RAIRO-Oper. Res. 59 (2025) 3729–3747
https://doi.org/10.1051/ro/2025124

RAIRO Operations Research
www.rairo-ro.org

ON λ-FORMAN-RICCI CURVATURE OF NETWORKS

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Abstract. Several discrete versions of Ricci curvature have been proposed since the geometrical properties of a network are used to understand important information associated with it. In this paper we obtain the main properties of the λ-Forman-Ricci curvature, a concept that generalizes and integrates the Forman-Ricci curvature and the augmented Forman-Ricci curvature. We show that this definition captures the essence of Ricci curvature in Riemannian manifolds, by proving discrete analogues of important results in geometry. Also, we study the integral λ-Forman-Ricci curvature, obtaining a kind of Gauss–Bonnet formula, and we study this integral curvature in the context of random networks.

Mathematics Subject Classification. 05C07, 05C38, 05C80.

Received August 25, 2024. Accepted September 8, 2025.

1. Introduction

The seminal work of Kanai [30] shows that some important geometric properties (*e.g.* isoperimetric inequalities, Liouville property, growth rate) of Riemannian manifolds are preserved under rough isometries (a weak version of bi-Lipschitz maps). In order to prove the stability of these properties, the researchers associate to each manifold with a lower bound of its Ricci curvature a (rough isometric) graph (see [19–23, 26, 27, 42]).

Ricci curvature is a basic tool in Riemannian geometry that characterizes the local geometric properties of Riemannian manifolds by relating the local rate of volume growth to geodesic dispersion. As Ricci curvature encapsulates essential information in geometry and related fields (*e.g.* Einstein's field equations in general relativity), there have been multiple attempts to profitably extend its definition to other domains. Thus, several discrete versions of Ricci curvature have been proposed (see [11, 17, 33, 34]), since the geometrical properties of a network are used to understand important information associated with it.

Gromov hyperbolicity on graphs is a discrete version of the concept of negative curvature. It has been basic to understanding the structural properties of a network, as this conceptualisation is related to the notion of the network backbone and its communication highways (see [31, 39, 41, 45]). Therefore, the study of Gromov hyperbolicity on graphs has become a subject of increasing interest (see, *e.g.* [5, 7, 36, 39, 40, 50]).

Keywords. λ-Forman-Ricci curvature – integral λ-Forman-Ricci curvature – random networks.

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The two initial discrete curvature versions are due to Ollivier (see [19,20]) and Forman (see [17]). Ollivier-Ricci curvature is a discrete analogue of Ricci curvature for graphs, based on optimal transport theory: the key idea in its definition is to use the Wasserstein distance. It provides insights into the structural properties of networks, such as expansion properties and community structures. Forman-Ricci curvature is another discrete analogue of Ricci curvature, but it is based on a combinatorial approach, using discrete differential forms: it follows from Bochner's method, which in the classical case allows to write the Laplacian as the sum of a Laplacian-like term and a term that depends on the Ricci curvature. Forman-Ricci curvature is useful for studying properties related to the graph's topological and geometric structure.

It is important remark that the Augmented Forman-Ricci curvature is an approach to the original conceptualisation of Forman-Ricci (see [17]). Variants of this concept have been previously studied in [49]. The Augmented Forman-Ricci curvature show relationships between community structure and curvature clusters in a network (see [6,28,46,47,51]). These curvature provides a tool for studying dynamic effects, resulting from information flow, in complex networks. Regions of curvature (high or low) describing the growth or contraction behaviour of the network (see [38,48]).

To define these curvatures, we always consider graphs which are connected and locally finite (*i.e.* each vertex has a finite number of neighbors). The vertex set of a graph G is denoted by $V(G)$, and the edge set is denoted by $E(G)$. If $u \in V(G)$, we denote by $N(u)$ the set of neighbours of u, *i.e.* $N(u) = \{v \in V(G) : uv \in E(G)\}$ and define the degree of u, $\deg(u)$, equals to the cardinality of $N(u)$.

The *Forman-Ricci curvature* and the *augmented Forman-Ricci curvature* of an edge uv of the graph G are defined, respectively, as

$$F(uv) := 4 - \deg(u) - \deg(v),$$
$$F_3(uv) := F(uv) + 3 \cdot \sharp\{\text{triangles} \ni uv\},$$

where $\sharp\{\text{triangles} \ni uv\}$ denotes the number of triangles (3-cycles) in which $uv \in E(G)$ is contained (see [16,17,37,38]).

The augmented Forman-Ricci curvature is a measure that is assigned to the edges of a graph to capture information about the local structure of the network, such as how "curved" or "flat" it is around a connection. Triangles are the basic units of local cohesion in networks, and play an important role in many structural properties, such as robustness, cohesion, clustering coefficient, information propagation, and Gromov hyperbolicity.

The Forman-Ricci curvature, in its most basic version, is calculated by taking into account only the edges and their end nodes, and the triangles that contain them. Thus, if an edge is in many triangles, it is interpreted as that part of the graph having negative curvature (there is a lot of local connectivity). If it is in few or no triangles, the curvature tends to be "positive" or flat. Also, in most real networks, triangles already capture much of the local connectivity, and so are usually sufficient for practical analysis, they are the simplest forms of closed local structure. Moreover, triangles are efficient to compute and provide a lot of useful information about the local geometry of a network.

In this paper we obtain the main properties of the λ-*Forman-Ricci curvature*. This concept allows to study in a unified way both the Forman-Ricci curvature and the augmented Forman-Ricci curvature, when $\lambda = 0$ and $\lambda = 3$, respectively. For a fixed $\lambda \in \mathbb{R}$, the λ-*Forman-Ricci curvature* of an edge uv of the graph G is defined as

$$F_\lambda(uv) := F(uv) + \lambda \cdot \sharp\{\text{triangles} \ni uv\}.$$

An interesting characteristic of the definition of this discrete curvature is that it is local, as the Ricci curvature in Riemannian geometry.

We show that this definition captures the essence of Ricci curvature in Riemannian manifolds, by proving discrete analogues of important results in Riemannian geometry, as Bonnet–Myers Theorem (see Thm. 3.2) and the theorem that states that Riemannian manifolds with non-negative Ricci curvature have polynomial volume growth (see Thm. 3.8).

We study also the integral λ-Forman-Ricci curvature, obtaining a kind of Gauss–Bonnet formula (see Prop. 4.9), and we study this integral curvature in the context of random networks.

2. On λ-Forman-Ricci curvature

Our first result allows to obtain closed formulas for the values of the λ-Forman-Ricci curvature on many families of classical graphs.

Proposition 2.1. *Given $\lambda \in \mathbb{R}$, we have:*

(1) $F_\lambda = 6 - 2n + \lambda(n-2)$ *on the complete graph* K_n *for any* $n \geq 2$.
(2) $F_\lambda = 4 - n - m$ *on the complete bipartite graph* $K_{m,n}$ *for any* $m, n \geq 1$.
(3) $F_\lambda = 4 - n$ *on the star graph* S_n *for any* $n \geq 2$.
(4) $F_\lambda = \lambda$ *on* C_3 *and* $F_\lambda = 0$ *on the cycle graph* C_n *for any* $n \geq 4$.
(5) $F_\lambda(e) = 1$ *if* e *is a pendant edge in the path graph* P_n *and* $F_\lambda(e) = 0$ *otherwise, for any* $n \geq 3$.
(6) $F_\lambda(e) = 2 - n + 2\lambda$ *if* e *is an edge incident to the central vertex* (*the vertex with degree* $n-1$) *in the wheel graph* W_n *and* $F_\lambda(e) = -2 + \lambda$ *otherwise, for any* $n \geq 4$.
(7) $F_\lambda = -2$ *on the Petersen graph.*
(8) $F_\lambda = 4 - 2n$ *on the hypercube graph* Q_n.

Proof. For each $n \geq 2$, we have

$$\begin{aligned} F_\lambda(uv) &= 4 - \deg(u) - \deg(v) + \lambda \cdot \sharp\{\text{triangles} \ni uv\} \\ &= 4 - (n-1) - (n-1) + \lambda(n-2) \\ &= 6 - 2n + \lambda(n-2) \end{aligned}$$

for every $uv \in E(K_n)$.

For each $m, n \geq 1$, we have

$$\begin{aligned} F_\lambda(uv) &= 4 - \deg(u) - \deg(v) + \lambda \cdot \sharp\{\text{triangles} \ni uv\} \\ &= 4 - n - m \end{aligned}$$

for every $uv \in E(K_{m,n})$.

