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SUSTAINABLE DYNAMIC PRICING, ADVERTISING, AND REPLENISHMENT STRATEGIES UNDER EMISSIONS CONSTRAINTS AND TIME-VARYING DEMAND

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Abstract. This study investigates how a profit-maximizing firm can jointly optimize its pricing, advertising efforts, and inventory replenishment strategies over a finite planning horizon under the cap-and-price regulatory framework. Our theoretical results demonstrate that cap-and-price regulation enables governments to effectively control emissions, offering an effective policy tool to balance economic development and environmental protection. Numerical examples reveal that goodwill and regulatory parameters significantly influence a firm's behavior. More frequent replenishment allows firms to fine-tune pricing and advertising strategies at the cost of higher replenishment and emission-related expenses. We alsofind that Cap-and-Trade yields the highest profit and moderate emissions, benefiting from trading flexibility and balanced incentives. In contrast, firms adopt aggressive strategies without regulation, producing the highest emissions and lowest profits. Furthermore, as carbon penalties internalize emission costs, firms may face higher operating costs, leading to rising prices or reduced advertising efforts. Such responses can affect brand image and weaken customer loyalty, underscoring the importance of integrated and adaptive decision-making in regulated markets. The proposed model offers valuable insights for high-emission sectors, supporting integrated decisions under environmental constraints. Finally, we conclude the study with insights derived from the theoretical analysis and numerical experiments and suggest directions for future research.

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1. Introduction

Products typically undergo a life cycle consisting of introduction, growth, maturity, and decline stages, each characterized by distinct patterns in demand and pricing. Technological advancement has significantly shortened product life cycles across various industries, and dynamic market conditions have accelerated the transition of products into maturity and even decline [3]. Rapid innovation leads to frequent product obsolescence as newer technologies quickly replace older ones. For instance, in the smartphone market, older models often lose consumer appeal within months of a new release. Evolving consumer preferences for novelty and innovation

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further compress product popularity cycles. For example, fast fashion brands such as ZARA adopt real-time design-to-market strategies, which deliberately shorten product life cycles to capitalize on emerging trends. Effectively managing products in the face of these shortened life cycles has become a critical challenge [14].

The Economic Order Quantity (EOQ) formula in classical inventory models simplifies decision-making by assuming a constant demand rate and a fixed selling price. These assumptions are particularly applicable during the maturity stage of a product's life cycle when demand stabilizes and price fluctuations are minimal. However, market demand and pricing vary significantly across different stages of the product life cycle. In response to this reality, Donaldson [17] introduced a classical inventory model incorporating a linear demand trend over a finite planning horizon. Many researchers, including [10, 33, 52], proposed inventory models that incorporate time-varying demand and dynamic pricing strategies. This growing body of research aims to provide more realistic and flexible decision-making frameworks that enable firms to adapt their inventory and pricing policies to changing market conditions.

In highly competitive industries like the 3C peripheral products market, firms often encounter considerable challenges in sustaining pricing power. Pricing products above competitors may lead to significant customer attrition, compelling firms to adopt alternative strategies to maintain their market position. Among these strategies, advertising plays a crucial role in generating goodwill and cultivating favorable consumer perceptions and emotional connections with the brand. This goodwill enhances a firm's intangible assets and stimulates demand by reinforcing both emotional resonance and rational trust with customers. Empirical marketing studies by Horsky [24] and Doganoglu and Klapper [16] have demonstrated that advertising serves not only as a reminder for existing customers but also as an effective tool for acquiring new ones, thereby boosting overall sales. As a result, firms operating in such competitive landscapes must invest substantial resources in multifaceted strategies, particularly advertising, to differentiate themselves from rivals and influence consumer choice. In response to this reality, Subramanyam and Kumaraswamy [46] extended the EOQ model by explicitly modeling the influence of advertising on demand. In contrast, Nerlove and Arrow [40] conceptualized advertising as an investment in accumulating a stock of goodwill, which affects both current and future demand.

As global warming significantly threatens the Earth's ecosystems and weather patterns, it also increases environmental and socio-economic challenges. To address these challenges, the 2015 Paris Agreement established a comprehensive global strategy to combat climate change. Its primary objective is to limit global temperature rise to well below 2 °C compared to pre-industrial levels, with a more ambitious goal of limiting it to 1.5 °C. To achieve these targets, developed nations have enforced stricter carbon emission control measures and enacted various regulations to curb greenhouse gas emissions across different sectors of the economy. These regulatory efforts have compelled firms to consider the implications of global warming on their business operations, particularly regarding resource availability, supply chain disruptions, and rising operational costs. In response to this reality, Benjaafar *et al.* [4] considered a lot-sizing problem involving production, transport, and inventory under several government emission regulations. Building on the work of Benjaafar *et al.* [4], several researchers have extended this framework to accommodate more general market conditions [9, 19, 30, 47].

As pointed out by Lodish *et al.* [31], the marginal effectiveness of advertising varies across different stages of the product life cycle. Incorporating the product life cycle into pricing and advertising models provides a more realistic depiction of the dynamic nature of strategic decision-making. It is well recognized that firms must adapt their decision-making processes and implement necessary operational adjustments to comply with increasingly stringent carbon emission regulations. However, most existing studies on dynamic pricing and advertising goodwill fail to consider the influence of the product life cycle. Although some inventory models address time-varying demand, they rarely account for the combined impacts of the advertising goodwill and regulatory constraints imposed by carbon emission policies, leaving critical gaps in the current literature. Given these considerations, we propose a dynamic optimization model integrating cap-and-price regulation with pricing, advertising, and time-varying demand, as demand patterns vary across life cycle stages and can substantially affect operational performance and environmental compliance. The remainder of this paper is organized as follows: Section 2 conducts a literature review on inventory models with pricing and time-varying demand, advertising, and emissions regulations. Section 3 presents a detailed description of the problem. In contrast, Section 4 outlines the solution procedure, explaining the rationale behind choosing the Particle Swarm Optimization (PSO) algorithm and detailing how it is utilized to solve the proposed problem. Additionally, Section 5 presents numerical examples and sensitivity analyses to illustrate the key features of the proposed model and evaluate the stability and effectiveness of the PSO algorithm in solving the proposed model. Finally, Section 6 concludes with a discussion of the results and directions for future research.

2. Literature review

This section reviews the relevant literature across four key topics: inventory models with pricing and time-varying demand, optimization models with advertising strategies, inventory models with emissions regulations, and inventory models with Particle Swarm Optimization. By examining and comparing prior studies within these domains, this review aims to identify gaps in the existing literature, thereby underscoring the contribution of the present study.

2.1. Inventory models with pricing and time-varying demand

As businesses face increasingly complex market environments, the assumption of constant demand in classical EOQ models has become less realistic, prompting researchers to develop models that account for time-varying demand. The fundamental result was first established by Donaldson [17], who introduced a classical inventory model incorporating a linear demand trend over a finite planning horizon. Inspired by Donaldson [17], several researchers extended this framework to accommodate more general market conditions. Notably, Goyal *et al.* [22], Yang *et al.* [54] and Massonnet *et al.* [36] proposed generalized inventory models that better reflect various market scenarios, accounting for different demand trends. Later, to offer a more realistic view of demand variations over a fixed time horizon, Chen *et al.* [8] constructed inventory models designed to capture demand patterns that mimic a Beta function, reflecting the product-life-cycle curve. Similarly, Wu *et al.* [53] employed a trapezoidal-type demand function to better represent the dynamics of product demand over its life cycle, providing a more flexible approach to modeling time-varying demand. These models are valuable for businesses looking to optimize inventory management by aligning replenishment decisions with fluctuating demand across different stages of the product life cycle, thereby broadening the applicability of the EOQ model and addressing the challenges posed by time-varying demand in real-world contexts.

In addition to time-varying demand, pricing is crucial for enhancing profitability and achieving a sustainable competitive advantage. Wee [52] addressed this issue by developing an inventory model for deteriorating items with a multiplicative demand function, in which demand declines exponentially over time but rises linearly with price. This model demonstrated the benefits of dynamic pricing strategies in inventory management, especially when the market demand fluctuates over time. Furthermore, customer purchase behavior is influenced by the selling price. In the short term, customers typically exhibit price-sensitive demand, where higher prices tend to discourage purchases while lower prices stimulate them. This relationship is captured by a price elasticity parameter embedded in the demand function, enabling the model to reflect realistic consumer responses to pricing strategies. Chen and Chen [7] employed dynamic programming to examine optimal dynamic pricing and replenishment strategies over a finite planning horizon, demonstrating that dynamic price adjustments can significantly enhance a firm's profitability. Tsao [50] further modeled the deteriorating inventory problem for a fashionable good under trade credit. His model optimizes pricing and replenishment decisions over in-stock and out-of-stock periods, demonstrating greater effectiveness in maximizing total profit than traditional one-phase strategies. Dye [18] expanded the classical EOQ model by integrating trade credit while accounting for price-dependent and time-varying demand. He aimed to determine the optimal replenishment and pricing strategies to maximize the discounted total profit. More recently, researchers such as Mahata *et al.* [32], Mukherjee and Mahata [38] and Chen *et al.* [10] have advanced models that include both price- and time-dependent demand to address a variety of business scenarios.

2.2. Optimization models with advertising strategies

Academic studies have recognized the causal link between current advertising efforts and subsequent demand. Advertising can enhance brand image and differentiate a firm from its competitors, increasing market share and attracting new customers. Based on this argument, Subramanyam and Kumaraswamy [46] extended the EOQ model by incorporating the impact of advertising on demand, modeling demand as a random variable influenced by the frequency of advertisements. Urban [51] studied finite replenishment inventory models where market demand is a deterministic function of time, price, and advertising expenditures. Following these contributions, Al-Amin Khan *et al.* [2], Bhunia *et al.* [6], Choudhury *et al.* [12] and San-José *et al.* [44] proposed joint advertising and replenishment models, addressing various business scenarios through a more integrated approach.

However, the studies mentioned above often assume that the impact of advertising on demand remains static over time, failing to account for the dynamic nature of advertising effects. Prior marketing studies have shown that immediate price sensitivity and long-term brand perception influence customer purchase behavior. Due to memory decay and reduced brand exposure, the stock of goodwill (SoG) gradually diminishes as consumers forget previously formed brand impressions. Therefore, firms must continuously invest in advertising to maintain their brand image and customer relationships. This accumulation enables firms to expand their customer base and retain loyal customers. Consequently, SoG accumulates over time and is dynamically affected by past and current advertising efforts. Recognizing goodwill as a capital asset that depreciates over time due to consumer forgetting, Nerlove and Arrow [40] were the first to propose an optimal dynamic pricing and advertising model for a monopolist in a continuous-time framework, treating advertising expenditures as investments in the SoG. Motivated by Nerlove and Arrow [40], several researchers, such as Jørgensen and Zaccour [27], Nair and Narasimhan [39], Crettez *et al.* [15] and Chenavaz and Eynan [11] have proposed differential game models to investigate how dynamic advertising affects market competition and firm performance under various scenarios. Recent studies, such as those by Huang *et al.* [25] and Jørgensen and Zaccour [28], provide extensive insights into new developments in dynamic pricing and advertising modeling. Readers are encouraged to consult the references cited in these works for a more in-depth exploration of dynamic pricing and advertising frameworks.

2.3. Inventory models with emissions regulations

Benjaafar *et al.* [4] first proposed a series of inventory models incorporating various environmental regulations and investigated their impact on operational decisions through several numerical examples. Chen *et al.* [9] proposed a method to achieve emission reductions through operational adjustments while minimizing the resulting increase in inventory costs under the EOQ framework. Dye and Yang [19], Mishra *et al.* [37] and Taleizadeh *et al.* [47] investigated sustainable inventory management for deteriorating items, focusing on how carbon emission parameters influence optimal replenishment strategy under different environmental regulatory frameworks. Their findings also indicated that increased regulatory pressure motivates firms to invest more in green technologies to reduce carbon emissions. Lee [30] examined the impact of abatement technology on the EOQ model under a cap-and-price regulation, in which firms are penalized for exceeding the carbon cap and rewarded for emitting below it. Researchers such as Shamayleh *et al.* [45], Ranjan and Jha [42], Pervin *et al.* [41] and Mardyana *et al.* [35] have analyzed the effects of emission mechanisms on inventory models with time-varying demand rates. Choudhury *et al.* [12] proposed joint optimization models for service facility investment, advertising, and replenishment under various carbon regulations. Mardyana and Mahata [34] developed a two-echelon supply chain comprising a manufacturer and a retailer. Under cap-and-trade regulation, the manufacturer adopts dual carbon reduction technologies to improve emission efficiency while both parties invest in reducing deterioration to enhance inventory freshness. However, these studies primarily focus on single-period or multi-period inventory problems with equal replenishment intervals, and they fail to incorporate the time-varying demand across different product life cycle stages into the model.

2.4. Inventory models with Particle Swarm Optimization

To better reflect real-world market conditions in inventory decision-making, numerous recent studies have adopted Particle Swarm Optimization (PSO) to solve complex inventory and operations-related problems. Akhtar *et al.* [1] and Dye [18] presented inventory models with partially backlogged shortages for deteriorating items over a finite planning horizon. In their studies, PSO was employed to determine the optimal selling prices and replenishment schemes that maximize total profit over the specified period. Bhunia and Shaikh [5], Tiwari *et al.* [49] and Jain and Singh [26] focused on analyzing single-period inventory models involving trade credit financing for deteriorating items. Saha *et al.* [43] formulated an inventory model in which pricing, investment in preservation technology, promotional efforts, and the replenishment cycle were considered decision variables to maximize the firm's total profit. These studies highlight the effectiveness of PSO in generating high-quality solutions for complex inventory management problems.

