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SOLVING MULTI-OBJECTIVE OPTIMIZATION PROBLEM IN BIPOLAR HESITANT FUZZY ENVIRONMENT

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Abstract. Multi-objective optimization problems are pervasive in various fields, ranging from engineering and economics to environmental management and decision-making processes. These problems involve the simultaneous optimization of multiple conflicting objectives, often leading to complex and non-linear relationships between decision variables. To tackle such intricate problems, this article introduces a novel approach: Bipolar Hesitant Fuzzy Optimization (BHFO) method. This method extends traditional fuzzy, hesitant fuzzy and bipolar fuzzy optimization techniques by incorporating bipolar hesitant fuzzy sets (BHFS), which allow decision-makers to assign degrees of hesitation and bipolarity to their preferences, reflecting the inherent uncertainty and ambiguity associated with real-world decision-making. This approach recognizes that decision-makers may not always be completely certain about their preferences, which is a common scenario in practical multi-objective optimization problems. In this article, we present the theoretical foundations of the BHFO method, including the representation of the parameter as generalized bipolar parabolic fuzzy numbers and operations on these numbers. The proposed approach empowers decision-makers to navigate the complexities of multi-objective optimization problems effectively, accommodating hesitant and bipolar preferences. Furthermore, we illustrate the application of the BHFO method by solving multi objective production planning problem and the result is compared with the other existing methods.

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1. Introduction

In recent years, the field of optimization has witnessed a paradigm shift towards solving complex real-world problems that involve multiple, often conflicting objectives. Multi-objective optimization problem (MOOP) differs from traditional single-objective optimization, which seeks a single optimal solution, by aiming to identify a set of compromise solutions known as the Pareto optimal solutions. These solutions represent trade-offs between objectives, enabling decision-makers to choose the one that aligns best with their preferences and priorities. In the field of MOOP, optimization techniques have proven to be invaluable for handling real-world complexities characterized by uncertainty, ambiguity, and vagueness.

Keywords. Hesitant fuzzy set, bipolar fuzzy set, bipolar hesitant fuzzy set, parabolic bipolar fuzzy number, bipolar hesitant fuzzy optimization.

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Numerous researchers express a keen interest in employing diverse optimization techniques for MOOP considering the uncertainty. As a Decision Maker (DM) involved in MOOP, the challenging task is to select an appropriate compromised solution set from a range of potential Pareto-optimal options. Within the existing body of literature, there exists a substantial volume of studies in the context of MOOP. The Multi-Objective Linear Programming Problem in a fuzzy environment extends the traditional linear programming framework to accommodate multiple conflicting objectives while addressing uncertainty using fuzzy logic. This interdisciplinary approach is particularly valuable in decision-making scenarios where real-world parameters are inherently vague and decision-makers seek a balanced and robust solution set. Many extensions of fuzzy sets such as intuitionistic fuzzy, hesitant fuzzy, neutrosophic fuzzy sets, are used to tackle the uncertainty of the parameters of optimization problem. Also, various optimization methods are developed to solve MOOP in fuzzy environments and its extensions.

A diverse range of human decision-making processes, especially in multi-agent choice and decision analysis, relies on both negative and positive aspects, known as bipolar judgemental reasoning. This concept of bipolarity, coupled with fuzziness, extends to various real-world applications, including impact and symptom assessment, collaboration and competition dynamics, feedback and feedforward mechanisms, companionship and enmity relationships, as well as mutual and conflicting benefits. In this domain, this study proposes an effective optimization strategy grounded in the application of bipolar hesitant fuzzy set (BHFS), demonstrating its enhanced effectiveness compared to existing methodologies.

The rest of the paper is organized in the following sections: Section 2 discusses the "literature review" of this research work. The preliminaries used in this paper are stated in Section 3. In Section 4 we have explained how MOOP are structured in bipolar hesitant fuzzy MOOP and then crisp MOOP by defuzzification. The solution method is discussed in Section 5. The result and discussion of the proposed method are presented in Section 6. In Section 7 the conclusion and future work of this research work are discussed.

2. Literature review

In numerous optimization scenarios, it has been noted that a minor deviation from specified constraints can result in a more effective solution. This occurrence is prevalent in real-life modelling, where accurate parameter fixing is often impractical due to approximations or human observations. Zadeh [1] introduced the concept of a fuzzy set (FS) to address challenges related to uncertainty and imprecision inherent in various real-life optimization problems. After that, Bellman and Zadeh [2] pioneered decision-making within a fuzzy environment, a groundbreaking contribution that significantly influenced the field. For solving MOOP, Zimmermann [3] first introduced an optimization method using the FS. Further, optimization within a fuzzy environment has been extensively investigated and applied across diverse fields by numerous researchers. Dubois and Prade [4] presented the coefficient of the system of linear constraints as fuzzy number. Several researchers, such as Tanaka and Asai [5], Luhandjula [6], Sakawa and Yano [7], among others, have extensively explored optimization in fuzzy environments, delving into intricate details and applying their discoveries across various fields. Bharti *et al.* [8] solved fully fuzzy MOOP by introducing a new distance function between two fuzzy numbers.

In addressing uncertainty and vagueness, the concept of fuzzy sets evolved into intuitionistic fuzzy (IF) sets (IFS) by Atanassov [9], considering the membership and non-membership with a sum greater than zero and less than or equal to one. Using this concept of IFS, Angelov [10] first proposed IF programming approach to solve MOOP. After that, a huge number of works were done by the researchers to solve MOOP in IF environment. Jana and Roy [11] investigated a MOOP of transportation model in IF environment. Mahapatra *et al.* [12] addressed multi-objective mathematical programming in IF environment and applied it to solve reliability optimization model. Nachammai and Thangaraj [13] introduced a novel method for solving Intuitionistic Fuzzy Linear Programming problems (IFLPP) by utilizing similarity measures of IFS to determine the composite relative degree of similarity and employing a score function to calculate the ranking function for the objective function. Nagoorgani and Ponnalagu [14] presented a novel approach involving division for Triangular Intuitionistic Fuzzy Numbers using α, β-cuts and a scoring function for ranking and proposed an accuracy function for defuzzification, facilitating IFLPP solutions. Bharati and Singh [15] provided a computational algorithm for solving MOOP using IF optimization method and conducted a comparative analysis of linear and nonlinear membership functions for optimization impact. Bharati and Mishra [16] applied the concept of conflict and non-conflict to solve IFLPP; after that they recently addressed a multi-objective LPP by simultaneously solving conflicting objectives using the concept of conflict and non-conflict in interval-valued IF objectives [17].

In situations where decision-makers hesitate to express preferences, traditional FS and IFS may prove inadequate; recognizing this limitation, Torra introduced the concept of hesitant fuzzy sets (HFS) [18] to offer a valuable resource for handling such scenarios. It permits the membership degree of an element to encompass a set of possible values within $[0, 1]$. In recent times, numerous scholars have explored hesitant fuzzy sets (HFSs), applying them across various fields of study. Bharati [19] pioneered the introduction of hesitant fuzzy numbers, including triangular hesitant fuzzy numbers and their expected values. Additionally, they employed the hesitant fuzzy linear programming approach to address MOOP. Bharati [20] defined the hesitant fuzzy membership function and non-membership function as innovative tools to address the challenges associated with uncertainty and hesitation in handling parameters. Roy and Jana [21] addressed a MOOP of production planning, where decision-makers collaborate in resource allocation for diverse product manufacturing with cooperative game theory and model parameters using triangular hesitant fuzzy numbers. Alcantud [22] explored ranked hesitant fuzzy sets, an innovative extension of hesitant fuzzy sets that is less rigorous than both probabilistic and proportional hesitant fuzzy sets. This novel extension integrates hierarchical information pertaining to the diverse evaluations provided for each alternative. Wang *et al.* [23] developed a novel hesitant fuzzy wind speed forecasting system and a multi-fuzzification methods to deal with the non-determinism problem. Rouhbakhsh *et al.* [24] presented a hesitant fuzzy multi-objective programming problem where decision makers' evaluations are given in a hesitant fuzzy set. Ahuja and Kumar [25], in solving hesitant fuzzy linear programming problems, proposed the Mehar approach to determine the optimal value. To support the creation of an innovative Multiple Attribute Decision Making algorithm, Ali *et al.* [26] formulated various averaging and geometric aggregation operators on the set of intuitionistic hesitant fuzzy connection numbers.

Human decision-making across various contexts, notably in multi objective optimization, heavily relies on bipolar judgmental reasoning that incorporates both negative and positive perspectives. Zhang [27] introduced the bipolar fuzzy set (BFS). Mehmood *et al.* [28] extend crisp linear programming to a bipolar fuzzy environment using bipolar fuzzy numbers. They define arithmetic operations and propose a method to solve fully bipolar fuzzy linear programming problems with equality constraints. Dubey *et al.* [29] explore multi-objective fuzzy linear programming (MOFLP) problems, within a heterogeneous bipolar framework. Also, a Pareto-optimality solution concept for MOFLP problems is introduced, along with an approach to identify the most feasible solution with the highest possible degree. Akram *et al.* [30] introduce key concepts like bipolar fuzzy numbers in parametric form, their distance, and bipolar fuzzy arithmetic, illustrated with examples, also explores solving linear systems with parametric bipolar fuzzy numbers and presents a new approach using α-cut expansion for fully bipolar fuzzy linear systems. Further incorporating hesitancy in bipolar valued fuzzy sets, Mandal and Ranadive [31] discussed bipolar hesitant fuzzy set (BHFS) and their application in MCDM problems. Recently, various works are done in bipolar and bipolar hesitant fuzzy environments in the field of MCDM [32–38]. To the best of our knowledge, there is no work on MOOP in a bipolar hesitant fuzzy environment still.

2.1. Novelties and contributions of these studies

The following are novelties and the key contributions presented in this study:

- This study introduces generalized non-linear bipolar fuzzy number (GNBFN) in particular, a parabolic bipolar fuzzy (PBFN) number to present the uncertainty of parameters.
- The defuzzified value of GNBFN is calculated by the centroid of area method, in particular for PBFN.
- Bipolar hesitant fuzzy optimization (BHFO) method is presented to solve MOOP.
- To show the applicability of the proposed method a multi objective production planning problem and a multi objective sustainable production planning problem is solved.

– To show the advancement of the method results are compared with other existing methods and the proposed method gives better results.

3. Preliminaries

Some preliminaries related to this work are discussed in this section.

3.1. Some basic definitions

Definition 3.1. X be an arbitrary set, then FS ($\mathcal{F}$) on X is defined as:

$$\mathcal{F} = \{\langle x, \mu_{\mathcal{F}}(x)\rangle \mid x \in X\}$$

where $\mu_{\mathcal{F}} : X \to [0,1]$ is a membership function.

Definition 3.2. X be an arbitrary set, then IFS ($\mathcal{J}$) on X is defined as:

$$\mathcal{J} = \{\langle x, \mu_j(x), \vartheta_j(x)\rangle \mid x \in X\}$$

where $\mu_j, \vartheta_j : X \to [0,1]$ are membership and non-membership functions respectively. And $0 \leq \mu_j(x)+\vartheta_j(x) \leq 1$ for all $x \in X$.

