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A THEORETICAL AND NUMERICAL STUDY OF AN INTERIOR-POINT ALGORITHM FOR CONVEX QUADRATIC SEMIDEFINITE OPTIMIZATION

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Abstract. In this paper, we present a theoretical and numerical study of a primal-dual path-following interior-point algorithm for solving convex quadratic semidefinite optimization problems (CQSDO). At each iteration, the algorithm uses only feasible full Nesterov–Todd steps for tracing approximately the central-path of CQSDO with the advantage that no line search is computed. Moreover, to ensure its well-definiteness and its locally quadratically convergence to an optimal solution and to enhance its numerical performances, new appropriate defaults are offered. Furthermore, we prove that the algorithm with short-update method has the currently best known polynomial complexity, namely, $\mathcal{O}\big(\sqrt{n+1}\log(n/\epsilon)\big)$. The efficiency of our algorithm is demonstrated through the numerical experiments on some CQSDO problems. Finally, a comparison between the efficiency of our proposed algorithm and existing ones is made.

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1. Introduction

Let $\mathcal{S}^n$ and $\mathcal{S}^n_+$ denote the cones of $n \times n$ real symmetric and symmetric positive semidefinite matrices, respectively. Consider the convex quadratic semidefinite optimization (CQSDO) problem in its standard format:

$$(\mathcal{P}) \qquad p^\star = \min_X \frac{1}{2} X \bullet \mathcal{Q}(X) + C \bullet X \ \textbf{s.t.}\ A_i \bullet X = b_i, \quad i = 1, \ldots, m, \quad X \succeq 0,$$

and its dual

$$(\mathcal{D}) \qquad d^\star = \max_{(X,y,Z)} b^\top y - \frac{1}{2} X \bullet \mathcal{Q}(X)\ \textbf{s.t.} \sum_{i=1}^m y_i A_i + Z = C + \mathcal{Q}(X), \quad Z \succeq 0,$$

where $b, y \in \mathcal{R}^m$, $X, Z \in \mathcal{S}^n_+$, C, $A_i \in \mathcal{S}^n$, $i = 1, \ldots, m$, $\mathcal{Q}(X)$ is a linear transformation on $\mathcal{S}^n$ and the inequality $M \succeq 0$ means that $M \in \mathcal{S}^n_+$. Recall that the notation $A \bullet B$, $A, B \in \mathcal{S}^n$ denotes the Frobenius inner product between A and B, which is equal to the $Tr(AB)$.

The CQSDO problem is an exciting topic of research in mathematical programming with wide applications in engineering and science fields. It also includes well known and important optimization problems such as linear optimization (LO) [10], semidefinite optimization (SDO) [11], standard convex quadratic optimization

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(CQO) and semidefinite least squares problems (SDLS) [16]. Further, the CQSDOs can be seen as a special case of semidefinite linear complementarity problems (SDLCP) [5]. As the cone $\mathcal{S}^n_+$ is a non polyhedral set, methods like simplicial type are not applicable for solving CQSDOs. Therefore, interior-point methods (IPMs) gained much more attention to solve this problem. Among the variants of them, the so-called primal-dual path-following algorithms (IPA) are the most used class of IPMs because they are more efficient from computational point of a view (*e.g.*, [23]). References [1–3, 12, 13, 17, 21] among others discuss variants of feasible primal-dual path-following IPMs from theoretical and practical point of a view.

In the last years, several primal-dual IPA designed for LO have been extended to SDO and CQSDO. The difficulty to extend them lies in acquiring a symmetric search direction with the desired properties. For this purpose, several symmetrization techniques are suggested to overcome this drawback. The Alizadeh, Haeberly, and Overton (AHO) [6], Kojima Shida and Shindoh [15] and Nesterov–Todd (NT) [19] are the most used symmetrization of search directions in semidefinite optimization. Further, in CQSDO the property of orthogonality between the scaled directions in the primal and dual spaces is not satisfied. This makes the analysis of the complexity a little difficult to the case of SDO.

For instance, there is a considerable number of primal-dual IPAs for solving CQSDO. Nie and Yuan [20] presented two methods such as a predictor-corrector and potential reduction. Toh [22] proposed an inexact primal-dual path-following algorithm for CQSDO. Later, Achache and Guerra [4] presented a full-Nesterov Todd interior-point method for solving CQSDO. Besides, Daili and Achache extended a primal-dual IPMs for the semidefinite least squares optimization [9] where the polynomial complexity is showed and some numerical results are reported. Subsequently, Bai *et al.* [7] presented an IPA for CQSDO based on a modified search direction. They proved its polynomial complexity. It is worth mentioning that their study is only concentrated on the separable monotone transformation, *i.e.*, $\mathcal{Q}(X) = QX$ where $Q \succeq 0$. In addition, no numerical results are reported to show the efficiency of their proposed algorithm. Very recently, Guerra presented a primal-dual IPA based also on new parametric kernel function for SDOs. She proved that the iteration bound depends on a parameter. So if the latter becomes very large the algorithm looses its polynomial complexity [14].

Also, it is worth to mention that the most primal-dual IPA are based on tracing approximately the central-path by restricting the feasible iterates in a neighborhood of it and decreasing the duality gap to zero for reaching an optimal solution. To do so, suitable defaults of the threshold τ which defines the size of this neighborhood and of the updating barrier parameter θ are suggested. These defaults are crucial in the convergence analysis and the numerical study of these algorithms.

In this paper, motivated by these works, we elaborate a full NT step feasible primal-dual path-following for solving the CQSDOs. Our contributions, is to treat on one hand a general form of the linear transformation $\mathcal{Q}(X)$ which is not necessarily separable and on the other hand, we offer new appropriate defaults of the parameters τ and θ which guarantee that the proposed algorithm is well-defined and converges locally quadratically to an optimal solution and they lead to good numerical results. Moreover, we establish its iteration bound with short-step method, namely, $\mathcal{O}\big(\sqrt{n+1}\log(n/\epsilon)\big)$ which coincides with the currently best known bound for these methods. Besides, we have implemented the algorithm of [7] which is based on a new NT search directions. Finally, a comparison is made between our numerical results and those obtained by Bai *et al.* [7].

Throughout the paper, the following notations are used. $\mathcal{R}^{n\times n}$ denotes the set of all $n \times n$ real matrices. The trace inner-product and the Frobenius norm in $\mathcal{S}^n$ are denoted, respectively, by: $A \bullet B = Tr(AB) = \sum_{i,j} a_{ij}b_{ij}$ and $||A||_F = (Tr(A^2))^{\frac{1}{2}}$, $A, B \in \mathcal{S}^n$. For a matrix A, $\lambda_i(A)$ denote its eigenvalues with $\lambda_{\min(A)}$ ($\lambda_{\max(A)}$) as the smallest (largest) one, respectively, and $\det(A)$ stand for its determinant. The square root matrix of any $X \in \mathcal{S}^n_+$ is denoted by $X^{1/2}$. If $f(x)$ and $g(x)$ are two positive real valued functions, then $g(x) = \mathcal{O}(f(x))$ if $g(x) \leq cf(x)$ for some positive constant c. The similarity between A and B denoted by $A \sim B$ which means $A = PBP^{-1}$ for some non singular matrix P and the identity matrix is denoted by I.

The paper is organized as follows. In Section 2, the solvability of CQSDO (strong duality), the concept of central-path and Nesterov–Todd symmetrization for search directions are discussed. Further, the generic primal-dual IPA is described. In Section 3, we show that the algorithm is well-defined and can solve the CQSDO problem in polynomial time. In Section 4, some numerical results are reported. Finally, in Section 5 a conclusion and future remarks end the paper.

