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RELIABILITY MEASURES OF DYNAMIC CIRCULAR SLIDING WINDOW SYSTEM

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Abstract. In this modern era of dynamic reliability analysis, this paper introduces the dynamic circular multi-state sliding window system (DCMSWS), an innovative mathematical model that extends the concept of a circular sliding window system into dynamic scenarios. DCMSWS is considered to have failed when the combined performance of r consecutive multistate elements (MEs) falls below a predefined threshold, W. Each component type has a number of distinct individual components with variable states that change over time, making the state probabilities of those components time-dependent. By employing an algorithm grounded in L_z-transform, dynamic reliability measures comprising reliability, availability, mean expected performance, and expected profit are derived for the water supply system. This study represents a pioneering effort in dynamic circular multi-state sliding window systems and their evaluation using the L_z-transform, intimately connected with queuing theory.

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1. Introduction

In today's hyper-connected world, the reliability of systems, services, and products has become an indispensable facet of our daily lives. Whether it is the uninterrupted operation of critical infrastructure, the dependable functioning of complex manufacturing processes, or the seamless execution of software applications, reliability is the bedrock upon which modern society thrives. Reliability, in this context, is not just a concept, but a mission-critical attribute that ensures the continuity and integrity of our systems and technologies. The concept of reliability can be traced back to the early days of engineering and has since evolved into a multi-faceted and interdisciplinary field. It encompasses an array of domains, such as electrical engineering, mechanical engineering, computer science, and beyond, and applies to systems as diverse as power grids, aircraft, medical devices, and digital platforms.

The concept of reliability in systems engineering and related fields is a fundamental and overarching principle that transcends specific domains and industries. It represents the bedrock upon which the functionality and performance of any system, whether it be mechanical, electronic, or software-based, rest. Reliability is a measure of the system's ability to perform its intended functions under specified conditions, consistently and without failure, over a designated period of time. It is the embodiment of trust, predictability, and depend-

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ability in our increasingly complex and interconnected world. Reliability, when applied to systems, extends far beyond the reliability of individual components. It encompasses the interplay and coordination of multiple components, subsystems, and processes that collectively contribute to the overall system's performance. As systems grow in complexity and size, they become more susceptible to various failure modes, making the study and implementation of reliability principles more critical than ever. In the realm of reliability engineering, out of various systems, the k-out-of-n system is considered to be the most effective system, which works if at least k out of n system works and finds extensive applications in various domains, such as spacecraft, traffic control systems, communication networks, and weapon systems [1,2]. Some remarkable work on the k-out-of-n system and its extensions can be found in [3–7]. Among the numerous extensions k-out-of-n system, the consecutive k-out-of-r-from-n system emerges as particularly noteworthy, owing to its extensive utility in realms such as quality control, inspection procedures, service systems, and radar detection challenges. The consecutive k-out-of-r-from-n: F-system has n ordered units and fails if at least k out of r consecutive units fail. This innovative system was first introduced by Griffith [8], further underscoring its significance in the field.

In the initial formulation of the consecutive k-out-of-r-from-n system, the individual components were modeled in a binary state configuration, signifying that each element could exist in one of two states: perfect functioning or complete failure. Subsequently, this foundational model evolved into a more versatile framework known as the multi-state sliding window system (MSWS). In this advanced system, the units were considered as multistate elements (MEs), allowing each unit to exhibit a spectrum of performance states, ranging from perfect functioning to various degrees of performance degradation, up to complete failure. The motivation behind the development of the MSWS (multi-state sliding window system) and its extended models is rooted in the abundance of practical applications, as highlighted in [9]. As an illustrative example, consider a heating system in a manufacturing line where the temperature at each point along the line is contingent on the cumulative heating contributions from a set of adjacent heaters. The system, in this context, encounters failure if the summation of radiation intensity at any point falls below a predetermined minimum threshold. Numerous real-world and engineering applications offer additional practical examples, including quality control schemes, sensor monitoring devices, and many others. Moreover, with regard to their topological configurations, the SWS can be categorized into two distinct classes: linear MSWS and circular MSWS.

A linear multistate sliding window system is composed of n linearly arranged multistate elements (MEs). Each ME, denoted as i, can exhibit a total of M_i different performance states. The system is deemed to have failed if the cumulative performance of any contiguous set of r MEs falls below a predefined threshold value, W. Some recent work done in reliability evaluation of linear MSWS includes [10–12]. The majority of previous research on MSWS has primarily focused on systems with linearly structured elements. However, it's important to note that there are numerous practical systems where the elements are arranged in a circular configuration. Some instances of such systems include the vacuum system of an electronic accelerator [13], the cyclic water supply systems of a thermo-electric plant [14], the microwave signal transmitting system [15], and the photographic system [16]. Hui *et al.* [17] and Wang *et al.* [18] introduced novel models for circular multi-state sliding window systems. Existing research in MSWS predominantly focused on multistate elements in a static, steady state. However, in several engineering contexts, the state of these multistate elements undergoes continuous changes over time. Consequently, this paper introduces a dynamic multi-state sliding window system (DCMSWS), where the state of each multistate element evolves over time.

Furthermore, various techniques like the stochastic process approach, supplementary variable technique [19], universal generating function method (UGF), and the Monte Carlo simulation technique have been employed to assess the reliability of various multi-state systems (MSS). Among these, the UGF approach has gained widespread popularity due to its simplicity and time-saving attributes. The UGF technique calculates the performance distribution of the multi-state system based on the performance distributions of its individual elements using straightforward algebraic operations. Initially proposed by I. Ushakov in the mid-1980s [20], engineers and scientists have since extensively utilized this method to evaluate the reliability of diverse systems [21–24] and networks [25–27]. However, a notable limitation of this vital technique is its applicability primarily to random variables, making it suitable only for evaluating the reliability of steady-state performance distributions in multi-state systems. As a result, it does not account for transient approaches in the multi-state systems, aging multi-state systems, and multi-state systems under increasing or decreasing stochastic demand, using fundamental UGF techniques. To overcome this limitation and extend the UGF technique's application to the reliability assessment of multi-state systems in dynamic states, where the entire structure and its components are well-defined by the stochastic processes (not just by random variables), the L_z-transform [28] was formulated for Markov processes with discrete states and continuous time. As such, the L_z-transform can be regarded as an extension of the universal generating function approach to encompass stochastic processes.

Over the years, many research articles have utilized the L_z-transform to dynamically estimate the reliability of diverse engineering systems. For instance, Frenkel *et al.* [29] employed the L_z-transform approach to evaluate the cooling effectiveness of an aging multi-state cooling system used in MRI equipment. Lisnianski and Benhaim [30] utilized the L_z-transform to derive reliability indices for a power station. In another study, Frenkel *et al.* [31] assessed critical aspects of operational sustainability in multi-state multi-power source traction drives of Norilsk icebreaker ships, including availability, power output, and power output deficit, using the L_z-transform. Additionally, a comprehensive book published in 2021 [32] encompassed theoretical explanations, real-world applications, and case studies from practical scenarios, offering insights into the area of dynamic reliability assessment of multi-state systems. Singh *et al.* [33] introduced an algorithm using the L_z-transform to assess reliability indices in dynamic multi-state m-out-of-r-within-k-out-of-n systems, accounting for time-varying component states and Markov processes. Bisht and Singh [34] proposed the Interval L_z-transform (IL_z), a novel method for estimating reliability indices of Multi-State systems with uncertain or insufficient data, demonstrated through a series–parallel system example. Some very recent work published in the domain of L_z-transform includes [35–39].