Definition 3.3. X be an arbitrary set, then HFS ($\mathcal{H}$) on X is defined as:

$$\mathcal{H} = \{\langle x, \mu_{\mathcal{H}}(x)\rangle \mid x \in X\}$$

where $\mu_{\mathcal{H}}(x)$ is a finite set of possible degree of acceptance in $[0,1]$.

Definition 3.4. X be an arbitrary set, then BFS ($\mathcal{B}$) on X is defined as:

$$\mathcal{B} = \{\langle x, \mu^p_{\mathcal{B}}(x), \mu^n_{\mathcal{B}}(x)\rangle \mid x \in X\}$$

where $\mu^p_{\mathcal{B}} : X \to [0,1]$ and $\mu^n_{\mathcal{B}} : X \to [-1,0]$ are positive and negative membership function respectively. And $-1 \leq \mu^p_{\mathcal{B}}(x) + \mu^n_{\mathcal{B}}(x) \leq 1$.

Example 3.5. A BFS could represent a customer's level of satisfaction with a new smartphone, capturing both positive and negative aspects of their experience. For instance, the phone's "battery life" might have a positive membership value of 0.9, reflecting high appeal. At the same time, the "camera quality" might have a negative membership value of -0.8, indicating notable dissatisfaction. This representation highlights a generally positive opinion of the phone, driven by its battery and camera quality.

Definition 3.6. X be an arbitrary set, then BHFS ($\mathcal{C}$) on X is defined as:

$$\mathcal{C} = \{\langle x, \mu^p_{\mathcal{C}}(x), \mu^n_{\mathcal{C}}(x)\rangle \mid x \in X\}$$

where $\mu^p_{\mathcal{C}}(x)$ is a finite set of positive degree of acceptance in $[0,1]$ and $\mu^n_{\mathcal{C}}(x)$ is a finite set of negative degree of acceptance in $[-1,0]$. And $-1 \leq \sup(\mu^p_{\mathcal{C}}(x)) + \inf(\mu^n_{\mathcal{C}}(x)) \leq 1$.

Example 3.7. A BHFS could represent the collective level of satisfaction of more than one (three) customers with a new smartphone, capturing both positive and negative aspects of their experiences. For instance, Customer A might assign the phone's battery life a positive membership value of 0.9, reflecting high appeal, while Customer B assigns it 0.6, and Customer C assigns it 0.7, showing varying degrees of satisfaction. Similarly, for camera quality, Customer A might give a negative membership value of -0.8, indicating dissatisfaction, while Customer B assigns -0.6, and Customer C assigns -0.7, reflecting consistent dissatisfaction among all three. By aggregating these values, the BHFS provides a nuanced view of the group's overall satisfaction, indicating a generally positive opinion due to good battery life but tempered by dissatisfaction with the camera quality.

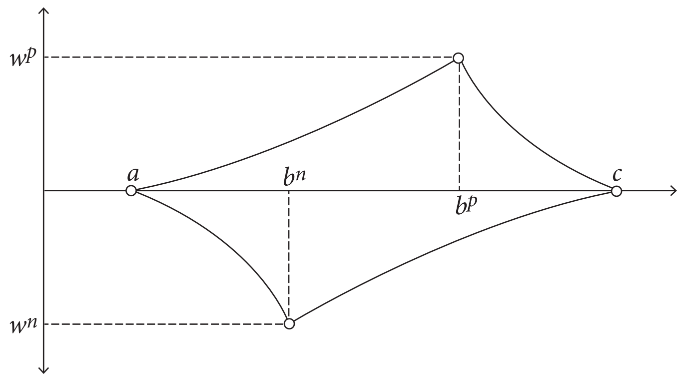

FIGURE 1. Generalized non-linear bipolar fuzzy number.

Definition 3.8 (Union and Intersection of BFS)**.** X be an arbitrary set, Let $\mathcal{C}_1 = \{\langle x, \mu^p_{\mathcal{C}_1}(x), \mu^n_{\mathcal{C}_1}(x)\rangle \mid x \in X\}, \mathcal{C}_2 = \{\langle x, \mu^p_{\mathcal{C}_2}(x), \mu^n_{\mathcal{C}_2}(x)\rangle \mid x \in X\}$ be two BFSs. Then the union of two sets can be defined as

$$D = \mathcal{C}_1 \cup \mathcal{C}_2 = \{\langle x, \mu^p_D(x), \mu^n_D(x)\rangle \mid x \in X\}$$

where $\mu^p_D(x) = \max\{\mu^p_{\mathcal{C}_1}(x), \mu^p_{\mathcal{C}_2}(x)\}$.

$$\mu^n_D(x) = \min\big\{\mu^n_{\mathcal{C}_1}(x), \mu^n_{\mathcal{C}_2}(x)\big\}.$$

And the intersection is defined as

$$E = \mathcal{C}_1 \cap \mathcal{C}_2 = \{\langle x, \mu^p_E(x), \mu^n_E(x)\rangle \mid x \in X\}$$

where $\mu^p_E(x) = \min\{\mu^p_{\mathcal{C}_1}(x), \mu^p_{\mathcal{C}_2}(x)\}$.

$$\mu^n_E(x) = \max\big\{\mu^n_{\mathcal{C}_1}(x), \mu^n_{\mathcal{C}_2}(x)\big\}.$$

Definition 3.9 (Generalized non-linear bipolar fuzzy number (GNBFN))**.** Let $\tilde{B} = (a, b^n, b^p, c)$ be a GNBFN. The positive and negative membership functions are defined as:

$$\mu^p(x) = \begin{cases} w^p\left(\frac{x-a}{b^p-a}\right)^t & \text{for } a \le x \le b^p \\ w^p & \text{for } x = b^p \\ w^p\left(\frac{c-x}{c-b^p}\right)^t & \text{for } b^p \le x \le c \\ 0 & \text{otherwise} \end{cases}$$

$$\mu^n(x) = \begin{cases} w^n\left(\frac{x-a}{b^n-a}\right)^t & \text{for } a \le x \le b^n \\ w^n & \text{for } x = b^n \\ w^n\left(\frac{c-x}{c-b^n}\right)^t & \text{for } b^n \le x \le c \\ 0 & \text{otherwise} \end{cases}$$

where $w^p \in (0, 1], w^n \in [-1, 0)$ and t be a positive real number.

The graphical representation of the GNBFN is presented in Figure 1.

3.2. Defuzzification of GNBFN

The defuzzification process aims to extract a single, precise value from the fuzzy set or fuzzy number, reflecting the most representative or crisp outcome. Several methods exist for defuzzification, each with its own strengths and weaknesses. Centroid Method, Mean of Maximum (MoM) Method, Bisector Method, Weighted Sum Method, graded mean integration method are few commonly used techniques. Choosing an appropriate defuzzification method depends on the specific characteristics of the fuzzy set and the application requirements. The goal is to obtain a single, meaningful output that captures the essential information embedded in the fuzzy representation, facilitating further analysis or decision-making in a real-world context. Here we have used centroid method for defuzzification. The centroid method calculates the centre of gravity of the fuzzy set by finding the weighted average of the fuzzy number's support. It considers the degree of membership of each value within the fuzzy set, providing a representative value that reflects the centre of the fuzzy set.

Since here the membership function is continuous, define

$$\begin{aligned}
D(\mu^p(x)) &= \frac{\int_a^c x\cdot\mu^p(x)\,\mathrm{d}x}{\int_a^c \mu^p(x)\,\mathrm{d}x} = \frac{\int_a^{b^p} x\cdot\mu^p(x)\,\mathrm{d}x + \int_{b^p}^c x\cdot\mu^p(x)\,\mathrm{d}x}{\int_a^{b^p} \mu^p(x)\,\mathrm{d}x + \int_{b^p}^c \mu^p(x)\,\mathrm{d}x}\\
&= \frac{\int_a^{b^p} x\cdot w^p\left(\frac{x-a}{b^p-a}\right)^t \mathrm{d}x + \int_{b^p}^c x\cdot w^p\left(\frac{c-x}{c-b^p}\right)^t \mathrm{d}x}{\int_a^{b^p} w^p\left(\frac{x-a}{b^p-a}\right)^t \mathrm{d}x + \int_{b^p}^c w^p\left(\frac{c-x}{c-b^p}\right)^t \mathrm{d}x}\\
&= \frac{\left[(b^p-a)\frac{\{b^p(t+1)+a\}}{(t+1)(t+2)} + (c-b^p)\frac{\{b^p(t+1)+c\}}{(t+1)(t+2)}\right]w^p}{\left[\left(\frac{b^p-a}{t+1}\right)+\left(\frac{c-b^p}{t+1}\right)\right]w^p}\\
&= \frac{(b^p-a)\{b^p(t+1)+a\}+(c-b^p)\{b^p(t+1)+c\}}{(t+2)(c-a)}\\
&= \frac{(a+tb^p+c)}{(t+2)}\ (\text{taking } c\neq a)
\end{aligned}$$

$$\begin{aligned}
D(\mu^n(x)) &= \frac{\int_a^c x\cdot\mu^n(x)\,\mathrm{d}x}{\int_a^c \mu^n(x)\,\mathrm{d}x} = \frac{\int_a^{b^n} x\cdot\mu^n(x)\,\mathrm{d}x + \int_{b^n}^c x\cdot\mu^n(x)\,\mathrm{d}x}{\int_a^{b^n} \mu^n(x)\,\mathrm{d}x + \int_{b^n}^c \mu^n(x)\,\mathrm{d}x}\\
&= \frac{\int_a^{b^n} x\cdot w^n\left(\frac{x-a}{b^n-a}\right)^t \mathrm{d}x + \int_{b^n}^c x\cdot w^n\left(\frac{c-x}{c-b^n}\right)^t \mathrm{d}x}{\int_a^{b^n} w^n\left(\frac{x-a}{b^n-a}\right)^t \mathrm{d}x + \int_{b^n}^c w^n\left(\frac{c-x}{c-b^n}\right)^t \mathrm{d}x}\\
&= \frac{\left[(b^n-a)\frac{\{b^n(t+1)+a\}}{(t+1)(t+2)} + (c-b^n)\frac{\{b^n(t+1)+c\}}{(t+1)(t+2)}\right]w^n}{\left[\left(\frac{b^n-a}{t+1}\right)+\left(\frac{c-b^n}{t+1}\right)\right]w^n}\\
&= \frac{(b^n-a)\{b^n(t+1)+a\}+(c-b^n)\{b^n(t+1)+c\}}{(t+2)(c-a)}\\
&= \frac{(a+tb^n+c)}{(t+2)}\ (\text{taking } c\neq a).
\end{aligned}$$

Accuracy function

The accuracy function is defined as

$$\begin{aligned}
\mathcal{A}\left(\tilde{B}\right) &= \lambda D(\mu^p(x)) + (1-\lambda)D(\mu^n(x)),\ \text{where } \lambda\in(0,1)\\
&= \left(\frac{D(\mu^p(x))+D(\mu^n(x))}{2}\right)\ (\text{in particular taking } \lambda=0.5)
\end{aligned}$$

$$= \frac{\frac{(a+tb^p+c)}{(t+2)} + \frac{(a+tb^n+c)}{(t+2)}}{2}$$
$$= \frac{2a + t(b^p + b^n) + 2c}{2(t+2)}.$$