2. Primal-dual IPA for CQSDO

Firstly, we briefly recall some known facts from Matrix Theory. For the details, we refer to the monograph [24]. Let $A, B \in \mathcal{R}^{n\times n}$, then

– $Tr(A) = \sum_{i=1}^{n} \lambda_i(A)$.
– $Tr(A+B) = Tr(A) + Tr(B)$.
– $Tr(AB) = Tr(BA)$.
– $Tr(A) = 0$ if $A = -A^\top$ "skew-symmetric matrix".
– $Tr(A) = Tr(A^\top)$.
– $A \sim B$, so $Tr(A) = Tr(B)$.
– If $A \in \mathcal{S}^n$: $\begin{array}{l}\|A\|_F = \sqrt{Tr(A^2)},\\ \|A\|_2 = |\lambda_{\max}(A)|.\end{array}$
  $\lambda_{\max}(A^{-1}) = \frac{1}{\lambda_{\min}(A)}$ with A^{-1} is the inverse of A.
– $\|AB\|_2 \leq \|A\|_2 \|B\|_F$.
– $\|A\|_2 \leqslant \|A\|_F \leqslant \sqrt{n}\|A\|_2,\ n \geqslant 1$.

Lemma 2.1 ([11], Lem. 2.1)**.** *Let $X, Z \in \mathcal{S}^n_+$. Then $X \bullet Z = 0$, if and only if $XZ = 0_{\mathcal{S}^n}$.*

Next, we deal with the solvability of CQSDO.

2.1. The solvability of CQSDO

In the sequel of the paper, we will denote by

$$\mathcal{F}_{\mathcal{P}} = \left\{X \in \mathcal{S}^n_+ \colon A_i \bullet X = b_i, i = 1, ..., m\right\},$$
$$\mathcal{F}^0_{\mathcal{P}} = \left\{X \in F_{\mathcal{P}} \colon X \in \mathcal{S}^n_{++}\right\},$$
$$\mathcal{F}_{\mathcal{D}} = \left\{(X, y, Z) \in \mathcal{S}^n_+ \times \mathcal{R}^m \times \mathcal{S}^n_+ \colon C + \mathcal{Q}(X) - \sum_{i=1}^{m} y_i A_i = Z\right\},$$
$$\mathcal{F}^0_{\mathcal{D}} = \left\{(X, y, Z) \in F_{\mathcal{D}} \colon X, Z \in \mathcal{S}^n_{++}\right\},$$

the feasible and the strictly feasible sets of $\mathcal{P}$ and $\mathcal{D}$, respectively.

Theorem 2.1 (Weak duality [8])**.** *Let $X \in \mathcal{F}_{\mathcal{P}}$ and $(X, y, Z) \in \mathcal{F}_{\mathcal{D}}$, then*

$$p^\star - d^\star = X \bullet Z \geq 0$$

where $p^\star - d^\star$ and $X \bullet Z$ are called the duality gap and the complementarity slackness condition for $\mathcal{P}$ and $\mathcal{D}$, respectively. Moreover, if $X \bullet Z = 0$, then X is an optimal solution of $\mathcal{P}$ and (y, Z) is an optimal solution of $\mathcal{D}$ with $p^\star = d^\star$.

Theorem 2.2 (Strong duality [8])**.** *If $\mathcal{F}^0_{\mathcal{P}} \times \mathcal{F}^0_{\mathcal{D}} \neq \emptyset$, then the sets of optimal solutions of $\mathcal{P}$ and $\mathcal{D}$ are nonempty bounded sets and we have $p^\star = d^\star$.*

Theorem 2.2 indicates that the strictly feasible CQSDO problem is solvable *i.e.*, the problems $\mathcal{P}$ and $\mathcal{D}$ have optimal solutions with $-\infty < p^\star = d^\star < +\infty$.

Consequently, solving the pair of problems $\mathcal{P}$ and $\mathcal{D}$ is equivalent to solving optimality conditions, *i.e.*, the KKT conditions:

$$\begin{cases} A_i \bullet X - b_i = 0, & \forall i = 1, \ldots, m, X \succeq 0, \\ C + \mathcal{Q}(X) - \sum_{i=1}^{m} y_i A_i - Z = 0_{\mathcal{S}^n}, & Z \succeq 0, \\ XZ = 0_{\mathcal{S}^n}. \end{cases} \tag{1}$$

2.2. The central-path of CQSDO

Throughout the paper, the following assumptions hold.

– **Strict feasibility (SF).** $\mathcal{F}^0_{\mathcal{P}} \times \mathcal{F}^0_{\mathcal{D}} \neq \emptyset$. This assumption is called the Interior-Point-Condition.
– **Independence.** The matrices $A_i, i = 1, \cdots, m$, are linearly independent.
– **Monotony and self-adjoint.** $\mathcal{Q}(X)$ is a monotone and self-adjoint linear transformation on $\mathcal{S}^n$, *i.e.*, $X \bullet \mathcal{Q}(X) \geq 0$ and $X \bullet \mathcal{Q}(Y) = \mathcal{Q}(X) \bullet Y, \ \forall X, Y \in \mathcal{S}^n$.

The basic idea behind primal-dual IPMs is to replace the third equation in (1) by the parameterized equation $XZ = \mu I$ with $\mu > 0$. Thus we consider the following system:

$$\begin{cases} A_i \bullet X - b_i = 0, & \forall i = 1, \cdots, m, \ X \succ 0, \\ C + \mathcal{Q}(X) - Z - \sum_{i=1}^m y_i A_i = 0_{\mathcal{S}^n}, & Z \succ 0, \\ XZ = \mu I, \ \mu > 0. \end{cases} \tag{2}$$

Under our assumptions, system (2) has a unique solution $(X(\mu), y(\mu), Z(\mu))$ for each $\mu > 0$. The set

$$\mathcal{C} = \{(X(\mu), y(\mu), Z(\mu)) : \mu > 0\}$$

of μ-centers is called the central-path of $\mathcal{P}$ and $\mathcal{D}$. If μ tends to zero then the limit of $\mathcal{C}$ exists and since the limit point satisfies the complementarity condition, the limit yields an approximated solution of $\mathcal{P}$ and $\mathcal{D}$ (*e.g.*, [6, 10]).

2.3. The Nesterov–Todd (NT) search directions for CQSDO

Next, we want to define search directions $(\Delta X, \Delta y, \Delta Z)$ that move in the direction of the central-path $\mathcal{C}$. Applying *Newton's method* for (2), for a strictly feasible point $(X \succ 0, y, Z \succ 0)$ such that $\mu I - XZ \neq 0$, we get the following system of equations:

$$\begin{cases} A_i \bullet \Delta X = 0_{\mathcal{R}}, & i = 1, \ldots, m, \\ \sum\limits_{i=1}^m (\Delta y)_i A_i + \Delta Z = \mathcal{Q}(\Delta X), & \\ \Delta X Z + X \Delta Z = \mu I - XZ, & X \succ 0, Z \succ 0, \end{cases}$$

or equivalently

$$\begin{cases} A_i \bullet \Delta X = 0_{\mathcal{R}}, & i = 1, \ldots, m, \\ \sum\limits_{i=1}^m (\Delta y)_i A_i + \Delta Z = \mathcal{Q}(\Delta X), & \\ \Delta X + X \Delta Z Z^{-1} = \mu Z^{-1} - X. & \end{cases} \tag{3}$$

A crucial observation from the third equation in (3) is that ΔX is not necessarily symmetric because the matrix $X \Delta Z Z^{-1}$ may not be symmetrical. Therefore, a remedy is to symmetrizing the latter. For that, we introduce a non singular matrix P and we replace the term $X \Delta Z Z^{-1}$ in the third equation in (3) by $P \Delta Z P^\top$, then we obtain

$$\begin{cases} A_i \bullet \Delta X = 0_{\mathcal{R}}, & i = 1, \ldots, m, \\ \sum\limits_{i=1}^m (\Delta y)_i A_i + \Delta Z = \mathcal{Q}(\Delta X), & \\ \Delta X + P \Delta Z P^\top = \mu Z^{-1} - X. & \end{cases}$$

Based on different symmetrization schemes, several search directions have been proposed in the literature of semidefinite optimization by Kojima *et al.* [15], Nesterov and Todd (NT) [19] and AHO (see [6]).