2. The proposed model: Dynamic circular sliding window system

The circular multistate sliding window system, in which the elements are MEs, generalizes the circular consecutive k-out-of-r-from-n system to multi-state scenarios. The Dynamic Circular Multistate Sliding Window System (DCMSWS) comprises n multistate elements interconnected in a circular configuration, where the nth element is adjacent to the first element as shown in Figure 1. These elements are characterized by their dynamic nature, with the state of each element undergoing continuous changes over time. Since there are n multistate elements, there are also n groups of r consecutive elements within the circular system. The first group encompasses $\text{ME } 1, \text{ME } 2, \ldots, \text{ME } r$, while the last group includes $\text{ME } n, \text{ME } n+1, \ldots, \text{ME } n+r-1$, where $\text{ME } n+1, \ldots, \text{ME } n+r-1$ corresponds to $\text{ME } 1, \ldots, \text{ME } r-1$, respectively. Each of these multistate elements denoted as ME i, can manifest M_i distinct states, which vary with time, ranging from complete failure to perfect functioning, with a unique performance rate associated with each state. The DCMSWS is considered to have failed if the summation of the performance of any set of r consecutive MEs falls below a predefined threshold value, W *i.e.*, the DCMSWS fails if

$$\prod_{f=1}^{n} \beta\left(\sum_{i=f}^{f=r-1} M_i \geq W\right) = 0.$$

In the context of the DCMSWS, every multistate element labeled as "i" displays a range of M_i unique states, extending from 1 to M_i. In the proposed DCMSWS, each ME is considered to be in four states. In this spectrum, state 4 represents the highest-performing state, state 1 signifies the state of failure, and the intermediate states between them encompass various degrees of performance. The system, as well as its multistate components, is regarded as repairable. Repairs are promptly initiated when these elements transition from their functional state to the state of failure, *i.e.*, to state 1. The repair process is conducted through two distinct mechanisms:

(a) **Minor repair:** a minor failure is a malfunction or failure of a system or component that does not result in catastrophic or severe consequences but still has some impact on performance or operation. Minor failures are typically less serious than major failures but can still result in inconvenience, decreased productivity, or

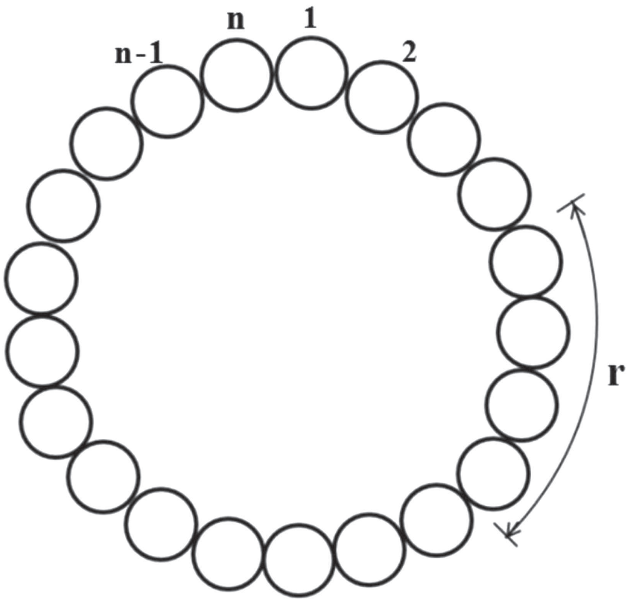

FIGURE 1. The Dynamic Circular Multistate Sliding Window System.

increased maintenance costs and are caused due to environmental factors, material degradation, etc. In the proposed model, the minor failure is repaired immediately as soon as it gets failed and is termed as a minor repair, for example, the minor failure caused from state 2 to state 1 (*i.e.*, λ_{21}) is repaired immediately (*i.e.*, transitioned back to state 2, μ_{12}) as soon as get failed.

(b) **Major repair:** a major failure refers to a significant malfunction or breakdown of a system or component that results in severe consequences, such as system outage, safety hazards, or substantial economic losses. Unlike minor failures, major failures have a substantial impact on the performance, functionality, and overall reliability of the system and are caused due to lack of maintenance, material defects, human error, fatigue, overload, etc. Major failures differ from minor failures in terms of their repair approach also.

In this study, major failures are not immediately repaired due to the significant time required for the repair process. Consequently, when a component experiences a major failure and reaches state 4, it forms a queue while waiting for repair. The repair is carried out by a single repairman, following a first-come, first-served basis following $M|M|1:\infty|FCFS$ queue, and is governed by the Erlang distribution. However, it's important to note that even after the repair, each multi-state element cannot be considered in a completely perfect condition. Instead, a more realistic perspective acknowledges that repair activities restore the multi-state element to a state somewhere between "as good as new" and "as bad as old". As a result, major repair transitions the component from state 1 to state 3, rather than directly to state 4 as shown in Figure 2. State 3 represents an intermediate condition, neither completely perfect nor entirely failed, but somewhere in between these two states. This consideration reflects the practical nature of repair outcomes in the reliability analysis of the system.

3. ALGORITHM FOR COMPUTING DYNAMIC RELIABILITY INDICES OF DYNAMIC CIRCULAR MULTI-STATE SLIDING WINDOW SYSTEM USING L_z-TRANSFORM

Consider a repairable dynamic circular multi-state sliding window system, comprising n multi-state components, each of these components exhibits M distinct performance states, spanning from 1 to M. In this case, the states between M, the greatest performing state, and 1, the state of failure, and the rest in between rep-

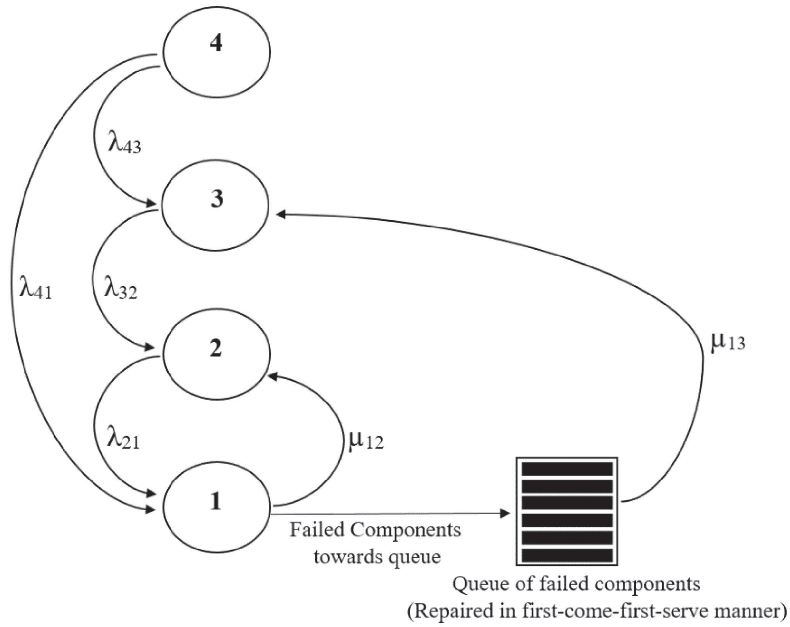

FIGURE 2. State transition diagram for MEs of DCMSWS.

resent intermediate states. Assume that each ME of the DCMSWS degrades according to the Markov process. For minor failures, immediate repairs are conducted. However, in the case of major failures, a first-come-first-serve (FCFS) queue with infinite capacity ($M|M|1:\infty$) is formed for repair, and the departure rate follows the Erlang distribution. A crucial aspect of this model is that after major repair, the component does not return to a state as good as new; rather, it remains in a condition between the "as good as new" and "completely failed" states. This realistic consideration acknowledges the limitations of achieving complete restoration after major repairs. The process of computing reliability measures for a dynamic circular multi-state sliding window system (DCMSWS) involves a series of steps:

Step 1. Begin by defining the system states, denoted as g, which comprises elements g_{ij} representing the performance states of any component i in state j. Here, j ranges from 1 to M_i.

Step 2. Determine the initial state probability distribution, referred to as p_0.

