

ON THE INDEPENDENCE NUMBER OF A VARIANT OF THE DIVISOR GRAPH

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Abstract. Let $n = p_1^{n_1} p_2^{n_2} \cdots p_t^{n_t}$ be a positive composite integer with distinct primes p_1, \dots, p_t . Consider the graph G'_n , introduced by Rather and Ganie [*RAIRO-Oper. Res.* **59** (2025) 1605–1616], whose vertices are the proper divisors of n and where two vertices are adjacent if and only if they are coprime. A complete description of maximal independent sets in G'_n is obtained *via* a correspondence with maximal intersecting families of subsets of $[t]$. This correspondence is applied to determine the independence number for various classes of n , including the square-free case, where the exact value and an explicit formula are established. For general n with at least one exponent $n_i > 1$, the independence number is expressed in terms of combinatorial properties of intersecting families, yielding a unified approach to both the structural characterization and enumeration of maximal independent sets.

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1. INTRODUCTION

The study of divisor graphs and their structural properties has gained attention due to their rich combinatorial structure and applications in number theory and graph theory. In particular, the graph G'_n was introduced in the work of Rather and Ganie [1], where several interesting results on connectivity, clique number, and independence number were established. Their contribution provided an elegant framework for exploring relationships between arithmetic functions and graph-theoretic parameters.

Rather and Ganie [1] A divisor graph G_n of a positive integer n is a simple graph with vertices as proper divisors (considering all divisors except 1 and n) of n , in which two distinct vertices are adjacent if and only if they are relatively prime. G'_n is a graph with vertices as divisors of n except n , such that two distinct vertices are adjacent if and only if they are relatively prime.

For a graph $G = (V, E)$, an *independent set* is a subset $S \subseteq V$ such that no two vertices in S are adjacent. The *independence number* $\alpha(G)$ is the cardinality of a largest independent set [2]. A family \mathcal{F} of subsets of a finite set X is called an *intersecting family* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$. An intersecting family \mathcal{F} is said to be *maximal* if it is not properly contained in any other intersecting family of subsets of X [3].

In this paper, the independence number of G'_n is determined when n is a product of distinct prime numbers, and the general structure of maximal independent sets is described when some prime factors appear with

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multiplicity. The approach builds on known combinatorial results, linking intersecting families to Independent sets in G'_n , and provides new insights into the independence number of this class of graphs.

1.1. Related work

The correspondence between independent sets in G'_n and intersecting families of subsets of $[t]$ connects our work to classical extremal set theory. The equivalence that characterizes maximal intersecting families was first established by Erdős and Kleitman [4], and later restated in a different form by Meyrowitz [3]. These classical results form the foundation of our approach.

Over the years, a large body of research has extended the scope of intersecting family theorems in various directions. One important line of work concerns multiset analogues. Meagher and Purdy [5] proved multiset versions of the Erdős-Rado and Hilton-Milner theorems, establishing sharp bounds for the largest t -intersecting families of k -multisets when the size of the ground set satisfies $m \leq 2k - t$. More recently, Wang and Hou [6] developed results for cross t -intersecting families of multisets, characterizing external configurations and generalizing classical EKR-type theorems.

Beyond multisets, other significant developments include the celebrated Ahlswede–Khachatrian complete intersection theorem [7], which determines the maximum size of a t -intersecting family of subsets and characterizes equality cases. Further refinements have also been obtained in related contexts; for instance, Ellis and Lifshitz [8] studied the union of several intersecting families and determined extremal conditions for maximizing their total size. These results demonstrate the depth and richness of extremal set theory and its many extensions.

The divisor graph G_n was studied by Rather and Ganie [1], where several structural properties such as connectivity, clique number, and independence number were investigated. In their work, they also introduced the variant G'_n , obtained by considering all divisors of n except n itself, with adjacency still defined by coprimality. While their analysis provided a valuable framework, some of the formulas they proposed for the independence number of G'_n in the square-free case turn out not to be valid for larger values of t . For instance, when $n = p_1 p_2 p_3 p_4$ (so $t = 4$), one can already construct an independent set of size $t + 1 = 5$, which goes beyond the earlier estimate. In the present paper, we refine this aspect establish the corrected value $\alpha(G'_n) = 2^{t-1} - 1$ when n is square-free.

In contrast to these combinatorial generalizations discussed above, our work does not attempt to extend the theory of intersecting families itself. Instead, we employ the classical case [3, 4] as a tool to analyze the graph G'_n . This approach allows us to determine the *general structure of maximal independent sets* and to establish the *independence number* of G'_n . Thus, while related to extremal set theory through the underlying techniques, our contribution lies in the domain of algebraic graph theory, refining and extending the study of divisor graphs initiated by Rather and Ganie [1].