Here, we utilize a non singular matrix P introduced by Nesterov–Todd (NT) where $P = X^{\frac{1}{2}}(X^{\frac{1}{2}}ZX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}} = Z^{-\frac{1}{2}}(Z^{\frac{1}{2}}XZ^{\frac{1}{2}})^{\frac{1}{2}}Z^{-\frac{1}{2}}$, $P \in \mathcal{S}^n_{++}$. To simplify matters, we set $D = P^{1/2}$, then D can be used to scale X and Z to the same symmetric positive definite matrix V [10] defined by,

$$V = \frac{1}{\sqrt{\mu}}D^{-1}XD^{-1} = \frac{1}{\sqrt{\mu}}DZD. \tag{4}$$

Further, the scaled search directions are given by

$$D_X = \frac{1}{\sqrt{\mu}}D^{-1}\Delta XD^{-1},\ D_Z = \frac{1}{\sqrt{\mu}}D\Delta ZD. \tag{5}$$

Next, due to (3) and (4), we get

$$\begin{cases} \bar{A}_i \ \bullet D_X = 0_{\mathcal{R}}, & i = 1, \dots, m, \\ \sum\limits_{i=1}^{m}(\Delta y)_i\bar{A}_i + D_Z - \bar{Q}(D_X) = 0_{\mathcal{S}^n}, \\ D_X + D_Z = P_V, \end{cases} \tag{6}$$

where $\bar{A}_i = \frac{1}{\sqrt{\mu}}DA_iD$, $\bar{Q}(D_X) = DQ(DD_XD)D$ and $P_V = V^{-1} - V$.

We notice from the first two equations of the system (6) that the scaled directions D_X and D_Z are symmetric matrices with the property

$$D_X \bullet D_Z = \frac{1}{\mu}\Delta X \bullet \Delta Z = \frac{1}{\mu}\Delta X \bullet \mathcal{Q}(\Delta X) \geq 0.$$

So it is clear that the scaled NT-directions (D_X, D_Z) in CQSDO are not orthogonal contrary to the case of SDO where we have $D_X \bullet D_Z = 0$. This makes the analysis of our proposed algorithm a little difficult.

2.4. The proximity and some bounds for the scaled NT-directions

Next, for controlling iterations during the Newton process and to keep them near to the central-path $\mathcal{C}$, we define a norm based proximity measure $\delta(XZ; \mu)$ as follows:

$$\delta(V) := \delta(XZ; \mu) = \frac{1}{2}\|P_V\|_F = \frac{1}{2}\left\|V^{-1} - V\right\|_F.$$

Note that $\delta(V) = 0 \Leftrightarrow V^{-1} = V \Leftrightarrow XZ = \mu I$. So the value of $\delta(V)$ is considered as a measure for the distance between the point (X, y, Z) and the central-path $\mathcal{C}$. Further, for the directions $(\Delta X, \Delta y, \Delta Z)$ and after a full-Newton step, the new iterate is defined as follows

$$\left(X^+, y^+, Z^+\right) = (X, y, Z) + (\Delta X, \Delta y, \Delta Z). \tag{7}$$

Next, we state some upper bounds for the scaled NT search directions and for the matrix norms of the symmetric matrix $D_{XZ} \in \mathcal{S}^n$ defined by

$$D_{XZ} = \frac{1}{2}(D_XD_Z + D_ZD_X).$$

Lemma 2.2 ([4], Lem. 3.3)**.** *Let (D_X, D_Z) be a solution of system (6) and $\mu > 0$. If $\delta = \delta(XZ; \mu)$, then*

$$0 \leq D_X \bullet D_Z \leq 2\delta^2,$$

and

$$\|D_{XZ}\|_2 \leq \frac{1}{4}\|D_X + D_Z\|_F^2 = \delta^2,\ \|D_{XZ}\|_F^2 \leq \frac{1}{8}\|P_V\|_F^4.$$

Input:
An accurancy paramter $\epsilon > 0$;
a threshold parameter τ, $0 < \tau < 1$ (default $\tau = \frac{1}{\sqrt{2}}$);
a barrier parameter θ, $0 < \theta < 1$ (default $\theta = \frac{1}{4\sqrt{n+1}}$);
a feasible point $(X^0 \succ 0, y^0, Z^0 \succ 0)$ and $\mu_0 > 0$ **s.t.** $\delta(X^0 Z^0; \mu_0) \leq \tau$;
begin
$k := 0$;
While $n\mu_k \geq \epsilon$ **do**
• $\mu_{k+1} = (1-\theta)\mu_k$;
• Compute $(\Delta X^k,\ \Delta y^k,\ \Delta Z^k)$ via system (6) and then use (5);
• Update $X^{k+1} = X^k + \Delta X^k$, $y^{k+1} = y^k + \Delta y^k$, $Z^{k+1} = Z^k + \Delta Z^k$;
$k := k+1$;
endWhile
end.

Figure 1. Algorithm 2.5. Primal-dual IPA for CQSDO.

2.5. The strategy of the algorithm for CQSDO

The proposed algorithm for solving CQSDO works as follows. A threshold (default) value $\tau > 0$ with $0 < \tau < 1$ is chosen and we suppose that a strictly feasible initial point $(X^0 \succ 0, y^0, Z^0 \succ 0)$ such that $\delta(X^0 Z^0; \mu_0) \leq \tau$ for certain $\mu_0 > 0$ is known. Using the obtained search directions computed from (6) and taking a full NT-step, the algorithm produces a new iterate (X^+, y^+, Z^+) from (7). Then, the barrier parameter μ is reduced by a factor $(1-\theta), (0 < \theta < 1)$ and solves system (6), and so target a new μ-center and so on. This process is repeated until the stopping criterion $n\mu \leq \epsilon$ is satisfied for a given positive accuracy parameter ϵ.

The generic IPA for CQSDO is stated in Figure 1 as follows.

3. The complexity analysis of Algorithm 2.5

In this section, we will prove that Algorithm 2.5 is well-defined and solves the CQSDO in polynomial time. Firstly, we quote two lemmas to achieve the proof of Lemma 3.3 (for more details, see *e.g.*, [11]).

Lemma 3.1. *Let* $X(\alpha) = X + \alpha\Delta X$ *and* $Z(\alpha) = Z + \alpha\Delta Z$ *such that* $X \succ 0$ *and* $Z \succ 0$*. If* $\det(X(\alpha)Z(\alpha)) > 0, \forall\, 0 \leq \alpha \leq \overline{\alpha}$*, then* $X(\overline{\alpha}) \succ 0$ *and* $Z(\overline{\alpha}) \succ 0$*.*

Lemma 3.2. *Let* $A \in \mathcal{S}^n$ *and let* $B \in \mathcal{R}^{n\times n}$ *a skew-symmetric matrix. If* $A \succ 0$ *then* $\det(A+B) > 0$*. Moreover, if* $\lambda_i(A+B) \in \mathcal{R}, \forall i = 1,\ldots,n$*, then* $0 < \lambda_{\min}(A) \leq \lambda_{\min}(A+B) \leq \lambda_{\max}(A+B) \leq \lambda_{\max}(B)$*.*

3.1. Feasibility and locally quadratically convergence of Algorithm 2.5

In this subsection, under the condition $\delta(XZ;\mu) < 1$, we show the strict feasibility of full NT-step. Moreover, if $\delta(XZ;\mu) \leq \frac{1}{\sqrt{2}}$ then the full NT-step converges locally quadratically to the target point $(X(\mu), y(\mu), Z(\mu))$.

Lemma 3.3 ([4], Lem. 3.5)**.** *Assume* $\delta := \delta(XZ;\mu) < 1$*, then the full NT-step is strictly feasible i.e.,* $X^+ \succ 0$ *and* $Z^+ \succ 0$*.*

In next lemmas, we show the locally quadratically convergence of the proximity measure, throughout the algorithm.