Step 3. Develop a set of differential equations that govern the transient state probabilities. These equations are based on the transition rates within the multi-state system. The following differential equations capture the evolution of state probabilities over time:

$$\left.\begin{aligned}
\frac{\mathrm{d}p_{iM_i}(t)}{\mathrm{d}t} &= \sum_{e=1}^{M_i-1}\mu_{eM_i}^{(i)}\cdot p_{ie}(t) - p_{iM}(t)\sum_{e=1}^{M_i-1}\lambda_{eM_i}^{(i)}(t)\\
\frac{\mathrm{d}p_{ij}(t)}{\mathrm{d}t} &= \sum_{e=j+1}^{M_i}\lambda_{ej}^{(i)}\cdot p_{ie}(t) + \sum_{e=1}^{j-1}\mu_{ej}^{(i)}\cdot p_{ie}(t) - p_{ij}(t)\left(\sum_{e=1}^{j-1}\lambda_{je}^{(i)}(t) + \sum_{e=1}^{M_i}\mu_{je}^{(i)}(t)\right)\\
\frac{\mathrm{d}p_{i1}(t)}{\mathrm{d}t} &= \sum_{e=2}^{M_i}\lambda_{e1}^{(i)}\cdot p_{ie}(t) - p_{i1}(t)\sum_{e=2}^{M_i}\mu_{ie}^{(i)}.
\end{aligned}\right\} \tag{1}$$

In equation (1), $\frac{\mathrm{d}p_{iM_i}(t)}{\mathrm{d}t}$ represents the rate of change of the state probability for component i being in its highest state, M_i. Conversely, $\frac{\mathrm{d}p_{ij}(t)}{\mathrm{d}t}$ signifies the rate of change of the state probability for component i

being in state j, where j ranges from 1 to $M_i - 1$. These terms are essential for tracking the evolution of the system's state probabilities over time. The transition rates λ and μ, responsible for describing the likelihood of transitions between different states, play a pivotal role in shaping these equations.

Step 4. Proceed with the resolution of the set of differential equations presented in equation (1). These equations are addressed while considering the initial state probability distribution, p_0, as defined in Step 2. The objective is to ascertain the transient state probabilities, denoted as p_{ij}, which provide valuable insights into the likelihood of the system residing in various states at different time intervals. By solving the differential equations as outlined in Step 3, we obtain these state probabilities, which serve as critical indicators for comprehending the system's dynamic behavior over time. Subsequently, these state probabilities become the foundation for computing the availability of the Dynamic Circular Multi-State Sliding Window System (DCMSWS), shedding light on its reliability and performance under diverse conditions.

Step 5. Replace p_{13} with the probability density function of the Erlang distribution, which is defined as follows:

$$f\left(t,\mu,\lambda\right)=\frac{\lambda^{\mu}t^{\mu-1}e^{-\lambda t}}{(\mu-1)!}.$$

Here, λ and μ correspond to the major failure and major repair rates, respectively.

Step 6. Calculate the L_z-transform for each multi-state component of DCMSWS. The L_z-transform of a Markov process $G\left(t\right)=\{g,A,p_0\}$, where g represents the set of states $G(t)\in\{g_{i1},g_{i2},\ldots,g_{iM}\}$ for $t\geq 0$, $\boldsymbol{A}=a_{ij}(t)$ is the transition intensity matrix with i and j ranging from 1 to M and p_0 is the initial state probability distribution p_0 is given by

$$L_z\left(G\left(t\right)\right)=f\left(z,t,p_0\right)=\sum_{j=1}^{M_i}p_{ij}\left(t\right)z^{g_{ij}} \tag{2}$$

where p_{ij} is the state probability of component i, which is in state j; $j=1,2,\ldots,M_i$; for any time $t\geq 0$, for any primary state probability distribution p_0 and z represents a complex variable.

It's important to note that the L_z-transform is not a UGF but rather a completely new type of entity because it depends on the preliminary conditions of the Markov process p_0 and time t. The property of L_z-transform, that shall be used is as follows: UGF operator "Ω_f" to L_z-transform in $M(t)$ and $N(t)$ for $t\geq 0$ can be used to generate the L_z-transform from two separate Markov processes, $f(M(t),N(t))$ as

$$L_z\left\{f(M(t),N(t))\right\}=\Omega_f\left(L_z\left\{M(t)\right\},L_z\left\{N(t)\right\}\right).$$

Step 7. In the context of the DCMSWS, where there exist n groups of r consecutive elements, our objective is to derive the L_z-transform for a single group. It's important to recognize that each possible combination of individual multistate element (ME) states corresponds to a potential state of the entire system. To determine the probability of the group being in any specific state, we calculate it as the product of the probabilities associated with the corresponding states of each individual ME. For instance, consider a group comprising two MEs, p and q, the joint L_z-transform for this group follows the pattern as outlined as in the property, ensuring that we capture the comprehensive state dynamics of the group, which can be obtained as:

$$\begin{aligned}L_z\left(Grp_1\left(t\right)\right)&=\otimes\left(L_z\left(E_p\left(t\right)\right),L_z\left(E_q\left(t\right)\right)\right)=\otimes\left(\sum_{i_p=1}^{M_p}p_{p,i_p}\cdot z^{g_{p,i_p}},\sum_{i_q=1}^{M_q}p_{q,i_q}\cdot z^{g_{q,i_q}}\right)\\&=\sum_{i_p=1}^{M_p}\sum_{i_q=1}^{M_q}p_{p,i_p}p_{q,i_q}\cdot z^{\left(g_{p,i_p},g_{q,i_q}\right)}.\end{aligned}$$

Similarly, the L_z-transform of the group of r consecutive elements of DCMSWS can be obtained as:

$$\begin{aligned} L_z\left(Grp_e(t)\right) &= \otimes\left(L_z\left(E_e(t)\right), L_z\left(E_{e+1}(t)\right), \ldots, L_z\left(E_{e+r-1}(t)\right)\right) \\ &= \sum_{j_e=1}^{M_e} \sum_{j_{e+1}=1}^{M_{e+1}} \cdots \sum_{j_{e+r-1}=1}^{M_{e+r-1}} \left(\prod_{i=e}^{e+r-1} p_{i,j_i}\right) \cdot z^{\left(g_{e,i_e}, g_{e+1,i_{e+1}}, \ldots, g_{e+r-1,i_{e+r-1}}\right)}. \end{aligned} \tag{3}$$

Step 8. Proceed to determine the dynamic reliability and availability of this initial group concerning various required performance levels and time intervals.

$$R(t) = A(t) = \sum_{j_e=1}^{M_e} \sum_{j_{e+1}=1}^{M_{e+1}} \cdots \sum_{j_{e+r-1}=1}^{M_{e+r-1}} \left(\prod_{i=e}^{e+r-1} p_{i,j_i}\right) \cdot \beta\left(\sum_{i=e}^{e+r-1} g_{i,j} \geq W\right). \tag{4}$$

The expressions of the system's reliability and availability are identical but both are not the same. This is due to the failure rate being taken into account when estimating availability, but not while estimating reliability.