2. INDEPENDENCE NUMBER OF G'_n

Proposition 2.5 in [1] presents the independence number of G'_n for $n = p_1 p_2 \cdots p_t$ with $t \geq 2$, where p_1, p_2, \dots, p_t are distinct prime numbers. The formula provided there is correct for certain values of t , but for $t \geq 4$ the set

$$S = \{p_1, p_1 p_2, p_1 p_3, \dots, p_1 p_t, p_1 p_2 p_3\}$$

forms an independent set in G'_n with $|S| = t + 1$, which shows that the expression in Proposition 2.5 does not hold in these cases. In this paper, it is established that $\alpha(G'_n) = 2^{t-1} - 1$ when $n = p_1 p_2 \cdots p_t$, using Proposition 2.1.

Furthermore, Theorem 2.4 describes the structure of maximal independent sets in G'_n when $n = p_1^{n_1} p_2^{n_2} \cdots p_t^{n_t}$, where p_1, p_2, \dots, p_t are distinct prime numbers and at least one exponent $n_i > 1$ for $1 \leq i \leq t$. This structural characterization complements the independence number result and provides deeper insight into the combinatorial properties of G'_n .

Proposition 2.1 ([3]). *Let \mathcal{F} be an intersecting family of subsets of X with $|X| = n$. Then, the following are equivalent*

- (i) \mathcal{F} is a maximal intersecting family.
- (ii) $|\mathcal{F}| = 2^{n-1}$.
- (iii) \mathcal{F} contains one of each pair of complementary subsets of X .

From Proposition 2.1, it follows readily that the cardinality of a maximal intersecting family of proper subsets of X , where $|X| = n$, is $2^{n-1} - 1$.

Theorem 2.2. *Let $n = p_1 p_2 \cdots p_t$, where $t \geq 2$ and p_1, p_2, \dots, p_t are distinct prime numbers. Then the independence number of the graph G'_n is $\alpha(G'_n) = 2^{t-1} - 1$.*

Proof. Consider a maximal independent set D in G'_n such that $D \neq \{1\}$. The goal is to show that $|D| = 2^{t-1} - 1$.

Define a function $f : D \rightarrow P([t])$, where $[t] = \{1, 2, \dots, t\}$ and $P([t])$ is the power set of $[t]$, such that for any $v \in D$, $f(v) = \{i \in [t] : p_i \text{ divides } v\}$. Let $\mathcal{F} = \{f(v) : v \in D\}$ be the image of D under f . The claim is that \mathcal{F} is a maximal intersecting family of proper subsets of $[t]$.

To show that \mathcal{F} is an intersecting family of proper subsets of $[t]$, consider any $F_1, F_2 \in \mathcal{F}$. There exist $v_1, v_2 \in D$ such that $F_1 = f(v_1)$ and $F_2 = f(v_2)$. Since D is an independent set in G'_n , the vertices v_1 and v_2 are not adjacent, meaning $\gcd(v_1, v_2) \neq 1$. Thus, some prime p_j (with $j \in [t]$) divides both v_1 and v_2 . Hence, $j \in f(v_1) \cap f(v_2) = F_1 \cap F_2$, so $F_1 \cap F_2 \neq \emptyset$. Therefore, \mathcal{F} is an intersecting family. Since no vertex $v \in D$ is divisible by all primes p_1, p_2, \dots, p_t (as $v \neq n$), each $f(v) \neq [t]$, so \mathcal{F} consists of proper subsets of $[t]$.

To prove that \mathcal{F} is maximal, assume the contrary. Suppose \mathcal{F} is not a maximal intersecting family of proper subsets of $[t]$. Then, there exists a non-empty proper subset $U \subset [t]$, $U \neq [t]$, such that $\mathcal{F} \cup \{U\}$ remains an intersecting family of proper subsets of $[t]$. Define the divisor $u = \prod_{i \in U} p_i$, which is a proper divisor of n since $U \neq [t]$. Since f is injective (distinct divisors $v_1, v_2 \in D$ produce distinct sets $f(v_1), f(v_2)$ due to unique prime factorizations), $u \notin D$. For every $v \in D$, since $f(v) \cap U \neq \emptyset$ (as $\mathcal{F} \cup \{U\}$ is intersecting), some $j \in f(v) \cap U$ exists, meaning p_j divides both v and u . Thus, $\gcd(v, u) \neq 1$, so u is not adjacent to any vertex in D . Hence, $D \cup \{u\}$ is an independent set containing D , contradicting the maximality of D . Therefore, \mathcal{F} is a maximal intersecting family of proper subsets of $[t]$.