2.5. Contribution to the field

As summarized in Table 1, the research landscape reveals that numerous recent studies have investigated joint decision-making in pricing, advertising, and inventory management under various operational and environmental contexts. While many studies have examined inventory models with time-varying demand and dynamic pricing, few have explicitly integrated dynamic advertising goodwill and environmental regulations into such decision-making frameworks. Since the impact of advertising on demand is mediated by the stock of goodwill, which accumulates over time and is shaped by past and current advertising efforts, firms must strategically plan their advertising activities to maintain competitiveness in the market. In light of these considerations, this study explores how government carbon emission regulations influence a firm's joint dynamic pricing and advertising strategies using a discrete-time dynamic framework over a finite planning horizon. This paper contributes to three key streams of literature:

(1) We develop a finite horizon inventory model that explicitly incorporates time-varying demand influenced by pricing and advertising goodwill.
(2) We incorporate demand fluctuations across different stages of the product life cycle and include the impact of emission regulations on pricing, advertising, and inventory decisions.
(3) We integrate a generalized carbon emission regulation into the firm's operational strategy, and analyze its effects on inventory replenishment, pricing, and advertising investment.

3. Problem description

To facilitate the formulation of the problem, we define the relevant notations, as summarized in Table 2. Consider a single-product inventory system operated by a profit-maximizing firm over a finite horizon H with unit purchasing cost c and zero lead time. The planning horizon H is divided into n replenishment periods, which can vary in length $t_i - t_{i-1}$, where $t_0 = 0$, $t_n = H$, and $i = 1, 2, \ldots, n$. Suppose that replenishment is instantaneous and shortages are not allowed. In each period, the demand rate is known with certainty and varies with time, selling price, and the SoG. The firm is allocated an initial carbon cap $\varpi \geq 0$. If the firm's emissions exceed this cap, it is charged E per unit to cover the excess between its actual emissions and the cap. Conversely, if the firm's emissions are below its carbon cap, it receives a subsidy of R per unit for the unused portion. For convenience, we assume $E \geq R$ as cases where $E < R$ are unrealistic.

At the beginning of each period, the firm determines the selling price and advertising efforts in order to maximize total profit over the planning horizon H. The selling price and advertising effort in period i are denoted by p_i and u_i, respectively, where $\boldsymbol{p} = \{p_1, p_2, p_3, \ldots, p_n\}$ and $\boldsymbol{u} = \{u_1, u_2, u_3, \ldots, u_n\}$. Let $G_i(t)$, $i = 1, 2, \ldots, n$, be the SoG at time t during $[t_{i-1}, t_i)$. Since the demand rate is influenced by price, SoG, and the stage of the product life cycle, we adopt a demand function consistent with the formulations in [10, 15, 50]. Specifically, the market demand during the ith replenishment period is defined as:

$$D_i(t) = g(t)\{\alpha - \beta p_i + K(G_i(t))\}, \quad t_{i-1} \leq t < t_i, \quad i = 1, 2, \ldots, n, \tag{1}$$

TABLE 1. Comparison of related research on emission-related pricing and advertising models.

Research	Single/ multi-period	Demand pattern	Selling price	Advertising	Carbon emission
[10,33]	Single-period	Price and time-varying	Fixed	No	No
[32,38]	Single-period	Time and credit period	No	No	No
[2,6,12]	Single-period	Price and advertisement	Fixed	Fixed	No
[44,51]	Single-period	Price, advertisement and time-varying	Fixed	Fixed	No
[12]	Single-period	Advertisement and service	No	No	Yes
[19,47]	Single-period	Price	Fixed	No	Yes
[9,30]	Single-period	Constant	Fixed	No	Yes
[41,45]	Single-period	Time-varying	Fixed	No	Yes
[8,17,22,36,54]	Multi-period	Time-varying	Fixed	No	No
[4]	Multi-period	Time-varying	No	No	Yes
[8,36,53]	Multi-period	Time-varying	Fixed	No	No
[7,18,50]	Multi-period	Price and time-varying	Fixed	No	No
[42]	Multi-period	Price and time-varying	Dynamic	No	Yes
[35]	Optimal control	Time-varying	No	No	Yes
[11,15,27,39,40]	Optimal control	Price and goodwill	Dynamic	Dynamic	No
This study	Multi-period	Price, goodwill and time-varying	Dynamic	Dynamic	Yes

TABLE 2. Notations.

Parameters	
H	Finite planing horizon
A	Ordering cost per order
c	Unit purchasing cost
ϖ	Initial carbon cap
E	Carbon penalty/ per unit emission
R	Carbon subsidy/ per unit emission
$G_i(t)$	The SoG at time t during the ith replenishment period, $i = 1, 2, 3, \ldots, n$
$D_i(t)$	The market demand at time t during the ith replenishment period, $i = 1, 2, 3, \ldots, n$
γ	The advertising eciency
δ	The depreciation rate of the SoG, $0 < \delta < 1$
G_0	The initial SoG at the beginning of the planning horizon
SR_i	The sales revenue in the ith replenishment period, $i = 1, 2, 3, \ldots, n$
PC_i	The purchasing cost in the ith replenishment period, $i = 1, 2, 3, \ldots, n$
HC_i	The holding cost in the ith replenishment period, $i = 1, 2, 3, \ldots, n$
AC_i	The advertising cost in the ith replenishment period, $i = 1, 2, 3, \ldots, n$
CE	The carbon emissions over the planning horizon
f	The carbon emission coefficient per unit sold
Π	Total profit over the planning horizon
Decision variables	
n	Number of replenishment period
p_i	The selling price in the ith replenishment period, $i = 1, 2, 3, \ldots, n$
u_i	The advertising efforts in the ith replenishment period, $i = 1, 2, 3, \ldots, n$
t_i	The ith replenishment time, $i = 1, 2, 3, \ldots, n$

where α denotes the baseline market demand, β reflects the price sensitivity of demand, $K(G_i(t)) > 0$ quantifies the positive influence of goodwill stock on latent demand, and $g(t)$ represents the time trend of demand. Meanwhile, similar to [27, 39], we assume that $K(G) = k_1G(t) - k_2\frac{[G(t)]^2}{2}$, implying the diminishing marginal effect of advertising goodwill on demand and the maxima of the SoG is $\frac{k_1}{k_2}$. The assumption behind $D_i(t)$ captures the dual effect of dynamic pricing and advertising strategies on customer behavior, namely immediate responses to price changes and long-term brand perception shaped by accumulated goodwill.

However, due to memory decay and reduced brand exposure, the firm's SoG gradually diminishes over time as consumers tend to forget previously formed brand impressions. The dynamics of the SoG result from the combined effects of advertising efforts and depreciation. Following Nerlove and Arrow [40], the variation of SoG with respect to time can be described by the differential equation:

$$\frac{\mathrm{d}G_i(t)}{\mathrm{d}t} = \gamma u_i - \delta G_i(t), \quad t_{i-1} < t < t_i, \quad i = 1, 2, \ldots, n,$$

with the initial condition $G_i(t_{i-1}) = G_{i-1}$, where $u_i \geq 0$ represents the advertising efforts during the ith replenishment period, $\gamma > 0$ denotes the advertising efficiency, and $0 < \delta < 1$ is the depreciation rate of the SoG, and G_{i-1} is the SoG at the beginning of the ith replenishment period.

Solving this differential equation with the given initial condition, the SoG at time t can be derived as follows:

$$G_i(t) = e^{-\delta(t-t_{i-1})}G_{i-1} + \frac{\gamma u_i}{\delta}\left[1 - e^{-\delta(t-t_{i-1})}\right], \quad t_{i-1} \leq t < t_i, \quad i = 1, 2, \ldots, n.$$

Assume that the initial SoG at the beginning of the planning horizon $G_1(0) = G_0$ is a known constant. The difference equation can then be solved explicitly as

$$\begin{aligned} G_{i+1}(0) &= G_i(t_i) \\ &= \frac{e^{-\delta(t_i-t_{i-1})}\left\{\gamma u_i\left[e^{\delta(t_i-t_{i-1})} - 1\right] + \delta G_{i-1}\right\}}{\delta} \\ &= e^{-\delta t_i}\left\{G_0 + \frac{\gamma\sum_{j=0}^{i-1} u_{j+1}\left(e^{\delta t_{j+1}} - e^{\delta t_j}\right)}{\delta}\right\}, \quad i = 1, 2, \ldots, n. \end{aligned}$$

By combining this with the demand function in (1), we can observe that the model allows advertising to influence demand indirectly through goodwill while simultaneously capturing the effects of pricing and the time trend associated with the product life cycle. Thus, the market demand can be represented as:

$$\begin{aligned} D_i(t) &= g(t)\left\{\alpha - \beta p_i + k_1G_i(t) - k_2\frac{[G_i(t)]^2}{2}\right\} \\ &= g(t)\Bigg\{\alpha - \beta p_i + k_1\left\{e^{-\delta(t-t_{i-1})}G_{i-1} + \frac{\gamma u_i}{\delta}\left[1 - e^{-\delta(t-t_{i-1})}\right]\right\} \\ &\quad -\frac{k_2}{2}\left\{e^{-\delta(t-t_{i-1})}G_{i-1} + \frac{\gamma u_i}{\delta}\left[1 - e^{-\delta(t-t_{i-1})}\right]\right\}^2\Bigg\}, \quad t_{i-1} \leq t < t_i, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{2}$$

According to the notation and assumptions mentioned above, the behavior of the inventory system over the planning horizon H can be exhibited in Figure 1. Using the above results, the sales revenue and purchasing cost in the ith replenishment period, denoted by SR_i and PC_i, are

$$\mathrm{SR}_i = p_i\int_{t_{i-1}}^{t_i} g(t)\left\{\alpha - \beta p_i + k_1G_i(t) - k_2\frac{[G_i(t)]^2}{2}\right\}\mathrm{d}t, \quad i = 1, 2, \ldots, n$$

and

$$\mathrm{PC}_i = c\int_{t_{i-1}}^{t_i} g(t)\left\{\alpha - \beta p_i + k_1G_i(t) - k_2\frac{[G_i(t)]^2}{2}\right\}\mathrm{d}t, \quad i = 1, 2, \ldots, n$$

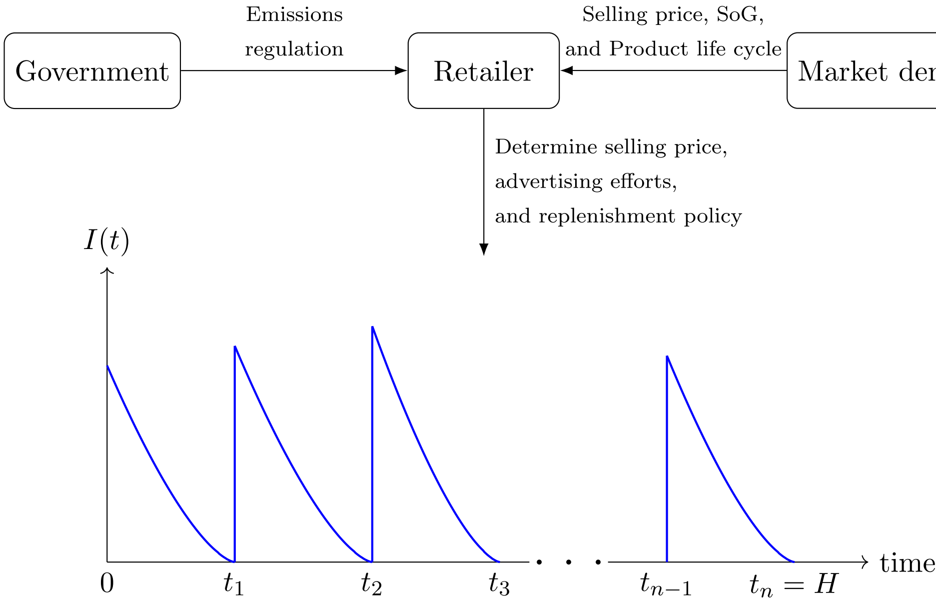

FIGURE 1. Graphical representation of the inventory system.

respectively. And, the holding cost, denoted by HC_i, can be calculated as

$$\mathrm{HC}_i = \int_{t_{i-1}}^{t_i} (t - t_{i-1}) g(t) \left\{ \alpha - \beta p_i + k_1 G_i(t) - k_2 \frac{[G_i(t)]^2}{2} \right\} \mathrm{d}t, \quad i = 1, 2, \ldots, n.$$

Additionally, implementing advertising strategies incurs operational costs. Without loss of generality, the cost of advertising effectiveness per unit time is assumed to be a quadratic function of u_i. Specifically, the advertising cost, denoted by AC_i, is assumed to be

$$\mathrm{AC}_i = \frac{w u_i^2}{2} (t_i - t_{i-1}), \quad i = 1, 2, \ldots, n,$$

which captures the increasing marginal costs associated with higher levels of advertising effort. Meanwhile, following the assumption in [30], the firm's carbon emissions is positively correlated with the number of products sold. This reflects the operational reality that higher sales volumes typically lead to increased production and transportation activities, resulting in greater emissions. Hence, the carbon emissions over the planning horizon H, denoted by CE, is then given by

$$\mathrm{CE} = f \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} D_i(t)\, \mathrm{d}t = f \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} g(t) \left\{ \alpha - \beta p_i + k_1 G_i(t) - \frac{k_2}{2} [G_i(t)]^2 \right\} \mathrm{d}t,$$

where f represents the carbon emission coefficient per unit sold.