Since the star graph with n vertices is $S_n = K_{1,n-1}$, we have $F_\lambda = 4 - n$ on S_n for every $n \geq 2$.

Note that $\sharp\{\text{triangles} \ni uv\} = 1$ for every $uv \in E(C_3)$ and $\sharp\{\text{triangles} \ni uv\} = 0$ for every $uv \in E(C_3)$ and $n > 3$. Since C_n is a 2-regular graph, we have $F_\lambda = \lambda$ on C_3 and $F_\lambda = 0$ on C_n for every $n > 3$.

Consider the path graph P_n. If e is a pendant edge in P_n, then $F_\lambda(e) = 4 - 1 - 2 = 1$; otherwise, $F_\lambda(e) = 4 - 2 - 2 = 0$.

Consider the wheel graph W_n. If e is an edge incident to the central vertex, then $F_\lambda(e) = 4 - 3 - (n-1) + 2\lambda = 2 - n + 2\lambda$; otherwise, $F_\lambda(e) = 4 - 3 - 3 + \lambda = -2 + \lambda$.

Consider the Petersen graph P. Since P is a triangle-free 3-regular graph, we have $F_\lambda(uv) = 4 - 3 - 3 = -2$ for every $uv \in E(P)$.

The hypercube Q_n is a triangle-free graph and every vertex has degree n. Then, $F_\lambda = 4 - 2n$. □

Proposition 2.1 has the following consequence.

Corollary 2.2. $F_\lambda = 0$ *on* C_n *for every* $n > 3$ *and* $\lambda \in \mathbb{R}$, *and* $F_\lambda = 0$ *on* K_n *for every* $n \geq 3$ *and* $\lambda = \frac{2n-6}{n-2}$.

It is well-known that for many purposes a δ-regular tree ($\delta \geq 3$) is a good model for the unit disk with its Poincaré metric with constant negative curvature (see *e.g.* [8]). As a consequence of the following result, we obtain that these trees also have constant negative λ-Forman-Ricci curvature.

Proposition 2.3. *Let* G *be a triangle-free graph and* $\lambda \in \mathbb{R}$.

(1) *If* G *has minimum degree* δ, *then* $F_\lambda \leq 4 - 2\delta$.
(2) *If* G *has* n *vertices, then* $F_\lambda \geq 4 - n$.

Proof. Since G is a triangle-free graph, we have

$$F_\lambda(uv) = 4 - \deg(u) - \deg(v) \leq 4 - 2\delta,$$

for every $uv \in E(G)$. Thus, $F_\lambda \leq 4 - 2\delta$.

Moreover, if G is a triangle-free graph with n vertices, then

$$\deg(u) + \deg(v) \leq n$$

for every $uv \in E(G)$. Thus, we have $4 - n \leq F_\lambda$. $\square$

The argument in the proof of Proposition 2.3 has the following consequence.

Corollary 2.4. *Let G be a δ-regular tree ($\delta \geq 3$) and $\lambda \in \mathbb{R}$. Then, G has constant negative λ-Forman-Ricci curvature: $F_\lambda = 4 - 2\delta$.*

Proposition 2.5. *Assume that $F_\lambda \geq k$ on a graph G for some real constants λ and k.*

If $0 \leq \lambda < 2$, then

$$\deg(u) + \deg(v) \leq \frac{2(4 - \lambda - k)}{2 - \lambda}$$

for every $uv \in E(G)$.

If $\lambda > 2$, then

$$\deg(u) + \deg(v) \geq \frac{2(k + \lambda - 4)}{\lambda - 2}$$

for every $uv \in E(G)$.

Proof. We have

$$F_\lambda(uv) = 4 - \deg(u) - \deg(v) + \lambda \cdot \sharp\,\{\text{triangles} \ni uv\} \geq k$$

for every $uv \in E(G)$. Since

$$\sharp\{\text{triangles} \ni uv\} \leq \min\{\deg(u), \deg(v)\} - 1 \leq \frac{1}{2}(\deg(u) + \deg(v)) - 1,$$

we conclude

$$\begin{aligned} 4 - \lambda - \left(1 - \frac{\lambda}{2}\right)(\deg(u) + \deg(v)) &= 4 - \deg(u) - \deg(v) + \frac{\lambda}{2}(\deg(u) + \deg(v)) - \lambda \\ &\geq 4 - \deg(u) - \deg(v) + \lambda \cdot \sharp\,\{\text{triangles} \ni uv\} \\ &\geq k \end{aligned}$$

for every $uv \in E(G)$, since $\lambda \geq 0$. Hence,

$$4 - \lambda - k \geq \frac{2 - \lambda}{2}(\deg(u) + \deg(v))$$

for every $uv \in E(G)$.

If $0 \leq \lambda < 2$, then

$$\deg(u) + \deg(v) \leq \frac{2(4 - \lambda - k)}{2 - \lambda}$$

for every $uv \in E(G)$.

If $\lambda > 2$, then

$$\deg(u) + \deg(v) \geq \frac{2(k + \lambda - 4)}{\lambda - 2}$$

for every $uv \in E(G)$. $\square$

Definition 2.6. An *automorphism* on a graph G is a bijection $\varphi : V(G) \to V(G)$ such that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(G)$.

Definition 2.7. An *edge-transitive graph* is a graph G such that, given any two edges u_1v_1 and u_2v_2 of G, there is an automorphism of G that maps u_1v_1 to u_2v_2. In other words, a graph is edge-transitive if its automorphism group acts transitively on its edges.

Proposition 2.8. *Every edge-transitive graph has constant Forman-Ricci curvature, augmented Forman-Ricci curvature and λ-Forman-Ricci curvature for every $\lambda \in \mathbb{R}$.*

Proof. Consider an edge-transitive graph G and $uv, u'v' \in E(G)$. Let T be an isomorphism of G with $T(uv) = T(u'v')$. Then, $\deg(u') + \deg(v') = \deg(u) + \deg(v)$ and $\sharp\{\text{triangles} \ni u'v'\} = \sharp\{\text{triangles} \ni uv\}$. Hence, G has constant λ-Forman-Ricci curvature for every $\lambda \in \mathbb{R}$. The other statements hold since the 0-Forman-Ricci curvature is the Forman-Ricci curvature, and the 3-Forman-Ricci curvature is the augmented Forman-Ricci curvature. □

Examples of edge-transitive graphs are the complete graphs K_n, the complete bipartite graphs $K_{m,n}$ (in particular, the star graphs $S_n = K_{1,n-1}$) and the cycle graph C_n.

Definition 2.9. We say that a triangle T in a graph G is *weakly isolated* if there exist adjacent vertices $u \in T$ and $u_1 \notin T$ such that u_1u does not belong to any triangle. A triangle T in a graph G is *isolated* if $T \cap T' = \emptyset$ for any triangle $T' \neq T$. It is clear that an isolated triangle is weakly isolated.

In particular, if a graph G has just a triangle T, then T is an isolated triangle. The following result characterizes the graphs with a weakly isolated triangle and $F_\lambda = 0$.

Proposition 2.10. *Let G be a connected graph with a weakly isolated triangle T and $\lambda \in \mathbb{R}$. If $F_\lambda = 0$, then $\lambda = 0$ or $\lambda = 2$.*

If $\lambda = 0$, then G is a cycle graph with three vertices.

If $\lambda = 2$ and T is isolated, then G is a cycle graph with three vertices and a pendant edge attached to each vertex in the cycle.

Proof. Let u, v, w be the vertices of a weakly isolated triangle T in G. Thus,

$$\sharp\{\text{triangles} \ni uv\} = \sharp\{\text{triangles} \ni uw\} = \sharp\{\text{triangles} \ni vw\} = 1$$

and so,

$$\begin{aligned} 0 = F_\lambda(uv) &= 4 - \deg(u) - \deg(v) + \lambda, \\ 0 = F_\lambda(uw) &= 4 - \deg(u) - \deg(w) + \lambda, \\ 0 = F_\lambda(vw) &= 4 - \deg(v) - \deg(w) + \lambda. \end{aligned}$$

Hence, $\deg(u) = \deg(v) = \deg(w)$ and

$$\deg(u) + \deg(v) + \deg(w) = 6 + \frac{3}{2}\lambda.$$

Therefore,

$$\deg(u) = \deg(v) = \deg(w) = 2 + \frac{1}{2}\lambda.$$

Since $u \in T$, we have $\deg(u) \geq 2$ and so, $\lambda \geq 0$. Since $\deg(u) = 2 + \lambda/2$ is a positive integer, λ is an even integer.