In particular, take $t = 2$, Then it becomes parabolic bipolar fuzzy number $\widetilde{B_p}$.

$$\mathcal{A}\left(\widetilde{B_p}\right) = \frac{a + b^p + b^n + c}{4}.$$

Theorem 3.10. *Accuracy function is linear.*

Proof. Let $\widetilde{B_{p1}} = (a_1, b_1^p, b_1^n, c_1), \widetilde{B_{p2}} = (a_2, b_2^p, b_2^n, c_2)$ be two parabolic bipolar fuzzy numbers and m, n are any real number.

$$\begin{aligned}\mathcal{A}\left(m \cdot \widetilde{B_{p1}} + n \cdot \widetilde{B_{p2}}\right) &= \mathcal{A}(ma_1 + na_2, mb_1^p + nb_2^p, mb_1^n + nb_2^n, mc_1 + nc_2) \\ &= \frac{(ma_1 + na_2) + (mb_1^p + nb_2^p) + (mb_1^n + nb_2^n) + (mc_1 + nc_2)}{4} \\ &= \frac{m(a_1 + b_1^p + b_1^n + c_1) + n(a_2 + b_2^p + b_2^n + c_2)}{4} \\ &= m\mathcal{A}\left(\widetilde{B_{p1}}\right) + n\mathcal{A}\left(\widetilde{B_{p2}}\right).\end{aligned}$$

So, $\mathcal{A}$ is a linear function. □

Theorem 3.11. *The relation* ($\mathcal{A}$, "$\succsim$") *defined as* $\mathcal{A}(\widetilde{B_{p1}}) \geq \mathcal{A}(\widetilde{B_{p2}}) \Leftrightarrow \widetilde{B_{p1}} \succsim \widetilde{B_{p2}}$ *is partial ordered.*

Proof. To prove "$\succsim$" is total ordered, we have to prove "$\succsim$" is (a) Reflexive, (b) Anti-symmetric, (c) Transitive. $\widetilde{B_{p1}}, \widetilde{B_{p2}}, \widetilde{B_{p3}}$ be any three BPFN.

(a) *Reflexive*: Since, $\mathcal{A}(\widetilde{B_{p1}}) \geq \mathcal{A}(\widetilde{B_{p1}}) \Leftrightarrow \widetilde{R_1} \succsim \widetilde{R_1}$, hence $\succsim$ is reflexive.

(b) *Anti-symmetric*: let $\widetilde{B_{p1}} \succsim \widetilde{B_{p2}}$ and $\widetilde{B_{p2}} \succsim \widetilde{B_{p1}}$.
$\widetilde{B_{p1}} \gtrsim \widetilde{B_{p2}} \Rightarrow \mathcal{A}(\widetilde{B_{p1}}) \geq \mathcal{A}(\widetilde{B_{p2}})$ and $\widetilde{B_{p2}} \succsim \widetilde{B_{p1}} \Rightarrow \mathcal{A}(\widetilde{B_{p2}}) \geq \mathcal{A}(\widetilde{B_{p1}})$.
Both implies that, $\mathcal{A}(\widetilde{B_{p1}}) = \mathcal{A}(\widetilde{B_{p2}}) \Rightarrow \widetilde{B_{p1}} \simeq \widetilde{B_{p2}}$.
Hence $\succsim$ is Anti-symmetric.

(c) *Transitive*: let $\widetilde{B_{p1}} \succsim \widetilde{B_{p2}}$ and $\widetilde{B_{p2}} \succsim \widetilde{B_{p3}}$.
$\widetilde{B_{p1}} \succsim \widetilde{B_{p2}} \Rightarrow \mathcal{A}(\widetilde{B_{p1}}) \geq \mathcal{A}(\widetilde{B_{p2}})$ and $\widetilde{B_{p2}} \succsim \widetilde{B_{p3}} \Rightarrow \mathcal{A}(\widetilde{B_{p2}}) \geq \mathcal{A}(\widetilde{B_{p3}})$.
Both implies that, $\mathcal{A}(\widetilde{B_{p1}}) \geq \mathcal{A}(\widetilde{B_{p3}}) \Rightarrow \widetilde{B_{p1}} \succsim \widetilde{B_{p3}}$.
Hence $\succsim$ is Transitive.

So, ($\mathcal{A}$, "$\succsim$") is totally ordered.
Similarly, ($\mathcal{A}$, "$\succsim$") and ($\mathcal{A}$, "$\simeq$") are ordered relation. □

$$\begin{aligned}\text{Hence, } \quad \widetilde{B_{p1}} \succsim \widetilde{B_{p2}} &\Leftrightarrow \mathcal{A}(\widetilde{B_{p1}}) \geq \mathcal{A}(\widetilde{B_{p2}}) \\ \widetilde{B_{p1}} \precsim \widetilde{B_{p2}} &\Leftrightarrow \mathcal{A}(\widetilde{B_{p1}}) \leq \mathcal{A}(\widetilde{B_{p2}}) \\ \widetilde{B_{p1}} \backsimeq \widetilde{B_{p2}} &\Leftrightarrow \mathcal{A}(\widetilde{B_{p1}}) = \mathcal{A}(\widetilde{B_{p2}}).\end{aligned}$$

4. Methodology

4.1. MOOP in Parabolic bipolar fuzzy environment

The MOOP is written in the form:

$$\begin{aligned}
\text{P1:}\quad & \text{Maximize } Z_i(x) = \sum_{k=1}^{r} c_{ik} \prod_{l=1}^{s} x_l^{\alpha_{ikl}} \quad \text{for } i = 1 \text{ to } t_1 \\
& \text{Minimize } Z_j(x) = \sum_{k=1}^{r'} d_{jk} \prod_{l=1}^{s} x_l^{\beta_{jkl}} \quad \text{for } j = 1 \text{ to } t_2 \\
& \text{Subject to} \\
& \qquad g_q(x) \le = \ge b_q \qquad \text{for } q = 1 \text{ to } t_3 \\
& \qquad x \ge 0
\end{aligned}$$

where $g_q(x) = \sum_{k=1}^{r''} e_{qk} \prod_{l=1}^{s} x_l^{\gamma_{ikl}}$.

MOOP in bipolar fuzzy environment is expressed by Taking the coefficient as PBFN, P1 becomes:

$$\begin{aligned}
\text{P2:}\quad & \text{Maximize } \widetilde{Z_i(x)} = \sum_{k=1}^{r} \widetilde{c_{ik}} \prod_{l=1}^{s} x_l^{\alpha_{ikl}} \qquad \text{for } i = 1 \text{ to } t_1 \\
& \text{Minimize } \widetilde{Z_j(x)} = \sum_{k=1}^{r'} \widetilde{d_{jk}} \prod_{l=1}^{s} x_l^{\beta_{jkl}} \qquad \text{for } j = 1 \text{ to } t_2 \\
& \text{Subject to} \\
& \qquad \sum_{k=1}^{r''} \widetilde{e_{qk}} \prod_{l=1}^{s} x_l^{\gamma_{ikl}} \le = \ge \widetilde{b_q} \qquad \text{for } q = 1 \text{ to } t_3 \\
& \qquad x \ge 0.
\end{aligned}$$

Crisp MOOP using accuracy function written as:

$$\begin{aligned}
\text{P3:}\quad & \text{Maximize } \mathcal{A}\left[\widetilde{Z_i(x)}\right] = \sum_{k=1}^{r} \mathcal{A}[\widetilde{c_{ik}}] \prod_{l=1}^{s} x_l^{\alpha_{ikl}} \qquad \text{for } i = 1 \text{ to } t_1 \\
& \text{Minimize } \mathcal{A}\left[\widetilde{Z_j(x)}\right] = \sum_{k=1}^{r'} \mathcal{A}\left[\widetilde{d_{jk}}\right] \prod_{l=1}^{s} x_l^{\beta_{jkl}} \qquad \text{for } j = 1 \text{ to } t_2 \\
& \text{Subject to} \\
& \qquad \sum_{k=1}^{r''} \mathcal{A}[\widetilde{e_{qk}}] \prod_{l=1}^{s} x_l^{\gamma_{ikl}} \le = \ge \mathcal{A}\left[\widetilde{b_q}\right] \qquad \text{for } q = 1 \text{ to } t_3 \\
& \qquad x \ge 0.
\end{aligned}$$

Which is equivalent to

$$\begin{aligned}
\text{P4:}\quad & \text{Maximize } Z_i'(x) = \sum_{k=1}^{r} c_{ik}' \prod_{l=1}^{s} x_l^{\alpha_{ikl}} \qquad \text{for } i = 1 \text{ to } t_1 \\
& \text{Minimize } Z_j'(x) = \sum_{k=1}^{r'} d_{jk}' \prod_{l=1}^{s} x_l^{\beta_{jkl}} \qquad \text{for } j = 1 \text{ to } t_2
\end{aligned}$$

Subject to

$$\sum_{k=1}^{r''} e'_{qk} \prod_{l=1}^{s} x_l^{\gamma_{ikl}} \leq = \geq b'_q \qquad \text{for } \; q = 1 \; \text{ to } \; t_3$$
$$x \geq 0$$

where $\mathcal{A}[\tilde{a}] = a'$.

Definition 4.1 (Efficient solution)**.** Efficient solution to a MOOP is a feasible solution x^* in feasible space of the problem (P1), if and only if $\nexists x$ such that $Z_i(x) \geq Z_i(x^*)$ for all $i = 1, 2, \dots t_1, Z_j(x) \leq Z_j(x^*)$ for all $j = 1, 2, \dots t_2$ and $Z_i(x) > Z_i(x^*)$ for at least one $i = 1, 2, \dots t_1$ or $Z_j(x) < Z_j(x^*)$ for at least one $j = 1, 2, \dots t_2$.

Theorem 4.2. *Efficient solution of problem* (P4) *is also an efficient solution of problem* (P2).

Proof. $x^* = (x_1^*, x_2^*, \dots, x_s^*)$ be an efficient solution of problem (P4). So, x^* is feasible solution of problem (P4). That implies

$$\sum_{k=1}^{r''} e'_{qk} \prod_{l=1}^{s} x_l^{*\gamma_{ikl}} \leq = \geq b'_q \qquad \text{for } q = 1 \text{ to } t_3$$

i.e.

$$\sum_{k=1}^{r''} \mathcal{A}[\widetilde{e_{qk}}] \prod_{l=1}^{s} x_l^{*\gamma_{ikl}} \leq = \geq \mathcal{A}\left[\widetilde{b_q}\right] \qquad \text{for } q = 1 \text{ to } t_3$$

since, $\mathcal{A}$ is linear and total ordered.

$$\sum_{k=1}^{r''} \widetilde{e_{qk}} \prod_{l=1}^{s} x_l^{*\gamma_{ikl}} \leq = \geq \widetilde{b_q} \qquad \text{for } q = 1 \text{ to } t_3.$$

So, x^* is a feasible solution of problem (P_2).

Now, $x^* \in F$ is an efficient solution of problem (P4). So, there does not exist any $x^{**} \in F$ such that $Z'_i(x^{**}) \geq Z'_i(x^*)$ for $1 \leq i \leq t_1$ and $Z'_j(x^{**}) \leq Z'_j(x^*)$ for $1 \leq j \leq t_2$ the strict inequality holds for at least one i or j.