Step 9. Establish a minimum required performance threshold, denoted as W, which increases by a certain amount as group increases. Select only the values that surpass this threshold (*i.e.*, values greater than W), or employ the $\wedge$ operator to eliminate the associated failure-related terms.

$$\begin{aligned} \widetilde{L_z}\left(Grp_e(t)\right) &= \wedge\left(L_z\left(Grp_e(t)\right)\right) \\ &= \sum_{j_e=1}^{M_e} \sum_{j_{e+1}=1}^{M_{e+1}} \cdots \sum_{j_{e+r-1}=1}^{M_{e+r-1}} \left(\prod_{i=e}^{e+r-1} p_{i,j_i}\right) \\ &\quad \cdot z^{\left(g_{e,i_e}, g_{e+1,i_{e+1}}, \ldots, g_{e+r-1,i_{e+r-1}}\right)} \cdot \beta\left(\sum_{i=e}^{e+r-1} g_{i,j} \geq W\right). \end{aligned} \tag{5}$$

Step 10. Upon acquiring the L_z-transform and corresponding availability for the first group, the L_z-transform for the subsequent group (Grp_{e+1}) comprising r consecutive elements can be obtained through the application of the operator $\overleftarrow{\otimes}$ on the L_z-transform of the first group (Grp_e) obtained by removing terms of failures and that of the element located at the $(e+r)$th position (E_{e+r}). It's important to note that Grp_{e+1} and Grp_e share $r-1$ common multistate elements (MEs). To derive the L_z-transform of Grp_{e+1}, the first element is removed from Grp_e, and the remaining $r-1$ elements are shifted one position to the left, with the element $e+r$ added to the last position. This transformation results in Grp_e evolving into Grp_{e+1}. Following this procedure, we can compute the L_z-transform of Grp_{e+1} by applying the $\overleftarrow{\otimes}$ operator.

$$\begin{aligned} L_z\left(Grp_{e+1}(t)\right) &= \widetilde{L_z}\left(Grp_e(t)\right) \overleftarrow{\otimes} L_z\left(E_{e+r}(t)\right) \\ &= \sum_{j_e=1}^{M_e} \sum_{j_{e+1}=1}^{M_{e+1}} \cdots \sum_{j_{e+r-1}=1}^{M_{e+r-1}} \left(\prod_{i=e}^{e+r-1} p_{i,j_i}\right) \cdot z^{\left(g_{e,i_e}, g_{e+1,i_{e+1}}, \ldots, g_{e+r-1,i_{e+r-1}}\right)} \\ &\quad \overleftarrow{\otimes} \sum_{j_{e+r}=1}^{M_{e+r}} p_{e+r,i_{e+r}} \cdot z^{g_{e+r,i_{e+r}}} \\ &= \sum_{j_e=1}^{M_e} \sum_{j_{e+1}=1}^{M_{e+1}} \cdots \sum_{j_{e+r-1}=1}^{M_{e+r-1}} \sum_{j_{e+r}=1}^{M_{e+r}} \left(\prod_{i=e}^{e+r-1} p_{i,j_i}\right) p_{e+r,i_{e+r}} \\ &\quad \cdot z^{\left(g_{e,i_e}, g_{e+1,i_{e+1}}, \ldots, g_{e+r-1,i_{e+r-1}}, g_{e+r,i_{e+r}}\right)}. \end{aligned} \tag{6}$$

Step 11. Obtain the dynamic reliability and availability of this group w.r.t. different performance levels and time intervals.

Step 12. Repeat the previous steps to calculate the L_z-transform and corresponding dynamic reliability and availability of all the groups of DCMSWS and the dynamic reliability measures of last group corresponds to that of entire system. The DCMSWS's dynamic reliability and availability can be obtained, as discussed in step 8 and the following indices can be obtained as:

– **Mean expected performance**
The Dynamic Circular Multi-State Sliding Window System's mean expected performance at time $t \geq 0$ is calculated as

$$E(t) = \sum_{\substack{j=1 \\ g_{ij}>0}}^{M} p_{ij}(t) \cdot g_{ij}. \tag{7}$$

– **Cost analysis**
If the service facility is always available, then the expected profit during the time interval $[0, t)$ is obtained by

$$E_p(t) = c_r \int_0^t A(t) - c_s t. \tag{8}$$

4. A numerical illustration: closed recurring water supply system

Consider a closed and recurring water supply system within a thermo-electric plant, consisting of four water pumps. This system serves as an excellent illustration of a Dynamic Circular Multi-State Sliding Window System. The primary function of this system is to pump water and expel steam from a turbine to a cooling tower *via* the four water pumps. Due to variations in water levels, the cooled water can be directed back to the boiler to generate steam for the turbine once more. In this water-cooling setup, each pump must possess sufficient power to transport water and steam to at least the subsequent three consecutive pumps. In the event that three or more consecutive pumps within the system experience failure, the entire system is considered to have failed. Consequently, this system can be aptly represented as a Dynamic Circular Multi-State Sliding Window System with parameters $n = 4$ and $r = 3$. Each water pump, constituting an individual element, operates across a spectrum of four distinct states, denoted as S4, S3, S2, and S1, corresponding to the fourth, third, second, and first states, respectively. Notably, S4 signifies the highest level of performance, while S1 represents complete failure. The intermediary states represent varying levels of intermediate performance, and these states undergo dynamic changes over time. Therefore, the closed and recurring water supply system aligns with the framework of the Dynamic Circular Multi-State Sliding Window System, characterized by $n = 4$ and $r = 3$. Importantly, this system is regarded as repairable, with repair procedures detailed in Section 2. For further insight into the system's performance and reliability, please refer to Table 1, which provides an overview of the states' transition rates and the performance characteristics of the four water pumps within the DCMSWS.

In Table 1, all the components pertain to the water pumps. Concentrating on the very first element within the water pump category, it is possible to construct a system of differential equations. These equations are established with the assistance of the transition rates outlined in Table 1.

$$\begin{aligned}
\frac{\mathrm{d}p_{11}(t)}{\mathrm{d}t} &= -3.14p_{11}(t) + 0.68p_{12}(t) + 0p_{13}(t) + 0.31p_{14}(t) \\
\frac{\mathrm{d}p_{12}(t)}{\mathrm{d}t} &= 1.91p_{11}(t) - 0.68p_{12}(t) + 0.54p_{13}(t) + 0p_{14}(t) \\
\frac{\mathrm{d}p_{13}(t)}{\mathrm{d}t} &= 1.23p_{11}(t) + 0p_{12}(t) - 0.54p_{13}(t) + 0.23p_{14}(t) \\
\frac{\mathrm{d}p_{14}(t)}{\mathrm{d}t} &= 0p_{11}(t) + 0p_{12}(t) + 0p_{13}(t) - 0.54p_{14}(t).
\end{aligned}$$

TABLE 1. Performance and transition rates of water supply system.

Water supply system		Transition rates (1/unit)				Performance Rates
Components	States	S1	S2	S3	S4	
1	S1	0.	1.91	1.23	0.0	0
	S2	0.68	0.0	0.0	0.0	1
	S3	0.	0.54	0.0	0.0	4
	S4	0.31	0.0	0.23	0.0	6
2	S1	0.	1.93	1.53	0.0	0
	S2	0.83	0.0	0.0	0.0	4
	S3	0.	0.71	0.0	0.0	5
	S4	0.29	0.0	0.45	0.0	7
3	S1	0.	1.74	1.36	0.0	0
	S2	0.55	0.0	0.0	0.0	2
	S3	0.	0.65	0.0	0.0	5
	S4	0.29	0.0	0.32	0.0	8
4	S1	0.	1.77	1.99	0.0	0
	S2	0.39	0.0	0.0	0.0	8
	S3	0.	0.12	0.0	0.0	10
	S4	0.21	0.0	0.19	0.0	12

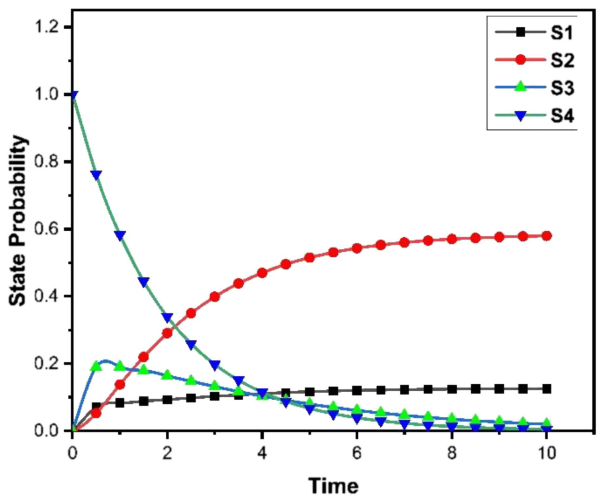

FIGURE 3. Probability of different states for first water pump.