Seeking for a contradiction assume that $\lambda \geq 4$. Thus, $\deg(u) \geq 4$. Since T is a weakly isolated triangle, we can assume that there exists $u_1 \notin T$ such that $u_1u \in E(G)$ does not belong to any triangle and so,

$$0 = F_\lambda(u_1u) = 4 - \deg(u_1) - \deg(u) \leq 4 - 1 - 4 = -1,$$

a contradiction. Consequently, we conclude that $\lambda = 0$ or $\lambda = 2$.

Assume that $\lambda = 0$. Thus, $\deg(u) = \deg(v) = \deg(w) = 2$ and so, G is a cycle graph with three vertices.

Assume that $\lambda = 2$ and T is isolated. Thus, $\deg(u) = \deg(v) = \deg(w) = 3$ and there exist $u_1, v_1, w_1 \in V(G)$ with $u_1u, v_1v, w_1w \in E(G)$. Since T is isolated, we have

$$\begin{aligned} 0 &= F_\lambda(u_1u) = 4 - \deg(u_1) - \deg(u), \\ 0 &= F_\lambda(v_1v) = 4 - \deg(v_1) - \deg(v), \\ 0 &= F_\lambda(w_1w) = 4 - \deg(w_1) - \deg(w), \end{aligned}$$

and we conclude $\deg(u_1) = \deg(v_1) = \deg(w_1) = 1$. Hence, G is a cycle graph with three vertices and a pendant edge attached to each vertex in the cycle. □

Remark 2.11. Note that to require a graph to be connected is not a restrictive hypothesis, since the results on connected graphs can be applied to each connected component.

Proposition 2.12. *Let G be a graph with minimum degree δ and maximum degree Δ, and $\lambda \in \mathbb{R}$.*

(1) *If $\lambda \geq 0$, then $F_\lambda \leq 4 - 2\delta + \lambda(\Delta - 1)$, and the equality is attained for the complete graph.*
(2) *If $\lambda < 0$, then $F_\lambda \leq 4 - 2\delta$, and the equality is attained for the δ-regular tree.*

Proof. We have $\deg(u) \geq \delta$ for every $u \in V(G)$ and

$$\sharp\,\{\text{triangles} \ni uv\} \leq \deg(u) - 1 \leq \Delta - 1$$

for every $uv \in E(G)$.

If $\lambda \geq 0$, then

$$\begin{aligned} F_\lambda(uv) &= 4 - \deg(u) - \deg(v) + \lambda \cdot \sharp\,\{\text{triangles} \ni uv\} \\ &\leq 4 - 2\delta + \lambda(\Delta - 1) \end{aligned}$$

for every $uv \in E(G)$.

By Proposition 2.1, the equality in this inequality is attained for the complete graphs.

If $\lambda < 0$, then

$$\begin{aligned} F_\lambda(uv) &= 4 - \deg(u) - \deg(v) + \lambda \cdot \sharp\,\{\text{triangles} \ni uv\} \\ &\leq 4 - \deg(u) - \deg(v) \leq 4 - 2\delta \end{aligned}$$

for every $uv \in E(G)$.

Since the δ-regular tree T_δ is a triangle-free graph, the equality in this inequality is attained for T_δ. □

3. Diameter and growth rate

In order to compute distances in graphs, we consider that every edge has length 1.

Definition 3.1. The *distance* $d_G(x, y)$ between two vertices x and y, in a connected graph G, is the minimum length of the paths joining x and y in G. The *diameter* of G, denoted by $\operatorname{diam}(G)$, is the maximum distance among all pairs of vertices of G.

An important result in Riemannian geometry is the Bonnet–Myers Theorem: If (M, g) is a complete connected Riemannian manifold with Ricci curvature bounded below by $1/r^2 > 0$, then M is a compact manifold with diameter at most πr. This implies that a manifold with a positive lower bound of its Ricci curvature has finite hyperbolicity constant. We have similar results for F_λ.

Theorem 3.2. *Assume that $F_\lambda \geq k$ on a connected graph G for some real constants satisfying either $\lambda \leq 0$ and $k > 0$ or $0 \leq \lambda < 2$ and $k > \lambda$. Then* $\operatorname{diam}(G) \leq 2$.

Proof. Assume first that $0 \le \lambda < 2$ and $k > \lambda$. Proposition 2.5 gives

$$\deg(u) + \deg(v) \le \frac{2(4-\lambda-k)}{2-\lambda} < \frac{2(4-2\lambda)}{2-\lambda} = 4$$

for every $uv \in E(G)$.

Assume now that $\lambda \le 0$ and $k > 0$. Then,

$$\begin{aligned} 0 < k \le F_\lambda(uv) &= 4 - \deg(u) - \deg(v) + \lambda \cdot \sharp \{\text{triangles} \ni uv\} \\ &\le 4 - \deg(u) - \deg(v), \\ \deg(u) &+ \deg(v) < 4, \end{aligned}$$

for every $uv \in E(G)$.

Consequently, we have in both cases $\deg(u) + \deg(v) = 2$ or $\deg(u) + \deg(v) = 3$.

If there exists $uv \in E(G)$ with $\deg(u) + \deg(v) = 2$, then $\deg(u) = \deg(v) = 1$ and so, G is the path graph P_2.

Otherwise, $\deg(u) + \deg(v) = 3$ for every $uv \in E(G)$. Therefore, $\deg(u) = 1$ and $\deg(v) = 2$ (or *vice versa*). Hence, G is the path graph P_3.

In both cases, $\operatorname{diam}(G) \le 2$. □

Definition 3.3. We say that a graph is *t-hyperbolic*, with $t \ge 0$, if any side in every geodesic triangle is contained in the t-neighborhood of the union of the other two sides. We define the *hyperbolicity constant* $\delta^*(G)$ of the graph G as the infimum of the constants $t \ge 0$ such that G is t-hyperbolic. Hyperbolicity on graphs is a discrete version of the concept of negative curvature.

Note that if $F_0 = 0$ on a connected graph G, then G is either a cycle graph with $n \ge 3$ vertices or a star graph with 4 vertices and so, we have either $\delta^*(G) = n/4$ or $\delta^*(G) = 0$.

The argument in the proof of Theorem 3.2 allows to relate λ-Forman-Ricci curvature and hyperbolicity.

Proposition 3.4. *If $F_\lambda \ge k$ on a graph G for some real constants satisfying either $\lambda \le 0$ and $k > 0$ or $0 \le \lambda < 2$ and $k > \lambda$, then $\delta^*(G) = 0$.*

Proof. Let G_0 be a fixed connected component of G. The argument in the proof of Theorem 3.2 gives that G_0 is a path graph. Since every tree has hyperbolicity constant equal to 0, we have $\delta^*(G_0) = 0$. Since every connected component of G has zero hyperbolicity constant, we conclude that $\delta^*(G) = 0$. □

Definition 3.5. Given any graph G, $u \in V(G)$ and $\rho > 0$, define

$$B_G(u,\rho) = \{v \in V(G) : \, d_G(v,u) < \rho\}, \quad \operatorname{vol}(B_G(u,\rho)) = \sharp\, B_G(u,\rho).$$

Note that $\operatorname{vol}\big(B_G(u,\rho)\big) = 1$ for every $0 < \rho \le 1$, and $\operatorname{vol}\big(B_G(u,\rho)\big) = 1 + \deg(u)$ for every $1 < \rho \le 2$.

Definition 3.6. The *polynomial growth order* of a graph G as

$$\inf \left\{ k \ge 0 : \, \limsup_{\rho\to\infty} \rho^{-k} \operatorname{vol}(B_G(u,\rho)) < \infty \right\}.$$

One can check that this definition does not depend on the choice of $u \in V(G)$.

Definition 3.7. We say that G has (at most) *polynomial volume growth* if its polynomial growth order is finite.

Polynomial volume growth is an interesting topic in Riemannian geometry, see *e.g.* [20, 30]. It is known that for complete Riemannian manifolds non-negative Ricci curvature implies (at most) polynomial volume growth. We obtain a similar result for F_λ.

