\Big[g_y^j - Dg_{\dot{y}}^j + \bar{r}^j + \nabla_{p^j}\mathfrak{K}_1^j\Big]\Big\}\Bigg] = 0, \qquad t \in \mathcal{I},$$

$$\theta^T\bar{\lambda}=0, \tag{21}$$
$$\mathcal{S}\Big(\bar{x}|\hat{B}^j\Big)=\bar{x}^T\beta^j,\quad \beta^j\in\hat{B}^j,\qquad j\in\mathcal{L},$$
$$\mathcal{S}\Big(\bar{x}|\hat{E}^j\Big)=\bar{x}^T\gamma^j,\quad \gamma^j\in\hat{E}^j,\qquad j\in\mathcal{L},$$
$$\chi^j\bar{y}^T+[\omega-\zeta\bar{y}]\bar{\lambda}^j\in N_{\hat{D}^j}(\bar{z}^j), \tag{22}$$
$$\bar{\hat{m}}^j\big[\chi^j\bar{y}^T+(\omega-\zeta\bar{y})\bar{\lambda}^j\big]\in N_{\hat{H}^j}(\bar{r}^j), \tag{23}$$
$$(\rho,\chi,\omega,\zeta,\theta)\neq 0,\qquad t\in\mathcal{I}. \tag{24}$$

The above relations hold throughout the interval $\mathcal{I}$, except at the corners of $(\dot{\bar{x}},\dot{\bar{y}})$. By corners we mean the points in $\mathcal{I}$ where $\dot{\bar{x}}$ or $\dot{\bar{y}}$ are discontinuous or non-differentiable. At such points, the conditions are interpreted in terms of the corresponding right- and left-hand limits, and equations (16) and (17) are valid with these unique one-sided limits. The piecewise smooth functions ω and ζ are continuously differentiable on $\mathcal{I}$, except possibly at the corners of $(\dot{\bar{x}},\dot{\bar{y}})$.

Now, equations (16)–(20), together with the observations on $Df^j_{\dot{y}}$ and $Dg^j_{\dot{y}}$ for $j\in\mathcal{L}$, as derived from equations (13)–(15) prior to the start of this theorem, reduce to

$$\begin{aligned}
&\Bigg[\sum_{j=1}^{k}\chi^j\bigg[\bigg(f^j_x+\beta^j+\Big(\mathfrak{H}^j_1-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{H}^j_1\Big)_x-\bar{\hat{m}}^j\Big(g^j_x-\gamma^j+\Big(\mathfrak{K}^j_1-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{K}^j_1\Big)_x\Big)\bigg)\\
&\quad-D\bigg(f^j_{\dot{x}}+\Big(\mathfrak{H}^j_1-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{H}^j_1\Big)_{\dot{x}}-\bar{\hat{m}}^j\Big(g^j_{\dot{x}}+\Big(\mathfrak{K}^j_1-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{K}^j_1\Big)_{\dot{x}}\Big)\bigg)+D^2\bigg(\Big(\mathfrak{H}^j_1-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{H}^j_1\Big)_{\ddot{x}}\\
&\quad-\bar{\hat{m}}\Big(\mathfrak{K}^j_1-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{K}^j_1\Big)_{\ddot{x}}\bigg)\bigg]+(\omega-\zeta\bar{y})^T\Bigg[\sum_{j=1}^{k}\bar{\lambda}^j\Big\{f^j_{yx}-Df^j_{\dot{y}x}+\Big(\nabla_{p^j}\mathfrak{H}^j_1\Big)_x-\bar{\hat{m}}^j\Big(g^j_{yx}-Dg^j_{\dot{y}x}\\
&\quad+\Big(\nabla_{p^j}\mathfrak{K}^j_1\Big)_x\Big)\Big\}\Bigg]-D\Bigg[(\omega-\zeta\bar{y})^T\Bigg[\sum_{j=1}^{k}\bar{\lambda}^j\Big\{f^j_{y\dot{x}}-Df^j_{\dot{y}\dot{x}}-f^j_{\dot{y}x}+\Big(\nabla_{p^j}\mathfrak{H}^j_1\Big)_{\dot{x}}-\bar{\hat{m}}^j\Big(g^j_{y\dot{x}}-Dg^j_{\dot{y}\dot{x}}\\
&\quad-g^j_{\dot{y}x}+\Big(\nabla_{p^j}\mathfrak{K}^j_1\Big)_{\dot{x}}\Big)\Big\}\Bigg]\Bigg]+D^2\Bigg[(\omega-\zeta\bar{y})^T\Bigg[\sum_{j=1}^{k}\bar{\lambda}^j\Big\{-f^j_{\dot{y}\dot{x}}+\Big(\nabla_{p^j}\mathfrak{H}^j_1\Big)_{\ddot{x}}-\bar{\hat{m}}^j\Big(-g^j_{\dot{y}\dot{x}}\\
&\quad+\Big(\nabla_{p^j}\mathfrak{K}^j_1\Big)_{\ddot{x}}\Big)\Big\}\Bigg]\Bigg]\Bigg](x(t)-\bar{x}(t))\geqq 0\qquad\forall\ x(t)>0,\ t\in\mathcal{I},
\end{aligned}\tag{25}$$