By the Proposition 2.1, the cardinality of a maximal intersecting family of proper subsets of $[t]$ is $2^{t-1} - 1$. Since f is injective, $|D| = |\mathcal{F}| = 2^{t-1} - 1$. Thus, every maximal independent set $D \neq \{1\}$ in G'_n has cardinality $2^{t-1} - 1$. Since the independence number $\alpha(G'_n)$ is the size of the largest independent set, and $\{1\}$ has size 1 (which is less than $2^{t-1} - 1$ for $t \geq 2$), it follows that $\alpha(G'_n) = 2^{t-1} - 1$. □

Corollary 2.3. *Let $n = p_1 p_2 \cdots p_t$, where $t \geq 2$ and p_1, p_2, \dots, p_t are distinct prime numbers. Then the graph G_n is well-covered, and its independence number is $\alpha(G_n) = 2^{t-1} - 1$.*

Proof. Since 1 is not a vertex in G_n , it follows from Theorem 2.2 that all maximal independent sets in G_n have the same cardinality, and thus $\alpha(G_n) = 2^{t-1} - 1$. □

Theorem 2.4 describes the structure of maximal independent sets in the graph G'_n , where $n = p_1^{n_1} p_2^{n_2} \cdots p_t^{n_t}$, with p_1, p_2, \dots, p_t being distinct prime numbers and at least one $n_i > 1$ for $1 \leq i \leq t$.

Theorem 2.4. *Let $n = p_1^{n_1} p_2^{n_2} \cdots p_t^{n_t}$, where p_1, p_2, \dots, p_t are distinct primes and at least one $n_i > 1$ for $1 \leq i \leq t$. Let A be the collection of all maximal intersecting families of subsets of $[t] = \{1, 2, \dots, t\}$. A set $D \neq \{1\}$ is a maximal independent set in G'_n if and only if $D = \bigcup_{F \in \mathcal{F}} \{x \in V(G'_n) : \prod_{i \in F} p_i \text{ divides } x\}$ for some $\mathcal{F} \in A$.*

Proof. Let $n = p_1^{n_1} p_2^{n_2} \cdots p_t^{n_t}$, where p_1, p_2, \dots, p_t are distinct primes and at least one $n_i > 1$. Let A denote the collection of all maximal intersecting families of subsets of $[t] = \{1, 2, \dots, t\}$. Suppose $D \neq \{1\} \subseteq V(G'_n)$ is a maximal independent set in G'_n . Define a function $f : D \rightarrow \mathcal{P}([t])$ by

$$f(v) = \{i \in [t] : p_i \text{ divides } v\}$$

for any $v \in D$. Let $\mathcal{F} = \{f(v) : v \in D\}$ be the image of D under f . The goal is to show that \mathcal{F} is a maximal intersecting family in $\mathcal{P}([t])$.

To show that \mathcal{F} is intersecting family of proper subsets of $[t]$, consider any $F_1, F_2 \in \mathcal{F}$. There exist $v_1, v_2 \in D$ such that $F_1 = f(v_1)$ and $F_2 = f(v_2)$. Since D is an independent set in G'_n , there is no edge between v_1 and v_2 , implying $\gcd(v_1, v_2) \neq 1$. Thus, there exists a prime p_j dividing both v_1 and v_2 . Hence, $j \in f(v_1) \cap f(v_2) = F_1 \cap F_2$, so $F_1 \cap F_2 \neq \emptyset$. Therefore, \mathcal{F} is an intersecting family.

To prove that \mathcal{F} is maximal, assume the contrary. Suppose \mathcal{F} is not maximal. Then, there exists a non-empty subset $U \subseteq [t]$ such that $\mathcal{F} \cup \{U\}$ is still an intersecting family. Define the divisor $u = \prod_{i \in U} p_i$, which is a proper divisor of n since at least one $n_i > 1$. If $u \in D$, then $f(u) = U \in \mathcal{F}$, contradicting the assumption that $U \notin \mathcal{F}$. Thus, $u \notin D$. For every $v \in D$, since $\mathcal{F} \cup \{U\}$ is intersecting, there exists $j \in [t]$ such that $j \in f(v) \cap U$. Hence, p_j divides both v and u , implying $\gcd(u, v) \neq 1$. Thus, $D \cup \{u\}$ is an independent set containing D , contradicting the maximality of D . Therefore, \mathcal{F} is a maximal intersecting family.

Now, construct a set $D' \subseteq V(G_n)$ using \mathcal{F} . For each $F \in \mathcal{F}$, define

$$V_F = \left\{ x \in V(G'_n) : \prod_{i \in F} p_i \text{ divides } x \right\},$$

and let $D' = \bigcup_{F \in \mathcal{F}} V_F$. To prove $D = D'$, first show that $D \subseteq D'$ and then prove D' is an independent set of G'_n . For any $v \in D$, since $f(v) \in \mathcal{F}$, it follows that $v \in V_{f(v)} \subseteq D'$. Thus, $D \subseteq D'$.