To compute the state probability for the first element of the water pump, we engage in the solution of the system of differential equations, as previously described. This solution is conducted with consideration of the initial conditions: $p_{11}(0)$, $p_{12}(0)$, $p_{13}(0)$, and $p_{14}(0)$, which are illustrated in Figure 3 and are set as follows: $p_{11}(0) = p_{12}(0) = p_{13}(0) = 0$ and $p_{14}(0) = 1$.

After determining the state probabilities for the initial water pump, the probability of the component being in the third state (p_{13}) is then replaced with the probability density function derived from the Erlang distribution. This substitution is made in light of the fact that the departure rate is governed by the Erlang distribution.

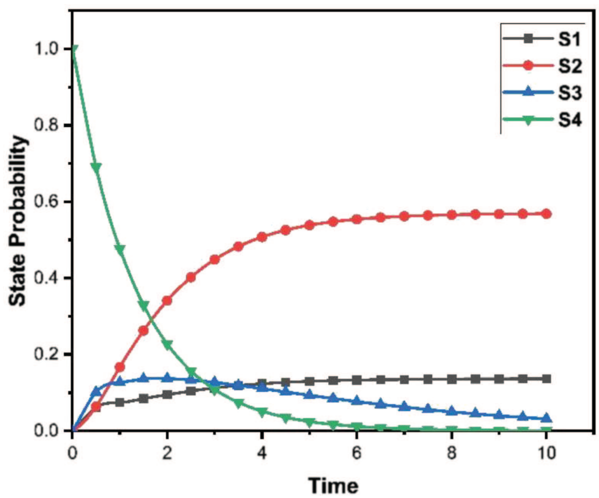

FIGURE 4. State probability of different states for second water pump.

Subsequently, the the L_z-transform for the first component, *i.e.*, the water pump, within the water supply system can be obtained by utilizing equation (2) as follows:

$$\begin{aligned} L_z\left(E_1\left(t\right)\right) &= \sum_{j=1}^{4} p_{1j}\left(t\right) z^{g_{1j}} = p_{11}(t)z^0 + p_{12}(t)\, z^1 + p_{13}(t)\, z^4 + p_{14}(t) z^6 \\ &= p_{11}(t)z^0 + p_{12}(t)z^1 + \frac{\lambda^{\mu} t^{\mu-1} e^{-\lambda t}}{(\mu-1)!} z^4 + p_{14}(t)z^6 \\ &= p_{11}(t)z^0 + p_{12}(t)z^1 + \left(\frac{0.31^{1.23} t^{0.23} e^{-0.31t}}{0.910754}\right) z^4 + p_{14}(t)z^6. \end{aligned}$$

Similarly, following a similar methodology, the differential equations for the second, third, and fourth water pumps within the water supply system are established by substituting $i = 2$, 3, and 4, and accordingly adjusting the state variables within equation (1). Systematically addressing these equations while adhering to the prescribed initial conditions allows us to deduce the state probabilities for the second, third, and fourth water pumps of the water supply system, which are presented in Figures 4, 5, and 6, respectively.

Then, the L_z-transform for second, third and fourth component of water supply system are found to be:

$$\begin{aligned} L_z\left(E_2\left(t\right)\right) &= \sum_{j=1}^{4} p_{2j}\left(t\right) z^{g_{2j}} = p_{21}(t)z^0 + p_{22}(t)z^4 + \left(\frac{0.29^{1.53} t^{0.53} e^{-0.29t}}{0.887567}\right) z^5 + p_{24}(t)z^7 \\ L_z\left(E_3\left(t\right)\right) &= \sum_{j=1}^{4} p_{3j}\left(t\right) z^{g_{3j}} = p_{31}(t)\, z^0 + p_{32}(t)z^2 + \left(\frac{0.29^{1.36} t^{0.36} e^{-0.29t}}{0.890184}\right) z^5 + p_{34}(t)z^8 \\ L_z\left(E_4\left(t\right)\right) &= \sum_{j=1}^{4} p_{3j}\left(t\right) z^{g_{3j}} = p_{41}(t)\, z^0 + p_{42}(t)z^8 + \left(\frac{0.21^{1.99} t^{0.99} e^{-0.21t}}{0.995813}\right) z^{10} + p_{44}(t)z^{12}. \end{aligned}$$

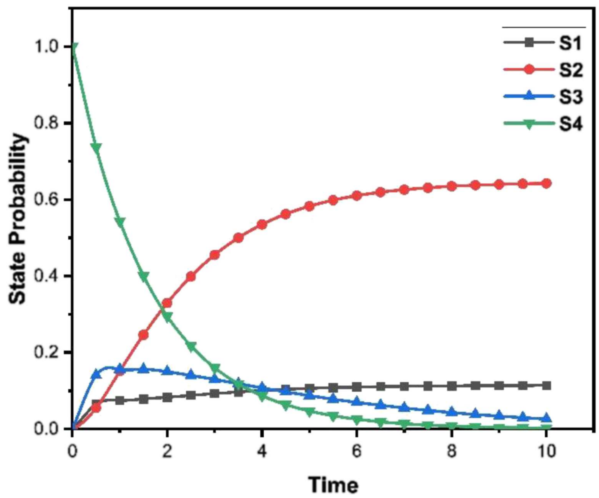

FIGURE 5. State probability of different states for third water pump.

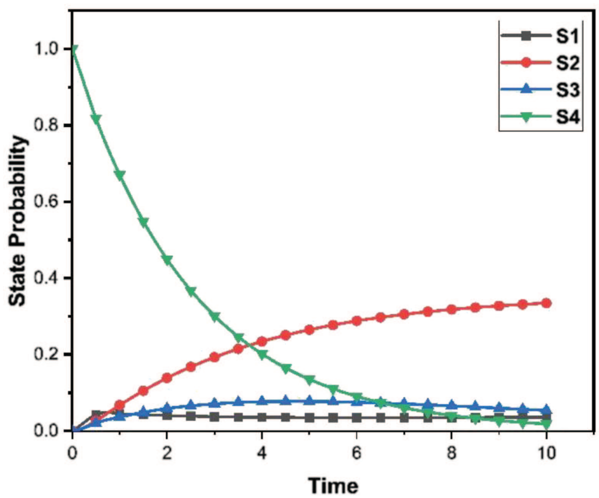

FIGURE 6. State probability of different states for forth water pump.

Subsequently, the L_z-transform of the first group, which comprises ME 1, ME 2, and ME 3, is computed as follows:

$$\begin{aligned} L_z\left(Grp_1(t)\right) &= \otimes\left\{L_z\left(E_1(t)\right), L_z\left(E_2(t)\right), L_z\left(E_3(t)\right)\right\} \\ &= \otimes\left\{\begin{array}{l} \left(p_{11}(t)\, z^0 + p_{12}(t)\, z^1 + \left(\frac{0.31^{1.23} t^{0.23} e^{-0.31t}}{0.910754}\right) z^4 + p_{14}(t)\, z^6\right), \\ \left(p_{21}(t)\, z^0 + p_{22}(t)\, z^4 + \left(\frac{0.29^{1.53} t^{0.53} e^{-0.29t}}{0.887567}\right) z^5 + p_{24}(t)\, z^7\right), \\ \left(p_{31}(t)\, z^0 + p_{32}(t)\, z^2 + \left(\frac{0.29^{1.36} t^{0.36} e^{-0.29t}}{0.890184}\right) z^5 + p_{34}(t)\, z^8\right) \end{array}\right\}. \end{aligned} \tag{9}$$

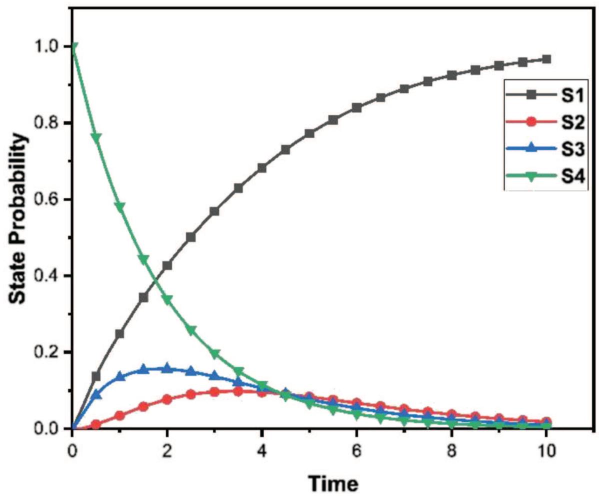

FIGURE 7. State probability of different states for water pump-1.