$$\begin{aligned}
&\sum_{j=1}^{k}\chi^j\bigg[\bigg(f^j_y-\bar{z}^j+\Big(\mathfrak{H}^j_1-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{H}^j_1\Big)_y-\bar{\hat{m}}^j\Big(g^j_y+\bar{r}^j+\Big(\mathfrak{K}^j_1-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{K}^j_1\Big)_y\Big)\bigg)-D\bigg(f^j_{\dot{y}}+\Big(\mathfrak{H}^j_1\\
&\quad-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{H}^j_1\Big)_{\dot{y}}-\bar{\hat{m}}^j\Big(g^j_{\dot{y}}+\Big(\mathfrak{K}^j_1-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{K}^j_1\Big)_{\dot{y}}\Big)\bigg)+D^2\bigg(\Big(\mathfrak{H}^j_1-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{H}^j_1\Big)_{\ddot{y}}-\bar{\hat{m}}\Big(\mathfrak{K}^j_1-(\bar{p}^j)^T\\
&\quad\times\nabla_{p^j}\mathfrak{K}^j_1\Big)_{\ddot{y}}\bigg)\bigg]+(\omega-\zeta\bar{y})^T\Bigg[\sum_{j=1}^{k}\bar{\lambda}^j\Big\{f^j_{yy}-Df^j_{\dot{y}y}+\Big(\nabla_{p^j}\mathfrak{H}^j_1\Big)_y-\bar{\hat{m}}^j\Big(g^j_{yy}-Dg^j_{\dot{y}y}+\Big(\nabla_{p^j}\mathfrak{K}^j_1\Big)_y\Big)\Big\}\Bigg]\\
&\quad-\zeta\Bigg[\sum_{j=1}^{k}\bar{\lambda}^j\Big\{\Big[f^j_y-Df^j_{\dot{y}}-\bar{z}^j+\nabla_{p^j}\mathfrak{H}^j_1\Big]-\bar{\hat{m}}^j\Big[g^j_y-Dg^j_{\dot{y}}+\bar{r}^j+\nabla_{p^j}\mathfrak{K}^j_1\Big]\Big\}\Bigg]-D\Bigg[(\omega-\zeta\bar{y})^T
\end{aligned}$$

$$\times\left[\sum_{j=1}^{k}\bar{\lambda}^j\left\{f^j_{y\dot{y}}-Df^j_{\dot{y}\dot{y}}-f^j_{\dot{y}y}+\left(\nabla_{p^j}\mathfrak{H}^j_1\right)_{\dot{y}}-\bar{\bar{m}}^j\left(g^j_{y\dot{y}}-Dg^j_{\dot{y}\dot{y}}-g^j_{\dot{y}y}+\left(\nabla_{p^j}\mathfrak{K}^j_1\right)_{\dot{y}}\right)\right\}\right]\Bigg]+D^2\Bigg[(\omega$$
$$-\zeta\bar{y})^T\left[\sum_{j=1}^{k}\bar{\lambda}^j\left\{-f^j_{\dot{y}\dot{y}}+\left(\nabla_{p^j}\mathfrak{H}^j_1\right)_{\ddot{y}}-\bar{\bar{m}}^j\left(-g^j_{\dot{y}\dot{y}}+\left(\nabla_{p^j}\mathfrak{K}^j_1\right)_{\ddot{y}}\right)\right\}\right]\Bigg]=0,\qquad j\in\mathcal{L},\ t\in\mathcal{I},\tag{26}$$