Lemma 3.4 ([4], Lem. 3.6)**.** *One has*

$$\lambda_{\min}(V_+^2) \geq 1 - \delta^2,$$

where $\lambda_{\min}(V_+^2)$ is the smallest eigenvalue of V_+^2.

Next, we show the locally quadratically convergence of the proximity measure near the central-path during the Newton process.

Lemma 3.5 ([4], Lem. 3.7)**.** *Assume $\delta < 1$, then*

$$\delta_+ := \delta(X^+Z^+;\mu) \leq \frac{\delta^2}{\sqrt{2(1-\delta^2)}}\cdot$$

Moreover, if $\delta \leq \frac{1}{\sqrt{2}}$, then $\delta_+ \leq \delta^2$, which shows the locally quadratically convergence of the proximity measure.

3.2. The influence of a full NT-step on the duality gap

Next lemma gives an upper bound for the duality gap $X^+ \bullet Z^+$ after a full NT-step.

Lemma 3.6 ([4], Lem. 3.8)**.** *Assume $\delta \leq \frac{1}{\sqrt{2}}$, after a full NT-step, then $X^+ \bullet Z^+ \leq \mu(n+1)$.*

3.3. The update of the barrier parameter

Next, we show the influence of updating $\mu_+ = (1-\theta)\mu$ on the proximity measure after a full NT-step. To simplify the calculus, we define the matrix U by

$$U = \frac{1}{\sqrt{\mu_+}} D^{-1}X^+D^{-1} = \frac{1}{\sqrt{\mu_+}} DZ_+D = \frac{1}{\sqrt{1-\theta}} V_+.$$

Theorem 3.1. *Assume $\delta \leq \frac{1}{\sqrt{2}}$ and $\mu_+ = (1-\theta)\mu$, where $0 < \theta < 1$, then*

$$\delta(X^+Z^+;\mu_+) \leq \delta_+ + \frac{\theta\sqrt{n+1}}{2\sqrt{1-\theta}}\cdot$$

Moreover, if $\theta = \frac{1}{4\sqrt{n+1}}$ and $n \geq 3$, then we have

$$\delta(X^+Z^+;\mu_+) \leq \frac{1}{\sqrt{2}}\cdot$$

Proof. For the proof of the first part of the lemma, we have

$$\begin{aligned} 2\delta\big(X^+Z^+;\mu_+\big) := 2\delta(U) = \big\|U^{-1} - U\big\|_F &= \left\|\sqrt{1-\theta}V_+^{-1} - \frac{1}{\sqrt{1-\theta}}V_+\right\|_F \\ &= \left\|\sqrt{1-\theta}\big(V_+^{-1} - V_+\big) - \frac{\theta}{\sqrt{1-\theta}}V_+\right\|_F \\ &\leq \sqrt{1-\theta}\big\|V_+^{-1} - V_+\big\|_F + \frac{\theta}{\sqrt{1-\theta}}\|V_+\|_F \\ &= 2\sqrt{1-\theta}\,\delta_+ + \frac{\theta}{\sqrt{1-\theta}}\|V_+\|_F \leq 2\delta_+ + \frac{\theta}{\sqrt{1-\theta}}\|V_+\|_F. \end{aligned}$$

As $V_+^2 = \dfrac{1}{\mu}D^{-1}X^+Z^+D$ and by Lemma 3.6, we have

$$\|V_+\|_F = \sqrt{Tr(V_+^2)} = \sqrt{\frac{1}{\mu}X^+ \bullet Z^+} \leq \sqrt{n+1}.$$

So we obtain

$$\delta(X^+Z^+;\mu_+) \leq \delta_+ + \frac{\theta\sqrt{n+1}}{2\sqrt{1-\theta}}.$$

Since $\delta \leq \frac{1}{\sqrt{2}}$, it follows that $\delta_+ \leq \delta^2 \leq \frac{1}{2}$, and consequently, we get

$$\delta(X^+Z^+;\mu_+) \leq \frac{1}{2} + \frac{\theta\sqrt{n+1}}{2\sqrt{1-\theta}}.$$

This proves the first part of it. For the second part, let $\theta = \frac{1}{4\sqrt{n+1}}$, then

$$\delta(X^+Z^+;\mu_+) \leq \frac{1}{2} + \frac{1}{8\sqrt{1-\theta}}.$$

Assume $n \geq 3$, so $0 \leq \theta \leq \frac{1}{8}$. Now as $\frac{1}{8\sqrt{1-\theta}} \leq \frac{1}{2\sqrt{14}}$ for all $0 \leq \theta \leq \frac{1}{8}$, therefore $\delta(X^+Z^+;\mu_+) \leq \frac{1}{2} + \frac{1}{8\sqrt{1-\theta}} \leq \frac{1}{2} + \frac{1}{2\sqrt{14}} < \frac{1}{\sqrt{2}}$. This completes the proof. □

Theorem 3.1 indicates that under our specific choice of our defaults, Algorithm 2.5 is well defined since the conditions $(X \succ 0, y, Z \succ 0)$ and $\delta(XZ;\mu) \leq \frac{1}{\sqrt{2}}$ are maintained throughout the algorithm.

3.4. The iteration bound

Next lemma provides an upper bound for the total number of iterations produced by Algorithm 2.5.

Lemma 3.7 ([4], Lem. 3.10)**.** *Let $\{(X^k, y^k, Z^k)\}$ be the sequence produced by Algorithm 2.5 with $\mu = \mu_k$, then the inequality $X^k \bullet Z^k \leq \epsilon$ holds if $k \geq \frac{1}{\theta}\log\left(\frac{2n\mu_0}{\epsilon}\right)$.*

Proof. According to Lemma 3.6, it follows that

$$X^k \bullet Z^k \leq (n+1)\mu_k$$

with $\mu_k = (1-\theta)\mu_{k-1} = (1-\theta)^k\mu_0$. This implies that

$$X^k \bullet Z^k \leq 2n(1-\theta)^k\mu_0$$

since $n+1 \leq 2n$ for all $n \geq 1$. Then the inequality $X^k \bullet Z^k \leq \epsilon$ holds if

$$2n(1-\theta)^k\mu_0 \leq \epsilon.$$

Taking the logarithms on both sides, we obtain

$$k\log(1-\theta) \leq \log\epsilon - \log(2n\mu_0).$$

Using $\log(1-\theta) \leq -\theta$ for $0 < \theta < 1$, then, we have $k\theta \geq \log\left(\frac{2n\mu_0}{\epsilon}\right)$. Finally,

$$k \geq \frac{1}{\theta}\log\left(\frac{2n\mu_0}{\epsilon}\right).$$

This proves the lemma. □

Theorem 3.2. *Let $\theta = \frac{1}{4\sqrt{n+1}}$, $\mu_0 = \frac{1}{2}$. Then Algorithm 2.5 requires at most*

$$\mathcal{O}\left(\sqrt{n+1}\log\left(\frac{n}{\epsilon}\right)\right)$$

iterations to obtain an $\epsilon-$ approximated optimal solution of $\mathcal{P}$ and $\mathcal{D}$.

Proof. Substituting $\theta = \frac{1}{4\sqrt{n+1}}$ and $\mu_0 = \frac{1}{2}$ in Lemma 3.7, the proof of the theorem is straightforward. This completes the proof. □

4. Numerical results

In this section, we present some numerical results of Algorithm 2.5 based on our new defaults $\theta = \frac{1}{4\sqrt{n+1}}$, $\tau = \frac{1}{\sqrt{2}}$ and $\mu_0 = \frac{1}{2}$. Our numerical results are obtained *via* the software *Matlab R2009b* environment and run it on an ordinary personal pc. In the implementation, we take $\epsilon = 10^{-6}$ as our tolerance and $(X^0 \succ 0, y^0, Z^0 \succ 0)$ stand for the initial point of the algorithm such that $\delta(X^0 Z^0; \mu_0) \leq \frac{1}{\sqrt{2}}$ is satisfied. In the below tables of the obtained numerical results, the number of iterations and the time obtained by the algorithm are denoted by "*Iter*" and "*CPU*", respectively. Besides, we implement the algorithm presented by Bai *et al.* [7] where we denote it by Algorithm 3 with $\theta = \frac{1}{2\sqrt{n}}$. Finally, a comparison is made between these two algorithms such that the initial point must be satisfied both required conditions for Algorithms 2.5 and 3, it means that

$$\delta_{\text{Alg } 2.5}(X_0 Z_0; \mu_0) \leq \frac{1}{\sqrt{2}} \quad \text{and} \quad \delta_{\text{Alg } 3}(X_0 Z_0; \mu_0) \leq \frac{1}{2}.$$

Now, we apply Algorithms 2.5 and 3 on some monotone CQSDO problems. These problems are reformulated from different well known optimization problems. For each example, $(X^\star, y^\star, Z^\star)$ denotes its obtained primal-dual optimal solution.