So, there does not exist any $x^{**} \in F$ such that

$$\max \sum_{k=1}^{r} c'_{ik} \prod_{l=1}^{s} (x^{**})_l^{\alpha_{ikl}} \geq \max \sum_{k=1}^{r} c'_{ik} \prod_{l=1}^{s} (x^*)_l^{\alpha_{ikl}} \qquad \text{for } 1 \leq i \leq t_1$$

and

$$\min \sum_{k=1}^{r'} d'_{jk} \prod_{l=1}^{s} (x^{**})_l^{\beta_{jkl}} \leq \min \sum_{k=1}^{r'} d'_{jk} \prod_{l=1}^{s} (x^*)_l^{\beta_{jkl}} \qquad \text{for } 1 \leq j \leq t_2,$$

and strict inequality holds for at least one i or j, where $c'_{ik} = \mathcal{A}(\widetilde{c_{ik}}), d'_{jk} = \mathcal{A}[\widetilde{d_{jk}}]$.

Thus there is no $x^{**} \in F$ such that

$$\max \sum_{k=1}^{r} \widetilde{c_{ik}} \prod_{l=1}^{s} (x^{**})_l^{\alpha_{ikl}} \geq \max \sum_{k=1}^{r} \widetilde{c_{ik}} \prod_{l=1}^{s} (x^*)_l^{\alpha_{ikl}} \qquad \text{for } 1 \leq i \leq t_1$$

and

$$\min \sum_{k=1}^{r'} \widetilde{d_{jk}} \prod_{l=1}^{s} (x^{**})_l^{\beta_{jkl}} \leq \min \sum_{k=1}^{r'} \widetilde{d_{jk}} \prod_{l=1}^{s} (x^*)_l^{\beta_{jkl}} \qquad \text{for } 1 \leq j \leq t_2,$$

and strict inequality holds for at least one i. Since $\mathcal{A}$ is linear and ordered. So, x^* is efficient solution of (P_2).
□

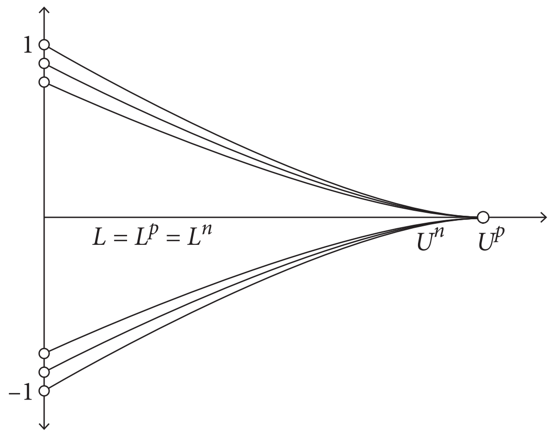

FIGURE 2. For Minimization.

4.2. Bipolar hesitant fuzzy optimization method

Bellman and Zadeh [2] initially introduced the idea of a fuzzy decision (D), a fuzzy goal (G), and a fuzzy constraint (C), which has since been widely applied to a variety of real-world problems with decision-making. The fuzzy decision set is defined as $D = G \cap C$. For bipolar hesitant fuzzy set, the decision set is defined as

$$D_{\mathcal{A}} = \left(\bigcap_{i=1}^{n} G_i\right) \bigcap \left(\bigcap_{k=1}^{l} C_k\right) = \left(x, \mu^p_{D_{\mathcal{A}}}(x), \mu^n_{D_{\mathcal{A}}}(x)\right)$$

where

$$\mu^p_{D_{\mathcal{A}}}(x) = \min\left\{\mu^p_{G_1}(x), \mu^p_{G_2}(x), \ldots, \mu^p_{G_n}(x); \mu^p_{C_1}(x), \mu^p_{C_2}(x), \ldots, \mu^p_{C_n}(x)\right\}$$
$$\mu^n_{D_{\mathcal{A}}}(x) = \max\left\{\mu^n_{G_1}(x), \mu^n_{G_2}(x), \ldots, \mu^n_{G_n}(x); \mu^n_{C_1}(x), \mu^n_{C_2}(x), \ldots, \mu^n_{C_n}(x)\right\}.$$

4.2.1. Bipolar hesitant fuzzy membership functions

Construction of the positive and negative MF are explained mathematically and for better understanding which are also presented graphically in Figures 2 and 3.

For maximization objectives

$$\mu^p_{E_m}[Z_i(x)] = \begin{cases} 0 & \text{for } Z_i(x) \leq L^p_i \\ \alpha_m \left(\frac{Z_i(x) - L^p_i}{U^p_i - L^p_i}\right)^t & \text{for } L^p_i \leq Z_i(x) \leq U^p_i \\ \alpha_m & \text{for } Z_i(x) \geq U^p_i \end{cases}$$

$$\mu^n_{E_m}[Z_i(x)] = \begin{cases} 0 & \text{for } Z_i(x) \leq L^n_i \\ -\beta_m \left(\frac{Z_i(x) - L^n_i}{U^n_i - L^n_i}\right)^t & \text{for } L^n_i \leq Z_i(x) \leq U^n_i \\ -\beta_m & \text{for } Z_i(x) \geq U^n_i \end{cases}$$

where $U^p_i = U^n_i = \max_{1 \leq j \leq t_1}\{Z_i(X^j)\}$, $L^p_i = \min_{1 \leq j \leq t_1}\{Z_i(X^j)\}$, $L^n_i = L^p_i + t(U^p_i - L^p_i)$.

For minimization objectives

$$\mu^p_{E_m}[Z_j(x)] = \begin{cases} \alpha_m & \text{for } Z_j(x) \leq L^p_j \\ \alpha_m \left(\frac{U^p_j - Z_j(x)}{U^p_j - L^p_j}\right)^t & \text{for } L^p_j \leq Z_j(x) \leq U^p_j \\ 0 & \text{for } Z_j(x) \geq U^p_j \end{cases}$$

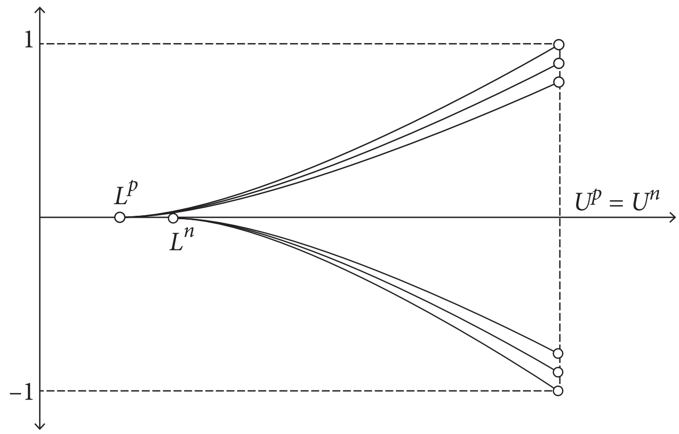

FIGURE 3. For Maximization.

$$\mu^n_{E_m}[Z_j(x)] = \begin{cases} -\beta_m & \text{for } Z_j(x) \le L^n_j \\ -\beta_m \left(\frac{U^n_j - Z_j(x)}{U^n_j - L^n_j}\right)^t & \text{for } L^n_j \le Z_j(x) \le U^n_j \\ 0 & \text{for } Z_j(x) \ge U^n_j \end{cases}$$

where $L^p_i = L^n_i = \min_{1\le j\le t_2}\{Z_i(X^j)\}$, $U^p_i = \max_{1\le j\le t_2}\{Z_i(X^j)\}$, $U^n_i = U^p_i - t(U^p_i - L^p_i)$.

Where $\mu^p_{E_m}$ is the positive membership function set by mth expert using parameter $\alpha_m \in (0,1]$ and $\mu^n_{E_m}$ is the negative membership function set by mth expert using parameter $\beta_m \in (0,1]$. and t be a positive real number.

4.2.2. Bipolar hesitant fuzzy optimization method

$$\begin{aligned} &\text{Maximize } \mu^p_{E_m}[Z_i(x)] \text{ and Maximize } \mu^p_{E_m}[Z_j(x)] \\ &\text{Minimize } \mu^n_{E_m}[Z_i(x)] \text{ and Minimize } \mu^n_{E_m}[Z_j(x)] \\ &\text{Subject to} \\ &\qquad \sum_{k=1}^{r''} e'_{qk} \prod_{l=1}^{s} x_l^{\gamma_{ikl}} \le=\ge b'_q \qquad \text{for } q = 1 \text{ to } t_3 \\ &\qquad x \ge 0. \end{aligned}$$

Using max–min method and arithmetic mean operator

$$\begin{aligned} \text{P5:}\quad &\text{Maximize } \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) - \left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \\ &\text{Subject to} \\ &\qquad \mu^p_{E_m}[Z_i(x)] \ge a_m; \mu^p_{E_m}[Z_j(x)] \ge a_m \qquad \text{for all } i, j \\ &\qquad \mu^n_{E_m}[Z_i(x)] \le b_m; \mu^n_{E_m}[Z_j(x)] \le b_m \qquad \text{for all } i, j \\ &\qquad \sum_{k=1}^{r''} e'_{qk} \prod_{\substack{l=1 \\ x\ge 0}}^{s} x_l^{\gamma_{ikl}} \le=\ge b'_q \qquad \text{for } q = 1 \text{ to } t_3 \end{aligned}$$

Theorem 4.3. *Optimal solution of problem* (P5) *is efficient solution of problem* (P4).

Proof. Let x^* be optimal solution of problem (P5). Assume, x^* is not efficient solution of problem (P4), so there exists a feasible solution x^{**} such that $Z_i(x^{**}) \geq Z_i(x^*)$ for $1 \leq i \leq t_1$ and $Z_j(x^{**}) \leq Z_j(x^*)$ for $1 \leq j \leq t_2$ and strict inequality holds for at least one i or j.

So for $1 \leq i \leq t_1$,

$$\alpha_m \left(\frac{Z_i(x^{**}) - L_i^p}{U_i^p - L_i^p} \right)^t \geq \alpha_m \left(\frac{Z_i(x^*) - L_i^p}{U_i^p - L_i^p} \right)^t \Rightarrow \mu_{E_m}^p [Z_i(x^{**})] \geq \mu_{E_m}^p [Z_i(x^*)]. \tag{4.1}$$

And

$$-\beta_m \left(\frac{Z_i(x^{**}) - L_i^n}{U_i^n - L_i^n} \right)^t \leq -\beta_m \left(\frac{Z_i(x^*) - L_i^n}{U_i^n - L_i^n} \right)^t \Rightarrow \mu_{E_m}^n [Z_i(x^{**})] \leq \mu_{E_m}^n [Z_i(x^*)]. \tag{4.2}$$

For $1 \leq j \leq t_2$,

$$\alpha_m \left(\frac{U_j^p - Z_j(x^{**})}{U_j^p - L_j^p} \right)^t \geq \alpha_m \left(\frac{U_j^p - Z_j(x^*)}{U_j^p - L_j^p} \right)^t \Rightarrow \mu_{E_m}^p [Z_j(x^{**})] \geq \mu_{E_m}^p [Z_j(x^*)]. \tag{4.3}$$

And

$$-\beta_m \left(\frac{Z_j(x^{**}) - L_j^n}{U_j^n - L_j^n} \right)^t \leq -\beta_m \left(\frac{Z_j(x^*) - L_j^n}{U_j^n - L_j^n} \right)^t \Rightarrow \mu_{E_m}^n [Z_j(x^{**})] \leq \mu_{E_m}^n [Z_j(x^*)]. \tag{4.4}$$

The strict inequality holds for at least one i or j.