Based on the above discussion, we can now formulate the total profit over the planning horizon H for a given positive integer n, denoted by $\Pi(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})$, as follows:

$$\Pi(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}) = \sum_{i=1}^{n} \{\mathrm{SR}_i - A - \mathrm{PC}_i - \mathrm{HC}_i - \mathrm{AC}_i\} - E \max\{\mathrm{CE} - \varpi, 0\} + R \max\{\varpi - \mathrm{CE}, 0\}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} [p_i - c - h(t - t_{i-1})] g(t) \left\{ \alpha - \beta p_i + k_1 G_i(t) - \frac{k_2}{2} [G_i(t)]^2 \right\} \mathrm{d}t \\
&\quad - nA - w \sum_{i=1}^{n} \frac{u_i^2}{2} (t_i - t_{i-1}) \\
&\quad - E \times \max\left\{ -\varpi + f \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} g(t) \left\{ \alpha - \beta p_i + k_1 G_i(t) - \frac{k_2}{2} [G_i(t)]^2 \right\} \mathrm{d}t, 0 \right\} \\
&\quad + R \times \max\left\{ \varpi - f \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} g(t) \left\{ \alpha - \beta p_i + k_1 G_i(t) - \frac{k_2}{2} [G_i(t)]^2 \right\} \mathrm{d}t, 0 \right\},
\end{aligned} \tag{3}
$$

where $t_0 = 0$, $t_n = H$, $G_0 = 0$,

$$
G_i(t) = \frac{e^{-\delta(t - t_{i-1})} \left\{ \gamma u_i \left[e^{\delta(t - t_{i-1})} - 1 \right] + \delta G_{i-1} \right\}}{\delta}, \quad i = 1, 2, \ldots, n,
$$

and

$$
G_{i+1}(0) = G_i(t_i) = e^{-\delta t_i} \left\{ G_0 + \frac{\gamma \sum_{j=0}^{i-1} u_{j+1} \left(e^{\delta t_{j+1}} - e^{\delta t_j} \right)}{\delta} \right\}, \quad i = 1, 2, \ldots, n.
$$

The goal of this paper is to determine the optimal replenishment, pricing, and advertising stragegies to maximize the total profit over H. The problem can be formulated as follows:

$$
\begin{aligned}
&\text{Maximize } \Pi(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u} \mid n) \\
&\text{subject to } p_i > c, && i = 1, 2, \ldots, n, \\
&\qquad u_i \geq 0, && i = 1, 2, \ldots, n, \\
&\qquad t_{i-1} < t_i, && i = 1, 2, \ldots, n, \\
&\qquad G_1(0) = G_0, \quad t_0 = 0, \quad t_n = H.
\end{aligned} \tag{4}
$$

Let

$$
\begin{aligned}
\Pi^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}) &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} [p_i - c - h(t - t_{i-1}) - fE] g(t) \left\{ \alpha - \beta p_i + k_1 G_i(t) - \frac{k_2}{2} [G_i(t)]^2 \right\} \mathrm{d}t \\
&\quad - w \sum_{i=1}^{n} \frac{u_i^2}{2} (t_i - t_{i-1}) + E\varpi - nA
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}) &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} [p_i - c - h(t - t_{i-1}) - fR] g(t) \left\{ \alpha - \beta p_i + k_1 G_i(t) - \frac{k_2}{2} [G_i(t)]^2 \right\} \mathrm{d}t \\
&\quad - w \sum_{i=1}^{n} \frac{u_i^2}{2} (t_i - t_{i-1}) + R\varpi - nA,
\end{aligned}
$$

be the total profits under carbon cap-and-trade regulation with carbon penalty E and R, respectively. After some algebra manipulation, (3) is equivalent to

$$
\Pi(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}) = \min\left\{ \Pi^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}), \Pi^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}) \right\}.
$$

Hence, the problem (4) can be rewritten as

$$
\text{Maximize } \min\left\{ \Pi^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}), \Pi^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}) \right\}
$$

$$\begin{aligned}
\text{subject to } & p_i > c, & i = 1, 2, \ldots, n, \\
& u_i \geq 0, & i = 1, 2, \ldots, n, \\
& t_{i-1} < t_i, & i = 1, 2, \ldots, n, \\
& G_1(0) = G_0, \quad t_0 = 0, \quad t_n = H. & \qquad (5)
\end{aligned}$$

For any given integer n, if we can show the concavity of both $\Pi^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})$ and $\Pi^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})$, it follows that $\Pi(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})$ in $(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})$ is concave since the minimum of two concave functions is also concave, ensuring the existence of a unique solution. However, we note that

$$\begin{aligned}
\frac{\partial^2 \Pi^x(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})}{\partial u_i^2} = & -\frac{k_2\gamma^2}{\delta^2} \int_{t_{i-1}}^{t_i} [p_i - c - h(t - t_{i-1}) - fx] g(t) \left[1 - e^{-\delta(t - t_{i-1})}\right]^2 \mathrm{d}t \\
& - \frac{k_2\gamma^2}{\delta^2} \int_{t_i}^{t_{i+1}} [p_{i+1} - c - h(t - t_i) - fx] g(t) e^{-2\delta t} \left[e^{\delta t_i} - e^{\delta t_{i-1}}\right]^2 \mathrm{d}t \\
& - \frac{k_2\gamma^2}{\delta^2} \int_{t_{i+1}}^{t_{i+2}} [p_{i+2} - c - h(t - t_{i+1}) - fx] g(t) e^{-2\delta t} \left[e^{\delta t_i} - e^{\delta t_{i-1}}\right]^2 \mathrm{d}t \\
& - \cdots \\
& - \frac{k_2\gamma^2}{\delta^2} \int_{t_{n-1}}^{t_n} [p_n - c - h(t - t_{n-1}) - fx] g(t) e^{-2\delta t} \left[e^{\delta t_i} - e^{\delta t_{i-1}}\right]^2 \mathrm{d}t \\
& - w(t_i - t_{i-1}), \ i = 1, 2, \ldots, n, \ \text{and } x = E \text{ or } R,
\end{aligned}$$

is indeterminate. Since the lengths of the replenishment periods are not equal, it is not possible to prove that the Hessian matrix of $\Pi^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})$ or $\Pi^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})$ with respect to $(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})$ is negative definite unless we assume that $p_i - c - hH - fE > 0$ for all $i = 1, 2, \ldots, n$.

The challenge in solving this nonlinear programming problem arises from the interaction between multiple decision variables, making the problem analytically complex. Since the objective function in (5), $\min\left\{\Pi^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}), \Pi^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})\right\}$, is non-concave and not differentiable at all points within its domain, the problem is difficult to solve analytically and cannot be addressed using traditional optimization methods.

To address this challenge, we employ the PSO algorithm, which is a population-based metaheuristic. Unlike traditional techniques such as gradient-based methods or genetic algorithms, PSO does not depend on gradient information or crossover operations. Given the structure of our model, which involves a nonlinear objective function with potential discontinuities in its derivative, PSO is particularly well-suited for identifying optimal pricing, advertising effort, and replenishment timing. Moreover, prior studies in inventory and operational decision-making have demonstrated the effectiveness and efficiency of PSO in solving inventory optimization problems under various complex constraints and dynamic environments (see, *e.g.*, [5,18,23,43]). In the following section, we demonstrate how PSO is implemented to solve the proposed optimization problem.

4. Solution procedure

4.1. The background of article swarm optimization

The Particle Swarm Optimisation algorithm, inspired by the social behavior observed in flocks of birds and schools of fish, explores complex search spaces by using interactions between individuals within a particle population to find optimal solutions. Since introduced by Eberhart and Kennedy [20] and Kennedy and Eberhart [29], PSO has been widely applied to optimization problems. In their comparative analysis of PSO variants, Eberhart and Shi [21] found that using a constriction factor along with a limit on the maximum speed, $V_{\max}$, gave the most effective results. In this paper, we apply the PSO algorithm to tackle the finite time horizon dynamic optimization problem involving pricing, advertising, and replenishment strategies to maximize the firm's profitability. To keep the focus on our application rather than the development of PSO variants, we use the basic PSO model with a constriction factor for our optimization framework.

In PSO with a constriction factor, each potential solution, referred to as a particle, explores the solution space by tracking the current best-performing particles. For a problem with d dimensions, the algorithm begins by initializing a population of I randomly distributed particles. These particles iteratively adjust their positions in the search space to approach the optimal solution. The size of the population reflects the total number of particles participating in the search process. At each step of the iteration, every particle determines its new position based on two key best values:

(1) The personal best (*pbest*): the best solution the particle has achieved so far.
(2) The global best (*gbest*): the best solution obtained so far by all particle in the population.

Once *pbest* and *gbest* have been determined, each particle adjusts its velocity based on its own best experience and the global best experience, directing its movement through the search space. In each generation, a particle updates its velocity and position based on the following equations:

$$v_{k+1}^i = \chi\left[v_k^i + \varphi_1 \times \text{rand} \times \left(pbest_k^i - x_k^i\right) + \varphi_2 \times \text{rand} \times \left(gbest_k - x_k^i\right)\right], \tag{6}$$

and

$$x_{k+1}^i = x_k^i + v_{k+1}^i, \tag{7}$$

where v_k^i, x_k^i, $pbest_k^i$, and $gbest_k$ represent the velocity, current position, personal best position of the ith particle, and global best position at iteration k, respectively. The terms φ_1 and φ_2 are acceleration constants, rand is a random number between 0 and 1, and χ is the constriction factor given by:

$$\chi = \frac{2}{\left|2 - \varphi - \sqrt{\varphi^2 - 4\varphi}\right|}, \tag{8}$$

where $\varphi = \varphi_1 + \varphi_2 > 4$.

In (8), χ represents the constriction factor, a crucial element in controlling particle exploration. It acts as a damping mechanism, preventing particles from venturing too far beyond the currently known best solutions. The parameters φ_1 and φ_2 are positive constants that balance the influence of a particle's personal best (*pbest*) against the global best (*gbest*) of the swarm. Clerc and Kennedy [13] suggest that using a constriction factor generally improves convergence. Typically, $\varphi = \varphi_1 + \varphi_2$ is set to 4.1, resulting in a constriction factor χ of approximately 0.7298. This configuration has been empirically shown to yield a good balance between exploration and exploitation. The optimization process continues iteratively, evaluating the fitness of each particle in every iteration, until a satisfactory solution is discovered or a predetermined termination criterion is met.

To further regulate particle movement, velocities are constrained within a predefined range, $[-V_{\max}, V_{\max}]$, for each dimension. This user-defined parameter, $V_{\max}$, acts as a speed limit. If a particle's calculated velocity in any dimension surpasses $V_{\max}$, it is capped at that limit. This velocity clamping mechanism promotes a more thorough exploration of the search space by preventing excessively large steps that might cause particles to overshoot promising regions or become trapped in unproductive areas. The complete PSO algorithm, incorporating these elements, is detailed in Algorithm 1.

4.2. Solving the optimal pricing, advertising and replenishment strategies problem

For any given replenishment and advertising strategies, we observe that

$$\frac{\partial^2 \Pi^x(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})}{\partial p_i^2} = -2\beta \int_{t_{i-1}}^{t_i} g(t)\,\mathrm{d}t < 0, \quad i = 1, 2, \ldots, n,$$

and

$$\frac{\partial^2 \Pi^x(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})}{\partial p_i \partial p_j} = 0, \quad i, j = 1, 2, \ldots, n,\ i \neq j,\ \text{and } x = E \text{ or } R,$$

Algorithm 1. The search procedure of PSO algorithm.

Step 1: Initialize I particles with random positions and velocities in d-dimensional search space.
Step 2: Evaluate the fitness of each particle.
Step 3: For each particle, keep track of its personal best position (*pbest*) where it achieved the highest fitness so far.
Step 4: Track the global best position (*gbest*) with the highest fitness among all particles.
Step 5: Update the velocity of each particle using (6) and (8).
Step 6: Update the position of each particle using the position update equation (7).
Step 7: Check the termination condition:
if the standard deviation of fitness values is less than ϵ (*e.g.*, 10^{-5}) or the maximum number of iterations (*e.g.*, 1000) is reached **then**
    Terminate the algorithm.
else
    Go back to Step 2.
end if

which implies that the Hessian matrixs of $\Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u})$ and $\Pi^{\mathrm{R}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u})$ with respect to $\boldsymbol{p}$ are negative definite, respectively. It follows that $\Pi(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u})$ is concave in $\boldsymbol{p}$ because the minimum of two concave functions is also concave. Furthermore, since the constraints $p_i - c \geq 0$ for $i = 1,2,\ldots,n$ are linear, there exists a unique $\boldsymbol{p}^* = \{p_1^*, p_2^*, \ldots, p_n^*\}$ that maximizes the profit function when the replenishment and advertising strategies are given.