4.1. Problem 1. The linear semidefinite optimization (SDO)

ExSDO. Consider the CQSDO problem whose primal-dual pair $\mathcal{P}$ and $\mathcal{D}$ have the following data:

$$\mathcal{Q}(X) = 0_{\mathcal{S}^4}, \quad A_1 = \begin{bmatrix} 0.9 & 0 & 0 & -1.5 \\ 0 & 3 & 0.75 & 0 \\ 0 & 0.75 & 1.098 & 0 \\ -1.5 & 0 & 0 & -0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.414 & 1.386 & 0 & 0 \\ 1.386 & 1.732 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1.333 & 0 \\ 0 & 0 & 0 & -0.333 \end{bmatrix}, \quad C = \begin{bmatrix} 1.7071 & 0.6931 & -0.1 & 0 \\ 0.6931 & 1.366 & -0.5 & 0.02 \\ -0.1 & -0.5 & 2 & 0 \\ 0 & 0.02 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 7.4986 \\ 4.7369 \\ 2.9 \end{bmatrix},$$

where the initial point is defined as follows

$$Z^0 = \begin{bmatrix} 1 & 0 & -0.1 & 0 \\ 0 & 0.5 & 0 & 0.02 \\ -0.1 & 0 & 1 & 0 \\ 0 & 0.02 & 0 & 1 \end{bmatrix}, \quad y^0 = \begin{bmatrix} 0 \\ 0.5 \\ 0 \end{bmatrix}, \quad X^0 = \begin{bmatrix} 0.9 & 0 & -0.1487 & 0 \\ 0 & 2 & 0 & 0.03 \\ -0.1487 & 0 & 1 & 0 \\ 0 & 0.03 & 0 & 1 \end{bmatrix}.$$

The obtained primal-dual optimal solution is

$$X^\star = \begin{bmatrix} 0.3968 & 0.6757 & 0.7576 & -0.0257 \\ 0.6757 & 1.1505 & 1.2899 & -0.0437 \\ 0.7576 & 1.2899 & 1.4463 & -0.0490 \\ -0.0257 & -0.0437 & -0.0490 & 0.0017 \end{bmatrix}, \quad y^\star = \begin{bmatrix} 0.0335 \\ 0.5372 \\ 0.6355 \end{bmatrix},$$

$$Z^\star = \begin{bmatrix} 0.2819 & -0.0515 & -0.1 & 0.0502 \\ -0.0515 & 0.0175 & 0.0121 & 0.02 \\ -0.1 & 0.0121 & 0.0416 & 0 \\ 0.0502 & 0.02 & 0 & 1.3029 \end{bmatrix}.$$

The optimal values for both problems $\mathcal{P}$ and $\mathcal{D}$ are $p^\star = d^\star = 6.8178$.

4.2. Problem 2. [16] The semidefinite least squares optimization (SDLS)

The SDLS is a convex optimization problem stated as follows

$$\begin{cases} \min_X f(X) = \frac{1}{2}\|BX - N\|_F^2 \\ \textbf{s.t. } A_i \bullet X = b_i, i = 1, \ldots, m, \\ \qquad X \succeq 0, \end{cases}$$

where $b \in \mathcal{R}^m$, and $N,\ B \in \mathcal{S}^n$. Because

$$\frac{1}{2}\|BX - N\|_F^2 = \frac{1}{2}B^2X \bullet X - (NB \bullet X + BN \bullet X)/2 + \frac{1}{2}N \bullet N,$$

then, the SDLS can be restated as a CQSDO problem with

$$f(X) = C \bullet X + \frac{1}{2}X \bullet \mathcal{Q}(X) + \frac{1}{2}N \bullet N,\ \mathcal{Q}(X) = B^2X \text{ and } C = -(NB + BN)/2.$$

ExSDLS. Consider the SDLS problem of size $(m = 5, n = 7)$ where its data is defined as follows

$$A_1 = \mathrm{diag}(6.4,\ 1.03,\ 1.03,\ 8.21,\ 1.69,\ 0,\ 4.2), \quad b = \begin{bmatrix}-4.794,\ 34.6310,\ 9.2640,\ 32.16,\ 32.3845\end{bmatrix}^\top,$$

$$A_3 = \begin{bmatrix} 0.66 & 4 & 5.6 & 0 & 0 & 0 & 5.9 \\ 4 & 0 & 0 & 0 & -1.6 & 0 & 0 \\ 5.6 & 0 & 8.88 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.93 & 0 & 0 & 0 \\ 0 & -1.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.03 & 0 \\ 5.9 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 3 & 5.5 & 0 & 0 & 3.6 & 0 & 0 \\ 5.5 & -6 & 0 & 4.4 & 0 & 2 & 0 \\ 0 & 0 & 2.1 & 0 & 0 & 7.08 & 0 \\ 0 & 4.4 & 0 & 4 & 0 & 0 & 0.3 \\ 3.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 7.08 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & -3.22 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0.1 & 10.2 & 0 & 0 & 0 & 2.828 & -1.4 \\ 10.2 & 2 & 6.7 & 0 & -1.5 & 0 & 0 \\ 0 & 6.7 & 7.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & 0 & 0 & 0 & 0 \\ 2.828 & 0 & 0 & 0 & 0 & 7.6 & 0 \\ -1.4 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -6.04 & 4.5 & 0.3 & 2.22 & 0 & 1.1 & 0 \\ 4.5 & 2 & 0 & 5.3 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 8.7 & 0 & 0 \\ 2.22 & 5.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8.7 & 0 & -1 & 0 & 0 \\ 1.1 & 0 & 0 & 0 & 0 & 3.5 & -3.7 \\ 0 & 0 & 0 & 0 & 0 & -3.7 & 0 \end{bmatrix},$$

$$N = \begin{bmatrix} -9 & 0 & 0 & 0 & -4.8 & 0 & 0 \\ 0 & -45.76 & 0 & -9.6 & 0 & 8 & -0.72 \\ 0 & 0 & -0.01 & 0 & 0 & 0 & 0 \\ 0 & -9.6 & 0 & -16.09 & 0 & 0 & -0.534 \\ -4.8 & 0 & 0 & 0 & -2.56 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & -8 & 0 \\ 0 & -0.72 & 0 & -0.534 & 0 & 0 & -5.0184 \end{bmatrix},$$

$$B = \begin{bmatrix} 2.3484 & -2.834 & 0 & 2 & 0 & 1.1 & 4.07 \\ -2.834 & 44.09 & 0 & -8 & 0 & -4.4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & -8 & 0 & 4 & 0 & 2.2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1.1 & -4.4 & 0 & 2.2 & 0 & 14.9 & 0 \\ 4.07 & 0 & 0 & 0 & 0 & 0 & 13.69 \end{bmatrix}.$$