Now $a_m^* = \min\{\mu_{E_m}^p [Z_i(x^*)], \mu_{E_m}^p [Z_j(x^*)] \mid 1 \leq i \leq t_1, 1 \leq j \leq t_2\}$

$a_m^{**} = \min\{\mu_{E_m}^p [Z_i(x^{**})], \mu_{E_m}^p [Z_j(x^{**})] \mid 1 \leq i \leq t_1, 1 \leq j \leq t_2\}$.

Then, by (4.1) and (4.3)

$$a_m^{**} \geq a_m^*. \tag{4.5}$$

Again, $b_m^* = \max\{\mu_{E_m}^n [Z_i(x^*)], \mu_{E_m}^n [Z_j(x^*)] \mid 1 \leq i \leq t_1, 1 \leq j \leq t_2\}$

$b_m^{**} = \max\{\mu_{E_m}^n [Z_i(x^{**})], \mu_{E_m}^n [Z_j(x^{**})] \mid 1 \leq i \leq t_1, 1 \leq j \leq t_2\}$

Then by (4.2) and (4.4)

$$b_m^{**} \leq b_m^*. \tag{4.6}$$

Now, (4.2) and (4.4) implies that,

$$\begin{aligned}(a_m^{**} - b_m^{**}) \geq (a_m^* - b_m^*) \Rightarrow &\left(\frac{a_1^{**} + a_2^{**} + \cdots + a_n^{**}}{n} \right) - \left(\frac{b_1^{**} + b_2^{**} + \cdots + b_n^{**}}{n} \right) \\ &\geq \left(\frac{a_1^* + a_2^* + \cdots + a_n^*}{n} \right) - \left(\frac{b_1^* + b_2^* + \cdots + b_n^*}{n} \right)\end{aligned}$$

which contradict that x^* is an optimal solution of problem (P5). So, the assumption is wrong. Hence x^* is an efficient solution of problem (P4). □

4.2.3. Solution algorithm

Step 1. Addressing each objective function independently within problem (P4) while disregarding the influence of others, adhere to the specified constraints and determine the optimal value for each objective function. Subsequently, ascertain the values of the remaining objective functions at the precise point where the optimal value of the individual objective function is attained.

Step 2. The pay-off matrix is constructed by Step 1,

$$\begin{bmatrix} Z_1^*(x^1) & Z_2(x^1) & \cdots & Z_{t_1+t_2}(x^1) \\ Z_1(x^2) & Z_2^*(x^2) & \cdots & Z_{t_1+t_2}(x^2) \\ \vdots & \vdots & \ddots & \vdots \\ Z_1(x^n) & Z_2(x^n) & \cdots & Z_{t_1+t_2}^*(x^n) \end{bmatrix}.$$

Step 3. The upper bound (U_r) and the lower bound (L_r) of each objective $Z_r(x)$ are set by,
For maximization objectives,

$$U_i = \max\{Z_i(x^r)\} = Z_i^*(x^r) \text{ and } L_i = \min\{Z_i(x^r)\}.$$

For minimization objectives,

$$U_r = \max\{Z_r(x^r)\} \text{ and } L_r = \min\{Z_r(x^r)\} = Z_r^*(x^r).$$

Now, upper and lower bound for positive and negative MF are defined by,
For maximization,

$$U_i^p = U_i^n = \max_{1\le j\le t_1}\{Z_i(X^j)\}, L_i^p = \min_{1\le j\le t_1}\{Z_i(X^j)\}, L_i^n = L_i^p + l(U_i^p - L_i^p).$$

For minimization,

$$L_i^p = L_i^n = \min_{1\le j\le t_2}\{Z_i(X^j)\}, U_i^p = \max_{1\le j\le t_2}\{Z_i(X^j)\}, U_i^n = U_i^p - l(U_i^p - L_i^p).$$

Step 4. The positive and negative MF are constructed by the lower and upper limit of the objective functions and the hesitancy is defined by varying the parameter (α) set by decision makers, where α belongs to $(0, 1]$.
Step 5. Construct single objective optimization model by max-min operator in bipolar hesitant fuzzy environment.
Step 6. Solve the single objective optimization problem (SOOP) by Lingo 18.0 software.

4.2.4. Flowchart

See Figure 4.

5. Numerical illustration

Example 5.1. To show the applicability and superiority of the proposed method, we have solved a MOOP in the field of production planning [8]. Here the parameter of the objectives and constraints are taken as bipolar parabolic fuzzy number.

Profit: Maximize $Z_1(x) = \widetilde{50}x_1 + \widetilde{100}x_2 + \widetilde{17.5}x_3$.

Quality: Maximize $Z_2(x) = \widetilde{92}x_2 + \widetilde{75}x_2 + \widetilde{50}x_3$.

Worker Satisfaction: Maximize $Z_3(x) = \widetilde{25}x_1 + \widetilde{100}x_2 + \widetilde{75}x_3$.

Subject to

$$\begin{aligned} &\widetilde{12}x_1 + \widetilde{17}x_2 \lesssim 1400 \\ &\tilde{3}x_1 + \tilde{9}x_2 + \tilde{8}x_3 \le 1000 \\ &\widetilde{10}x_1 + \widetilde{13}x_2 + \widetilde{15}x_3 \le \widetilde{1750} \end{aligned}$$

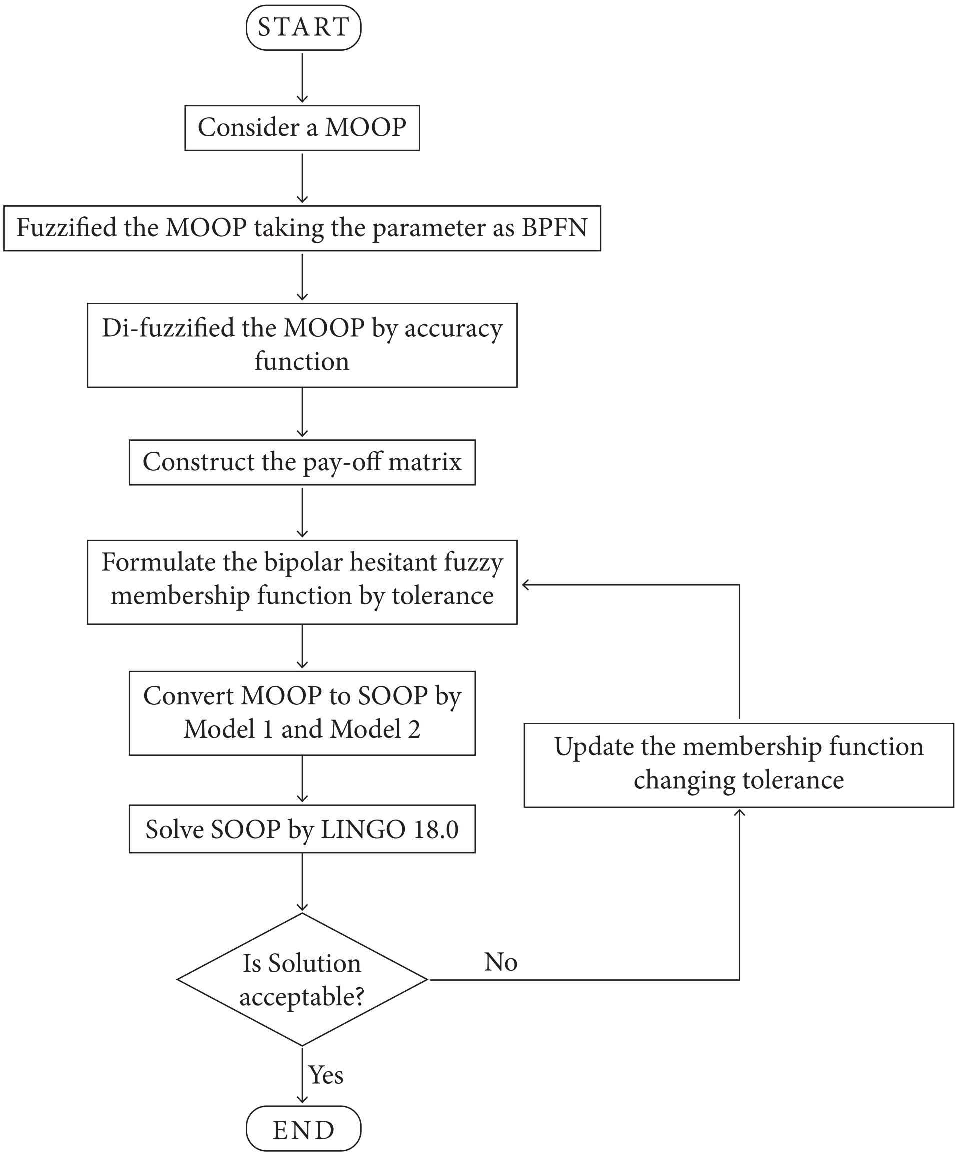

FIGURE 4. Flowchart of solution procedure.

$$\begin{aligned}
&\widetilde{6}x_1 + \widetilde{16}x_3 \le \widetilde{1325}\\
&\widetilde{12}x_1 + \tilde{7}x_3 \le \widetilde{900}\\
&\widetilde{9.5}x_1 + \widetilde{9.5}x_2 + \widetilde{4}x_3 \le \widetilde{1075}\\
&x_1, x_2, x_3 \ge 0
\end{aligned} \tag{5.1}$$

$\widetilde{50} = (48, 49, 50, 52)$; $\widetilde{100} = (98, 99, 101, 103)$;
$\widetilde{17.5} = (16, 17.5, 18, 19)$; $\widetilde{92} = (90, 91, 93, 95)$;
$\widetilde{75} = (73, 74, 76, 78)$; $\widetilde{25} = (23, 25, 27, 28)$;
$\widetilde{12} = (10, 11, 13, 14)$; $\widetilde{17} = (14, 16, 18, 19)$;
$\widetilde{1400} = (1400, 1400, 1410, 1430)$; $\widetilde{3} = (1.5, 2.5, 3.5, 4)$;
$\widetilde{9} = (7.5, 8, 9, 9.5)$; $\widetilde{8} = (6.5, 7, 8, 8.5)$;
$\widetilde{1000} = (1000, 1000, 1010, 1040)$; $\widetilde{10} = (8, 9.5, 10, 11)$;
$\widetilde{13} = (11.5, 12, 13.5, 14.5)$; $\widetilde{15} = (13, 14, 15.5, 17)$;
$\widetilde{1750} = (1750, 1750, 1760, 1780)$; $\widetilde{6} = (5, 6, 7, 8)$;
$\widetilde{16} = (14, 15.5, 17, 18)$; $\widetilde{1325} = (1325, 1325, 1335, 1350)$;
$\widetilde{7} = (5, 6.5, 7, 8.5)$; $\widetilde{900} = (900, 900, 910, 940)$;
$\widetilde{9.5} = (8, 8.5, 9.5, 10)$; $\widetilde{4} = (3, 4, 4.5, 5)$;
$\widetilde{1075} = (1075, 1075, 1090, 1110)$.