To determine the optimal pricing strategy, it is evident from (5) that

$$\min\big\{\Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u}),\Pi^{\mathrm{R}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u})\big\} = \begin{cases} \Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u}), & \mathrm{CE} > \varpi, \\ \Pi^{\mathrm{R}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u}), & \mathrm{CE} < \varpi, \\ \Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u}) = \Pi^{\mathrm{R}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u}), & \mathrm{CE} = \varpi. \end{cases}$$

If there is an interior optimal solution for $\min\big\{\Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u}),\Pi^{\mathrm{R}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u})\big\} = \Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u})$, we can determine this by taking the first-order derivatives of $\Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u})$ with respect to p_i and setting them to zero, yielding the following condition:

$$\begin{aligned} \frac{\partial \Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u})}{\partial p_i} &= \int_{t_{i-1}}^{t_i} g(t)\left\{(t-t_{i-1})h + \alpha + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t \\ &\quad - \beta(2p_i - c - fE)\int_{t_{i-1}}^{t_i} g(t)\,\mathrm{d}t \\ &= 0, \end{aligned}$$

implying the optimal selling price, denoted by p_i^{E}, is

$$p_i^{\mathrm{E}} = \frac{1}{2}\left\{c + fE + \frac{\int_{t_{i-1}}^{t_i} g(t)\left\{(t-t_{i-1})\beta h + \alpha + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t}{\beta\int_{t_{i-1}}^{t_i} g(t)\,\mathrm{d}t}\right\}, \quad i = 1,2,\ldots,n. \tag{9}$$

Following a similar approach, we can obtain the interior optimal selling price, denoted by p_i^{R}, for the case of $\min\big\{\Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u}),\Pi^{\mathrm{R}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u})\big\} = \Pi^{\mathrm{R}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u})$ as

$$p_i^{\mathrm{R}} = \frac{1}{2}\left\{c + fR + \frac{\int_{t_{i-1}}^{t_i} g(t)\left\{(t-t_{i-1})\beta h + \alpha + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t}{\beta\int_{t_{i-1}}^{t_i} g(t)\,\mathrm{d}t}\right\}, \quad i = 1,2,\ldots,n. \tag{10}$$

On the other hand, if the optimal selling price, denoted by p_i^{ER}, is bounded, it implies that

$$f\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} g(t)\left\{\alpha - \beta p_i + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t - \varpi = 0.$$

Recall that $\Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u})$ is concave in $\boldsymbol{p}$, and the constraints $p_i - c \geq 0$, $i = 1, 2, \ldots, n$, and

$$f\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} g(t)\left\{\alpha - \beta p_i + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t - \varpi = 0$$

are linear, a unique $\boldsymbol{p}^* = \{p_1^*, p_2^*, \ldots, p_n^*\}$ exists to maximize the profit function when the replenishment and advertising strategies are given. To determine the optimal pricing strategy, let

$$\begin{aligned}\mathcal{L}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u},\lambda) = &\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}[p_i - c - h(t-t_{i-1})]g(t)\left\{\alpha - \beta p_i + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t\\ &- nA - w\sum_{i=1}^{n}\frac{u_i^2}{2}(t_i - t_{i-1})\\ &-\lambda\left\{f\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} g(t)\left\{\alpha - \beta p_i + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t - \varpi\right\},\end{aligned}$$

where Lagrange multiplier $\lambda \geq 0$. The first-order necessary conditions for $\mathcal{L}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u},\lambda)$ are

$$\begin{aligned}\frac{\partial\mathcal{L}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u},\lambda)}{\partial p_i} &= \int_{t_{i-1}}^{t_i} g(t)\left\{(t-t_{i-1})h + \alpha + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t\\ &\quad - \beta(2p_i - c - f\lambda)\int_{t_{i-1}}^{t_i} g(t)\,\mathrm{d}t\\ &= 0, \quad i = 1, 2, \ldots, n,\end{aligned}$$

and

$$\frac{\partial\mathcal{L}(\boldsymbol{t},\boldsymbol{p},\boldsymbol{u},\lambda)}{\partial\lambda} = f\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} g(t)\left\{\alpha - \beta p_i + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t - \varpi = 0.$$

Solving the above equations algebraically yields the optimal selling price, denoted by p_i^{ER}, and the Lagrange multiplier λ given by

$$p_i^{\mathrm{ER}} = \frac{1}{2}\left\{c + f\lambda + \frac{\int_{t_{i-1}}^{t_i} g(t)\left\{(t-t_{i-1})\beta h + \alpha + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t}{\beta\int_{t_{i-1}}^{t_i} g(t)\,\mathrm{d}t}\right\}, \quad i = 1, 2, \ldots, n, \tag{11}$$

and

$$\lambda = \frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} g(t)\left\{\alpha - \beta c - (t-t_{i-1})\beta h + \left\{k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\right\}\mathrm{d}t - \varpi}{\beta f\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} g(t)\,\mathrm{d}t},$$

respectively. It should be also noted that $\Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p}^{\mathrm{ER}},\boldsymbol{u}) = \Pi^{\mathrm{R}}(\boldsymbol{t},\boldsymbol{p}^{\mathrm{ER}},\boldsymbol{u})$ because $\mathrm{CE} = 0$.

Further, let $\mathrm{CE}^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{u})$ and $\mathrm{CE}^{\mathrm{R}}(\boldsymbol{t},\boldsymbol{u})$ denote the carbon emissions for $\Pi^{\mathrm{E}}(\boldsymbol{t},\boldsymbol{p}^{\mathrm{E}},\boldsymbol{u})$ and $\Pi^{\mathrm{R}}(\boldsymbol{t},\boldsymbol{p}^{\mathrm{R}},\boldsymbol{u})$, respectively. Utilizing (9) and (10), one can easy to see that

$$\begin{aligned}\mathrm{CE}^{\mathrm{R}}(\boldsymbol{t},\boldsymbol{u}) &= f\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} g(t)\left\{\alpha - \beta p_i^{\mathrm{R}} + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t\\ &\geq f\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} g(t)\left\{\alpha - \beta p_i^{\mathrm{E}} + k_1 G_i(t) - \frac{k_2}{2}[G_i(t)]^2\right\}\mathrm{d}t\end{aligned}$$

$$= \mathrm{CE}^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{u}) \tag{12}$$

because $E \geq R$. This result, along with the relationships among $\mathrm{CE}^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{u})$, $\mathrm{CE}^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{u})$, and ϖ, yields

$$\max\min\left\{\Pi^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}), \Pi^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})\right\} = \begin{cases} \Pi^{\mathrm{E}}\left(\boldsymbol{t}, \boldsymbol{p}^{\mathrm{E}}, \boldsymbol{u}\right), & \text{if } \mathrm{CE}^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{u}) - \varpi > 0, \\ \Pi^{\mathrm{R}}\left(\boldsymbol{t}, \boldsymbol{p}^{\mathrm{R}}, \boldsymbol{u}\right), & \text{if } \mathrm{CE}^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{u}) - \varpi < 0, \\ \Pi^{\mathrm{E}}\left(\boldsymbol{t}, \boldsymbol{p}^{\mathrm{ER}}, \boldsymbol{u}\right), & \text{if } \mathrm{CE}^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{u}) \leq \varpi \leq \mathrm{CE}^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{u}), \end{cases} \tag{13}$$

and so

$$\underset{\boldsymbol{p}}{\operatorname{argmax}} \min\left\{\Pi^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}),\ \Pi^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u})\right\} = \begin{cases} \boldsymbol{p}^{\mathrm{E}}, & \text{if } \mathrm{CE}^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{u}) - \varpi > 0, \\ \boldsymbol{p}^{\mathrm{R}}, & \text{if } \mathrm{CE}^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{u}) - \varpi < 0, \\ \boldsymbol{p}^{\mathrm{ER}}, & \text{if } \mathrm{CE}^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{u}) \leq \varpi \leq \mathrm{CE}^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{u}). \end{cases} \tag{14}$$

Equations (13) and (14) show that $\min\left\{\Pi^{\mathrm{E}}\left(\boldsymbol{t}, \boldsymbol{p}^{\mathrm{E}}, \boldsymbol{u}\right), \Pi^{\mathrm{R}}\left(\boldsymbol{t}, \boldsymbol{p}^{\mathrm{R}}, \boldsymbol{u}\right)\right\}$ can be represented by $(\boldsymbol{t}, \boldsymbol{u})$, and this simplifies the $3n-1$ dimensional problem of determining optimal pricing, advertising, and replenishment strategies to a $2n-1$ dimensional problem. Therefore, the problem (4) can be reformulated as follows:

$$\begin{aligned} &\text{Maximize } \Pi(\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{u}|n) = \min\left\{\Pi^{\mathrm{E}}(\boldsymbol{t}, \boldsymbol{p}^{\mathrm{E}}, \boldsymbol{u}), \Pi^{\mathrm{R}}(\boldsymbol{t}, \boldsymbol{p}^{\mathrm{R}}, \boldsymbol{u})\right\} \\ &\text{subject to } p_i^{\mathrm{E}} > c, && i = 1, 2, \ldots, n, \\ &\qquad p_i^{\mathrm{R}} > c, && i = 1, 2, \ldots, n, \\ &\qquad p_i^{\mathrm{ER}} > c, && i = 1, 2, \ldots, n, \\ &\qquad u_i \geq 0, && i = 1, 2, \ldots, n, \\ &\qquad t_{i-1} < t_i, && i = 1, 2, \ldots, n, \\ &\qquad G_1(0) = G_0, \quad t_0 = 0, \quad t_n = H. \end{aligned} \tag{15}$$

Based on the characteristics of our optimization problem, we selected the PSO algorithm with boundary constraints as our solution approach. The constraints in our model are addressed through the implementation of an exterior penalty method, which transforms the constrained optimization problem into an unconstrained one. The pseudo-objective function is then given as follows:

$$\begin{aligned} \phi(\mathbf{t}, \boldsymbol{u}|n) = \min&\left\{\Pi^{\mathrm{E}}\left(\boldsymbol{t}, \boldsymbol{p}^{\mathrm{E}}, \boldsymbol{u}\right),\ \Pi^{\mathrm{R}}\left(\boldsymbol{t}, \boldsymbol{p}^{\mathrm{R}}, \boldsymbol{u}\right)\right\} - \mu_1 \sum_{i=1}^{n}\left[\min\left(0,\ p_i^{\mathrm{E}} - c\right)\right]^2 \\ &- \mu_2 \sum_{i=1}^{n}\left[\min\left(0,\ p_i^{\mathrm{R}} - c\right)\right]^2 - \mu_3 \sum_{i=1}^{n}\left[\min\left(0,\ p_i^{\mathrm{ER}} - c\right)\right]^2 \\ &- \mu_4 \sum_{i=1}^{n}[\min(0,\ u_i)]^2 - \mu_5 \sum_{i=1}^{n}[\max(0,\ t_{i-1} - t_i)]^2, \end{aligned} \tag{16}$$

where μ_1, μ_2,..., μ_5 are large positive constants, commonly referred to as penalty factors. The pseudo-objective function (16) is then used to assess the fitness of individuals within the population. Therefore, the problem can be reformulated as:

$$\begin{aligned} &\text{Maximize } \phi(\mathbf{t}, \boldsymbol{u}|n) \\ &\text{subject to } G_1(0) = G_0, \quad t_0 = 0, \quad t_n = H. \end{aligned} \tag{17}$$

The solution procedure for finding optimal pricing, advertising, and replenishment strategies for any given integer of n is provided as Algorithm 2.

Algorithm 2. PSO Algorithm for pricing, advertising, and replenishment optimization.