Theorem 3.8. *Let G be a graph and $\lambda < 4/3$ with $F_\lambda \geq 0$. Then, G is a finite graph or its polynomial growth order is* 1.

Proof. Assume that $\sharp\{\text{triangles} \ni uv\} = r$ for some $uv \in E(G)$. Then, $\deg(u), \deg(v) \geq r+1$ and we obtain

$$\begin{aligned} 0 \leq F_\lambda(uv) &= 4 - \deg(u) - \deg(v) + \lambda r \\ &< 4 - (r+1) - (r+1) + \frac{4}{3} r, \\ \frac{2}{3} r &< 2, \\ r &\leq 2. \end{aligned}$$

(1) Assume first that $r = 2$. Thus, $\deg(u), \deg(v) \geq 3$.
Seeking for a contradiction assume that $\deg(u) > 3$. Then,

$$\begin{aligned} 0 \leq F_\lambda(uv) &= 4 - \deg(u) - \deg(v) + 2\lambda \\ &< 4 - 4 - 3 + \frac{8}{3} = -\frac{1}{3}, \end{aligned}$$

a contradiction. Hence, we conclude that $\deg(u) = 3$. The same argument gives that $\deg(v) = 3$.
Consider $w \in N(u) \cap N(v)$. Then, $1 \leq \sharp\{\text{triangles} \ni uw\} \leq 2$.

(1.1) Assume first that $\sharp\{\text{triangles} \ni uw\} = 1$. Seeking for a contradiction assume that $\deg(w) > 2$. Thus,

$$\begin{aligned} 0 \leq F_\lambda(uw) &= 4 - \deg(u) - \deg(w) + \lambda \\ &< 4 - 3 - 3 + \frac{4}{3} = -\frac{2}{3}, \end{aligned}$$

a contradiction. Therefore, $\deg(w) = 2$. Then, G is a complete graph with 4 vertices without an edge and so, it is a finite graph.

(1.2) Assume that $\sharp\{\text{triangles} \ni uw\} = 2$. If $z = N(u) \cap N(v) \setminus \{w\}$, then $zw \in E(G)$ (u, v, w, z is a clique).
Seeking for a contradiction assume that $\deg(w) > 3$. Therefore,

$$\begin{aligned} 0 \leq F_\lambda(uw) &= 4 - \deg(u) - \deg(w) + 2\lambda \\ &< 4 - 3 - 4 + \frac{8}{3} = -\frac{1}{3}, \end{aligned}$$

a contradiction. Thus, $\deg(w) = 3$. The same argument gives that $\deg(z) = 3$. Then, G is a complete graph with 4 vertices and so, it is a finite graph.

(2) Assume now that $r = 1$, and consider $w = N(u) \cap N(v)$. If $\sharp\{\text{triangles} \ni uw\} = 2$, then the previous argument gives that G is a finite graph. Hence, we can assume that $\sharp\{\text{triangles} \ni uw\} = 1$. Thus, $\deg(u), \deg(v), \deg(w) \geq 2$.

(2.1) Assume that $\deg(u) \geq 3$ or $\deg(v) \geq 3$ or $\deg(w) \geq 3$. Without loss of generality we can assume that $\deg(u) \geq 3$. Therefore,

$$\begin{aligned} 0 \leq F_\lambda(uv) &= 4 - \deg(u) - \deg(v) + \lambda \\ &< 4 - 3 - 2 + \frac{4}{3} = \frac{1}{3} \end{aligned}$$

and so, $\deg(u) = 3$ and $\deg(v) = 2$. The same argument gives $\deg(w) = 2$. If $u' = N(u) \setminus \{v, w\}$, then $\sharp\{\text{triangles} \ni uu'\} = 0$, since otherwise $\deg(u) \geq 4$ or $\deg(v) \geq 3$ or $\deg(w) \geq 3$, a contradiction. Hence,

$$\begin{aligned} 0 \leq F_\lambda(uu') &= 4 - \deg(u) - \deg(u') \\ &\leq 4 - 3 - 1 = 0 \end{aligned}$$

and hence, $\deg(u') = 1$. Since $\deg(u) = 3$, $\deg(v) = \deg(w) = 2$ and $\deg(u') = 1$; we conclude that G is a triangle with an attached edge at u and so, it is a finite graph.

(2.2) Assume that $\deg(u) = \deg(v) = \deg(w) = 2$. Then, G is the cycle graph with 3 vertices, and it is a finite graph.

(3) Finally, assume that $r = 0$. If $\sharp\{\text{triangles} \ni zw\} > 0$ for some edge $zw \in E(G)$, then the previous argument gives that G is a finite graph. Hence, we can assume that $\sharp\{\text{triangles} \ni zw\} = 0$ for every $zw \in E(G)$ and so,

$$0 \le F_\lambda(zw) = 4 - \deg(z) - \deg(w),$$
$$\deg(z) + \deg(w) \le 4$$

for every $zw \in E(G)$. Consequently, $\deg(z) \le 3$ for every $z \in V(G)$.

(3.1) If $\deg(z) = 3$ for some $z \in V(G)$, then $\deg(w) = 1$ for every $w \in N(z)$ and so, G is a star graph with 4 vertices and it is a finite graph.

(3.2) Therefore, we can assume that $\deg(z) \le 2$ for every $z \in V(G)$.

(3.2.1) If $\deg(z) = 2$ for every $z \in V(G)$, then G is either a cycle graph or isomorphic to the Cayley graph of the group $\mathbb{Z}$. In the first case, G is a finite graph. In the second one, $\sharp\, B_G(u, n) = 2n - 1$ for every $u \in V(G)$ and $n \in \mathbb{Z}^+$, and the polynomial growth order of G is 1.

(3.2.2) If there is a single vertex with degree 1, then G is an infinite tree with a vertex of degree 1 and infinitely many vertices with degree 2. Thus, $n \le \sharp\, B_G(u, n) \le 2n - 1$ for every $u \in V(G)$ and $n \in \mathbb{Z}^+$, and the polynomial growth order of G is 1.

(3.2.3) If there are at least two vertices with degree 1, then G is a path graph and it is a finite graph.

□

4. Integral curvature

A topological descriptor is a single number that represents a chemical structure in graph-theoretical terms *via* the molecular graph, they play a significant role in mathematical chemistry (see *e.g.* [9, 25]).

Definition 4.1. The *first and second Zagreb indices*, denoted by M_1 and M_2, respectively, and introduced by Gutman *et al.* in 1972 (see [24]), are defined as

$$M_1(G) = \sum_{u \in V(G)} \big(\deg(u) + \deg(v)\big), \quad M_2(G) = \sum_{uv \in E(G)} \deg(u)\deg(v).$$

Proposition 4.2. *Assume that $F_\lambda \le k$ on a graph G for some real constants λ and k. If $\lambda \le 0$, then*

$$\deg(u) + \deg(v) \ge \frac{2(4 - \lambda - k)}{2 - \lambda}$$

for every $uv \in E(G)$.

Proof. Since $\lambda \le 0$, we have

$$\sharp\,\{\text{triangles} \ni uv\} \le \min\{\deg(u), \deg(v)\} - 1 \le \frac{1}{2}(\deg(u) + \deg(v)) - 1,$$
$$\lambda \cdot \sharp\,\{\text{triangles} \ni uv\} \ge \frac{\lambda}{2}(\deg(u) + \deg(v)) - \lambda,$$

and we conclude

$$\begin{aligned} 4 - \lambda - \left(1 - \frac{\lambda}{2}\right)(\deg(u) + \deg(v)) &= 4 - \deg(u) - \deg(v) + \frac{\lambda}{2}(\deg(u) + \deg(v)) - \lambda \\ &\le 4 - \deg(u) - \deg(v) + \lambda \cdot \sharp\{\text{triangles} \ni uv\} \\ &\le k \end{aligned}$$

for every $uv \in E(G)$. Hence,

$$4-\lambda-k \le \frac{2-\lambda}{2}\left(\deg(u)+\deg(v)\right),$$
$$\deg(u)+\deg(v) \ge \frac{2(4-\lambda-k)}{2-\lambda},$$

for every $uv \in E(G)$. □

Proposition 4.2 has the following consequence, relating the λ-Forman-Ricci curvature and the first Zagreb index.

Corollary 4.3. *Assume that $F_\lambda \le k$ on a graph G with m edges for some real constants λ and k. If $\lambda \le 0$, then*

$$M_1(G) \ge \frac{2(4-\lambda-k)}{2-\lambda}\, m.$$

Proposition 2.5 has the following consequence.