$$(\omega-\zeta\bar{y})^T\left[\left[f^j_y-Df^j_{\dot{y}}-\bar{z}^j+\nabla_{p^j}\mathfrak{H}^j_1\right]-\bar{\bar{m}}^j\left[g^j_y-Dg^j_{\dot{y}}+\bar{r}^j+\nabla_{p^j}\mathfrak{K}^j_1\right]\right]-\theta^j=0,\tag{27}$$

$$\rho^j-\chi^j\left(g^j-\mathcal{S}\left(\bar{x}|\hat{E}^j\right)+\bar{y}^T\bar{r}^j+\mathfrak{K}^j_1-(\bar{p}^j)^T\nabla_{p^j}\mathfrak{K}^j_1\right)-(\omega-\zeta\bar{y})^T\left[\bar{\lambda}^j\left(g^j_y-Dg^j_{\dot{y}}+\bar{r}^j+\nabla_{p^j}\mathfrak{K}^j_1\right)\right]=0,\tag{28}$$

$$\left((\omega-\zeta\bar{y})\bar{\lambda}^j-\chi^j\bar{p}^j\right)\left[\nabla_{p^jp^j}\mathfrak{H}^j_1-\bar{\bar{m}}^j\nabla_{p^jp^j}\mathfrak{K}^j_1\right]=0,\qquad j\in\mathcal{L},\ t\in\mathcal{I}.\tag{29}$$

Since $\theta\geqq 0$ and $\bar{\lambda}>0$, equation (21) implies

$$\theta=0.\tag{30}$$

Substituting $\theta=0$ into equation (27) leads to

$$(\omega-\zeta\bar{y})^T\left[\left[f^j_y-Df^j_{\dot{y}}-\bar{z}^j+\nabla_{p^j}\mathfrak{H}^j_1\right]-\bar{\bar{m}}^j\left[g^j_y-Dg^j_{\dot{y}}+\bar{r}^j+\nabla_{p^j}\mathfrak{K}^j_1\right]\right]=0,\qquad\forall j\in\mathcal{L},\quad t\in\mathcal{I}.\tag{31}$$

From hypothesis (i), the matrices $\nabla_{p^jp^j}\mathfrak{H}^j_1-\bar{\bar{m}}^j\nabla_{p^jp^j}\mathfrak{K}^j_1$ are nonsingular. Using these conditions in equation (29), we obtain

$$(\omega-\zeta\bar{y})^T\bar{\lambda}^j=\chi^j\bar{p}^j,\quad j\in\mathcal{L}.\tag{32}$$

Additionally, hypothesis (ii) states that the set of vectors $\{[f^j_y-Df^j_{\dot{y}}-\bar{z}^j+\nabla_{p^j}\mathfrak{H}^j_1]-\bar{\bar{m}}^j[g^j_y-Dg^j_{\dot{y}}+\bar{r}^j+\nabla_{p^j}\mathfrak{K}^j_1]\}$ are linearly independent. Therefore, from equation (31), we deduce

$$\omega-\zeta\bar{y}=0.\tag{33}$$

Substituting equation (33) into equation (32) gives

$$\chi^j\bar{p}^j=0,\qquad j\in\mathcal{L}.\tag{34}$$

Since $\chi^j\in\mathbb{R}$ and $\bar{p}^j\in\mathbb{R}^m$, this implies either

$$\chi^j=0\ \text{ or }\ \bar{p}^j=0,\qquad j\in\mathcal{L}.\tag{35}$$

Combining equations (26), (33), (35), and hypothesis (iii), we arrive at

$$\sum_{j=1}^{k}\left(\chi^j-\zeta\bar{\lambda}^j\right)\left[f^j_y-Df^j_{\dot{y}}-\bar{z}^j+\nabla_{p^j}\mathfrak{H}^j_1-\bar{\bar{m}}^j\left(g^j_y-Dg^j_{\dot{y}}+\bar{r}^j+\nabla_{p^j}\mathfrak{K}^j_1\right)\right]=0,\qquad j\in\mathcal{L}.\tag{36}$$