Initialize:
Set dimension $d = 2n - 1$
Set population size $I = 20d$
Set $V_{\max}(t_i) = \frac{H}{n}$ for $i = 1, 2, \ldots, n-1$
Set $V_{\max}(u_i) = 100$ for $i = 1, 2, \ldots, n$
Set $\varphi_1 = \varphi_2 = 2.05$, $\mu_1 = \mu_2 = 10^9$
Set $\text{iter}_{\max} = 1000$, $k = 0$
for each particle $i = 1, 2, \ldots, I$ **do**
  Step 1: Initialize particle position $x_0^i = \{t_1^0, t_2^0, \ldots, t_{n-1}^0, u_1^0, u_2^0, \ldots, u_n^0\}$.
    Randomly generate $t_1^0, t_2^0, \ldots, t_{n-1}^0$ from from $(0, H)$, and sort in ascending order.
    Randomly generate $u_1^0, u_2^0, \ldots, u_n^0$ from $(0, \overline{u})$.
  Step 2: Initialize particle velocity v_0^i, randomly generate d points in the range $(-V_{\max}, V_{\max})$.
  Step 3: Evaluate fitness $\phi(x_0^i|n)$ for all particles using the objective function (16).
  Step 4: Update personal best position for each particle:
  if $\phi(x_k^i|n) > \phi(pbest_{k-1}^i|n)$ **then**
    $pbest_k^i = x_k^i$
  else
    $pbest_k^i = pbest_{k-1}^i$
  end if
  Step 5: Update global best position:
  if $\max_{1 \le i \le I} \phi(x_k^i|n) > \phi(gbest_{k-1}|n)$ **then**
    $gbest_k = \arg\max_{1 \le i \le I} \phi(x_k^i|n)$
  else
    $gbest_k = gbest_{k-1}$
  end if
  Step 6: Update velocity for each particle v_k^i using the velocity update rule:

$$v_{k+1}^i = \chi\big[v_k^i + \varphi_1 \times \text{rand} \times \big(pbest_k^i - x_k^i\big) + \varphi_2 \times \text{rand} \times \big(gbest_k - x_k^i\big)\big]$$

  Constrain: Ensure velocity v_{k+1}^i is within the range $[-V_{\max}, V_{\max}]$:

$$v_{k+1}^i = \max\big(\min\big(v_{k+1}^i, V_{\max}\big), -V_{\max}\big)$$

  Step 7: Update particle position x_{k+1}^i:

$$x_{k+1}^i = x_k^i + v_k^i$$

  Step 8: Check termination condition:
  if $\text{std}(\phi(x_k|n)) < 10^{-5}$ or $k = \text{iter}_{\max}$ **then**
    Terminate the algorithm.
  else
    Increment $k = k + 1$ and go back to Step 3.
  end if
end for

4.3. Estimation of the replenishment number

To determine the optimal number of replenishments, denoted by n^*, we developed an efficient search methodology instead of using exhaustive enumeration. Our approach begins with a strategic initial value selection for n^* to enhance computational efficiency. To facilitate the analysis, we consider a simplified scenario where the SoG maintains a constant value of G_0 throughout the planning period H. The product's selling price is approximated by taking the arithmetic mean of the regular price p^{R} and the emergency price p^{E}, which are derived from equations (9) and (10) respectively. Following the approach of Teng *et al.* [48] and Yang *et al.* [54], we can estimate the number of replenishments by substituting the relevant parameters into the classical EOQ formula:

$$n = \text{rounded integer of } \sqrt{\frac{hH \int_0^H g(t)\Big\{\alpha - \beta\Big(\frac{p^{\mathrm{E}}+p^{\mathrm{R}}}{2}\Big) + k_1 G_0 - \frac{k_2}{2} G_0^2\Big\}\,\mathrm{d}t}{2A}}. \tag{18}$$

Beginning the search for the optimal number of replenishments with the estimate from (18), rather than starting with $n = 1$, significantly reduces computational complexity. Based on these considerations, the following Algorithm 3 to efficiently solve the optimal number of replenishments.

Algorithm 3. The search procedure for finding the optimal number of replenishments n^*.

Step 1: Choose two initial trial values of n^*, say n as in (18) and $n + 1$.
Compute $\phi(\mathbf{t}^*, \boldsymbol{u}^*|n)$ and $\phi(\mathbf{t}^*, \boldsymbol{u}^*|n + 1)$.
if $\phi(\mathbf{t}^*, \boldsymbol{u}^*|n) \leq \phi(\mathbf{t}^*, \boldsymbol{u}^*|n + 1)$ **then**
  repeat
    Increment n
    Compute $\phi(\mathbf{t}^*, \boldsymbol{u}^*|n)$ and $\phi(\mathbf{t}^*, \boldsymbol{u}^*|n + 1)$
  until $\phi(\mathbf{t}^*, \boldsymbol{u}^*|n) < \phi(\mathbf{t}^*, \boldsymbol{u}^*|n + 1)$
  Set $n^* = n$ and stop.
else
  repeat
    Decrement n
    Compute $\phi(\mathbf{t}^*, \boldsymbol{u}^*|n)$ and $\phi(\mathbf{t}^*, \boldsymbol{u}^*|n - 1)$
  until $\phi(\mathbf{t}^*, \boldsymbol{u}^*|n) > \phi(\mathbf{t}^*, \boldsymbol{u}^*|n - 1)$
  Set $n^* = n$ and stop.
end if

5. Computational results

In this section, we provide several numerical examples to verify the proposed model. Next, a sensitivity analysis is conducted to gain meaningful insights into how changes in major parameters impact the optimal solution.

5.1. Numerical examples

We describe numerical examples based on a project period of one year, with $H = 12$ representing the 12 months. The carbon emission quota for the year is $\varpi = 5000$. Other relevant parameters are as follows: $A = 250$ per order, $d(p) = 140 - 3p$, $K(G) = G - 0.005G^2$, $G_0 = 20$, $c = 15$ per unit, $h = 1$ per month, $E = 3$ per emission unit, $R = 2$ per emission unit, $f = 2$, $\delta = 0.2$, $\gamma = 0.7$, and $w = 10$, with the conditions $p_i > c$ and $u_i > 0$ holding for $i = 1, 2, 3, \ldots, n$.

The solution procedure is executed using Wolfram Mathematica 13.3.1 on a computer with an Apple M2 Max processor running Mac OS X 15.0. Unless otherwise specified, these parameter values remain constant throughout the analysis. Furthermore, three types of time trends for the market demand, namely: $g(t) = 2\mathrm{e}^{0.08t}$, $g(t) = 5\mathrm{e}^{-0.05t}$, and $g(t) = \frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$ are considered, where $\mathcal{B}(2, 3)$ denotes the Beta function. The first time trend function, $g(t) = 2\mathrm{e}^{0.08t}$, represents exponential growth in demand, often associated with the early stages of a product's life cycle, where demand increases at an accelerating rate, suggesting the product is experiencing rapid adoption or gaining popularity. In contrast, the second function, $g(t) = 5\mathrm{e}^{-0.05t}$, reflects a declining demand trend, typically observed when a product enters the maturity or decline phase of its life cycle, where demand diminishes over time as market saturation occurs, and fewer new customers enter the market. Lastly, the third function, $g(t) = \frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$, captures a seasonal demand pattern, where demand initially rises, peaks at a certain point during the planning horizon, and subsequently declines. Such seasonal demand trends are common for products whose sales fluctuate based on specific periods or events.

By using (9), (10) and (18), we derive the initial numbers of replenishments for the three cases: $g(t) = 2\mathrm{e}^{0.08t}$, $g(t) = 5\mathrm{e}^{-0.05t}$, and $g(t) = \frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$, which yield $n = 5$, $n = 6$, and $n = 5$, respectively. Subsequently, applying Algorithms 2 and 3, we obtain that the optimal numbers of replenishments are 7, 8, and 7 for the three respective cases, as shown in Table 3. The optimal decision results for various market demand time trends are shown as in Table 4. The following observations can be made from the results in Tables 3 and 4.

TABLE 3. Optimal total profit and corresponding carbon emissions as functions of n under different market demand time trends with carbon cap-and-price regulation.

	$g(t) = 2e^{0.08t}$		$g(t) = 5e^{-0.05t}$		$g(t) = \frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$	
n	Π	CE	Π	CE	Π	CE
1	29 639.1	3383.05	35 046.0	3958.21	33 904.1	3704.64
2	36 692.8	3962.45	41 162.6	4470.21	40 157.0	4237.42
3	38 880.5	4134.25	43 207.0	4634.62	41 990.5	4385.24
4	39 821.7	4214.67	44 133.9	4715.25	42 749.3	4453.38
5	40 274.6	4261.07	44 598.5	4763.00	43 094.8	4492.61
6	40 487.6	4291.21	44 828.1	4794.54	43 238.3	4518.11
7	40 565.4*	4312.35	44 922.2	4816.92	43 267.8*	4536.04
8	40 559.6	4327.98	44 931.0*	4833.62	43 226.8	4549.33
9	40 498.6	4340.02	44 882.8	4846.55	43 139.1	4559.59
10	40 399.1	4349.56	44 794.4	4856.86	43 018.7	4567.74

Notes. The shaded cells indicate the values identified by Algorithm 3 as the optimal number of replenishments that maximize the total profit. The asterisk (*) denotes the optimal value of n that yields the highest total profit across all cases.

TABLE 4. Optimal strategies for various market demand time trends.

Case	i	t_i	u_i	p_i	$t_i - t_{i-1}$	Q_i
$g(t) = 2e^{0.08t}$	1	2.1500	11.2997	36.8723	2.1500	249.35
	2	4.1112	11.7774	37.5123	1.9612	286.70
	3	5.9187	11.7825	37.8994	1.8076	321.03
	4	7.5978	11.1497	38.0919	1.6791	351.33
	5	9.1655	9.6469	38.0967	1.5677	375.33
	6	10.6318	6.9676	37.8824	1.4663	389.35
	7	12.0000	2.7450	37.3776	1.3682	388.70
$g(t) = 5e^{-0.05t}$	1	1.3277	15.5359	36.6929	1.3277	346.25
	2	2.6881	13.9421	37.4831	1.3604	357.41
	3	4.0877	12.4783	37.9169	1.3996	358.48
	4	5.5398	10.9948	38.1072	1.4521	352.05
	5	7.0519	9.4117	38.1101	1.5121	339.44
	6	8.6301	7.4874	37.9411	1.5782	321.50
	7	10.2772	5.1356	37.5782	1.6471	298.57
	8	12.0000	1.9458	36.9521	1.7228	269.45
$g(t) = \frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$	1	1.5571	16.1649	37.1824	1.5571	192.86
	2	2.7978	16.2467	37.7553	1.2407	369.85
	3	3.9588	14.4044	38.1877	1.1610	436.74
	4	5.1336	11.7447	38.3598	1.1748	453.40
	5	6.4085	8.5971	38.2947	1.2749	432.03
	6	7.9545	5.0914	37.9676	1.5460	374.87
	7	12.0000	0.9476	37.2257	4.0455	261.47

(1) Increasing the number of replenishment cycles enhances strategic flexibility. Since the SoG positively influences market demand and advertising efforts directly contribute to SoG accumulation, more frequent replenishment points allow the firm to fine-tune its advertising and pricing strategies. This enables the firm to better align with shifting market conditions, proactively shape customer perceptions, and boost demand over time. Nonetheless, such an aggressive market engagement approach also leads to higher product sales, increasing

cumulative carbon emissions due to the positive relationship between emissions and total units sold. Consequently, the firm must weigh the benefits of more responsive demand management against the potential rise in environmental costs, especially under cap-and-price regulations. This trade-off underscores the importance of integrated decision-making where marketing effectiveness and emission responsibilities are jointly considered.

(2) Under an increasing demand trend, the firm adopts shorter replenishment intervals $t_i - t_{i-1}$ to ensure timely product availability and avoid missed sales opportunities. This strategy reflects the firm's response to growing market dynamics, where more frequent replenishment allows the firm to maintain sufficient stock to meet escalating demand. From an operational perspective, front-loading advertising efforts and implementing relatively lower prices in the early stages help the firm capture consumer attention and accelerate goodwill accumulation. As a result, the order quantity Q_i increases over time, ensuring that the firm can continuously meet demand growth without excessive inventory build-up. However, this aggressive replenishment and promotion strategy increases production and logistics activities, leading to higher carbon emissions. Therefore, the firm must carefully balance profitability and market responsiveness against the environmental costs induced by stricter carbon regulations. This highlights the dual role of pricing and advertising as tactical levers for demand stimulation and contributors to the firm's carbon emissions.

(3) Under a decreasing demand trend, the firm adopts longer replenishment intervals $t_i - t_{i-1}$ and reduces the order quantity Q_i in each cycle to avoid overstocking and minimize unnecessary operational costs. This strategy reflects a conservative response to weakening market demand, where less frequent replenishments help prevent excess inventory and reduce holding costs. From a managerial perspective, the firm also scales back advertising efforts and avoids aggressive pricing discounts, as the marginal return on these strategies diminishes in a contracting market. This conservative adjustment safeguards profitability under decreasing demand and supports environmental compliance by limiting production and logistics activities, thereby reducing cumulative carbon emissions.

Furthermore, since PSO is a stochastic search technique, each run may yield slightly different results. Although its design helps avoid local optima, convergence to the global optimum is not always guaranteed. To assess the performance of the PSO algorithm, each example was implemented 300 times. Additionally, we compared PSO with three alternative algorithms provided by Wolfram Mathematica: Nelder Mead, Differential Evolution, and Simulated Annealing. The statistical results from these simulations are presented in Table 5. As shown in Table 5, PSO consistently achieves the best or near-best objective values across all demand trend scenarios, highlighting its strong stability and convergence reliability. Moreover, PSO requires significantly less computational time and fewer convergence generations than Nelder Mead, Differential Evolution, and Simulated Annealing. The computational results demonstrate that PSO generally outperforms other algorithms in both computational efficiency and solution robustness. Although PSO performs slightly below Differential Evolution regarding the best solution quality in the third scenario, it significantly outperforms Differential Evolution regarding computational time and convergence speed. These findings suggest that PSO is an efficient algorithm and a robust method for solving the proposed nonlinear joint pricing and advertising optimization model under carbon emissions regulation.

To further understand the impact of different carbon emission mechanisms on firm decision-making, we conduct a comparative analysis of three alternative regulation schemes: Cap-and-Trade, Carbon Offsetting, and No Regulation. Considering the product's life cycle, we focus on the case where $g(t) = \frac{t^1(H-t)^2}{500\,\mathcal{B}(2,3)}$. All computed results of the numerical experiments are presented in Table 6; the differences across these schemes provide practical insights into how firms should adjust their pricing, advertising, and replenishment strategies under various regulatory environments.

In the Cap-and-Trade scenario, the firm achieves a higher level of total profit ($\Pi = 43\,871.5$) while maintaining carbon emissions at a manageable level (CE = 4256.29). This outcome is primarily attributed to the firm's flexibility under the trading mechanism, which allows it to exceed initial emission allowances by purchasing additional credits in the carbon market. As a result, the firm can adopt slightly more aggressive advertising strategies and fulfill higher demand without being constrained by a fixed emission cap. This result is also

TABLE 5. Experimental results of PSO for different time trends of demand.