Corollary 4.4. *Assume that $F_\lambda \ge k$ on a graph G with m edges for some real constants λ and k.*

If $0 \le \lambda < 2$, then

$$M_1(G) \le \frac{2(4-\lambda-k)}{2-\lambda}\, m.$$

If $\lambda > 2$, then

$$M_1(G) \ge \frac{2(k+\lambda-4)}{\lambda-2}\, m.$$

The following result relates the first Zagreb index of a graph G with the cardinality of the triangles in G.

Proposition 4.5. *If G is a graph with m edges, then*

$$M_1(G) \ge 2\, m\left(C_e(G)+1\right).$$

Proof. Since

$$\sharp\left\{\text{triangles} \ni uv\right\} \le \min\left\{\deg(u),\deg(v)\right\}-1 \le \frac{1}{2}\left(\deg(u)+\deg(v)\right)-1,$$

we have

$$\sum_{uv\in E(G)} \sharp\left\{\text{triangles} \ni uv\right\} \le \frac{1}{2}\sum_{uv\in E(G)}\left(\deg(u)+\deg(v)\right)-2\,m,$$
$$3\cdot\sharp\left\{\text{triangles in } G\right\} \le \frac{1}{2}\, M_1(G)-m,$$
$$m\, C_e(G) \le \frac{1}{2}\, M_1(G)-m.$$

□

Definition 4.6. The *integral Forman curvature* and the *integral λ-Forman curvature* of a finite graph G are

$$IF(G) = \sum_{uv\in E(G)} F(uv),$$
$$IF_\lambda(G) = \sum_{uv\in E(G)} F_\lambda(uv),$$

respectively.

Proposition 2.1 allows to compute the integral λ-Forman curvature for many important graphs.

Proposition 4.7. (1) *$IF_\lambda(K_n) = \frac{1}{2}\, n(n-1)(6-2n+\lambda(n-2))$ for the complete graph with $n \geq 2$.*
(2) *$IF_\lambda(K_{m,n}) = nm(4-n-m)$ for the complete bipartite graph with $m, n \geq 1$.*
(3) *$IF_\lambda(S_n) = (n-1)(4-n)$ for the star graph with $n \geq 2$.*
(4) *$IF_\lambda(C_3) = 3\lambda$ and $IF_\lambda(C_n) = 0$ for the cycle graph with $n \geq 4$.*
(5) *$IF_\lambda(P_n) = 2$ with $n \geq 2$.*
(6) *$IF_\lambda(W_n) = (n-1)(3\lambda - n)$ for the wheel graph with $n \geq 4$.*
(7) *$IF_\lambda(G) = -30$ for the Petersen graph.*
(8) *$IF_\lambda(Q_n) = (4-2n)2^n$ for the hypercube graph.*

An important property of a network is the presence of clusters and the number of them, see [6,29,47,51]. In this regard, it is essential to note that the more triangles a given edge contains, the greater the overlap of its neighbourhoods. This idea suggests an analogy with the notion of Ricci curvature in Riemannian geometry.

Definition 4.8. If G is a graph with m edges, let us define the *edge clustering* of an edge $uv \in E(G)$ by

$$C_e(uv) = \frac{1}{m} \cdot \sharp\{\text{triangles} \ni uv\}$$

and the *edge clustering coefficient* of G as

$$C_e(G) = \sum_{uv\in E(G)} C_e(uv) = \frac{1}{m} \sum_{uv\in E(G)} \sharp\,\{\text{triangles} \ni uv\} = \frac{3}{m} \cdot \sharp\,\{\text{triangles in } G\}\,.$$

Different types of edge clustering and vertex clustering are introduced and studied in several papers. An edge clustering coefficient is a measure of the density of an edge in a graph.

Gauss–Bonnet formula is a main result in Riemannian geometry. The following result can be viewed as a kind of Gauss–Bonnet formula.

Proposition 4.9. *If G is a graph with m edges, then*

$$IF_\lambda(G) = 4\,m - M_1(G) + \lambda\, m\, C_e(G).$$

Proof. We have

$$\begin{aligned} IF_\lambda(G) &= \sum_{uv\in E(G)} 4 \;-\; \sum_{uv\in E(G)} \big(\deg(u)+\deg(v)\big) \;+\; \lambda \sum_{uv\in E(G)} m\, C_e(uv) \\ &= 4\,m - M_1(G) + \lambda\, m\, C_e(G). \end{aligned}$$

□

Let A be the adjacency matrix of G, and denote by a_{ii} the i-entry in the main diagonal of A^2. We can write $IF_\lambda(G)$ in terms of the adjacency matrix.

Proposition 4.10. *If G is a graph with n vertices, then*

$$IF_\lambda(G) = 2\,\text{trace}(A^2) - \sum_{i=1}^{n} a_{ii}^2 + \frac{3}{2}\,\lambda\,\text{trace}\left(A^3\right).$$

Proof. Consider $V(G) = \{v_1, \dots, v_n\}$. It is well-known that $a_{ii} = \deg(v_i)$ for $i = 1, \dots, n$. If G has m edges, then

$$2\,m = \sum_{u\in V(G)} \deg(u) = \sum_{i=1}^{n} a_{ii} = \text{trace}\left(A^2\right),$$

$$M_1(G) = \sum_{uv\in E(G)} \big(\deg(u)+\deg(v)\big) = \sum_{u\in V(G)} \deg(u)^2 = \sum_{i=1}^{n} a_{ii}^2.$$

Also, it is known that

$$\mathrm{trace}(A^3) = 2 \cdot \sharp\{\text{triangles in } G\}.$$

Hence, Proposition 4.9

$$\begin{aligned} IF_\lambda(G) &= 4\,m - M_1(G) + \lambda\, m\, C_e(G) \\ &= 2\,\mathrm{trace}(A^2) - \sum_{i=1}^{n} a_{ii}^2 + \frac{3}{2}\,\lambda\,\mathrm{trace}(A^3). \end{aligned}$$

□

5. Integral λ-Forman-Ricci curvature of random networks

The computation of the Forman-Ricci curvature for some models of random networks has been recently reported in reference [16]. There, moreover, the λ-Forman-Ricci curvature with $\lambda = 2, 3$ was also reported for random networks. Thus, to complete our study, below we compute the integral λ-Forman-Ricci curvature $IF_\lambda(G)$ of three models of random networks:

(i) Erdös-Rényi (ER) networks, introduced by Solomonoff and Rapoport [43] and investigated later in great detail by Erdös and Rényi [12, 13],
(ii) random geometric (RG) graphs, introduced by Gilbert [18] and often used to study the structure and dynamics of spatially embedded complex systems [10, 35], and
(iii) bipartite random (BR) networks, see *e.g.* [32].

These random network models are defined as follows:

Definition 5.1. ER networks $G_{\text{ER}}(n, p)$ are formed by n vertices connected independently with probability $p \in [0, 1]$.

Definition 5.2. RG graphs $G_{\text{RG}}(n, r)$ consist of n vertices uniformly and independently distributed on the unit square, where two vertices are connected by an edge if their Euclidean distance is less or equal than the connection radius $r \in [0, \sqrt{2}]$.

Definition 5.3. BR networks $G_{\text{BR}}(n_{\text{A}}, n_{\text{B}}, p)$ are composed by two disjoint sets, set A and set B, with n_{A} and n_{B} vertices each such that there are no adjacent vertices within the same set, being $n = n_{\text{A}} + n_{\text{B}}$ the total number of vertices in the bipartite network. The vertices of the two sets are connected randomly with probability $p \in [0, 1]$.

We add that the computational study of $IF_\lambda(G)$ we perform below is justified by the random nature of the graph models we want to explore. Since a given parameter set $[(n, p), (n, r), \text{ or } (n_{\text{A}}, n_{\text{B}}, p)]$ represents an infinite-size ensemble of random (ER, RG, or BR) graphs, the computation of $IF_\lambda(G)$ on a single graph is irrelevant. In contrast, the computation of $IF_\lambda(G)$ on a large ensemble of random graphs, all characterized by the same parameter set, may provide useful average information about the full ensemble.