Further, using assumption (ii) and equation (36), we have

$$\chi^j=\zeta\bar{\lambda}^j,\quad j\in\mathcal{L},$$
$$\text{i.e.,}\ \ \chi=\zeta\bar{\lambda}.\tag{37}$$

Now, if $\chi^j=0$ for some $j\in\mathcal{L}$, then from equation (37) we have $\zeta=0$, since $\bar{\lambda}>0$. This further implies that $\chi^j=0$ for all $j\in\mathcal{L}$. Substituting these values into the equation (33) gives $\omega=0$. In addition to this, the equations (33) and (28) gives $\rho=0$. As a result, $(\rho,\chi,\omega,\zeta,\theta)=0$ indicating from equation (30), which contradicts equation (24). Thus $\chi^j\neq 0\ \forall\ j\in\mathcal{L}$, *i.e.*, $\chi\neq 0$. Hence equation (34) implies

$$\bar{p}^j=0\ \text{ for all }\ j.\tag{38}$$

Using equations (33), (38), hypothesis (iii) and $\bar{\lambda} > 0$, the equations (25) and (26) reduce to

$$\sum_{j=1}^{k} \chi^j \left[f_x^j + \beta^j - Df_{\dot{x}}^j + \nabla_x \mathfrak{H}_1^j - \bar{\bar{m}}^j \left(g_x^j - \gamma^j - Dg_{\dot{x}}^j + \nabla_x \mathfrak{K}_1^j \right) \right] (x(t) - \bar{x}(t)) \geqq 0, \qquad t \in \mathcal{I}, \tag{39}$$

and

$$\sum_{j=1}^{k} (\chi^j - \zeta \bar{\lambda}^j) \left[f_y^j - \bar{z}^j - Df_{\dot{y}}^j + \nabla_{p^j} \mathfrak{H}_1^j - \bar{\bar{m}}^j \left(g_y^j + \bar{r}^j - Dg_{\dot{y}}^j + \nabla_{p^j} \mathfrak{K}_1^j \right) \right] = 0, \qquad t \in \mathcal{I}, \tag{40}$$

respectively.

By invoking hypothesis (ii), the set of vectors $\{[f_y^j - Df_{\dot{y}}^j - \bar{z}^j + \nabla_{p^j} \mathfrak{H}_1^j] - \bar{\bar{m}}^j [g_y^j - Dg_{\dot{y}}^j + \bar{r}^j + \nabla_{p^j} \mathfrak{K}_1^j]\}$ is linearly independent. Applying this to equation (40) gives

$$\chi^j = \zeta \bar{\lambda}^j \quad \text{or} \quad \chi = \zeta \bar{\lambda}. \tag{41}$$

As $\chi \neq 0$ and $\bar{\lambda} > 0$ we conclude that $\zeta \neq 0$ that is $\zeta > 0$, equation (33) becomes

$$\bar{s} = \frac{\omega}{\zeta} \in \mathbb{R}^m. \tag{42}$$

Utilizing equations (38), (41), hypothesis (iii) and $\zeta > 0$, equation (39) becomes

$$\sum_{j=1}^{k} \bar{\lambda}_j \left[f_x^j + \beta^j - Df_{\dot{x}}^j + \nabla_{q^j} \mathfrak{H}_2^j - \bar{\bar{m}}^j \left(g_x^j - \gamma^j - Dg_{\dot{x}}^j + \nabla_{q^j} \mathfrak{K}_2^j \right) \right] (x(t) - \bar{x}(t)) \geqq 0, \qquad t \in \mathcal{I}. \tag{43}$$