As previously mentioned, in the pursuit of determining the state probabilities for assessing reliability, the equation (1) pertaining to the first water pump takes the form:

$$\begin{aligned}
\frac{\mathrm{d}p_{11}(t)}{\mathrm{d}t} &= 0p_{11}(t) + 0.68p_{12}(t) + 0p_{13}(t) + 0.31p_{14}(t)\\
\frac{\mathrm{d}p_{12}(t)}{\mathrm{d}t} &= 0p_{11}(t) - 0.68p_{12}(t) + 0.54p_{13}(t) + 0p_{14}(t)\\
\frac{\mathrm{d}p_{13}(t)}{\mathrm{d}t} &= 0p_{11}(t) + 0p_{12}(t) - 0.54p_{13}(t) + 0.23p_{14}(t)\\
\frac{\mathrm{d}p_{14}(t)}{\mathrm{d}t} &= 0p_{11}(t) + 0p_{12}(t) + 0p_{13}(t) - 0.54p_{14}(t).
\end{aligned}$$

Upon solving this system of differential equations with the specified initial conditions, we derive the state probabilities for the first water pump, as visualized in Figure 7. A similar procedure is employed to ascertain the state probabilities for each water pump of Closed Recurring Water Supply System

Upon successfully acquiring the L_z-transform for the first group, as guided by the determined state probabilities, the cumulative L_z-transform of the water supply system is evaluated by solving the equation (9). Ultimately, the dynamic reliability for the closed recurring water supply system is derived through equation (9), considering the system's minimum stipulated performance threshold denoted as W. The visual depiction of the system's reliability across time, encompassing different levels of mandatory minimum performance, is vividly illustrated by a three-dimensional graph as illustrated in Figure 8.

Likewise, the dynamic availability of the initial group of elements within the water supply system can be determined using the L_z-transform, which is obtained by solving equation (9) with state probabilities graphed in Figures 3 to 6. This dynamic availability is visualized in a three-dimensional representation concerning minimum performance and time, as depicted in Figure 9.

Once the L_z-transform and the corresponding dynamic reliability and availability metrics for the first group have been acquired, the $\wedge$ operator is applied to the L_z-transform of this group, effectively removing terms associated with failures. In this instance, let us consider $W = 18$, with incremental increments of 4 as the number of groups increases. This choice is made to account for the evolving dynamic characteristics of each

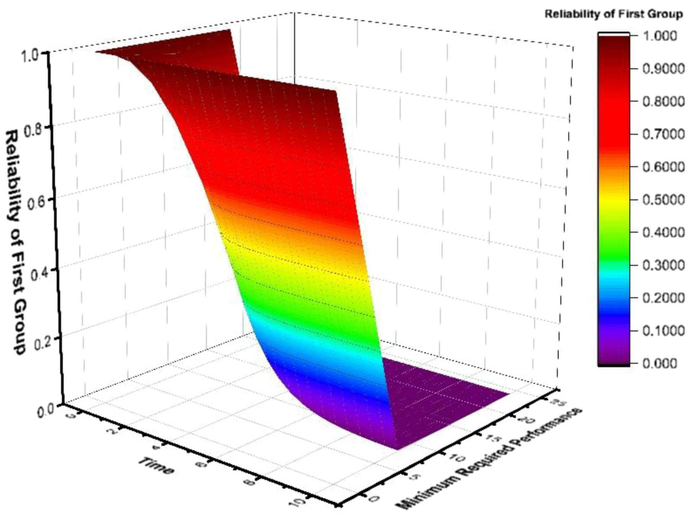

FIGURE 8. Dynamic reliability of first group of water supply system.

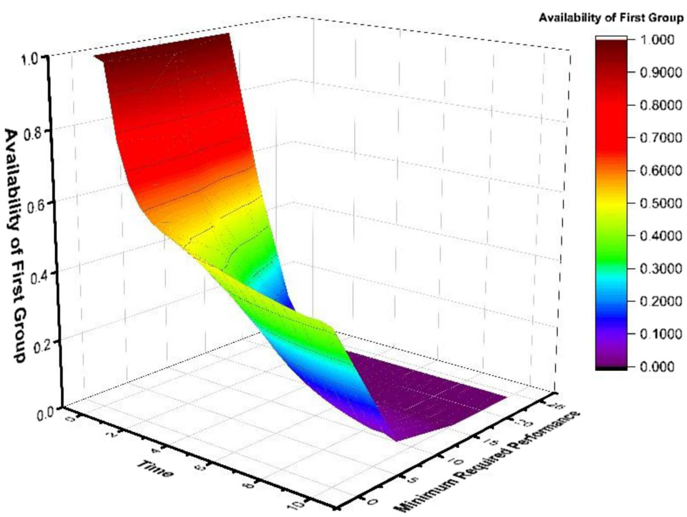

FIGURE 9. Dynamic availability of first group of water supply system.

group. Hence, the L_z-transform of first group after removing terms related to failures can be obtained as

$$\widetilde{L_z}\left(Grp_1\left(t\right)\right) = \wedge\left(L_z\left(Grp_1\left(t\right)\right)\right)$$
$$\widetilde{L_z}\left(Grp_1\left(t\right)\right) = P_{14}P_{23}P_{34}\cdot z^{6,5,8} + P_{14}P_{24}P_{33}\cdot z^{6,5,8} + P_{14}P_{24}P_{34}\cdot z^{6,7,8}.$$

Subsequently following this, the L_z-transform for the second group of the water supply system, encompassing ME 2, ME 3, and ME 4, is derived. This is accomplished by employing the operator $\overleftarrow{\otimes}$ on the L_z-transform of the first group, from which the failure-related terms have been removed, and combining it with the L_z-transform

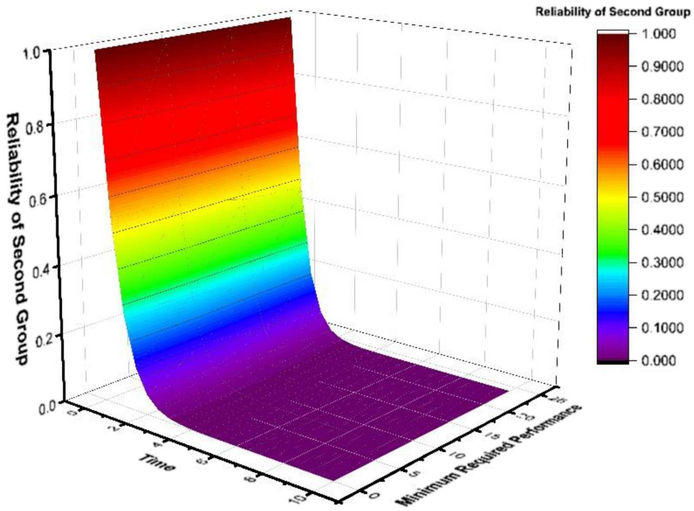

FIGURE 10. Dynamic reliability of second group of water supply system.

of the subsequent element in the preceding group, as outlined in step 10.

$$\begin{aligned} L_z\left(Grp_2\left(t\right)\right) &= \widetilde{L_z}\left(Grp_1\left(t\right)\right) \overleftarrow{\otimes} L_z\left(E_4\left(t\right)\right) \\ &= \left(P_{14}P_{23}P_{34}\cdot z^{6,5,8} + P_{14}P_{24}P_{33}\cdot z^{6,5,8} + P_{14}P_{24}P_{34}\cdot z^{6,7,8}\right) \overleftarrow{\otimes} \Bigg(p_{41}\left(t\right)z^0 + p_{42}(t)z^8 \\ &\quad + \left(\frac{0.21^{1.99}t^{0.99}e^{-0.21t}}{0.995813}\right)z^{10} + p_{44}(t)z^{12}\Bigg). \end{aligned} \tag{10}$$

Subsequently, the dynamic reliability and availability of the second group within the water supply system can be determined by solving equation (10) while considering the respective state probabilities. This analysis is conducted across varying time intervals and different performance requirements, and the results for dynamic reliability and availability are visually presented in a three-dimensional graph, as showcased in Figures 10 and 11, respectively.