The equivalent crisp MOOP using the defuzzification of the fuzzy numbers becomes,

$$\begin{aligned}
&\text{Maximize } Z_1(x) = 49.75x_1 + 100.25x_2 + 17.625x_3\\
&\text{Maximize } Z_2(x) = 92.25x_1 + 75.25x_2 + 49.75x_3\\
&\text{Maximize } Z_3(x) = 25.75x_1 + 100.25x_2 + 75.25x_3\\
&\text{Subject to}\\
&\qquad 12x_1 + 16.75x_2 \le 1410\\
&\qquad 2.875x_1 + 8.5x_2 + 7.5x_3 \le 1012.5\\
&\qquad 10.125x_1 + 12.875x_2 + 14.875x_3 \le 1760\\
&\qquad 6.5x_1 + 16.125x_3 \le 1333.75\\
&\qquad 12x_1 + 6.625x_3 \le 912.5\\
&\qquad 9x_1 + 9x_2 + 4.125x_3 \le 1087.5\\
&\qquad x_1, x_2, x_3 \ge 0.
\end{aligned} \tag{5.2}$$

Pay-off matrix:

	Z_1	Z_2	Z_3
X^1	9068.933	8583.575	11 353.98
X^2	8020.579	10 457.97	9191.140
X^3	9019.175	8276.772	11 396.69

$$\begin{aligned}
&X^1 = (4.678730, 80.82718, 41.60235)\\
&X^2 = (52.70763, 46.41841, 42.26542)\\
&X^3 = (0.000000, 82.71318, 41.25840)\\
&U_1^p = 9068.933, U_2^p = 10457.97, U_3^p = 11396.69
\end{aligned}$$

$$U_1^n = 9068.933, U_2^n = 10457.97, U_3^n = 11396.69$$
$$L_1^p = 8020.579, L_2^p = 8276.772, L_3^p = 9191.140.$$

Take $l = 0.2, L_1^n = 8230.2498, L_2^n = 8713.0116, L_3^n = 9632.25$.

Now for positive hesitant membership function,

$$\mu_{E_m}^p[Z_1(x)] = \begin{cases} 0 & \text{for } Z_1(x) \leq 8020.579 \\ \alpha_m\left(\frac{Z_1(x)-8020.579}{9068.933-8020.579}\right)^t & \text{for } 8020.579 \leq Z_1(x) \leq 9068.933 \\ \alpha_m & \text{for } Z_1(x) \geq 9068.933 \end{cases}$$

$$\mu_{E_m}^p[Z_2(x)] = \begin{cases} 0 & \text{for } Z_2(x) \leq 8276.772 \\ \alpha_m\left(\frac{Z_2(x)-8276.772}{10457.97-8276.772}\right)^t & \text{for } 8276.772 \leq Z_2(x) \leq 10457.97 \\ \alpha_m & \text{for } Z_2(x) \geq 10457.97 \end{cases}$$

$$\mu_{E_m}^p[Z_3(x)] = \begin{cases} 0 & \text{for } Z_3(x) \leq 9191.140 \\ \alpha_m\left(\frac{Z_3(x)-9191.140}{11396.69-9191.140}\right)^t & \text{for } 9191.140 \leq Z_3(x) \leq 11396.69 \\ \alpha_m & \text{for } Z_3(x) \geq 11396.69. \end{cases}$$

Now for negative hesitant membership function,

$$\mu_{E_m}^n[Z_1(x)] = \begin{cases} 0 & \text{for } Z_1(x) \leq 8230.2498 \\ -\beta_m\left(\frac{Z_1(x)-8230.2498}{9068.933-8230.2498}\right)^t & \text{for } 8230.2498 \leq Z_1(x) \leq 9068.933 \\ -\beta_m & \text{for } Z_1(x) \geq 9068.933 \end{cases}$$

$$\mu_{E_m}^n[Z_2(x)] = \begin{cases} 0 & \text{for } Z_2(x) \leq 8713.0116 \\ -\beta_m\left(\frac{Z_2(x)-8713.0116}{10457.97-8713.0116}\right)^t & \text{for } 8713.0116 \leq Z_2(x) \leq 10457.97 \\ -\beta_m & \text{for } Z_2(x) \geq 10457.97 \end{cases}$$

$$\mu_{E_m}^n[Z_3(x)] = \begin{cases} 0 & \text{for } Z_3(x) \leq 9632.250 \\ -\beta_m\left(\frac{Z_3(x)-9632.250}{11396.690-9632.250}\right)^t & \text{for } 9632.250 \leq Z_3(x) \leq 11396.690 \\ -\beta_m & \text{for } Z_3(x) \geq 11396.690. \end{cases}$$

Taking the decision of three expert E_1, E_2, E_3 and $\alpha_1 = 1$, $\alpha_2 = 0.98$, $\alpha_3 = 0.96$, $\beta_1 = 1$, $\beta_2 = 0.98$, $\beta_3 = 0.96$.

$$\text{Maximize } \left(\frac{a_1 + a_2 + a_3}{3}\right) - \left(\frac{b_1 + b_2 + b_3}{3}\right)$$

Subject to

$$\alpha_1\left(\frac{Z_1(x) - 8020.579}{9068.933 - 8020.579}\right)^t \geq a_1; \quad \alpha_1\left(\frac{Z_2(x) - 8276.772}{10457.97 - 8276.772}\right)^t \geq a_1; \quad \alpha_1\left(\frac{Z_3(x) - 9191.140}{11396.69 - 9191.140}\right)^t \geq a_1$$
$$\alpha_2\left(\frac{Z_1(x) - 8020.579}{9068.933 - 8020.579}\right)^t \geq a_2; \quad \alpha_2\left(\frac{Z_2(x) - 8276.772}{10457.97 - 8276.772}\right)^t \geq a_2; \quad \alpha_2\left(\frac{Z_3(x) - 9191.140}{11396.69 - 9191.140}\right)^t \geq a_2$$
$$\alpha_3\left(\frac{Z_1(x) - 8020.579}{9068.933 - 8020.579}\right)^t \geq a_3; \quad \alpha_3\left(\frac{Z_2(x) - 8276.772}{10457.97 - 8276.772}\right)^t \geq a_3; \quad \alpha_3\left(\frac{Z_3(x) - 9191.140}{11396.69 - 9191.140}\right)^t \geq a_3$$
$$-\beta_1\left(\frac{Z_1(x) - 8230.2498}{9068.933 - 8230.2498}\right)^t \leq b_1; \quad -\beta_1\left(\frac{Z_2(x) - 8713.0116}{10457.97 - 8713.0116}\right)^t \leq b_1;$$
$$-\beta_1\left(\frac{Z_3(x) - 9632.250}{11396.690 - 9632.250}\right)^t \leq b_1; \quad -\beta_2\left(\frac{Z_1(x) - 8230.2498}{9068.933 - 8230.2498}\right)^t \leq b_2;$$

$$-\beta_2\left(\frac{Z_2(x)-8713.0116}{10457.97-8713.0116}\right)^t \leq b_2;\quad -\beta_2\left(\frac{Z_3(x)-9632.250}{11396.690-9632.250}\right)^t \leq b_2$$
$$-\beta_3\left(\frac{Z_1(x)-8230.2498}{9068.933-8230.2498}\right)^t \leq b_3;\quad -\beta_3\left(\frac{Z_2(x)-8713.0116}{10457.97-8713.0116}\right)^t \leq b_3;$$
$$-\beta_3\left(\frac{Z_3(x)-9632.250}{11396.690-9632.250}\right)^t \leq b_3$$

$0 \leq a_m \leq 1; -1 \leq b_m \leq 0$ and constraints in problem (5.2).

Example 5.2 (Multi-Objective Optimization in Sustainable Production Planning)**.** Sustainable production planning aims to balance economic, environmental, and social objectives. Multi-objective optimization (MOO) is a powerful tool to address such problems, as it allows decision-makers to optimize multiple conflicting objectives simultaneously. Below is a mathematical formulation of a sustainable production planning problem using MOO.

Problem description

A manufacturing company wants to optimize its production plan to:

(1) Maximize profit (economic objective).
(2) Minimize environmental impact (environmental objective).
(3) Maximize social benefit (social objective).

The company produces n products, and the production process involves m resources. The goal is to determine the optimal production quantities x_i for each product i.

Mathematical formulation

Decision variables

x_i Quantity of product i to produce $(i = 1, 2, \ldots, n)$.

Parameters

p_i Unit sell price of product i.
c_i Unit production cost of product i.
e_i Environmental impact (*e.g.* carbon emissions) per unit of product i.
s_i Social benefit (*e.g.* jobs created) per unit of product i.
r_j Available capacity of resource $j (j = 1, 2, \ldots, m)$.
a_{ij} Amount of resource j required to produce one unit of product i.

Objectives

Maximize profit:

$$\text{Maximize } Z_1 = \sum_{i=1}^{n} (p_i - c_i) x_i.$$

Minimize environmental impact:

$$\text{Minimize } Z_2 = \sum_{i=1}^{n} e_i x_i.$$

Maximize social benefit:

$$\text{Maximize } Z_3 = \sum_{i=1}^{n} s_i x_i.$$

Constraints

Resource constraints:

$$\sum_{i=1}^{n} a_{ij}x_i \leq r_j \qquad \forall\, j = 1, 2, \ldots, m.$$

Non-negativity constraints:

$$x_i \geq 0 \qquad \forall\, i = 1, 2, \ldots, n.$$

Example data

Consider a case with $n = 3$ products and $m = 2$ resources.