Let $x(t) \geq 0$, then $x(t) + \bar{x}(t) \geq 0$. So, from equation (43), we have

$$\sum_{j=1}^{k} \bar{\lambda}_j \left[f_x^j + \beta^j - Df_{\dot{x}}^j + \nabla_{q^j} \mathfrak{H}_2^j - \bar{\bar{m}}^j \left(g_x^j - \gamma^j - Dg_{\dot{x}}^j + \nabla_{q^j} \mathfrak{K}_2^j \right) \right] x(t) \geqq 0, \qquad t \in \mathcal{I},$$

that is

$$\sum_{j=1}^{k} \bar{\lambda}_j \left[f_x^j + \beta^j - Df_{\dot{x}}^j + \nabla_{q^j} \mathfrak{H}_2^j - \bar{\bar{m}}^j \left(g_x^j - \gamma^j - Dg_{\dot{x}}^j + \nabla_{q^j} \mathfrak{K}_2^j \right) \right] \geqq 0, \qquad t \in \mathcal{I}, \tag{44}$$

Also, let $x(t) = 0$ and $x(t) = 2\bar{x}(t)$ simultaneously in equation (43) results in

$$\bar{x}(t)^T \sum_{j=1}^{k} \bar{\lambda}_j \left[f_x^j + \beta^j - Df_{\dot{x}}^j + \nabla_{q^j} \mathfrak{H}_2^j - \bar{\bar{m}}^j \left(g_x^j - \gamma^j - Dg_{\dot{x}}^j + \nabla_{q^j} \mathfrak{K}_2^j \right) \right] = 0, \qquad t \in \mathcal{I}. \tag{45}$$

From equations (22), (33) and (41) with $\zeta > 0$, gives

$$\bar{\lambda}^j \bar{y} \in N_{\hat{D}^j}(\bar{z}^j), \text{ for } j \in \mathcal{L},$$
$$\text{or} \quad \bar{y} \in N_{\hat{D}^j}(\bar{z}^j), \text{ using } \bar{\lambda}^j > 0.$$

Since $\hat{D}^j$ is compact convex set in $\mathbb{R}^m$, we have $\bar{y}^T \bar{z}^j = \mathcal{S}(\bar{y}|\hat{D}^j)$, $j \in \mathcal{L}$.

Moreover, from equations (23), (33), and (41) with $\zeta > 0$, we have

$$\bar{\bar{m}}^j \bar{\lambda}^j \bar{y} \in N_{\hat{H}^j}(\bar{r}^j),$$
$$\text{or} \quad \bar{y} \in N_{\hat{H}^j}(\bar{r}^j), \text{ using } \bar{\lambda}^j > 0, \qquad j \in \mathcal{L}.$$

Since $\hat{H}^j$ is compact convex set in $\mathbb{R}^m$, $\bar{y}^T\bar{r}^j = \mathcal{S}(\bar{y}|\hat{H}^j)$, $j \in \mathcal{L}$.

Therefore, from equations (42), (44), (45), it concludes that $(\bar{x}, \bar{y}, \bar{\bar{m}}, \bar{w} = \beta, \bar{s} = \gamma, \bar{q} = 0)$ is a feasible solution for the dual problem $(\text{SSHFVD})_{\bar{\lambda}}$. Thus, (SSHFVP) and $(\text{SSHFVD})_{\bar{\lambda}}$ have equal objectives values (*i.e.*, $\bar{\bar{m}} = \bar{\bar{n}}$).

Further, if $(\bar{x}, \bar{y}, \bar{\bar{m}}, \bar{w}, \bar{s}, \bar{q} = 0)$ is not an efficient solution of $(\text{SSHFVD})_{\bar{\lambda}}$ then there exists $(\bar{u}, \bar{v}, \bar{\bar{n}}, \bar{w}, \bar{s}, \bar{q} = 0)$ feasible solution for $(\text{SSHFVD})_{\bar{\lambda}}$ such that

$$\bar{\bar{m}} \leq \bar{\bar{n}},$$

which contradicts weak duality theorem (Thm. 3.1). Thus $(\bar{x}, \bar{y}, \bar{\bar{m}}, \bar{w}, \bar{s}, \bar{q} = 0)$ is an efficient solution of $(\text{SSHFVD})_{\bar{\lambda}}$. Hence it completes the proof. □

Theorem 3.3 (Converse duality theorem)**.** *Let $(\bar{u}, \bar{v}, \bar{\bar{n}}, \bar{\lambda}, \bar{w}, \bar{s}, \bar{q})$ be a weakly efficient solution for (SSHFVD). Assume the following conditions hold:*