Applying the $\wedge$ operator with a threshold value of $W = 26$ to the L_z-transform of the second group in order to eliminate failure-related terms is represented as follows:

$$\begin{aligned} \widetilde{L_z}\left(Grp_2\left(t\right)\right) &= \wedge\left(L_z\left(Grp_2\left(t\right)\right)\right) \\ &= P_{14}P_{23}P_{34}P_{42}\cdot z^{5,8,10} + P_{14}P_{23}P_{34}P_{42}\cdot z^{5,8,12} + P_{14}P_{24}P_{33}P_{44}\cdot z^{7,5,12} \\ &\quad + P_{14}P_{24}P_{34}P_{43}\cdot z^{7,8,10} + P_{14}P_{24}P_{34}P_{43}\cdot z^{7,8,12}. \end{aligned}$$

Continuing with the process, the L_z-transform for the third group of the water supply system, comprising ME 3, ME 4, and ME 1, is calculated using the $\overleftarrow{\otimes}$ operator as follows:

$$\begin{aligned} L_z\left(Grp_3\left(t\right)\right) &= \widetilde{L_z}\left(Grp_2\left(t\right)\right) \overleftarrow{\otimes} L_z\left(E_1\left(t\right)\right) \\ &= \Big(P_{14}P_{23}P_{34}P_{42}\cdot z^{5,8,10} + P_{14}P_{23}P_{34}P_{42}\cdot z^{5,8,12} + P_{14}P_{24}P_{33}P_{44}\cdot z^{7,5,12} \\ &\quad + P_{14}P_{24}P_{34}P_{43}\cdot z^{7,8,10} + P_{14}P_{24}P_{34}P_{43}\cdot z^{7,8,12}\Big) \\ &\overleftarrow{\otimes} \left(p_{11}(t)z^0 + p_{12}(t)z^1 + \left(\frac{0.31^{1.23}t^{0.23}e^{-0.31t}}{0.910754}\right)z^4 + p_{14}(t)z^6\right). \end{aligned} \tag{11}$$

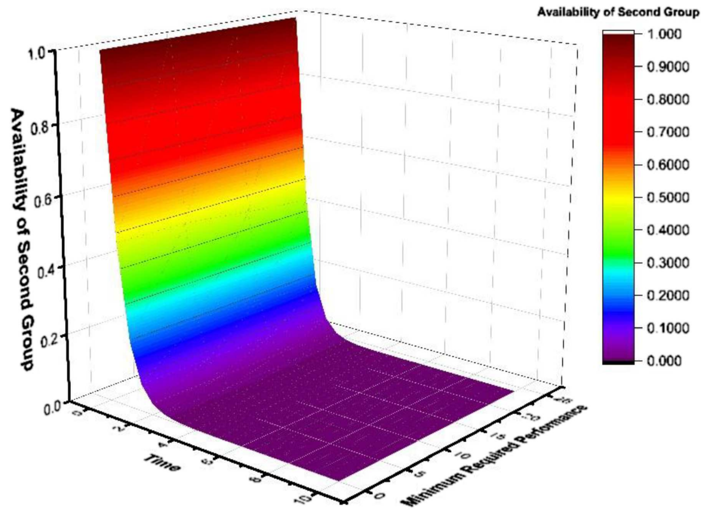

FIGURE 11. Dynamic availability of second group of water supply system.

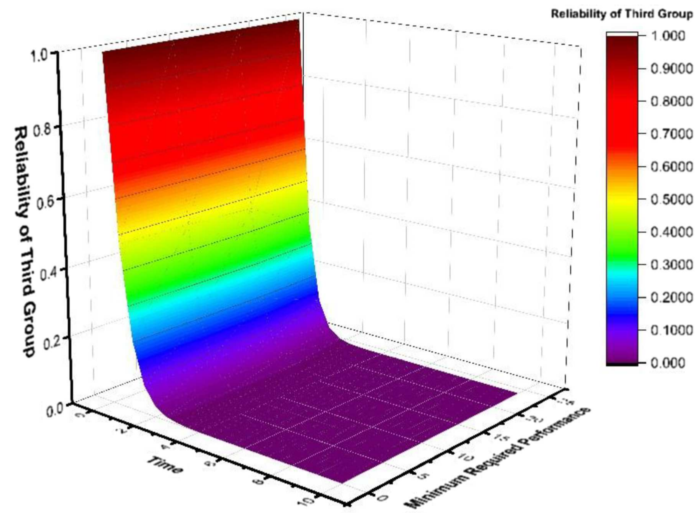

FIGURE 12. Dynamic reliability of third group of water supply system.

Following this, the dynamic reliability and availability of the third group within the water supply system can be ascertained by solving equation (11) while accounting for the relevant state probabilities. This comprehensive analysis encompasses diverse time intervals and varying performance criteria. The outcomes are visually depicted in three-dimensional graphs, presented in Figures 12 and 13.

Finally, the L_z-transform of fourth group which consists of ME 4, ME 1, ME 2 which is equal to that of entire system, can be obtained by following the same procedure. After L_z-transform of whole system is obtained, the following dynamic reliability characteristics can be obtained.

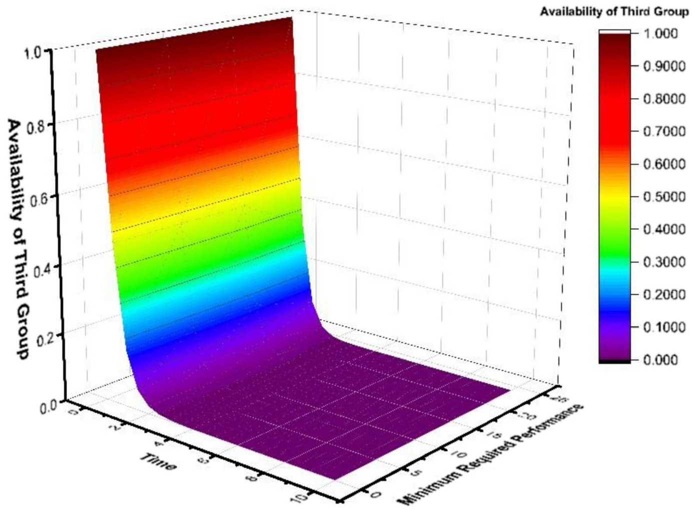

FIGURE 13. Dynamic availability of third group of water supply system.

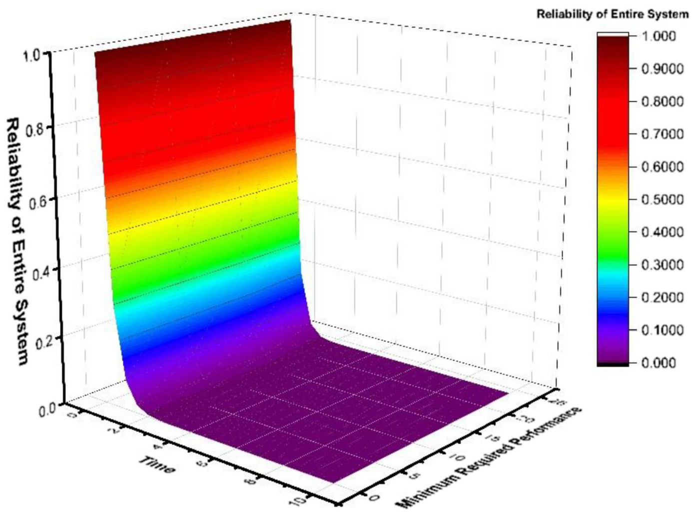

FIGURE 14. Dynamic reliability of entire water supply system.