Algorithm	Case	ACT	ACG	Best	Q_3	Median	Q_1	Worst	Mean	Std
Particle swarm optimization	$g(t)=2e^{0.08t}$	3.0812	147.8	40 565.4	40 565.4	40 565.4	40 565.4	40 565.4	40 565.4	0.0000
	$g(t)=5e^{-0.05t}$	3.9278	157.4	44 931.0	44 931.0	44 931.0	44 931.0	44 931.0	44 931.0	0.0000
	$g(t)=\frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$	6.7232	159.0	43 267.8	43 267.8	43 267.8	43 267.8	42 987.5	43 266.9	16.1860
Nelder Mead	$g(t)=2e^{0.08t}$	38.8286	1237.8	40 565.4	40 565.3	40 565.4	40 565.4	40 237.2	40 561.4	22.6840
	$g(t)=5e^{-0.05t}$	70.3469	1637.7	44 931.0	44 920.7	44 930.2	44 930.9	44 498.0	44 898.2	80.5379
	$g(t)=\frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$	103.4490	1494.5	43 267.8	42 987.9	43 266.5	43 267.8	42 355.5	43 154.2	179.3300
Differential evolution	$g(t)=2e^{0.08t}$	163.5460	1005.3	40565.4	40 565.4	40 565.4	40 565.4	40 565.4	40 565.4	0.0000
	$g(t)=5e^{-0.05t}$	236.0180	1003.1	44 931.0	44 931.0	44 931.0	44 931.0	44 931.0	44 931.0	0.0000
	$g(t)=\frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$	528.1080	1004.5	43 267.8	43 267.8	43 267.8	43 267.8	43 267.8	43 267.8	0.0000
Simulated annealing	$g(t)=2e^{0.08t}$	52.4288	122 543.0	40 545.8	40 480.8	40 498.8	40 517.0	40 406.1	40 497.1	25.6653
	$g(t)=5e^{-0.05t}$	69.3386	147 271.0	44 912.2	44 778.7	44 815.1	44 843.1	44 596.0	44 808.6	52.0172
	$g(t)=\frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$	70.6888	132 706.0	43 242.5	43 086.1	43 137.7	43 180.4	42 877.1	43 127.1	70.8035

Notes. Q_1 and Q_3 represent the first and third quartiles, respectively. ACT represents average CPU time (second). ACG represents average convergence generations.

TABLE 6. Optimal strategies for various emission mechanisms.

Emission mechanism	i	t_i	u_i	p_i	$t_i - t_{i-1}$	Q_i	Π	CE
Cap-and-Trade $E=3$ and $R=3$	1	1.5554	15.4344	38.1249	1.5554	180.87	43 871.5	4256.29
	2	2.7945	15.5502	38.6535	1.2392	347.05		
	3	3.9540	13.8028	39.0613	1.1594	409.32		
	4	5.1271	11.2565	39.2215	1.1732	424.97		
	5	6.4002	8.2286	39.1514	1.2731	405.15		
	6	7.9442	4.8627	38.8261	1.5440	351.49		
	7	12.0000	0.9002	38.1017	4.0559	245.89		
Carbon offsetting $E=3$ and $R=0$	1	1.5589	17.3273	35.5995	1.5589	184.33	42 880.4	5000
	2	2.8016	17.3497	36.2409	1.2427	358.17		
	3	3.9653	15.3434	36.7095	1.1637	425.85		
	4	5.1435	12.5107	36.8989	1.1782	443.60		
	5	6.4219	9.1733	36.8410	1.2785	422.42		
	6	7.9710	5.4515	36.5116	1.5491	364.04		
	7	12.0000	1.0230	35.7443	4.0290	249.55		
Without emissions regulation $E=0$ and $R=0$	1	1.5593	17.5462	35.2895	1.5593	216.30	42 895.1	5090.11
	2	2.8023	17.5556	35.9434	1.2430	415.29		
	3	3.9664	15.5186	36.4185	1.1641	490.54		
	4	5.1452	12.6533	36.6108	1.1787	509.74		
	5	6.4242	9.2813	36.5542	1.2791	485.92		
	6	7.9739	5.5195	36.2246	1.5497	420.89		
	7	12.0000	1.0374	35.4528	4.0261	292.20		

expected, given that the subsidy rate R equals the penalty rate E, providing sufficient incentives to operate efficiently under emission regulations.

Under the Carbon Offsetting mechanism, the firm achieves moderate profitability ($\Pi = 42\,880.4$) while generating the highest level of emissions among all scenarios (CE = 5000). This outcome is primarily attributable to the absence of emission subsidies ($R = 0$), which eliminates direct incentives for the firm to reduce emissions through operational adjustments. Consequently, the firm adopts a more aggressive market strategy by increasing advertising and lowering prices, resulting in greater demand and elevated carbon emissions. Since the offset mechanism allows firms to compensate for rather than reduce their emissions, it may encourage short-term

TABLE 7. Effects of changes in the model parameters on the optimal number of replenishments, total profit and carbon emissions.

Parameter		n^*	$\Pi(\boldsymbol{t}^*, \boldsymbol{p}^*, \boldsymbol{u}^*)$	CE*
G_0 $\varpi = 4000$	10	7	38 818.0	4152.66
	20	7	40 871.5	4256.29
	30	7	42 851.8	4350.92
	40	7	44 746.3	4437.13
E $\varpi = 4000$	2.0	7	41 267.8	4536.04
	2.5	7	41 034.7	4396.41
	3.5	7	40 778.5	4115.65
	4.0	6	40 759.1	4000.00
R $\varpi = 5000$	0	7	42 880.4	5000.00
	1	7	42 943.0	4813.93
	2	7	43 267.8	4536.04
	3	7	43 871.5	4256.29
ϖ	4000	7	40 871.5	4256.29
	4500	7	42 265.5	4500.00
	5000	7	43 267.8	4536.04
	5500	7	44 267.8	4536.04

market expansion at the expense of long-term environmental responsibility. From a managerial perspective, while carbon offsetting provides strategic flexibility, decision-makers must carefully evaluate whether the cost of purchasing offsets is economically justified by the gains in revenue, particularly in the absence of direct regulatory subsidies.

Finally, in the Without Emissions Regulation scenario, the firm focuses entirely on maximizing demand by adopting aggressive pricing and advertising strategies throughout the planning horizon. As a result, it offers lower selling prices. It allocates higher advertising efforts, which drive up sales but also lead to the highest level of carbon emissions across all scenarios (CE = 5090.11). However, this approach results in a decline in total profit ($\Pi = 42\,895.1$), indicating that aggressive market expansion without emission constraints may not yield better financial outcomes. This finding highlights the value of regulatory mechanisms such as Cap-and-Price, which can effectively guide firms to make strategic, operational adjustments that balance profitability with environmental responsibility.

5.2. Sensitivity analysis

In this subsection, we conduct several numerical experiments to investigate further how the model's optimal decisions change with the model parameters regarding carbon emissions. The same parameters as in Section 4.1 are used. Considering the product's life cycle, we only consider the case where $g(t) = \frac{t^1(H-t)^2}{500\,\mathcal{B}(2,3)}$. All computed results of the numerical experiments are presented in Table 7 and Figures 2–5. From Table 7 and Figure 2, the numerical results first illustrate that a higher initial value of G_0 leads to an increased SoG, market demand rate, and selling prices but decreased advertising efforts, which results in higher total profits and carbon emissions. This happens because consumers are more likely to choose products from firms with higher levels of trust and reputation. With a higher initial SoG, the firm benefits from enhanced customer loyalty and brand trust, leading to a higher SoG and market demand rate. In response, the firm can charge a higher selling price to boost unit marginal profit but reduce advertising efforts to lower relevant advertising costs. Thus, the total profits increase as G_0 increases. However, as demand grows, the firm needs to increase production and inventory replenishment to meet this demand, which raises carbon emissions associated with logistics activities. Therefore, businesses should recognize the dual role of SoG. While it enables cost-effective pricing and advertising strategies, it also requires tighter control over emission-related operations to comply with carbon regulations. Firms with substantial brand equity should actively invest in green logistics and carbon-efficient practices to align profitability with sustainability goals.

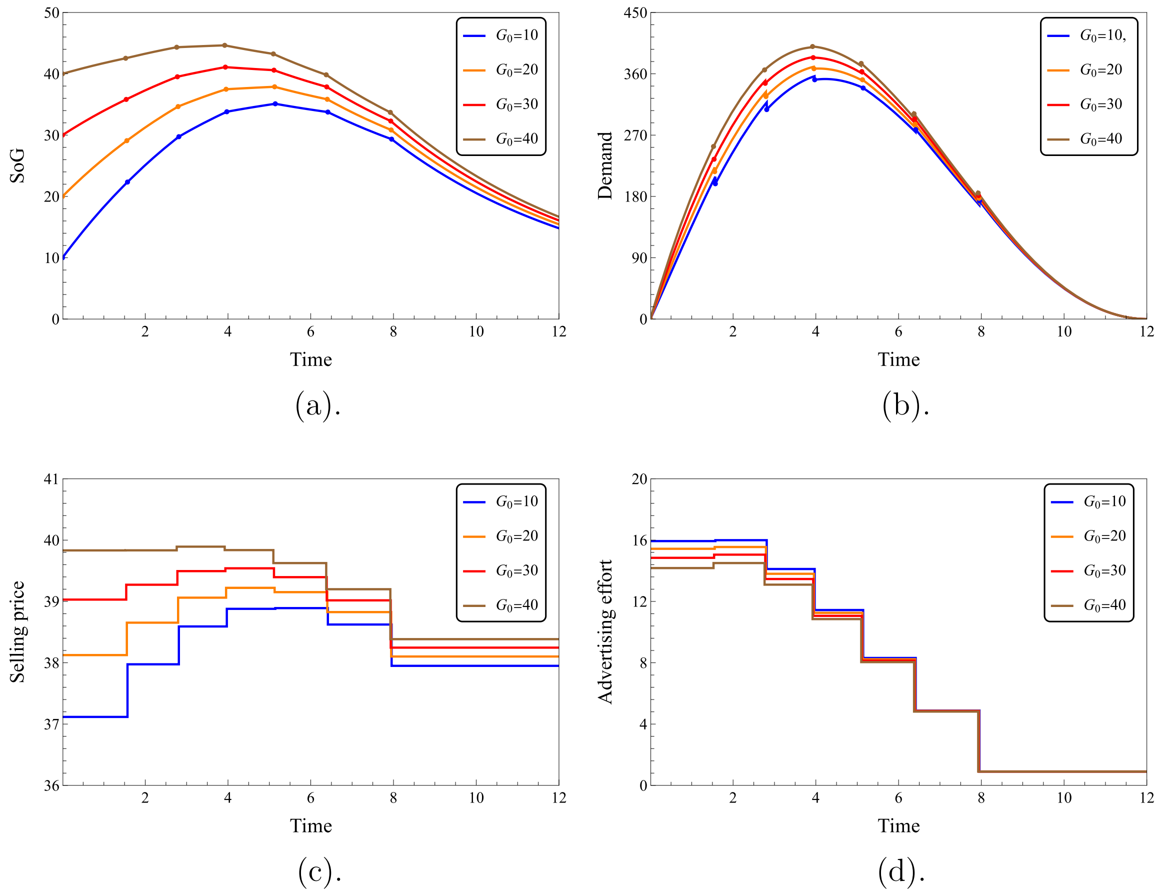

FIGURE 2. The impact of G_0 on the optimal SoG, market demand rate, selling prices, and advertising efforts under the market demand trend $g(t) = \frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$. (a) Optimal SoG. (b) Optimal demand rate. (c) Optimal selling prices. (d) Optimal advertising efforts.