5.1. Integral λ-Forman-Ricci curvature of Erdös–Rényi random networks

We note from Proposition 4.9 that $IF_\lambda(G_{\text{ER}})$ is given in terms of $M_1(G_{\text{ER}})$ and $C_e(G_{\text{ER}})$, so we compute them first. In Figures 1a and 1b we present the averages of $M_1(G_{\text{ER}})$ and $C_e(G_{\text{ER}})$ as a function of the probability p of ER networks of sizes $n = \{50, 100, 200, 400, 800\}$. All averages are computed over ensembles of $10^6/n$ random networks where each ensemble is characterized by a fixed parameter pair (n, p). We observe that both quantities, $\langle M_1(G_{\text{ER}})\rangle$ and $\langle C_e(G_{\text{ER}})\rangle$, are increasing functions of both p and n, as expected. Here and below, $\langle \cdot \rangle$ means average over an ensemble of random graphs characterized by the same set of parameters.

Notice that in the dense limit, *i.e.* when $\langle \deg(G_{\text{ER}})\rangle \gg 1$, $\deg(u) \approx \deg(v) \approx \langle \deg(G_{\text{ER}})\rangle$ with

$$\langle \deg(G_{\text{ER}})\rangle = (n-1)p. \tag{1}$$

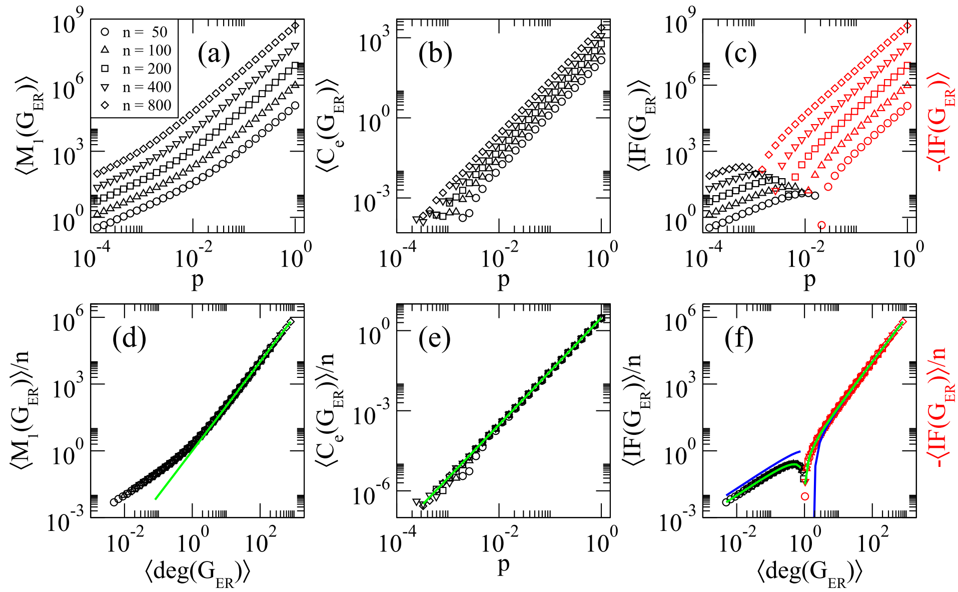

FIGURE 1. Average (a) $M_1(G)$, (b) $C_e(G)$, and (c) $IF(G)$ as a function of the probability p of Erdös-Rényi networks of size n. (d) $\langle M_1(G_{\mathrm{ER}})\rangle/n$ *vs.* $\langle \deg(G_{\mathrm{ER}})\rangle$, (e) $\langle C_e(G_{\mathrm{ER}})\rangle/n$ *vs.* p, and (f) $\langle IF(G_{\mathrm{ER}})\rangle/n$ *vs.* $\langle \deg(G_{\mathrm{ER}})\rangle$. All averages are computed over $10^6/n$ networks. Green lines in panels (d), (e), and (f) are, respectively, equations (2), (3), and (5). Blue lines in panel (f) are equation (4).

Thus, when $\langle \deg(G_{\mathrm{ER}})\rangle \gg 1$, $M_1(G_{\mathrm{ER}})$ can be approximated as

$$M_1(G_{\mathrm{ER}}) = \sum_{uv\in E(G_{\mathrm{ER}})} (\deg(u)+\deg(v)) \approx 2\sum_{uv\in E(G_{\mathrm{ER}})} \langle \deg(G_{\mathrm{ER}})\rangle \approx n\,\langle \deg(G_{\mathrm{ER}})\rangle^2,$$

which leads us to

$$\frac{\langle M_1(G_{\mathrm{ER}})\rangle}{n} \approx \langle \deg(G_{\mathrm{ER}})\rangle^2. \tag{2}$$

Indeed, equation (2) describes well the behavior of $\langle M_1(G_{\mathrm{ER}})\rangle$ in the dense limit, as can be seen in Figure 1d where we observe good correspondence between equation (2) (see the green line) and the numerical data already for $\langle \deg(G_{\mathrm{ER}})\rangle \geq 10$.

On the other hand, the ratio $\langle C_e(G_{\mathrm{ER}})\rangle/n$ can be written as

$$\frac{\langle C_e(G_{\mathrm{ER}})\rangle}{n} = 3p^2, \tag{3}$$

that we verify in Figure 1e.

It is important to stress that while the ratio $\langle M_1(G_{\mathrm{ER}})\rangle/n$ is a function of the average degree only, the ratio $\langle C_e(G_{\mathrm{ER}})\rangle/n$ is a function of the probability p only; see equations (2) and (3), respectively. That is, $\langle \deg(G_{\mathrm{ER}})\rangle$ is the scaling parameter of $\langle M_1(G_{\mathrm{ER}})\rangle/n$ (as well as of many other topological indices, see *e.g.* Refs. [1–4]) while p is the scaling parameter of $\langle C_e(G_{\mathrm{ER}})\rangle/n$. The fact that the ratios $\langle M_1(G_{\mathrm{ER}})\rangle/n$ and $\langle C_e(G_{\mathrm{ER}})\rangle/n$ accept different scaling parameters is of particular relevance here because since $IF_\lambda(G_{\mathrm{ER}})$ is a function of both $M_1(G_{\mathrm{ER}})$ and $C_e(G_{\mathrm{ER}})$, we do not expect to find a scaling parameter for $IF_\lambda(G_{\mathrm{ER}})$ itself.

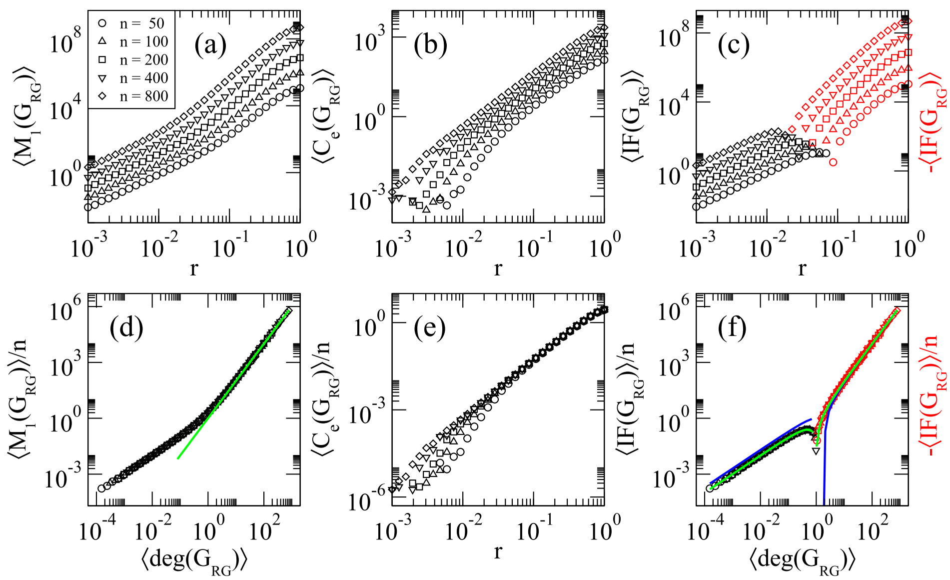

FIGURE 2. Average (a) $IF_\lambda(G)$, (b) $IF_{-3}(G)$, and (c) $IF_3(G)$ as a function of the probability p of Erdös-Rényi networks. In (a) $IF_\lambda(G)$ *vs.* p is shown for integer values of λ in the interval $[-5, 5]$ for networks of size $n = 200$. In (b, c) several sizes are considered as indicated in panel (b). All averages are computed over $10^6/n$ graphs.