(i) *The Hessian matrices $\nabla_{q^jq^j}\mathfrak{H}_2^j - \bar{\bar{n}}^j\nabla_{q^jq^j}\mathfrak{K}_2^j$, $j \in \mathcal{L}$, are nonsingular.*

(ii) *The set of vectors $\{[f_x^j - Df_{\dot{x}}^j + \bar{w}^j + \nabla_{q^j}\mathfrak{H}_2^j] - \bar{\bar{n}}^j[g_x^j - Dg_{\dot{x}}^j - \bar{s}^j + \nabla_{q^j}\mathfrak{K}_2^j];\ j \in \mathcal{L}\}$ is linearly independent.*

(iii) *The following gradient conditions hold for all $j \in \mathcal{L}$:*

$$\nabla_\theta\mathfrak{H}_2^j(t, \bar{u}, \dot{\bar{u}}, \bar{v}, \dot{\bar{v}}, 0) = 0, \quad \nabla_\theta\mathfrak{K}_2^j(t, \bar{u}, \dot{\bar{u}}, \bar{v}, \dot{\bar{v}}, 0) = 0, \qquad \forall\theta \in \{x, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}, q^j\}.$$

Additionally,

$$\nabla_y\mathfrak{H}_2^j(t, \bar{u}, \dot{\bar{u}}, \bar{v}, \dot{\bar{v}}, 0) = \nabla_{p^j}\mathfrak{H}_1^j(t, \bar{u}, \dot{\bar{u}}, \bar{v}, \dot{\bar{v}}, 0),$$
$$\nabla_y\mathfrak{K}_2^j(t, \bar{u}, \dot{\bar{u}}, \bar{v}, \dot{\bar{v}}, 0) = \nabla_{p^j}\mathfrak{K}_1^j(t, \bar{u}, \dot{\bar{u}}, \bar{v}, \dot{\bar{v}}, 0).$$

Then:

(i) *There exists $\bar{z}^j \in \hat{D}^j$ and $\bar{r}^j \in \hat{H}^j$, $j \in \mathcal{L}$, such that $(\bar{u}, \bar{v}, \bar{\bar{n}}, \bar{z}, \bar{r}, \bar{p} = 0)$ is feasible for* $(\text{SSHFVP})_{\bar{\lambda}}$.

(ii) *The objective values of (SSHFVD) and* $(\text{SSHFVP})_{\bar{\lambda}}$ *are equal.*

Furthermore, if the hypotheses of a weak duality theorem are satisfied for all feasible solutions of $(\text{SSHFVP})_{\bar{\lambda}}$ *and* $(\text{SSHFVD})_{\bar{\lambda}}$, *then $(\bar{u}, \bar{v}, \bar{\bar{n}}, \bar{z}, \bar{r}, \bar{p} = 0)$ is an efficient solution of* $(\text{SSHFVP})_{\bar{\lambda}}$.

Proof. The proof follows similarly to Theorem 3.2 and is therefore omitted. □

4. Application of weak duality theorem

In this section, we will be explaining the applicability of the weak duality theorem using the following considered practical model.

In a textile industry, efficient production scheduling is crucial for optimizing resources, minimizing costs, and ensuring smooth operations. The given functional models the industry's goal of designing a production plan over a one-year period while balancing production efforts and operational constraints.

$$\frac{\int_0^1 \left(x^2 + y^2 + 10\right) \mathrm{d}t}{\int_0^1 (y^2 + 5)\, \mathrm{d}t}.$$

Objective function

– The numerator $\int_0^1 (x^2 + y^2 + 10)\, \mathrm{d}t$ represents the total production cost, which depends on two key factors:
  - $x(t)$: Represents the production rate of one type of textile product (*e.g.*, cotton fabric).
  - $y(t)$: Represents the production rate of another product (*e.g.*, synthetic fiber).
  - The term 10 accounts for fixed costs such as machinery maintenance, labor, and overhead costs.

– The denominator $\int_0^1 (y^2 + 5)\, \mathrm{d}t$ serves as a normalization factor, representing the total availability of production resources, with y^2 indicating material usage and 5 capturing base operational capacity.