Next, we observe from Table 7 and Figures 3 and 4 that higher values of E or R lead to decreased SoG, market demand rate, advertising efforts, and increased selling prices, resulting in lower total profits and carbon emissions. An economic interpretation of the results is as follows. As the carbon emission penalty increases, the firm faces higher operating costs due to additional expenses associated with carbon emissions. To maintain profit margins in the face of higher costs, the firm raises selling prices to pass these costs onto consumers and reduces advertising efforts to cut expenses. However, increasing the product's selling prices and lowering advertising efforts damage the firm's brand reputation and erode customer loyalty, decreasing SoG and market demand rate as the carbon price increases. Conversely, increased subsidies for reducing emissions incentivize the firm to lower its advertising efforts to reduce emissions, leading to a decrease in SoG and market demand rate. To compensate for the reduced sales volume and maintain profit margins, the firm increases the product's selling prices to enhance unit marginal profits. Therefore, selling prices rise as the firm adjusts its strategy in response to emission reduction subsidies. These findings suggest that firms must strategically balance profit margins with market demand under stricter carbon emission penalties. Over-reliance on price increases and advertising cuts may reduce demand and harm long-term brand value. Managers should explore alternative ways to maintain

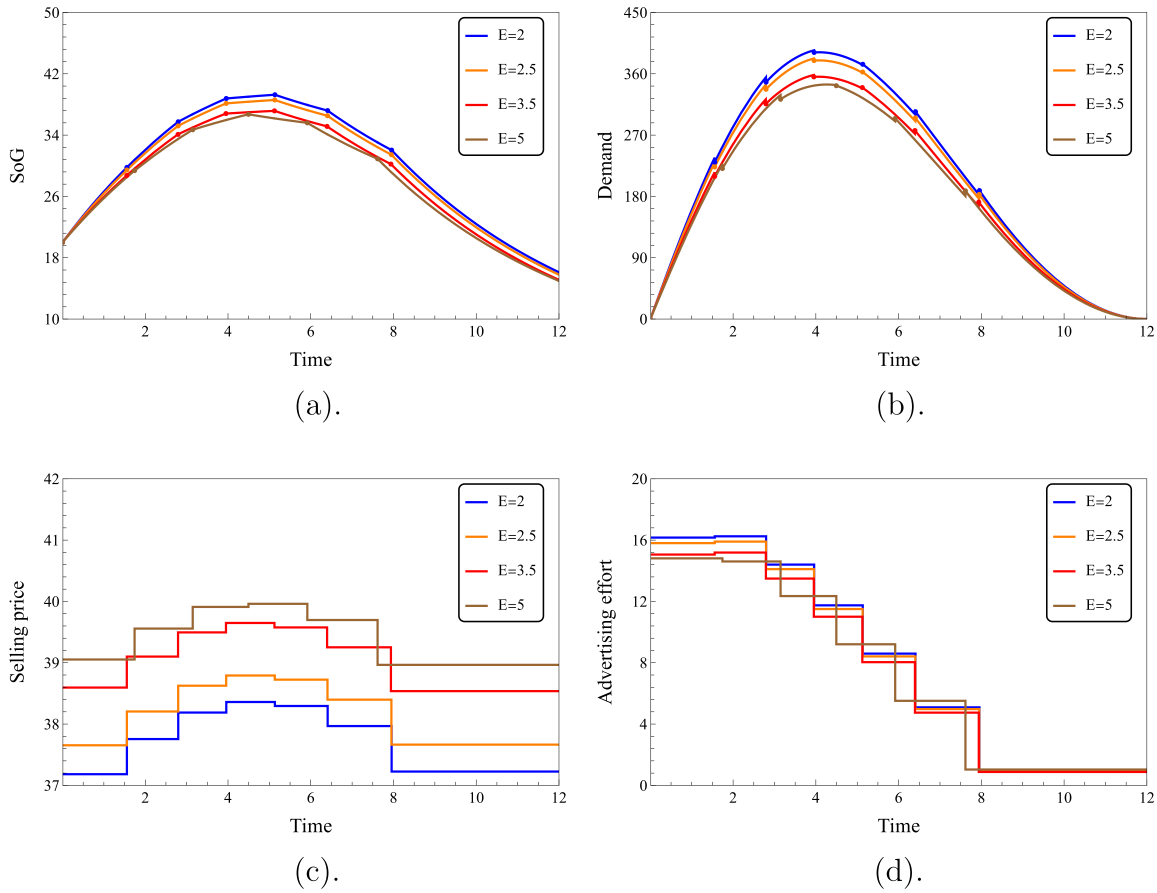

FIGURE 3. The impact of E on the optimal SoG, market demand rate, selling prices, and advertising efforts under the market demand trend $g(t) = \frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$. (a) Optimal SoG. (b) Optimal demand rate. (c) Optimal selling prices. (d) Optimal advertising efforts.

competitiveness, such as investing in cleaner technologies to reduce emission costs without compromising market appeal.

6. Concluding remarks

Emission mechanisms aim to reduce greenhouse gas emissions from anthropogenic activities by imposing limits and associated costs on carbon outputs. In today's competitive business environment, firms face the dual challenge of maximizing profits while complying with increasingly stringent environmental regulations. Consequently, firms must consider both government carbon management regulations and the stage of the product life cycle in which their products are made when setting selling prices and advertising strategies. This paper proposes a dynamic optimization model integrating cap-and-price regulation with pricing, advertising, and time-varying demand. It provides firms with a comprehensive tool to optimize their pricing and advertising strategies over a finite planning horizon. To develop a straightforward methodology to analyze the problem in a discrete dynamic optimization framework, we assume that the demand function follows a quadratic function form of selling price and SoG. The proposed model is flexible and can be adapted to various emission regulation

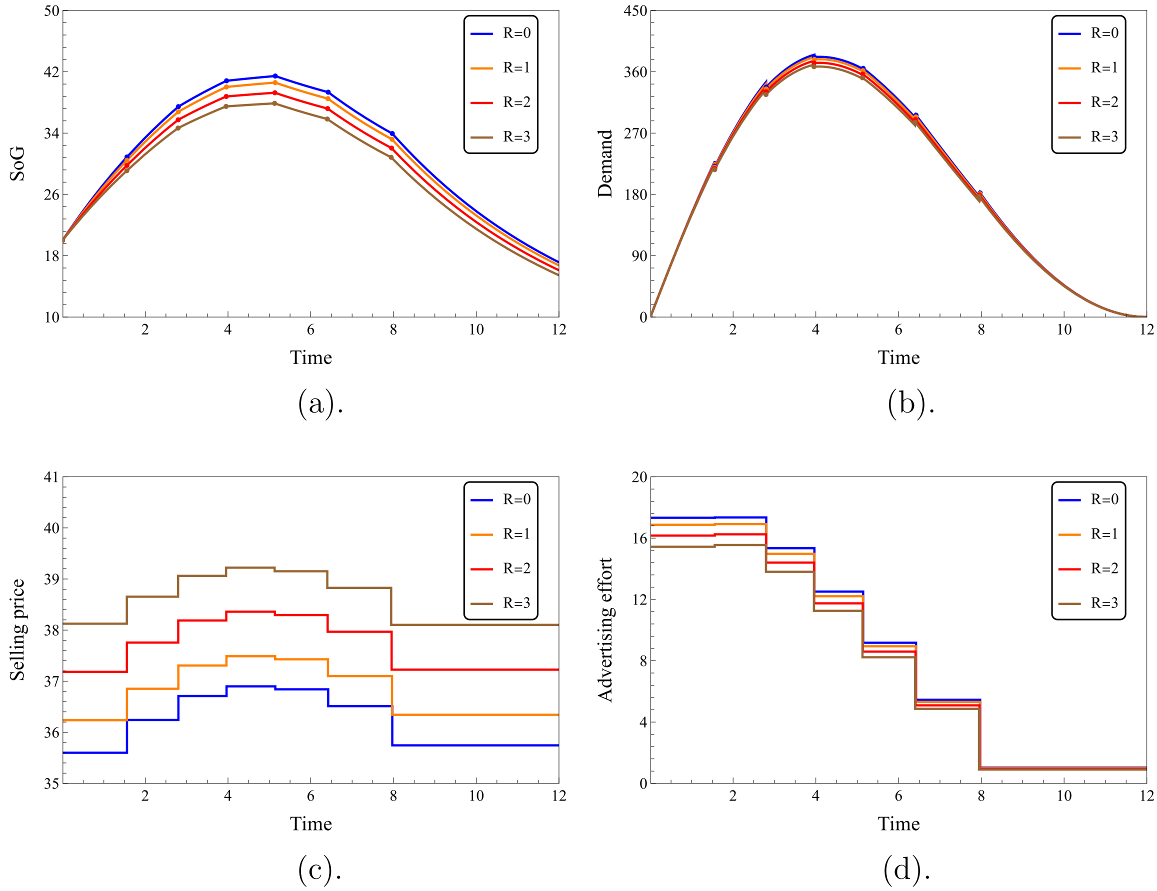

FIGURE 4. The impact of R on the optimal SoG, market demand rate, selling prices, and advertising efforts under the market demand trend $g(t) = \frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$. (a) Optimal SoG. (b) Optimal demand rate. (c) Optimal selling prices. (d) Optimal advertising efforts.

mechanisms, including cap-and-trade (*i.e.*, $E = R$), carbon taxes (*i.e.*, $\varpi = 0$), cap-and-offset (*i.e.*, $R = 0$), and stringent carbon caps (*i.e.*, $E \to \infty$ and $R = 0$). Furthermore, we selected the PSO algorithm due to its robustness, simplicity, and ease of implementation. The computational results also demonstrate that the PSO algorithm with a constriction factor offers acceptable efficiency and accurate search capabilities, effectively solving the optimization problem in different scenarios.

Our study shows that carbon cap-and-price regulations can effectively balance economic growth and emissions control. Integrating strategic decision-making across pricing, advertising, and replenishment strategies is essential for firms to maintain profitability while effectively managing carbon emissions and sustaining customer loyalty. Our theoretical results show that the government can constrain a firm's carbon emissions within a specific range under the carbon cap-and-price regulation. Thus, the cap-and-price regulation offers governments an effective policy tool for balancing economic development with environmental protection. The numerical results also demonstrate how frequently advertising effectively manages the SoG by adjusting advertising efforts as replenishments increase, influencing market demand. However, production and logistics activities inevitably increase as demand rises, leading to higher carbon emissions. This finding underscores the importance of determining

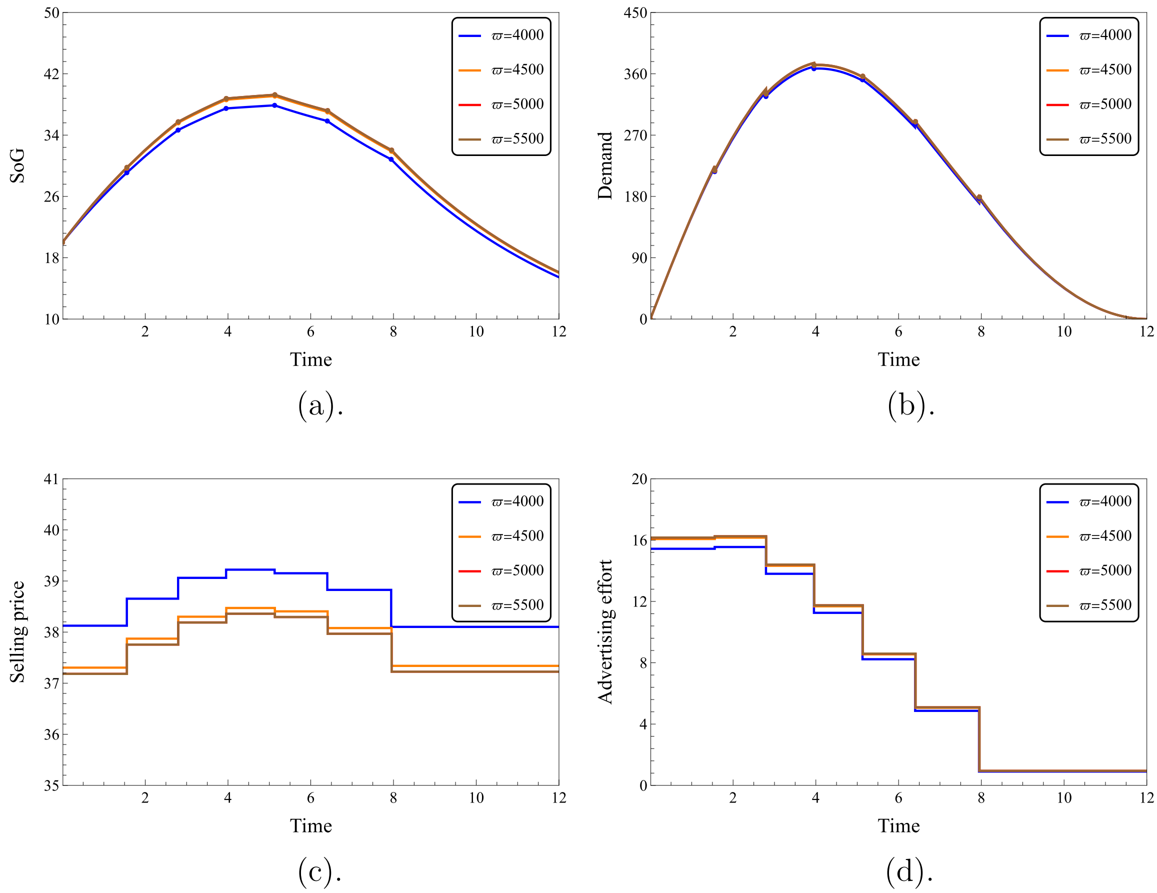

FIGURE 5. The impact of ϖ on the optimal SoG, market demand rate, selling prices, and advertising efforts under the market demand trend $g(t) = \frac{t^1(H-t)^2}{500\mathcal{B}(2,3)}$. (a) Optimal SoG. (b) Optimal demand rate. (c) Optimal selling prices. (d) Optimal advertising efforts.

optimal replenishment strategies within a finite planning horizon to balance demand growth with emission control.