However, in contrast with $IF_\lambda(G_{\rm ER})$, since $\langle IF(G_{\rm ER})\rangle$ is just a function of the vertex degrees, the ratio $\langle IF(G_{\rm ER})\rangle/n$ should scale with $\langle \deg(G_{\rm ER})\rangle$ as $\langle M_1(G_{\rm ER})\rangle/n$ does. Specifically, in the dense limit we can write

$$\frac{\langle IF(G_{\rm ER})\rangle}{n} \approx \langle \deg(G_{\rm ER})\rangle\left[2 - \langle \deg(G_{\rm ER})\rangle\right], \tag{4}$$

which demonstrates the scaling of $\langle IF(G_{\rm ER})\rangle/n$ with $\langle \deg(G_{\rm ER})\rangle$.

Then, in Figure 1c we present the average $IF(G_{\rm ER})$ as a function of the probability p of ER networks of different sizes n. Here since $\langle IF(G_{\rm ER})\rangle$ could be negative, we plot $\langle IF(G_{\rm ER})\rangle > 0$ in black and $-\langle IF(G_{\rm ER})\rangle > 0$ in red, so we can see all data together in a single log–log plot. In Figure 1f we now plot $\langle IF(G_{\rm ER})\rangle/n$ as a function of $\langle \deg(G_{\rm ER})\rangle$ and observe, as expected, that indeed the average degree is the scaling parameter of the ratio $\langle IF(G_{\rm ER})\rangle/n$; that is, all data for different n fall on top of the same curve. Moreover, in Figure 1f we also plot equation (4) (see the blue lines) and verify that it describes well the numerical data already for $\langle \deg(G_{\rm ER})\rangle \geq 10$. Evidently, equation (4) does not describe $\langle IF(G_{\rm ER})\rangle/n$ for small $\langle \deg(G_{\rm ER})\rangle$, as expected, since it was obtained in the dense limit. However, a detailed inspection of the numerical data allows us to propose

$$\frac{\langle IF(G_{\rm ER})\rangle}{n} \approx \langle \deg(G_{\rm ER})\rangle\left[1 - \langle \deg(G_{\rm ER})\rangle\right], \tag{5}$$

which describes the numerical data in the complete range of $\langle \deg(G_{\rm ER})\rangle$ remarkably well; see the good match of equation (4) (green lines) and the numerical data (symbols) in Figure 1f.

Therefore, in Figure 2a we finally present the average of $IF_\lambda(G)$ as a function of the probability p of ER networks. In fact $\langle IF_\lambda(G_{\rm ER})\rangle$ is shown for integer values of λ in the interval $[-5, 5]$ for networks of size $n = 200$. Indeed, the behavior of $\langle IF_\lambda(G_{\rm ER})\rangle$ *vs.* p depends in a highly non-trivial way on all network parameters: n, λ, and p. This, in fact, is not easily seen in Figure 2a. Therefore, in Figures 2b and 2c we plot, as examples, $\langle IF_{-3}(G_{\rm ER})\rangle$ and $\langle IF_3(G_{\rm ER})\rangle$. There it can be seen that even for positive λ, $\langle IF_\lambda(G_{\rm ER})\rangle$ may be negative in a wide range of p, see Figure 2c.

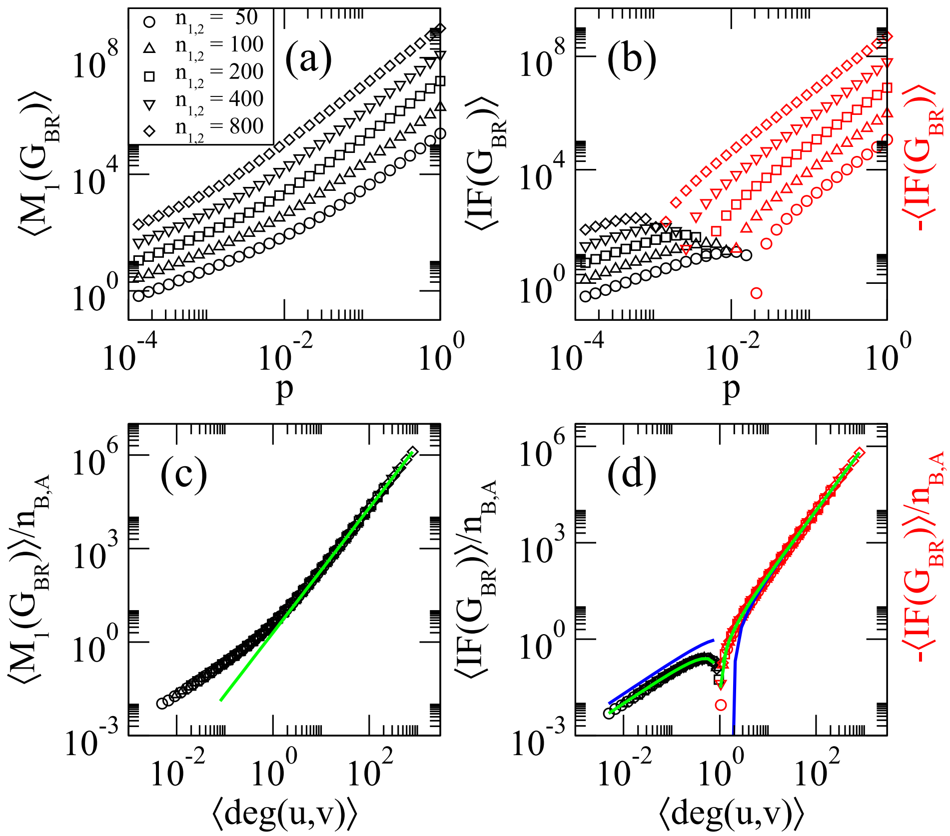

FIGURE 3. Average (a) $M_1(G)$, (b) $C_e(G)$, and (c) $IF(G)$ as a function of the connection radius r of random geometric graphs of size n. (d) $\langle M_1(G_{\rm RG})\rangle/n$ *vs.* $\langle \deg(G_{\rm RG})\rangle$, (e) $\langle C_e(G_{\rm RG})\rangle/n$ *vs.* r, and (f) $\langle IF(G_{\rm RG})\rangle/n$ *vs.* $\langle \deg(G_{\rm RG})\rangle$. All averages are computed over $10^6/n$ networks. Green lines in panels (d) and (f) are, respectively, equations (7) and (9). Blue lines in panel (f) are equation (8).

5.2. Integral λ-Forman-Ricci curvature of random geometric graphs

We now turn our attention to RG graphs. As well as for ER networks, before computing $IF_\lambda(G_{\rm RG})$, we first look at $M_1(G_{\rm RG})$, $C_e(G_{\rm RG})$, and $IF(G_{\rm RG})$. Then, in Figures 3a–3c we present the averages of $M_1(G_{\rm ER})$, $C_e(G_{\rm ER})$, and $IF(G_{\rm RG})$ as a function of the connection radius r of RG graphs of sizes $n = \{50, 100, 200, 400, 800\}$. All averages are computed over ensembles of $10^6/n$ random graphs where each ensemble is characterized by a fixed parameter pair (n, r). For the three quantities on RG graphs we observe a scenario similar to that reported for ER networks; compare Figures 3a–3c with Figures 1a–1c.

As in the previous Subsection, here we also explore the dense limit. Indeed, for RG graphs in the dense limit, *i.e.* when $\langle \deg(G_{\rm RG})\rangle \gg 1$, we can approximate $\deg(u) \approx \deg(v) \approx \langle \deg(G_{\rm RG})\rangle$, where [15]

$$\langle \deg(G_{\rm RG})\rangle = (n-1) \times \begin{cases} r^2\left[\pi - \frac{8}{3}r + \frac{1}{2}r^2\right], & 0 \le r \le 1\,, \\ \frac{1}{3} - 2r^2\left[1 - \arcsin(1/r) + \arccos(1/r)\right] & \\ \qquad + \frac{4}{3}(2r^2+1)\sqrt{r^2-1} - \frac{1}{2}r^4, & 1 \le r \le \sqrt{2}\,. \end{cases} \tag{6}$$

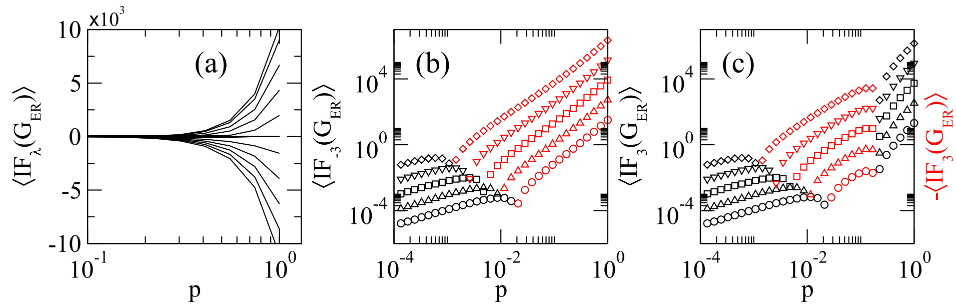

FIGURE 4. Average (a) $IF_\lambda(G)$, (b) $IF_{-3}(G)$, and (c) $IF_3(G)$ as a function of the connection radius r of random geometric graphs. In (a) $IF_\lambda(G)$ *vs.* r is shown for integer values of λ in the interval $[-5, 5]$ for graphs of size $n = 200$. In (b, c) several sizes are considered as indicated in panel (b). All averages are computed over $10^6/n$ graphs.