4.1. Reliability

The dynamic reliability of the entire water supply system is determined and depicted in a three-dimensional graph, relating to both required performance and time. This visualization is presented in Figure 14.

4.2. Availability

The dynamic availability of the entire water supply system is calculated and presented in a three-dimensional graph, illustrating its relationship with both the required performance and time. This visualization is showcased in Figure 15.

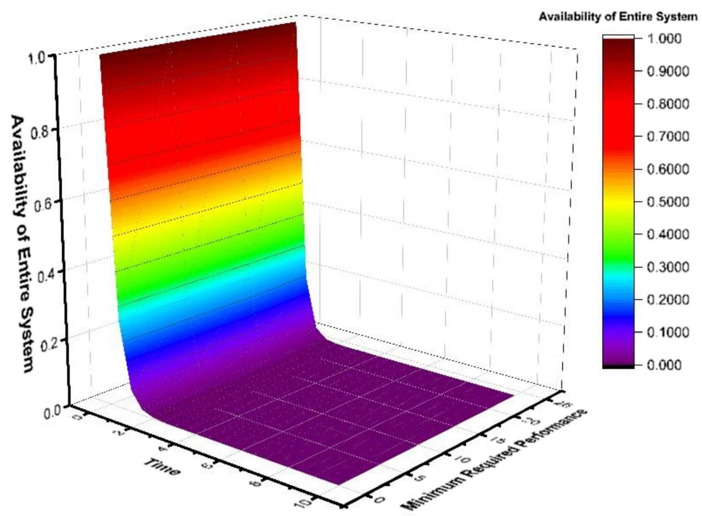

FIGURE 15. Dynamic availability of entire water supply system.

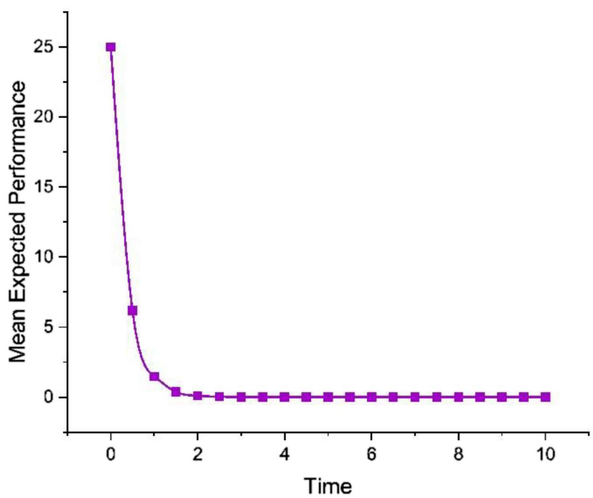

FIGURE 16. Mean expected performance of the water supply system.

Both reliability ($R(t)$) and availability ($A(t)$) exhibit exponential decay over time, with reliability initially higher ($R(t) > A(t)$) before a crossover occurs and availability becomes more resilient ($A(t) > R(t)$). This trend persists across all operational conditions, with complete system exhaustion occurring by $t = 4$. The findings highlight the importance of maintenance scheduling before $t = 1$ to sustain performance above critical thresholds.

4.3. Mean expected performance

The mean expected performance of the entire water supply system is determined using equation (7) in combination with the final L_z-transform of the complete system. This calculated mean expected performance is then presented in Figure 16.

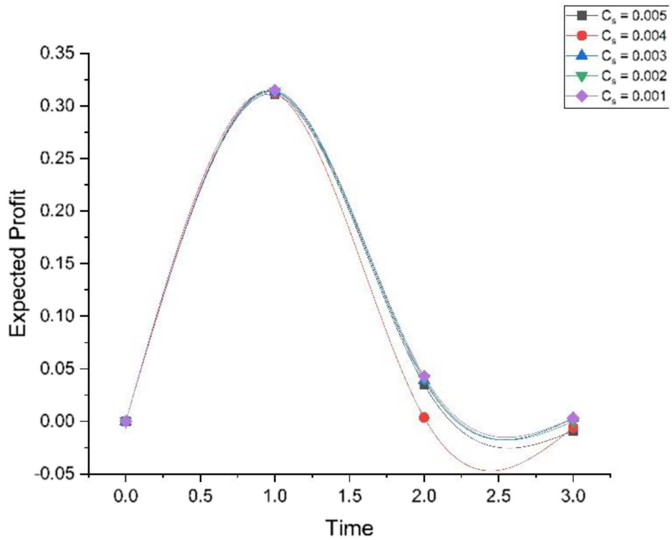

FIGURE 17. Expected profit of the water supply system.

The graph depicts the exponential decay of mean expected performance over time, starting at 25 ($t = 0$) and sharply declining to near-zero values by $t = 5$. The rapid drop suggests significant early-phase degradation or convergence to equilibrium. Performance stabilizes at negligible levels beyond $t = 5$, indicating long-term asymptotic behavior. This trend highlights the critical role of initial conditions in the system's dynamics.

4.4. Cost analysis

Setting revenue cost (c_r) at 5 and changing service cost (c_s) as, 0.001, 0.002, 0.003, 0.004, 0.05 and varying time from 0 to 3 units, the expected profit is obtained and is depicted in Figure 17.

The results demonstrate a consistent profit peak ($\sim$0.31$-$0.32) at $t = 1$ across all cost scenarios ($C_s = 0.001 - 0.005$), followed by exponential decay. Higher cost parameters ($C_s \geq 0.004$) exhibit improved resilience, maintaining non-negative profits through $t = 3$, while lower values lead to earlier losses.

5. Conclusion

This paper presents a novel mathematical model to study the dynamic circular multi-state sliding window system (DCMSWS), an extension of the circular sliding window system that operates in a dynamic context. One distinguishing feature of the proposed model is that it incorporates the continuous evolution of each component's state over time, making their probabilities time-dependent. The DCMSWS is deemed to have failed when the combined performance of any set of r consecutive multistate elements (MEs) falls below a predefined threshold value, W. Reparability is a central aspect of this system, encompassing immediate repairs for minor failures and queuing for major failures in an $M|M|1:\infty|FCFS$ fashion while awaiting repair. Importantly, after major repairs, components do not revert to their optimal state but instead settle in a condition between "as good as new" and "as bad as old". Employing an algorithm rooted in the L_z-transform, this paper calculates dynamic reliability indices, including reliability, availability, mean expected performance, and expected profit. The considered dynamic model holds significant relevance for practical engineering systems, offering an enhanced means of evaluating the system's dynamic reliability measures. The study's findings are applied to a Water Supply System, illustrating the real-world applicability of the proposed model.

The reliability and availability of each group within the water supply system have been determined. For the first group, reliability experiences a decline as w changes at time t, but it regains a value of 1 when W equals 0. In contrast, the availability of the first group steadily decreases for each W, and the rate of decline in reliability is greater than that of availability, except in cases where the required performance is zero. For the other groups and the entire system, both reliability and availability decrease over time and eventually stabilize. Notably, at the outset, reliability is slightly higher than availability in relation to time and various required performance levels. However, as time progresses, availability surpasses reliability. This shift is attributed to the repair process, which follows an $M|M|1:\infty|FCFS$ queue in the availability calculations. The mean required performance and the expected cost is also determined for the water supply system. To the best of our knowledge, this study marks the pioneering effort in the realm of dynamic circular multi-state sliding window systems and their evaluation of reliability indices through the utilization of the L_z-transform, intricately connected to queueing theory. The innovative approach introduced here exhibits significant potential for application in a range of complex engineering systems that adhere to the specified criteria. This promises to expand its practical relevance and impact within various engineering domains.

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Data availability statement

No specific data has been used in the preparation of manuscript.

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