Data Table

Product (i)	p_i	c_i	e_i	s_i	a_{i1}	a_{i2}
1	50	30	2	4	3	2
2	80	40	3	2	4	1
3	60	35	1	3	2	3

Resource capacities

$$r_1 = 100 \text{ (Resource 1)}$$
$$r_2 = 80 \text{ (Resource 2)}.$$

Hence, the MOOP becomes,

$$\begin{aligned} &\text{Maximize } Z_1 = (50-30)x_1 + (80-40)x_2 + (60-35)x_3 \\ &\text{Minimize } Z_2 = 2x_1 + 3x_2 + 1x_3 \\ &\text{Maximize } Z_3 = 4x_1 + 2x_2 + 3x_3 \\ &\text{Subject to} \\ &\qquad 3x_1 + 4x_2 + 2x_3 \leq 100 \\ &\qquad 2x_1 + 1x_2 + 3x_3 \leq 80 \\ &\qquad x_1, x_2, x_3 \geq 0. \end{aligned}$$

Taking the parameter as bipolar parabolic fuzzy number, above problem becomes

$$\begin{aligned} &\text{Maximize } Z_1 = \left(\widetilde{50} - \widetilde{30}\right)x_1 + \left(\widetilde{80} - \widetilde{40}\right)x_2 + \left(\widetilde{60} - \widetilde{35}\right)x_3 \\ &\text{Minimize } Z_2 = \tilde{2}x_1 + \tilde{3}x_2 + \tilde{1}x_3 \\ &\text{Maximize } Z_3 = \tilde{4}x_1 + \tilde{2}x_2 + \tilde{3}x_3 \\ &\text{Subject to} \\ &\qquad \tilde{3}x_1 + \tilde{4}x_2 + \tilde{2}x_3 \lesssim \widetilde{100} \\ &\qquad \tilde{2}x_1 + \tilde{1}x_2 + \tilde{3}x_3 \lesssim \widetilde{80} \\ &\qquad x_1, x_2, x_3 \geq 0 \end{aligned}$$

where $\tilde{1} = (0.8, 0.9, 1, 1.2)$; $\tilde{2} = (1.7, 1.9, 2, 2.1)$; $\tilde{3} = (2.9, 3, 3.1, 3.2)$; $\tilde{4} = (3.8, 4, 4.1, 4.2)$; $\widetilde{30} = (27, 29, 32, 34)$; $\widetilde{35} = (32, 34, 36, 37)$; $\widetilde{40} = (38, 40, 42, 43)$; $\widetilde{50} = (47, 49, 51, 52)$; $\widetilde{60} = (57, 59, 60, 62)$; $\widetilde{80} = (77, 79, 82, 83)$; $\widetilde{100} = (98, 100, 102, 103)$.

After de fuzzifying the above bipolar parabolic fuzzy number by using the formula,

$$\mathcal{A}\left(\widetilde{B_p}\right) = \frac{a + b^p + b^n + c}{4}.$$

The crisp MOOP becomes

$$\begin{aligned}
&\text{Maximize } Z_1 = 19.25x_1 + 39.5x_2 + 24.75x_3\\
&\text{Minimize } Z_2 = 1.925x_1 + 3.05x_2 + 0.975x_3\\
&\text{Maximize } Z_3 = 4.025x_1 + 1.925x_2 + 3.05x_3\\
&\text{Subject to}\\
&\qquad 3.05x_1 + 4.025x_2 + 1.925x_3 \le 100.75\\
&\qquad 1.925x_1 + 0.975x_2 + 3.05x_3 \le 80.25\\
&\qquad x_1, x_2, x_3 \ge 0.
\end{aligned} \tag{5.3}$$

Pay-off matrix:

	Z_1	Z_2	Z_3
X^1	1115.358	65.89	115.2830
X^2	0	0	0
X^3	750.2909	61.4093	146.4372

$$\begin{aligned}
&X^1 = (0, 14.69379, 21.61428)\\
&X^2 = (0, 0, 0)\\
&X^3 = (27.3021, 0, 9.0798)\\
&U_1^p = 1115.358, U_2^p = 65.89, U_3^p = 146.4372.
\end{aligned}$$

Take $l = 0.2$, $U_1^n = 1115.358$, $U_2^n = 52.712$, $U_3^n = 146.4372$

$$L_1^p = 0, L_2^p = 0, L_3^p = 0.$$

Take $l = 0.2$, $L_1^n = 223.0716$, $L_2^n = 0$, $L_3^n = 29.28744$.

Now for positive hesitant membership function,

$$\mu_{E_m}^p[Z_1(x)] = \begin{cases} 0 & \text{for } Z_1(x) \le 0\\ \alpha_m\left(\frac{Z_1(x)}{1115.358}\right)^t & \text{for } 0 \le Z_1(x) \le 1115.358\\ \alpha_m & \text{for } Z_1(x) \ge 1115.358 \end{cases}$$

$$\mu_{E_m}^p[Z_2(x)] = \begin{cases} \alpha_m & \text{for } Z_2(x) \le 0\\ \alpha_m\left(\frac{65.89 - Z_2(x)}{65.89}\right)^t & \text{for } 0 \le Z_2(x) \le 65.89\\ 0 & \text{for } Z_2(x) \ge 65.89 \end{cases}$$

$$\mu_{E_m}^p[Z_3(x)] = \begin{cases} 0 & \text{for } Z_3(x) \le 0\\ \alpha_m\left(\frac{Z_3(x)}{146.4372}\right)^t & \text{for } 0 \le Z_3(x) \le 146.4372\\ \alpha_m & \text{for } Z_3(x) \ge 146.4372. \end{cases}$$

Now for negative hesitant membership function,

$$\mu_{E_m}^n[Z_1(x)] = \begin{cases} 0 & \text{for } Z_1(x) \le 223.0716\\ -\beta_m\left(\frac{Z_1(x) - 223.0716}{1115.358 - 223.0716}\right)^t & \text{for } 223.0716 \le Z_1(x) \le 1115.358\\ -\beta_m & \text{for } Z_1(x) \ge 1115.358 \end{cases}$$

$$\mu^n_{E_m}[Z_2(x)] = \begin{cases} -\beta_m & \text{for } Z_2(x) \le 0 \\ -\beta_m\left(\frac{52.712-Z_2(x)}{52.712}\right)^t & \text{for } 0 \le Z_2(x) \le 52.712 \\ 0 & \text{for } Z_2(x) \ge 52.712 \end{cases}$$

$$\mu^n_{E_m}[Z_3(x)] = \begin{cases} 0 & \text{for } Z_3(x) \le 29.28744 \\ -\beta_m\left(\frac{Z_3(x)-29.28744}{146.4372-29.28744}\right)^t & \text{for } 29.28744 \le Z_3(x) \le 146.4372 \\ -\beta_m & \text{for } Z_3(x) \ge 146.4372. \end{cases}$$

Taking the decision of three expert E_1, E_2, E_3 and $\alpha_1 = 1$, $\alpha_2 = 0.98$, $\alpha_3 = 0.96$, $\beta_1 = 1$, $\beta_2 = 0.98$, $\beta_3 = 0.96$.

$$\text{Maximize } \left(\frac{a_1+a_2+a_3}{3}\right) - \left(\frac{b_1+b_2+b_3}{3}\right)$$

Subject to

$$\alpha_1\left(\frac{Z_1(x)}{1115.358}\right)^t \ge a_1;\quad \alpha_1\left(\frac{65.89-Z_2(x)}{65.89}\right)^t \ge a_1;\quad \alpha_1\left(\frac{Z_3(x)}{146.4372}\right)^t \ge a_1$$

$$\alpha_2\left(\frac{Z_1(x)}{1115.358}\right)^t \ge a_2;\quad \alpha_2\left(\frac{65.89-Z_2(x)}{65.89}\right)^t \ge a_2;\quad \alpha_2\left(\frac{Z_3(x)}{146.4372}\right)^t \ge a_2$$

$$\alpha_3\left(\frac{Z_1(x)}{1115.358}\right)^t \ge a_3;\quad \alpha_3\left(\frac{65.89-Z_2(x)}{65.89}\right)^t \ge a_3;\quad \alpha_3\left(\frac{Z_3(x)}{146.4372}\right)^t \ge a_3$$

$$-\beta_1\left(\frac{Z_1(x)-223.0716}{1115.358-223.0716}\right)^t \le b_1;\quad -\beta_1\left(\frac{52.712-Z_2(x)}{52.712}\right)^t \le b_1;\quad -\beta_1\left(\frac{Z_3(x)-29.28744}{146.4372-29.28744}\right)^t \le b_1$$

$$-\beta_2\left(\frac{Z_1(x)-223.0716}{1115.358-223.0716}\right)^t \le b_2;\quad -\beta_2\left(\frac{52.712-Z_2(x)}{52.712}\right)^t \le b_2;\quad -\beta_2\left(\frac{Z_3(x)-29.28744}{146.4372-29.28744}\right)^t \le b_2$$

$$-\beta_3\left(\frac{Z_1(x)-223.0716}{1115.358-223.0716}\right)^t \le b_3;\quad -\beta_3\left(\frac{52.712-Z_2(x)}{52.712}\right)^t \le b_3;\quad -\beta_3\left(\frac{Z_3(x)-29.28744}{146.4372-29.28744}\right)^t \le b_3.$$

$0 \le a_m \le 1; -1 \le b_m \le 0$ and constraints in problem (5.3).

6. Results snd discussion

Solving the Example 1, results achieved by applying the proposed method and other existing methods are presented in Table 1. Here the efficient solutions with the objective values by existing methods and proposed method are listed. Second column shown the results obtained by fuzzy optimization method with specific satisfaction level of $\alpha = 0.5$. The results obtained by IF optimization method with linear and non-linear MF are shown in third and fourth column respectively. In fifth column the results are obtained by goal programming approach and the objective weight are calculated by conflict and nonconflict of the objectives in IF environment.

In the last column we have shown the results obtained by proposed bipolar hesitant fuzzy optimization method. It has been seen that by the proposed method $Z_1 = 8625.242$, $Z_2 = 9534.830$, $Z_3 = 10\,514.20$, with the efficient solution $(26.57541, 64.14001, 43.84850)$. From the optimal values of the objective by proposed method it has been conclude that slight depletion of the production quality the profit and worker satisfaction are increased by the solution of the proposed method. Also, to show the superiority of the proposed method total of the optimal objective values are calculated. It has been seen that the total optimal objective value by the proposed method is 28 674.272 which is better than the other existing method.

For better representation, optimal objective values by the proposed method and existing methods are presented graphically in Figure 5. In Figure 6, the total optimal objective values are presented.

As a result, the efficiency of the proposed methodology in addressing MOOP is highlighted.

TABLE 1. Comparison of the solutions by existing methods and proposed method for Example 1.

Decision variables and objective function	Fuzzy optimization ($\alpha = 0.5$)	IF optimization (Linear MF)	IF optimization (Non-Linear MF)	Conflict nonConflict in IF environment	Proposed method
x_1	65.2571	58.4833	49.8906	31.2808	26.57541
x_2	26.9187	34.5907	47.1360	60.2933	65.14001
x_3	49.8324	47.6992	42.5550	43.3375	43.84850
Z_1	6826.7920	7217.9710	7952.8425	8453.6955	8625.242
Z_2	10 514.1757	10 359.7261	10 252.8852	9595.2274	9534.830
Z_3	8060.7275	8498.5925	9152.4900	10 170.5729	10 514.20
Total	25 401.6902	26 076.2896	27 358.2177	28 229.4958	28 674.272

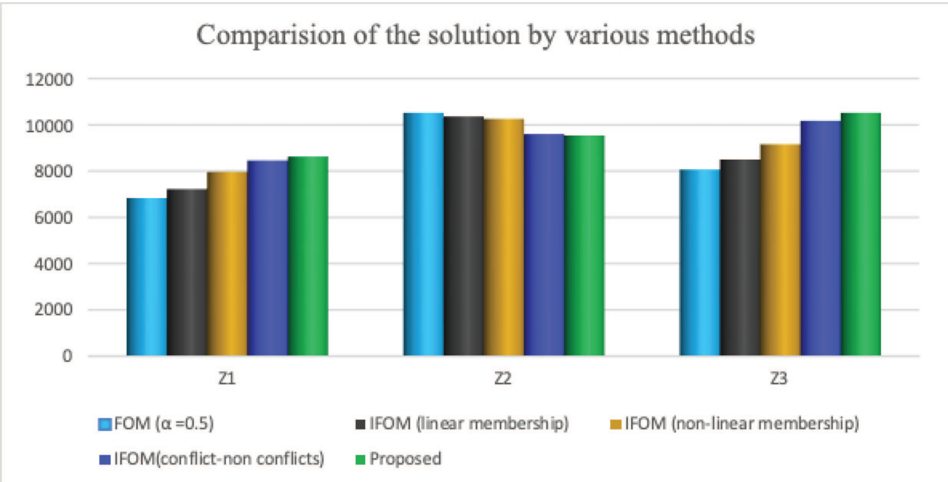

FIGURE 5. Graphical presentation the optimal objective values by various methods for Example 1.