The initial point is defined as follows

$$X^0 = \begin{bmatrix} 1.2 & 0 & -0.25 & 0 & 0 & 0 & 0 \\ 0 & 0.55 & 0 & 0.02 & 0 & 0.001 & 0 \\ -0.25 & 0 & 1.8 & 0 & 0 & 0 & 0 \\ 0 & 0.02 & 0 & 1.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.001 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.1 \end{bmatrix}, \quad y^0 = \begin{bmatrix} 0 \\ 0.75 \\ 0 \\ 0.75 \\ 0 \end{bmatrix},$$

$$Z^0 = \begin{bmatrix} 1.4 & 0 & -0.017 & 0 & 0 & 0.5 & 0 \\ 0 & 1.1818 & 0 & 0.0016 & 0 & 0 & 0 \\ -0.017 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0.0016 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.5 \end{bmatrix}.$$

The obtained primal-dual optimal solution is

$$X^\star = \begin{bmatrix} 1.2653 & -0.0021 & -0.2553 & -0.0378 & -0.0685 & -0.0011 & -0.0208 \\ -0.0021 & 0.5516 & 0.0062 & 0.0325 & 0.0009 & -0.0007 & 0.0004 \\ -0.2553 & 0.0062 & 1.7801 & 0.0326 & -0.0534 & -0.0035 & 0.0008 \\ -0.0378 & 0.0325 & 0.0326 & 1.8843 & 0.0302 & -0.0071 & 0.0096 \\ -0.0685 & 0.0009 & -0.0534 & 0.0302 & 0.3369 & 0.0009 & 0.0207 \\ -0.0011 & -0.0007 & -0.0035 & -0.0071 & 0.0009 & 0.9994 & 0.0005 \\ -0.0208 & 0.0004 & 0.0008 & 0.0096 & 0.0207 & 0.0005 & 1.1071 \end{bmatrix},$$

$$Z^\star = 10^{-6} \times \begin{bmatrix} 0.1139 & 0.0001 & 0.0171 & 0.0016 & 0.0256 & 0.0002 & 0.0016 \\ 0.0001 & 0.2503 & -0.0008 & -0.0043 & -0.0004 & 0.0001 & -0.0001 \\ 0.0171 & -0.0008 & 0.0804 & -0.0013 & 0.0164 & 0.0003 & 0 \\ 0.0016 & -0.0043 & -0.0013 & 0.0734 & -0.0064 & 0.0005 & -0.0005 \\ 0.0256 & -0.0004 & 0.0164 & -0.0064 & 0.4182 & -0.0004 & -0.0073 \\ 0.0002 & 0.0001 & 0.0003 & 0.0005 & -0.0004 & 0.1380 & -0.0001 \\ 0.0016 & -0.0001 & 0 & -0.0005 & -0.0073 & -0.0001 & 0.1247 \end{bmatrix},$$

$$y^\star = \begin{bmatrix} -0.0061, & 0.7501, & -0.0148, & 0.8435, & 0.1955 \end{bmatrix}^\top.$$

The optimal values for both problems $\mathcal{P}$ and $\mathcal{D}$ are $p^\star = d^\star = 3473.9$.

Particular case. If $B = I$, then the SDLS is reduced to the Nearest Correlation Matrix Problem (NCMP):

$$(\mathcal{P}) \quad \begin{cases} \min_X f(X) = \frac{1}{2}\|X - N\|_F^2 \\ \textbf{s.t. } A_i \bullet X = b_i, \quad i = 1, \dots, m, \\ \qquad X \succeq 0. \end{cases}$$

Here, $f(X) = -N \bullet X + \frac{1}{2} X \bullet \mathcal{Q}(X) + \frac{1}{2} N \bullet N$, $\mathcal{Q}(X) = X$ and $C = -N$.

ExNCMP. Consider the Nearest Correlation Matrix Problem with size $m = 4$, $n = 8$ where its data is defined as follows

$$A_1 = \begin{bmatrix} -1.2 & 0 & 0.3 & 0.22 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 5.3 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 1.87 & 0 & 0 & 0 \\ 0.22 & 5.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.87 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.5 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.66 & 4 & 5.6 & 0 & 0 & 0 & 9.5 & 0 \\ 4 & 0 & 0 & 0 & -10.6 & 0 & 0 & 0 \\ 5.6 & 0 & 8.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.3 & 0 & 0 & 0 & 0 \\ 0 & -10.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.03 & 0 & 0 \\ 9.5 & 0 & 0 & 0 & 0 & 0 & -6.9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 6 & 2.55 & 0 & 0 & 6.3 & 0 & 0 & 0 \\ 2.55 & -5 & 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 1.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.4 & 0 & 0 & 6.03 & 0 \\ 6.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6.03 & 0 & 0 & -0.22 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0.1 & 10.2 & 0 & 0 & 0 & 2.828 & -1.4 & -3 \\ 10.2 & 2 & 6.7 & 0 & -1.5 & 0 & 0 & 0 \\ 0 & 0 & 7.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.7321 & 0 & 0 & 0 & 0 & 7.6 & 0 & 0 \\ -1.4 & 0 & 0 & 0 & 0 & 0 & 1.05 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} -0.466 \\ 4.676 \\ 11.999 \\ 14.525 \end{bmatrix}, \quad C = -N,$$

$$N = \begin{bmatrix} -0.772 & -10.65 & -4.201 & 0 & 0 & -1.304 & -6.075 & 2.25 \\ -10.65 & -2.768 & 0 & 0.0005 & 9.075 & 0.001 & 0 & 0 \\ -4.201 & 0 & -12.286 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0005 & 0 & -2.461 & 0 & 0 & 0 & 0 \\ 0 & 9.075 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.304 & 0.001 & 0 & 0 & 0 & -7.972 & 0 & 0 \\ -6.075 & 0 & 0 & 0 & 0 & 0 & 4.446 & 0 \\ 2.25 & 0 & 0 & 0 & 0 & 0 & 0 & 5.25 \end{bmatrix}.$$

The initial point is defined as follows

$$X^0 = \begin{bmatrix} 0.904 & 0 & -0.0025 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.55 & 0 & 0.0019 & 0 & 0.001 & 0 & 0 \\ -0.0025 & 0 & 1.007 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0019 & 0 & 1.007 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.03 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad y^0 = \begin{bmatrix} 0 \\ 0.25 \\ 0 \\ 0.3 \end{bmatrix},$$

$$Z^0 = \begin{bmatrix} 1.106 & 0 & -0.0013 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 1.818 & 0 & 0.0014 & 0 & 0 & 0 & 0 \\ -0.0013 & 0 & 0.993 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0014 & 0 & 0.993 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.9708 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The obtained optimal solution is:

$$X^\star = \begin{bmatrix} 0.4253 & 0.1903 & -0.0251 & -0.0633 & 0.1488 & 0.1709 & -0.0891 & -0.1536 \\ 0.1903 & 0.0932 & -0.0044 & -0.0419 & 0.056 & 0.1339 & -0.0371 & -0.0617 \\ -0.0251 & -0.0044 & 0.6 & -0.0006 & -0.0543 & 0.0008 & -0.0001 & -0.0014 \\ -0.0633 & -0.0419 & -0.0006 & 0.2219 & 0.0107 & -0.0016 & 0.1956 & -0.0009 \\ 0.1488 & 0.056 & -0.0543 & 0.0107 & 0.0741 & -0.0006 & -0.018 & -0.0600 \\ 0.1709 & 0.1339 & 0.0008 & -0.0016 & -0.0006 & 0.5576 & 0.1037 & -0.0169 \\ -0.0891 & -0.0371 & -0.0001 & 0.1956 & -0.018 & 0.1037 & 0.2044 & 0.0237 \\ -0.1536 & -0.0617 & -0.0014 & -0.0009 & -0.0600 & -0.0169 & 0.0237 & 0.0644 \end{bmatrix},$$

$$Z^\star = \begin{bmatrix} 0.3486 & -0.6608 & 0.0008 & -0.0569 & -0.1009 & 0.0466 & 0.0423 & 0.0997 \\ -0.6608 & 1.4689 & -0.0044 & 0.1134 & 0.1379 & -0.1446 & -0.0371 & -0.0617 \\ 0.0008 & -0.0044 & 0.0001 & -0.0006 & 0.0007 & 0.0008 & -0.0001 & -0.0014 \\ -0.0569 & 0.1134 & -0.0006 & 0.0429 & 0.0107 & -0.0016 & -0.0434 & -0.0009 \\ -0.1009 & 0.1379 & 0.0007 & 0.0107 & 0.0447 & -0.0006 & -0.018 & -0.06 \\ 0.0466 & -0.1446 & 0.0008 & -0.0016 & -0.0006 & 0.0225 & -0.0139 & -0.0169 \\ 0.0423 & -0.0371 & -0.0001 & -0.0434 & -0.018 & -0.0139 & 0.0559 & 0.0237 \\ 0.0997 & -0.0617 & -0.0014 & -0.0009 & -0.06 & -0.0169 & 0.0237 & 0.1096 \end{bmatrix},$$

$$y^\star = \begin{bmatrix} -0.0294, & 0.2472, & 0.0396, & 0.3746 \end{bmatrix}^\top.$$