Next, verify that D' is an independent set in G'_n . For any $x_1, x_2 \in D'$, there exist $F_1, F_2 \in \mathcal{F}$ such that $x_1 \in V_{F_1}$ and $x_2 \in V_{F_2}$. Since \mathcal{F} is intersecting, there exists $j \in F_1 \cap F_2$. Thus, p_j divides every element in V_{F_1} and V_{F_2} , so p_j divides both x_1 and x_2 . Hence, $\gcd(x_1, x_2) \neq 1$, and there is no edge between x_1 and x_2 in G'_n . Therefore, D' is an independent set containing D . Since D is maximal, $D = D'$. Thus,

$$D = \left\{ x \in V(G'_n) : \prod_{i \in F} p_i \text{ divides } x \text{ for some } F \in \mathcal{F} \right\},$$

where $\mathcal{F} \in A$.

Conversely, suppose $D = \{x \in V(G'_n) : \prod_{i \in F} p_i \text{ divides } x \text{ for some } F \in \mathcal{F}\}$, where $\mathcal{F} \in A$ is a maximal intersecting family. Note that $D \neq \{1\}$. We need to prove that D is a maximal independent set in G'_n .

First, we show that D is an independent set. For any $x_1, x_2 \in D$, there exist $F_1, F_2 \in \mathcal{F}$ such that $x_1 \in V_{F_1}$ and $x_2 \in V_{F_2}$. Since \mathcal{F} is intersecting, there exists $j \in F_1 \cap F_2$. Thus, p_j divides both x_1 and x_2 , so $\gcd(x_1, x_2) \neq 1$. Hence, x_1 and x_2 are not adjacent in G'_n , and D is an independent set.

To prove maximality, assume, for contradiction, that D is not maximal. Then, there exists $w \in V(G'_n) \setminus D$ such that $D \cup \{w\}$ is an independent set. Define $F_w = \{i \in [t] : p_i \text{ divides } w\}$. If $F_w \in \mathcal{F}$, then $w \in V_{F_w} \subseteq D$, contradicting $w \notin D$. Thus, $F_w \notin \mathcal{F}$. Consider the family $\mathcal{F} \cup \{F_w\}$. For any $F \in \mathcal{F}$, let $u = \prod_{i \in F} p_i \in V_F \subseteq D$. Since $D \cup \{w\}$ is an independent set, $\gcd(u, w) \neq 1$. Thus, there exists a prime p_j dividing both u and w , so $j \in F \cap F_w$. Hence, $\mathcal{F} \cup \{F_w\}$ is an intersecting family containing \mathcal{F} , contradicting the maximality of \mathcal{F} . Therefore, D is a maximal independent set.

Thus, $D \neq \{1\} \subseteq V(G'_n)$ is a maximal independent set in G'_n if and only if

$$D = \left\{ x \in V(G'_n) : \prod_{i \in F} p_i \text{ divides } x \text{ for some } F \in \mathcal{F} \right\},$$

where $\mathcal{F} \in A$. □

Corollary 2.5. *Let $n = p_1^{n_1} p_2^{n_2} \cdots p_t^{n_t}$, where p_1, p_2, \dots, p_t are distinct primes and at least one $n_i > 1$ for $1 \leq i \leq t$. Let A be the collection of all maximal intersecting families of subsets of $[t] = \{1, 2, \dots, t\}$. Then, independence number $\alpha(G'_n) = \max_{\mathcal{F} \in A} \{|D| : D = \{x \in V(G'_n) : \prod_{i \in F} p_i \text{ divides } x \text{ for some } F \in \mathcal{F}\}$.*

Using Theorem 2.2 and Corollary 2.5, the independence number of G'_n can be determined for any positive composite integer n .

3. CONCLUSION

This article gives a detailed study of the independence number of G'_n . It has been shown that for $n = p_1 p_2 \cdots p_t$, where p_1, p_2, \dots, p_t are distinct prime numbers, the independence number is $\alpha(G'_n) = 2^{t-1} - 1$. In addition, the structure of maximal independent sets has been characterized in the case where at least one prime factor of n appears with multiplicity greater than one. These findings refine earlier observations and provide further insight into the combinatorial structure of divisor graphs, offering directions for future work on related graph parameters.

CONFLICTS OF INTEREST

The author declare that they have no competing interests.

DATA AVAILABILITY STATEMENT

No new data/codes were created or analyzed in this study.

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