Here a multi objective sustainable production planning problem (Example 2) is solved by Fuzzy optimization method with satisfaction level $\alpha = 0.5$, Bipolar fuzzy optimization method and bipolar hesitant fuzzy optimization method (proposed). The results as efficient solution (optimal decision variables) and the optimal objective values are listed in Table 2. From the optimal values of the objective by proposed method it has been conclude that slight depletion of the environmental objective the profit and social benefit are maximum by the propose method with $Z_1 = 657.8530$, $Z_2 = 28.05131$, $Z_3 = 84.09459$, since here the second objective is minimization and other two are maximization, the total optimal value of the objectives is not calculated to compare (Fig. 7).

7. CONCLUSION AND FUTURE SCOPE

In this paper, we presented a methodological approach for addressing a MOOP and to illustrate the method a production planning problem (Example 1) and a multi objective sustainable production planning problem is solved by taking the parameter of the objectives and constraints as non-linear bipolar hesitant fuzzy number. Also, a solution methodology in bipolar hesitant fuzzy environment named the bipolar hesitant fuzzy optimization approach is applied to get a compromise solution. The solution obtained by the proposed method is

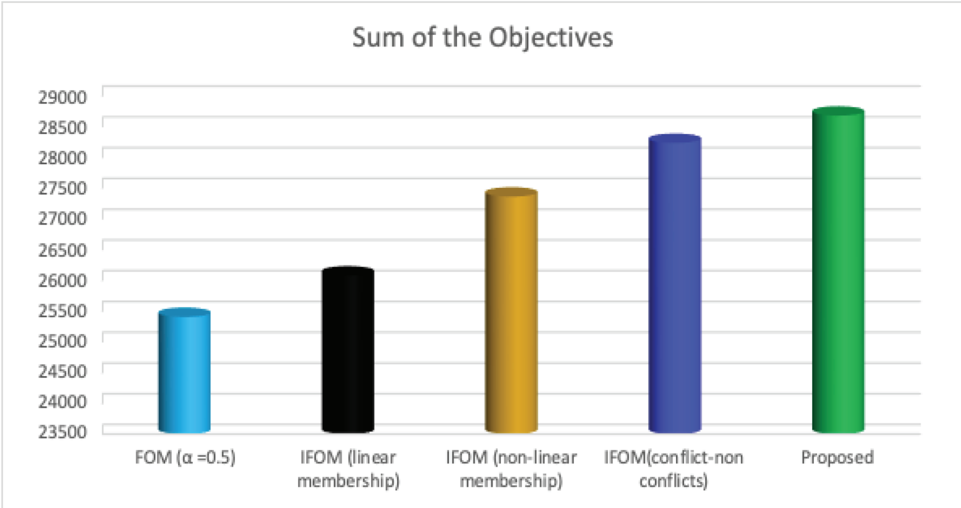

FIGURE 6. Graphical presentation of the total optimal value by various methods for Example 1.

TABLE 2. Comparison of the solutions by existing methods and proposed method for Example 2.

Decision variables and objective function	Fuzzy optimization with level of satisfaction $\alpha = 0.5$	Bipolar fuzzy optimization method	Proposed method (Bipolar hesitant fuzzy optimization method)
x_1	2.719374	0	1.830757
x_2	0	0	0
x_3	20.41742	21.56550	25.15600
Z_1	557.6790	533.7461	657.8530
Z_2	25.14178	21.02636	28.05131
Z_3	73.21860	65.77477	84.09459

compared with the other existing methods in the literature. It has been seen that the proposed method gives better results compared to others. For better understanding, the results are presented graphically.

As MOOP are inherent to real-world scenarios, they continuously evolve over time, necessitating an ongoing quest for enhanced methodologies. The scope for improvement remains perpetual, particularly with the introduction of more versatile parameters such as bipolar q-rung and bipolar q-rung hesitant fuzzy. Consequently, this methodology exhibits the potential to adapt and perform differently across diverse environments. Therefore, the exploration of MOOP solutions under varying conditions remains a viable avenue for future research.

Advantages of the proposed method

– Captures both positive and negative preferences, allowing for a more flexible decision-making process.
– Accounts for decision makers' uncertainty by incorporating multiple membership values.
– Provides a more refined and realistic evaluation in complex multi-objective problems.
– Suitable for uncertain and ambiguous environments such as production planning, finance, engineering, and healthcare.
– Can be integrated with other optimization techniques for better performance.

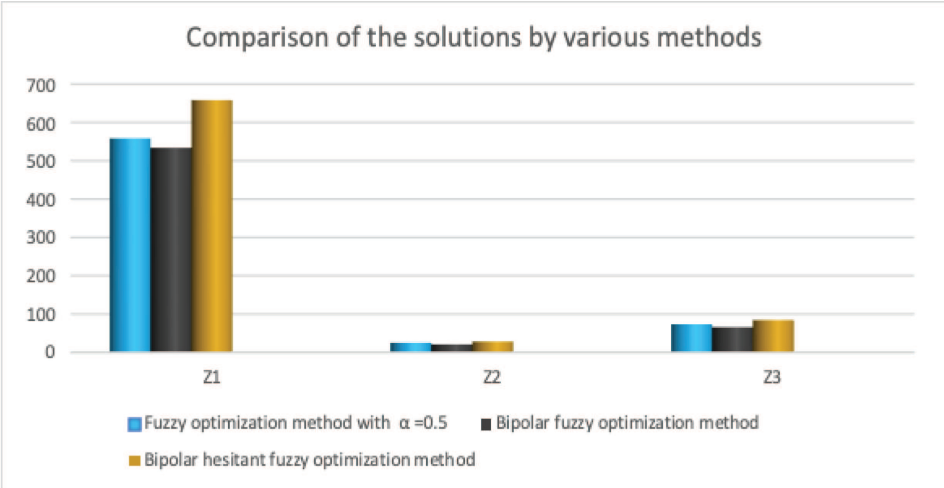

FIGURE 7. Graphical presentation the optimal objective values by various methods for Example 2.

Disadvantages of the proposed method

– Requires extensive calculations due to multiple membership values and bipolar considerations.
– Results may be harder to interpret compared to traditional fuzzy methods.
– Effectiveness depends on proper selection of parameters, which may not always be straightforward.
– Lacks widely accepted frameworks or benchmarks, making validation challenging.
– May struggle with high-dimensional problems due to increased computational overhead.

Lingo programme for Example 1

```
Max = (((a1+a2+a3)/3)-((b1+b2+b3)/3));

z1=19.25*x1+39.5*x2+24.75*x3;
z2=1.925*x1+3.05*x2+0.975*x3;
z3=4.025*x1+1.925*x2+3.05*x3;

p1*((z1)/(1115.358))^t >=a1;
p2*((z1)/(1115.358))^t >=a2;
p3*((z1)/(1115.358))^t >=a3;

p1*((65.89-z2)/(65.89))^t >=a1;
p2*((65.89-z2)/(65.89))^t >=a2;
p3*((65.89-z2)/(65.89))^t >=a3;

p1*((z3)/(146.4372))^t >=a1;
p2*((z3)/(146.4372))^t >=a2;
p3*((z3)/(146.4372))^t >=a3;

-q1*((z1-223.0716)/(1115.358-223.0716))^t <=b1;
-q2*((z1-223.0716)/(1115.358-223.0716))^t <=b2;
-q3*((z1-223.0716)/(1115.358-223.0716))^t <=b3;
```

```
-q1*((52.712-z2)/(52.712))^t <=b1;
-q2*((52.712-z2)/(52.712))^t <=b2;
-q3*((52.712-z2)/(52.712))^t <=b3;

-q1*((z3-29.28744)/(146.4372-29.28744))^t <=b1;
-q2*((z3-29.28744)/(146.4372-29.28744))^t <=b2;
-q3*((z3-29.28744)/(146.4372-29.28744))^t <=b3;
a1>=0;
a1<=1;
a2>=0;
a2<=1;
a3>=0;
a3<=1;
b1<=0;
b1>=-1;
b2<=0;
b2>=-1;
b3<=0;
b3>=-1;
p1=1;
p2=0.96;
p3=0.94;
q1=1;
q2=0.96;
q3=0.94;
t=3;

3.05*x1+4.025*x2+1.925*x3<=100.75;
1.925*x1+0.975*x2+3.05*x3<=80.25;
x1>=0;
x2>=0;
x3>=0;
end
```

Lingo programme for Example 2

```
Max = (((a1+a2+a3)/3)-((b1+b2+b3)/3));

z1=19.25*x1+39.5*x2+24.75*x3;
z2=1.925*x1+3.05*x2+0.975*x3;
z3=4.025*x1+1.925*x2+3.05*x3;

p1*((z1)/(1115.358))^t >=a1;
p2*((z1)/(1115.358))^t >=a2;
p3*((z1)/(1115.358))^t >=a3;

p1*((65.89-z2)/(65.89))^t >=a1;
p2*((65.89-z2)/(65.89))^t >=a2;
p3*((65.89-z2)/(65.89))^t >=a3;
```

```
p1*((z3)/(146.4372))^t >=a1;
p2*((z3)/(146.4372))^t >=a2;
p3*((z3)/(146.4372))^t >=a3;

-q1*((z1-223.0716)/(1115.358-223.0716))^t <=b1;
-q2*((z1-223.0716)/(1115.358-223.0716))^t <=b2;
-q3*((z1-223.0716)/(1115.358-223.0716))^t <=b3;

-q1*((52.712-z2)/(52.712)) ^t <=b1^(1/t);
-q2*((52.712-z2)/(52.712)) ^t <=b2^(1/t);
-q3*((52.712-z2)/(52.712)) ^t <=b3^(1/t);

-q1*((z3-29.28744)/(146.4372-29.28744))^t <=b1;
-q2*((z3-29.28744)/(146.4372-29.28744))^t <=b2;
-q3*((z3-29.28744)/(146.4372-29.28744))^t <=b3;
a1>=0;
a1<=1;
a2>=0;
a2<=1;
a3>=0;
a3<=1;
b1<=0;
b1>=-1;
b2<=0;
b2>=-1;
b3<=0;
b3>=-1;
p1=1;
p2=0.96;
p3=0.94;
q1=1;
q2=0.96;
q3=0.94;
t=3;

3.05*x1+4.025*x2+1.925*x3<=100.75;
1.925*x1+0.975*x2+3.05*x3<=80.25;
x1>=0;
x2>=0;
x3>=0;
end
```

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Data availability statement

The research data associated with this article are included in the article.

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