The optimal values for both problems $\mathcal{P}$ and $\mathcal{D}$ are $p^\star = d^\star = 58.9843$**.**

4.3. Problem 3. The convex semidefinite quadratic optimization (CQSDO)

ExCQSDO1. Consider the CQSDO with size $n = 5$, $m = 4$ where

$$\mathcal{Q}(X) = H_1 X H_1 + H_2 X H_2,$$

with H_1 and H_2 are given arbitrary matrices in $\mathcal{S}^n$ and the data of whose primal-dual pair $\mathcal{P}$ and $\mathcal{D}$ is defined as follows

$$C = \begin{bmatrix} 1.6229 & -0.2228 & -0.03 & 0 & 0.0903 \\ -0.2228 & -7.4138 & -1.355 & 0.002 & 0.75 \\ -0.03 & -1.355 & -6.0941 & 0 & 0 \\ 0 & 0.002 & 0 & 1 & 0 \\ 0.0903 & 0.75 & 0 & 0 & -10.0625 \end{bmatrix}, \quad b = \begin{bmatrix} 240.4839 \\ -5.7808 \\ 5.6 \\ 4.21 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 3.9087 & 0.8914 & 0 & 0 & -0.3328 \\ 0.8914 & 33.9557 & 5.42 & 0 & -3 \\ 0 & 5.42 & 29.3764 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.3328 & -3 & 0 & 0 & 42.25 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2.3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 0 & -4 & -4 & 0 \\ 0 & -2 & 0 & 8 & 0 \\ -4 & 0 & 0 & 0 & 5.667 \\ -4 & 8 & 0 & 0 & 0 \\ 0 & 0 & 5.667 & 0 & -3 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0.3 & 0.4 & 0 & 0 & 0 \\ 0.4 & 2 & 0 & 0 & -1.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & 0 & 0 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.333 & 0.111 & 0 & 0 & 0 \\ 0.111 & 2 & 0 & 0 & 0.75 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0.75 & 0 & 0 & 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & -2.71 & 0 & 0 \\ 0 & -2.71 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

The initial point is defined as follows

$$X^0 = \begin{bmatrix} 0.7 & 0 & -0.0149 & 0 & 0 \\ 0 & 2 & 0 & 0.3 & 0 \\ -0.0149 & 0 & 1 & 0 & 0 \\ 0 & 0.3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad Z^0 = \begin{bmatrix} 1 & 0 & -0.03 & 0 & 0.007 \\ 0 & 0.5 & 0 & 0.002 & 0 \\ -0.03 & 0 & 1 & 0 & 0 \\ 0 & 0.002 & 0 & 1 & 0 \\ 0.007 & 0 & 0 & 0 & 0.5 \end{bmatrix},$$

$$y^0 = \begin{bmatrix} 0.25, 0, 0.25, 0 \end{bmatrix}^\top.$$

The obtained primal-dual optimal solution is

$$X^\star = \begin{bmatrix} 0.0346 & 0.1737 & 0.0928 & -0.0742 & -0.0389 \\ 0.1737 & 1.9639 & -0.0435 & 0.3211 & -0.0443 \\ 0.0928 & -0.0435 & 1.0831 & -0.063 & -0.0682 \\ -0.0742 & 0.3211 & -0.063 & 0.9916 & 0 \\ -0.0389 & -0.0443 & -0.0682 & 0 & 2.0114 \end{bmatrix}, \quad y^\star = \begin{bmatrix} 0.267 \\ 0.0637 \\ 0.4806 \\ 0.0112 \end{bmatrix},$$

$$Z^\star = \begin{bmatrix} 0.693 & -0.0741 & -0.0575 & 0.0722 & 0.0098 \\ -0.0741 & 0.0079 & 0.0062 & -0.0077 & -0.001 \\ -0.0575 & 0.0062 & 0.0048 & -0.006 & -0.0008 \\ 0.0722 & -0.0077 & -0.006 & 0.0075 & 0.001 \\ 0.0098 & -0.001 & -0.0008 & 0.001 & 0.0001 \end{bmatrix}.$$

The optimal values for both problems $\mathcal{P}$ and $\mathcal{D}$ are $p^\star = d^\star = 13.0945$.

ExCQSDO2 (Stein transformation). Consider the following CQSDO problem with size $(m = 5, n = 6)$ where

$$\mathcal{Q}(X) = X - LXL \;\text{ with }\; \|L\|_F \leqslant 1,\; L \in \mathcal{S}^n,$$

(the condition $\|L\|_F \leqslant 1$ ensures the monotony of $\mathcal{Q}(X)$),

$$C = \begin{bmatrix} 1.4648 & 1.3881 & 1.3808 & 1.376 & 0.9841 & 1.6538 \\ 1.3881 & 1.5898 & 1.4659 & 1.442 & 1.0509 & 1.7054 \\ 1.3808 & 1.4659 & 1.5574 & 1.5318 & 1.0294 & 1.7956 \\ 1.376 & 1.442 & 1.5318 & 1.7456 & 1.242 & 1.9587 \\ 0.9841 & 1.0509 & 1.0294 & 1.242 & 0.971 & 1.387 \\ 1.6538 & 1.7054 & 1.7956 & 1.9587 & 1.387 & 2.2591 \end{bmatrix}, \quad b = \begin{bmatrix} 9.327 \\ 14.3084 \\ 11.1363 \\ 12.7695 \\ 9.5005 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 1.4805 & 1.3813 & 1.3748 & 1.376 & 0.4497 & 1.6538 \\ 1.3813 & -4.028 & 1.4659 & 1.442 & 1.0516 & 1.7054 \\ 1.3748 & 1.4659 & 2.6062 & 1.5318 & 1.0294 & 1.7956 \\ 1.376 & 1.442 & 1.5318 & -1.2544 & 1.242 & 1.9587 \\ 0.4497 & 1.0516 & 1.0294 & 1.242 & 2.1366 & 1.387 \\ 1.6538 & 1.7054 & 1.7956 & 1.9587 & 1.387 & 3.0461 \end{bmatrix},$$

$$L = \begin{bmatrix} 0.5 & 0.01 & 0 & 0 & 0 & 0 \\ 0.01 & 0 & 0.601 & 0 & 0 & 0 \\ 0 & 0.601 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.125 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad \begin{aligned} A_2 &= \operatorname{diag}(-1.3,\ 0,\ -2.1,\ 0.5,\ 8,\ 1.2) \\ A_3 &= \operatorname{diag}(0.3,\ 1.2,\ -2,\ 4.4,\ 3,\ 1.5) \\ A_4 &= \operatorname{diag}(3.03,\ -0.3,\ -1,\ 0,\ 5.03,\ -0.3) \\ A_5 &= \operatorname{diag}(6.033,\ 0.03,\ 0,\ 0,\ 0, 1). \end{aligned}$$

The initial point is defined as follows

$$X^0 = \begin{bmatrix} 1.3543 & 0 & 0 & 0 & -0.57 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.71 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -0.57 & 0 & 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.3 \end{bmatrix}, \quad \begin{aligned} Z^0 &= \operatorname{diag}(1,\ 1,\ 1,\ 2,\ 1,\ 1), \\ y^0 &= \begin{bmatrix} 1, 0, 0, 0, 0 \end{bmatrix}^\top. \end{aligned}.$$

TABLE 1. Number of iterations and CPU time for Problems 1, 2, 3 and 4.