Thus, we can write:

$$\frac{\langle M_1(G_{\rm RG})\rangle}{n} \approx \langle \deg(G_{\rm RG})\rangle^2 \tag{7}$$

and

$$\frac{\langle IF(G_{\rm RG})\rangle}{n} \approx \langle \deg(G_{\rm RG})\rangle \left[2 - \langle \deg(G_{\rm RG})\rangle\right]. \tag{8}$$

Remarkably, the approximate equations (7) and (8) for RG graphs are exactly the same as the corresponding equations for ER graphs, see equations (2)–(4); however note that the definition of $\langle \deg(G)\rangle$ is different for both models, *i.e.* compare equations (1) and (6).

Then in Figures 3d and 3f we validate equations (7) and (8) by plotting them together with the corresponding numerical data. Indeed, we observe a very good correspondence of these equations and the data in the dense limit, *i.e.* for $\langle \deg(G_{\rm RG})\rangle \geq 10$, as expected. Moreover, as for ER networks, here for RG graphs we observe that

$$\frac{\langle IF(G_{\rm RG})\rangle}{n} \approx \langle \deg(G_{\rm RG})\rangle \left[1 - \langle \deg(G_{\rm RG})\rangle\right] \tag{9}$$

describes well $\langle IF(G_{\rm RG})\rangle$ for all values of $\langle \deg(G_{\rm RG})\rangle$; see the green lines in Figure 3f.

We want to add that the ratio $\langle C_e(G_{\rm RG})\rangle/n$ seems to be a function of the connection radius only, as seen in Figure 3e. There, the data for small r does not fall on top of a single curve due to poor statistics.

Thus, in Figure 4a we present the average of $IF_\lambda(G)$ as a function of the connection radius r of RG graphs. $\langle IF_\lambda(G_{\rm RG})\rangle$ is shown for integer values of λ in the interval $[-5, 5]$ for networks of size $n = 200$. In Figures 4b and 4c we plot, as examples, $\langle IF_{-3}(G_{\rm RG})\rangle$ and $\langle IF_3(G_{\rm RG})\rangle$.

For comparison purposes, Figure 4 for RG graphs is equivalent to Figure 2 for ER networks. Indeed, by comparing these two figures we can conclude that $\langle IF_\lambda(G)\rangle$, for given values of λ, may serve to distinguish between random graph models, a task almost imposible when using other quantities. That is, while the curves of $\langle M_1(G)\rangle/n$ *vs.* $\langle \deg(G)\rangle$ and $\langle IF(G)\rangle/n$ *vs.* $\langle \deg(G)\rangle$ are indistinguishable when computed for ER networks and RG graphs, $\langle IF_3(G)\rangle$ is not; compare Figures 2c and 4c.

5.3. Integral λ-Forman-Ricci curvature of bipartite random networks

Here we consider BR networks. Recall that BR networks $G_{\rm BR}(n_{\rm A}, n_{\rm B}, p)$ are composed by two disjoint sets, set A and set B, with $n_{\rm A}$ and $n_{\rm B}$ vertices each. However, since for bipartite networks $C_e(G) = 0$ then $IF_\lambda(G_{\rm BR}) = IF(G_{\rm BR})$, so we explore $M_1(G_{\rm BR})$ and $IF(G_{\rm BR})$ only.

In Figures 5a and 5b we present the averages of $M_1(G_{\rm BR})$ and $IF(G_{\rm BR})$ as a function of the probability p of BR networks of sizes $n_{\rm A} = n_{\rm B} = \{50, 100, 200, 400, 800\}$. All averages are computed over ensembles of $10^6/n$,

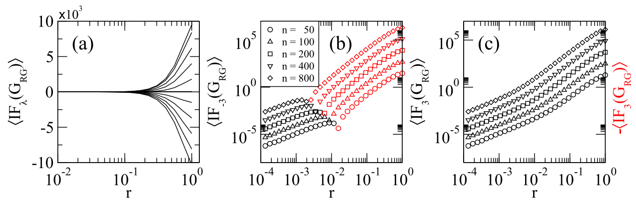

FIGURE 5. Average (a) $M_1(G)$ and (b) $IF(G)$ as a function of the probability p of random bipartite graphs of size $n_A = n_B$. (c) $\langle M_1(G_{BR})\rangle / n_{B,A}$ *vs.* $\langle \deg(u,v)\rangle$ and (d) $\langle IF(G_{BR})\rangle / n_{B,A}$ *vs.* $\langle \deg(u,v)\rangle$ with $\langle \deg(u,v)\rangle$ given in equation (10). All averages are computed over $10^6/n$ networks. Green lines in panels (c) and (d) are, respectively, equations (11) and (13). Blue lines in panel (f) are equation (12).

$n = n_A + n_B$, random graphs where each ensemble is characterized by a fixed parameter set (n_A, n_B, p). For these two quantities on BR networks we observe a scenario similar to that reported for ER networks and RG graphs; compare Figures 5a and 5b with Figures 1a, 1c and 3a, 3c.

Now, as for ER networks and RG graphs we write approximate expressions for $\langle M_1(G_{BR})\rangle$ and $\langle IF(G_{BR})\rangle$ in the dense limit. Moreover, we label the vertices of set A [set B] in the bipartite network as u [v]. Thus, when $n_A p \gg 1$ and $n_B p \gg 1$, we can approximate $\deg(u) \approx \langle \deg(u)\rangle$ and $\deg(v) \approx \langle \deg(v)\rangle$ with

$$\langle \deg(u,v)\rangle = n_{B,A} p. \tag{10}$$

Therefore, in the dense limit, we can write:

$$\frac{\langle M_1(G_{BR})\rangle}{n_{B,A}} \approx \langle \deg(u,v)\rangle \left[\langle \deg(u)\rangle + \langle \deg(v)\rangle\right] \tag{11}$$

and

$$\frac{\langle IF(G_{BR})\rangle}{n_{B,A}} \approx \langle \deg(u,v)\rangle \left[4 - \langle \deg(u)\rangle - \langle \deg(v)\rangle\right]. \tag{12}$$

It is remarkable to notice that in the case of $n_A = n_B = n/2$, where $\langle \deg(u)\rangle = \langle \deg(v)\rangle = np/2$, equations (11), (12) for BR networks reproduce equations (2), (4) for ER networks.

Then in Figures 5c and 5d we validate equations (11) and (12) by plotting them together with the corresponding numerical data. Indeed, we observe a very good correspondence of these equations and the data in the dense limit, *i.e.* for $\langle \deg(u,v)\rangle \geq 10$, as expected. Moreover, as for ER networks and RG graphs, here we observe that

$$\frac{\langle IF(G_{BR})\rangle}{n_{B,A}} \approx \langle \deg(u,v)\rangle \left[2 - \langle \deg(u)\rangle - \langle \deg(v)\rangle\right] \tag{13}$$

describes well $\langle IF(G_{BR})\rangle$ for all values of $\langle \deg(u,v)\rangle$; see the green lines in Figure 5d.

Acknowledgments

We would like to thank the referee for his/her careful reading of the manuscript and several useful comments which have helped us to improve the paper.

Funding

J.A.M.-B. thanks support from VIEP-BUAP (Grant No. 100405811-VIEP2024) Mexico.

Data availability statement

The research data associated with this article are included in the article.

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