The industry aims to minimize this ratio, ensuring that production costs remain low while optimizing resource utilization.

Boundary conditions

- $x(0) = 0 = x(1)$, $y(0) = 0 = y(1)$: The production starts and ends at zero within the planning period, reflecting a cycle-based production schedule where output gradually increases and decreases.
- $\dot{x}(0) = 0 = \dot{x}(1)$, $\dot{y}(0) = 0 = \dot{y}(1)$: These conditions ensure smooth transitions in production without abrupt changes, which aligns with practical scheduling constraints in textile manufacturing.

Constraints interpretation

$$-\left(2y - 2y\frac{\int_0^1 (x^2 + y^2 + 10)\,\mathrm{d}t}{\int_0^1 (y^2 + 5)\,\mathrm{d}t}\right) \geq 0.$$

- Ensures that production levels $y(t)$ remain feasible by maintaining a balance between demand and cost-effectiveness.
-

$$y\left(2y - 2y\frac{\int_0^1 (x^2 + y^2 + 10)\,\mathrm{d}t}{\int_0^1 (y^2 + 5)\,\mathrm{d}t}\right) \geq 0.$$

- Ensures that production decisions depend on market demand and available resources, preventing overproduction.

Problem formulation

The optimization problem is as follows:

$$\begin{aligned}
\text{(EP) Minimize } & \frac{\int_0^1 \left(x^2 + y^2 + 10\right)\mathrm{d}t}{\int_0^1 (y^2 + 5)\,\mathrm{d}t} \\
\text{Subject to } & x(0) = 0 = x(1), \quad y(0) = 0 = y(1), \\
& \dot{x}(0) = 0 = \dot{x}(1), \quad \dot{y}(0) = 0 = \dot{y}(1), \\
& -\left(2y - 2y\frac{\int_0^1 \left(x^2 + y^2 + 10\right)\mathrm{d}t}{\int_0^1 (y^2 + 5)\,\mathrm{d}t}\right) \geqq 0, \\
& y\left(2y - 2y\frac{\int_0^1 \left(x^2 + y^2 + 10\right)\mathrm{d}t}{\int_0^1 (y^2 + 5)\,\mathrm{d}t}\right) \geqq 0, \\
& \text{where } x\colon \mathcal{I} \to [0,\infty) \text{ and } y\colon \mathcal{I} \to [0,\infty) \text{ are smooth functions.}
\end{aligned}$$

To discuss the implementation of the weak duality theorem, we first simplify the problem (EP) using a parametric prospective as follows:

$$\begin{aligned}
\text{(SEP) Minimize } & \hat{m}^1 \\
\text{Subject to } & x(0) = 0 = x(1), \quad y(0) = 0 = y(1), \\
& \dot{x}(0) = 0 = \dot{x}(1), \quad \dot{y}(0) = 0 = \dot{y}(1), \\
& \int_0^1 \left(x^2 + y^2 + 10\right)\mathrm{d}t - \hat{m}^1 \int_0^1 \left(y^2 + 5\right)\mathrm{d}t = 0, \\
& -\left(2y - 2y\hat{m}^1\right) \geqq 0, \\
& y\left(2y - 2y\hat{m}^1\right) \geqq 0, \\
& x(t) \in [0,\infty), \quad t \in [0,1].
\end{aligned}$$

By considering the higher-order functions, $\mathfrak{H}_2(t,x,\dot{x},y,\dot{y},q)=q^4$, and $\mathfrak{K}_2(t,x,\dot{x},y,\dot{y},q)=0$, we can define its the dual model in the following manner:

$$
\begin{aligned}
\text{(SED) Maximum } & \hat{n}^1\\
\text{Subject to } & u(0)=0=u(1),\quad v(0)=0=v(1),\\
& \dot{u}(0)=0=\dot{v}(1),\quad \dot{u}(0)=0=\dot{v}(1),\\
& \int_0^1\left(u^2+v^2+10-3q^4\right)\mathrm{d}t-\hat{n}^1\int_0^1\left(v^2+5\right)\mathrm{d}t=0,\\
& \left(2u+4q^3\right)\geqq 0,\\
& u\left(2u+4q^3\right)\leqq 0,\\
& v(t)\in[0,\infty),\quad t\in[0,1],
\end{aligned}
$$