		Algorithm 2.5			Algorithm 3		
	$(n, m) \downarrow$	ITER	CPU	δ_1	ITER	CPU	δ_2
ExSDO	$(4, 3)$	19	0.1212	0.1944	53	0.2504	0.1876
ExSDLS	$(7, 5)$	14	0.6764	0.5095	78	2.0653	0.4712
ExNCMP	$(8, 4)$	12	0.5387	0.0051	82	3.5961	0.0051
ExCQSDO1	$(5, 4)$	17	0.2477	0.2655	61	0.6332	0.2642
ExCQSDO2	$(6, 5)$	15	0.0871	0.4685	70	0.2807	0.4242

TABLE 2. Number of iterations and CPU time for the previous problems.

		Ameliorated Algorithm								
		$\theta = 0.5$		$\theta = 0.65$		$\theta = 0.88$		$\theta = 0.92$		$\delta_1(X^0Z^0; \mu_0)$
Problem	$(n, m) \downarrow$	ITER	CPU	ITER	CPU	ITER	CPU	ITER	CPU	
ExSDO	$(4, 3)$	22	0.1138	15	0.0859	8	0.055	7	0.0547	0.1944
ExSDLS	$(7, 5)$	23	0.9394	16	0.604	8	0.3356	7	0.3032	0.5095
ExNCMP	$(8, 4)$	23	0.9394	16	0.604	8	0.3356	7	0.3032	0.0051
ExCQSDO1	$(5, 4)$	23	0.2615	15	0.1985	8	0.1438	7	0.1313	0.2655
ExCQSDO2	$(6, 5)$	24	0.1223	16	0.0925	8	0.0686	7	0.0598	0.4685

The obtained primal-dual optimal solution is

$$X^\star = \begin{bmatrix} 1.3758 & 0.1480 & 0.1347 & 0.1092 & -0.5297 & 0.1380 \\ 0.1480 & 0.0621 & 0.0572 & 0.1138 & 0.0861 & 0.1432 \\ 0.1347 & 0.0572 & 0.0529 & 0.1123 & 0.0739 & 0.1423 \\ 0.1092 & 0.1138 & 0.1123 & 0.8115 & 0.0988 & 0.1544 \\ -0.5297 & 0.0861 & 0.0739 & 0.0988 & 0.7956 & 0.1115 \\ 0.1380 & 0.1432 & 0.1423 & 0.1544 & 0.1115 & 1.1981 \end{bmatrix}, \quad y^\star = \begin{bmatrix} 1.0786 \\ -0.0554 \\ 0.4046 \\ -0.0521 \\ 0.1427 \end{bmatrix},$$

$$Z^\star = \begin{bmatrix} 0.0021 & -0.0014 & -0.0122 & 0.0011 & 0.0024 & 0.0010 \\ -0.0014 & 0.0774 & -0.0795 & 0.0005 & -0.0021 & 0.0005 \\ -0.0122 & -0.0795 & 0.1726 & -0.0081 & -0.0135 & -0.0073 \\ 0.0011 & 0.0005 & -0.0081 & 0.0006 & 0.0013 & 0.0006 \\ 0.0024 & -0.0021 & -0.0135 & 0.0013 & 0.0027 & 0.0012 \\ 0.0010 & 0.0005 & -0.0073 & 0.0006 & 0.0012 & 0.0005 \end{bmatrix}.$$

The optimal values for both problems $\mathcal{P}$ and $\mathcal{D}$ are $p^\star = d^\star = 13.4275$.

In Table 1, the obtained numerical results for *Problems 1, 2, 3, and 4* are summarized. Here, $\delta_1 = \delta_{\text{Alg } 2.5}(X^0Z^0; \mu_0)$ corresponds to the Algorithm 2.5 with $\theta = \frac{1}{4\sqrt{n+1}}$, while $\delta_2 = \delta_{\text{Alg } 3}(X^0Z^0; \mu_0)$ corresponds to the Algorithm 3 with $\theta = \frac{1}{2\sqrt{n}}$.

Comment. From Table 1, we see that for all testing problems, the number of iterations and the CPU time produced by Algorithm 2.5 are less than those obtained by Algorithm 3. However, we presume across these results, if the size of n becomes large, the number of iterations growth with it, because the theoretical defaults decrease to zero as the latter increases. Therefore, to improve the numerical performances of Algorithm 2.5, we use constant values of the default θ. So the number of iterations and the elapsed time are significantly decreased, namely when $\theta = 0.92$. These new results are stated in Table 2.

ExCQSDO3. Consider the CQSDO of large size ($n = 2m$) with

$$\mathcal{Q}(X) = QXQ \text{ with } Q \in \mathcal{S}^n_+,$$

$$Q[i,j] = \begin{cases} 3 & \text{if } i = j \\ 1 & \text{if } i = j+1 \text{ or } i = j-1, \\ 0 & \text{otherwise} \end{cases} \quad A_i[j,k] = \begin{cases} 1 & \text{if } j = k = i \text{ or } j = k = i + m \\ 0 & \text{otherwise} \end{cases}$$

$$b[i] = 2,\ i = 1, \ldots, m, \quad C = \sum_{i=1}^{m} y^0 A_i + Z^0 - \mathcal{Q}(X^0).$$

The triplet starting point is taken as

$$X^0[i,j] = \begin{cases} \frac{11}{6} & \text{if } i = j = 1, \ldots, m \\ 1.34 & \text{if } i = j = m+1, \ldots, n, \\ 0 & \text{if not} \end{cases} \quad Z^0[i,j] = \begin{cases} 1.43 & \text{if } i = j = 1, \ldots, m \\ 2.22 & \text{if } i = j = m+1, \ldots, n \\ 0 & \text{if not} \end{cases}$$

$$y^0[i] = \frac{-7}{12}, \quad i = 1, \ldots, m.$$

The obtained approximated primal-dual optimal solution is given by:

$$X^\star[i,j] = \begin{cases} 1.93 & \text{if } i = j = 1, \ldots, m \\ 1.23 & \text{if } i = j = m+1, \ldots, n, \\ 0 & \text{if not.} \end{cases}$$

$$Z^\star = 0_{n\times n}, \quad y^\star[i] = 1.25, \quad i = 1, \ldots, m.$$

The details of our obtained numerical results for different size of (m, n) are summarized in Table 3.

TABLE 3. Number of iterations and CPU time for the *ExCQSDO3* with different values of θ.

↓ Size/θ →	$\theta = 0.5$		$\theta = 0.88$		$\theta = 0.92$		$\theta = 1/4\sqrt{n+1}$			$\theta = 1/2\sqrt{n}$		
(m, n)	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	δ_1	Iter	CPU	δ_2
$(5, 10)$	22	1.945	7	0.673	6	0.54	9	1.005	0.0928	87	8.146	0.0463
$(10, 20)$	23	48.269	8	12.844	7	10.54	7	6.320	0.1312	132	145.72	0.065
$(25, 50)$	24	2302.05	8	653.76	7	618.06	–	–	0.2075	–	–	0.1036

5. Conclusion and future work

In this paper, we presented a feasible primal-dual short-step IPA for solving general form of monotone CQSDO problems based on Nesterov–Todd full-Newton steps. Moreover, we obtained its polynomial time complexity, namely, $\mathcal{O}\big(\sqrt{n+1}\log\big(\frac{n}{\epsilon}\big)\big)$ which is the currently best known complexity for such short-step methods. The efficiency of the algorithm is proven by reporting some numerical results. The analysis of the algorithm with other search directions remains a good topic of research in the future.

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Data availability statement

The research data associated with this article are included in the article.

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