Furthermore, to better understand the impact of emission regulations on firm behavior, we compare three mechanisms: Cap-and-Trade, Carbon Offsetting, and No Regulation. We find that the Cap-and-Trade scenario yields the highest profit and moderate emissions, benefiting from the flexibility of carbon credit trading and balanced incentives ($R = E$). The firm achieves moderate profitability under the Carbon Offsetting mechanism while generating relatively high emissions. This outcome is primarily attributable to the absence of emission subsidies ($R = 0$), which removes incentives for the firm to reduce emissions through operational adjustments. The firm stimulates demand through increased advertising and lower prices. Since the offsetting approach allows firms to compensate for emissions instead of reducing them at the source, it may encourage short-term expansion at the cost of long-term sustainability. From a managerial perspective, while carbon offsetting offers flexibility, firms must carefully evaluate whether the cost of purchasing offsets is justified by revenue gains, primarily when no direct subsidy supports emission reductions. The firm adopts aggressive pricing and advertising strategies without regulation, leading to higher emissions and lower profits. These results emphasize the importance of regulatory design in aligning environmental goals with economic performance. Meanwhile, our sensitivity analysis indicates that initial goodwill and regulatory parameters significantly influence demand and emissions. Carbon penalty internalizes carbon emission activities, meaning that firms must bear the costs associated with their emissions. However, rising carbon penalties increase the firm's operating costs and have become a pressing concern for many industries, particularly in the steel, cement, and aviation sectors. These increased costs are likely reflected in higher product costs, which will be passed on to consumers. For example, some airlines have begun to impose carbon emission surcharges on tickets to offset their carbon emission costs. Additionally, raising selling prices and reducing advertising efforts can damage the firm's brand reputation and weaken customer loyalty.

This study has several limitations that provide avenues for future research. First, the proposed model does not incorporate investment in abatement technologies or green innovation. As a result, carbon emissions are primarily reduced through adjustments in pricing, advertising, or replenishment strategies. While such strategies can help firms comply with emission regulations, they may also negatively affect customer perception and long-term brand loyalty. With growing consumer awareness of environmental sustainability, investments in abatement and green innovation have become increasingly important. These efforts directly reduce emissions and enhance a firm's public image and brand reputation. Future research could enrich the current model by explicitly incorporating green innovation or abatement investments to better capture economic and environmental objectives. Second, this study focuses solely on a single firm's decision-making. However, effective carbon reduction often requires coordination across the entire supply chain. Future studies may consider incorporating joint decision-making mechanisms under cost-sharing or revenue-sharing contracts, such as cooperative advertising and shared abatement strategies. Such an integrated approach could enhance both environmental performance and supply chain profitability. Finally, while the model captures the direct effects of price and goodwill on demand, it does not incorporate more nuanced customer behaviors under dynamic pricing, such as strategic purchasing, reference price effects, or perceived price fairness. These aspects fall beyond the scope of the current study but represent promising directions for future research to more accurately reflect real-world consumer psychology.

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References

[1] M. Akhtar, A.K. Manna and A.K. Bhunia, Optimization of a non-instantaneous deteriorating inventory problem with time and price dependent demand over finite time horizon via hybrid DESGO algorithm. *Exp. Syst. Appl.* **211** (2023) 118676.

[2] M. Al-Amin Khan, A.A. Shaikh, I. Konstantaras, A.K. Bhunia and L.E. Cárdenas-Barrón, Inventory models for perishable items with advanced payment, linearly time-dependent holding cost and demand dependent on advertisement and selling price. *Int. J. Prod. Econ.* **230** (2020) 107804.

[3] B.L. Bayus, Are product life cycles really getting shorter? *J. Prod. Innov. Manag.* **11** (1994) 300–308.

[4] S. Benjaafar, Y. Li and M. Daskin, Carbon footprint and the management of supply chains: insights from simple models. *IEEE Trans. Autom. Sci. Eng.* **10** (2013) 99–116.

[5] A.K. Bhunia and A.A. Shaikh, An application of PSO in a two-warehouse inventory model for deteriorating item under permissible delay in payment with different inventory policies. *Appl. Math. Comput.* **256** (2015) 831–850.

[6] A.K. Bhunia, A.A. Shaikh, V. Dhaka, S. Pareek and L.E. Cárdenas-Barrón, An application of genetic algorithm and PSO in an inventory model for single deteriorating item with variable demand dependent on marketing strategy and displayed stock level. *Sci. Iran.* **25** (2018) 1641–1655.

[7] J.M. Chen and L.T. Chen, Pricing and lot-sizing for a deteriorating item in a periodic review inventory system with shortages. *J. Oper. Res. Soc.* **55** (2004) 892–901.

[8] C.K. Chen, C. Liao and T.C. Weng, Optimal replenishment policies for the case of a demand function with product-life-cycle shape in a finite planning horizon. *Exp. Syst. Appl.* **32** (2007) 65–76.

[9] X. Chen, S. Benjaafar and A. Elomri, The carbon-constrained EOQ. *Oper. Res. Lett.* **41** (2013) 172–179.

[10] L. Chen, X. Chen, M.F. Keblis and G. Li, Optimal pricing and replenishment policy for deteriorating inventory under stock-level-dependent, time-varying and price-dependent demand. *Comput. Ind. Eng.* **135** (2019) 1294–1299.

[11] R.Y. Chenavaz and A. Eynan, Advertising, goodwill, and the Veblen effect. *Eur. J. Oper. Res.* **289** (2021) 676–682.

[12] M. Choudhury, S. Das, G.W. Weber and G.C. Mahata, Carbon-regulated inventory pricing and ordering policies with marketing initiatives under order volume discounting scheme. *Ann. Oper. Res.* (2025)

[13] M. Clerc and J. Kennedy, The particle swarm–explosion, stability, and convergence in a multidimensional complex space. *IEEE Trans. Evol. Comput.* **6** (2002) 58–73.

[14] R. Crawford, Life Cycle Assessment in the Built Environment. Routledge, London (2011).

[15] B. Crettez, N. Hayek and G. Zaccour, Existence and uniqueness of optimal dynamic pricing and advertising controls without concavity. *Oper. Res. Lett.* **46** (2018) 199–204.

[16] T. Doganoglu and D. Klapper, Goodwill and dynamic advertising strategies. *Quant. Mark. Econ.* **4** (2006) 5–29.

[17] W.A. Donaldson, Inventory replenishment policy for a linear trend in demand-an analytical solution. *Oper. Res. Q.* **28** (1977) 663–670.

[18] C.Y. Dye, A finite horizon deteriorating inventory model with two-phase pricing and time-varying demand and cost under trade credit financing using particle swarm optimization. *Swarm Evol. Comput.* **5** (2012) 37–53.

[19] C.Y. Dye and C.T. Yang, Sustainable trade credit and replenishment decisions with credit-linked demand under carbon emission constraints. *Eur. J. Oper. Res.* **244** (2015) 187–200.

[20] R.C. Eberhart and J. Kennedy, A new optimizer using particle swarm theory, in Proceedings of the Sixth International Symposium on Micromachine and Human Science, Nagoya, Japan (1995) 39–43.

[21] R.C. Eberhart and Y. Shi, Comparing inertia weights and constriction factors in particle swarm optimization, in Proceedings of the 2000 Congress on Evolutionary Computation, CEC00 (Cat. No. 00TH8512). Vol. of 1. IEEE (2000) 84–88.

[22] S.K. Goyal, D. Morrin and F. Nebebe, The finite horizon trended inventory replenishment problem with shortages. *J. Oper. Res. Soc.* **43** (1992) 1173–1178.

[23] M. Honarvar, M. Alimohammadi Ardakani and M. Modarres, A particle swarm optimization algorithm for solving pricing and lead time quotation in a dual-channel supply chain with multiple customer classes. *Adv. Oper. Res.* **2020** (2020) 5917126.

[24] D. Horsky, An empirical analysis of the optimal advertising policy. *Manag. Sci.* **23** (1977) 1037–1049.

[25] J. Huang, M. Leng and L. Liang, Recent developments in dynamic advertising research. *Eur. J. Oper. Res.* **220** (2012) 591–609.

[26] M. Jain and P. Singh, Optimal inspection and advance payment policy for deteriorating items using differential evolution metaheuristic. *Appl. Soft Comput.* **128** (2022) 109475.

[27] S. Jørgensen and G. Zaccour, Equilibrium pricing and advertising strategies in a marketing channel. *J. Optim. Theor. Appl.* **102** (1999) 111–125.

[28] S. Jørgensen and G. Zaccour, A survey of game-theoretic models of cooperative advertising. *Eur. J. Oper. Res.* **237** (2014) 1–14.

[29] J. Kennedy and R.C. Eberhart, Particle swarm optimization, in Proceedings of IEEE International Conference on Neural Networks. Vol. 4. Piscataway, NJ (1995) 1942–1948.

[30] J.Y. Lee, Investing in carbon emissions reduction in the EOQ model. *J. Oper. Res. Soc.* **71** (2020) 1289–1300.

[31] L.M. Lodish, M. Abraham, S. Kalmenson, J. Livelsberger, B. Lubetkin, B. Richardson and M.E. Stevens, How TV advertising works: a meta-analysis of 389 real world split cable TV advertising experiments. *J. Mark. Res.* **32** (1995) 125–139.

[32] P. Mahata, G.C. Mahata and S. Kumar De, Optimal replenishment and credit policy in supply chain inventory model under two levels of trade credit with time- and credit-sensitive demand involving default risk. *J. Ind. Eng. Int.* **14** (2018) 31–42.

[33] P. Mahata, G.C. Mahata and A. Mukherjee, An ordering policy for deteriorating items with price-dependent iso-elastic demand under permissible delay in payments and price inflation. *Math. Comput. Model. Dyn. Syst.* **25** (2019) 575–601.

[34] R. Mardyana, and G.C. Mahata, Impacts of dual carbon emission reduction technology and technology spillovers of deterioration reduction on supply chain system's performances considering government incentives and contract design. *J. Clean. Prod.* **468** (2024) 142977.

[35] R. Mardyana, S.K. De and G.C.M., Investigation of an imperfect production system with controllable deterioration under emission and service constraint via optimal control theory. *Int. J. Manag. Sci. Eng. Manag.* **20** (2025) 1–23.

[36] G. Massonnet, J.P. Gayon and C. Rapine, Approximation algorithms for deterministic continuous-review inventory lot-sizing problems with time-varying demand. *Eur. J. Oper. Res.* **234** (2014) 641–649.
[37] U. Mishra, J.Z. Wu and B. Sarkar, Optimum sustainable inventory management with backorder and deterioration under controllable carbon emissions. *J. Clean. Prod.* **279** (2021) 123699.
[38] A. Mukherjee and G.C. Mahata, Optimal replenishment and credit policy in an inventory model for deteriorating items under two-levels of trade credit policy when demand depends on both time and credit period involving default risk. *RAIRO-Oper. Res.* **52** (2018) 1175–1200.
[39] A. Nair and R. Narasimhan, Dynamics of competing with quality-and advertising-based goodwill. *Eur. J. Oper. Res.* **175** (2006) 462–474.
[40] M. Nerlove and K.J. Arrow, Optimal advertising policy under dynamic conditions. *Economica* **29** (1962) 129–142.
[41] M. Pervin, S.K. Roy, P. Sannyashi and G.W. Weber, Sustainable inventory model with environmental impact for non-instantaneous deteriorating items with composite demand. *RAIRO-Oper. Res.* **57** (2023) 237–261.
[42] A. Ranjan and J. Jha, Multi-period dynamic pricing model for deteriorating products in a supply chain with preservation technology investment and carbon emission. *Comput. Ind. Eng.* **174** (2022) 108817.
[43] S. Saha, B. Sarkar and M. Sarkar, Application of improved meta-heuristic algorithms for green preservation technology management to optimize dynamical investments and replenishment strategies. *Math. Comput. Simul.* **209** (2023) 426–450.
[44] L.A. San-José, J. Sicilia and B. Abdul-Jalbar, Optimal policy for an inventory system with demand dependent on price, time and frequency of advertisement. *Comput. Oper. Res.* **128** (2021) 105169.
[45] A. Shamayleh, M. Hariga, R. As'ad and A. Diabat, Economic and environmental models for cold products with time varying demand. *J. Clean. Prod.* **212** (2019) 847–863.
[46] E.S. Subramanyam and S. Kumaraswamy, EOQ Formula under varying marketing policies and conditions. *AIIE Trans.* **13** (1981) 312–314.
[47] A.A. Taleizadeh, L. Aliabadi and P. Thaichon, A sustainable inventory system with price-sensitive demand and carbon emissions under partial trade credit and partial backordering. *Oper. Res.* 22 (2022) 4471–4516.
[48] J.T. Teng, M.S. Chern, H.L. Yang and Y.J. Wang, Deterministic lot-size inventory models with shortages and deterioration for fluctuating demand. *Oper. Res. Lett.* **24** (1999) 65–72.
[49] S. Tiwari, C.K. Jaggi, A.K. Bhunia, A.A. Shaikh and M. Goh, Two-warehouse inventory model for non-instantaneous deteriorating items with stock-dependent demand and inflation using particle swarm optimization. *Ann. Oper. Res.* **254** (2017) 401–423.
[50] Y.C. Tsao, Two-phase pricing and inventory management for deteriorating and fashion goods under trade credit. *Math. Methods Oper. Res.* **72** (2010) 107–127.
[51] T.L. Urban, Deterministic inventory models incorporating marketing decisions. *Comput. Ind. Eng.* **22** (1992) 85–93.
[52] H.M. Wee, Joint pricing and replenishment policy for deteriorating inventory with declining market. *Int. J. Prod. Econ.* **40** (1995) 163–171.
[53] J. Wu, J.T. Teng and K. Skouri, Optimal inventory policies for deteriorating items with trapezoidal-type demand patterns and maximum lifetimes under upstream and downstream trade credits. *Ann. Oper. Res.* **264** (2018) 459–476.
[54] H.L. Yang, J.T. Teng and M.S. Chern, A forward recursive algorithm for inventory lot-size models with power-form demand and shortages. *Eur. J. Oper. Res.* **137** (2002) 394–400.