where $n=m=1$, $f^1=x^2+y^2+10$, $g^1=y^2+5$, $\hat{B}^1=\hat{E}^1=\hat{D}^1=\hat{H}^1=0$,

$$
\hat{m}^1=\frac{\mathcal{F}^1(x,y)}{\mathcal{G}^1(x,y)}=\frac{\int_0^1\left(x^2+y^2+10\right)\mathrm{d}t}{\int_0^1(y^2+5)\,\mathrm{d}t},
$$

$$
\hat{n}^1=\frac{\mathcal{M}^1(u,v)}{\mathcal{N}^1(u,v)}=\frac{\int_0^1\left(u^2+v^2+10-3q^4\right)\mathrm{d}t}{\int_0^1(v^2+5)\,\mathrm{d}t}.
$$

We can easily verify that $(x,y,\hat{m}^1,\lambda,z,r,p)=(t^2(t-1)^2,0,\hat{m}^1,1,0,0,t)$ and $(u,v,\hat{n}^1,\lambda,w,x,q)=(0,0,\hat{n}^1,1,0,0,t)$ are the feasible solutions of (SEP) and (SED), respectively, where $\hat{m}^1=\frac{6301}{3150}$ and $\hat{n}^1=\frac{47}{25}$.

Next, assume that $\eta_1(t,x,\dot{x},y,\dot{y})=(x+y+1)^2$, and $\eta_2(t,s,\dot{s},y,\dot{y})=(s+y+1)^4$ for establishing the following weak duality assumptions of Theorem 3.1:

(i) $\int_0^1\left((x^2+v^2+10)-\hat{n}^1(v^2+5)\right)\mathrm{d}t$ is higher-order pseudoinvex at $u(t)=0$ with respect to η_1^T, and $v(t)=t$. We have

$$
\int_0^1\eta_2(t,x,\dot{x},u,\dot{u})\left(2u+4q^3\right)\mathrm{d}t=\int_0^1(x+u+1)^4\left(2u+4q^3\right)\mathrm{d}t\geqq 0
$$

$$
\Longrightarrow \int_0^1\left(x^2+v^2+10-\hat{n}^1\left(v^2+5\right)-u^2-v^2-10+\hat{n}^1\left(v^2+5\right)-q^4+4q^4\right)\mathrm{d}t
$$

$$
=\int_0^1\left(x^2+3q^4\right)\mathrm{d}t\geqq 0.
$$

(ii) Analogously, we can show that $\int_0^1\left(-(x^2+v^2+10)+\hat{m}^1(v^2+5)\right)\mathrm{d}t$ is higher-order pseudoinvex at $y(t)=0$ with respect to η_2^T, and $x(t)=t^2(t-1)^2$. Thus, the explanation is omitted.

(iii) $\eta_1(t,x,\dot{x},u,\dot{u})+u=(x+u+1)^2\geqq 0$,

(iv) $\eta_2(t,v,\dot{v},y,\dot{y})+y=(v+y+1)^4\geqq 0$,

(v) $\int_0^1(g^1(t,x,\dot{x},v,\dot{v})-\mathcal{S}\left(x|\hat{E}^j\right)+v^Tr^j)\,\mathrm{d}t=\int_0^1(v^2+5)\,\mathrm{d}t>0$.

Since, all hypothesis of Theorem 3.1 are satisfied, therefore, we have, $\hat{m}\nleq\hat{n}$, that is $\frac{6301}{3150}\nleq\frac{47}{25}$.

5. Conclusion

In this manuscript, we have formulated a Mond-Weir type higher-order multiobjective fractional variational symmetric dual program involving support functions. Theoretical relations between the primal and dual models have been provided using duality theorems. Notably, we established weak duality and strong duality results under the supposition of higher-order pseudoinvexity. In addition, we provided an illustration of a non-trivial functional, which is a higher-order pseudoinvex but not a first-order pseudoinvex. Furthermore, a detailed discussion of the weak duality theorem is furnished by using the practical problem of the proposed class model.

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Conflict of Interest

The authors declare that they have no conflict of interest.

Data availability statement

No new data or code were created or analyzed in this study.

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