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SUSTAINABLE DUAL-CHANNEL SUPPLY CHAIN MODEL WITH CIRCULAR ECONOMIC INDEX, GREEN AND PRICE SENSITIVE DEMAND

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Abstract. A non-green manufacturing process can destroy the environment rapidly and it creates global warming. On the other hand, green production processes and circular economic products can reduce environmental pollution. Waste management through reuse, recycling, etc., is equally crucial for a healthy ecosystem and for saving natural resources. So, circular economic product based green manufacturing model development is very important in the current situation which can apply to the manufacturing industry. For this, a circular economic and green product based dual supply chain model is developed in this study. Here, three supply chain participants (manufacturer, online retailer and offline retailer) are considered where the manufacturer produces circular economic green products and fulfils the customers' demand by retailers. For the online channel, the retailer purchases the products from the manufacturer and sales to the customers through online mode. But in the offline channel, the retailer completed the same tasks through offline mode. Also, it is assumed that the market demand is dependent on retailers' selling price, product's green level and circular economic index. Moreover, the retail price is influenced by the circular economic index. Then four different problems are proposed that maximizes the profits of manufacturer, retailers (online and offline) and integrated supply chain system and solved by centralized and decentralized methods. The objective of this study is to determine the optimal selling prices, product's green level and circular economic index which maximizes the profits of the supply chain members. Finally, sensitivity analysis is carried out to determine the effects of model parameters on the optimal policy and draw a fruitful conclusion from this study.

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1. Introduction

As industries face increasing pressure from environmental regulations, consumer awareness, and corporate social responsibility, integrating sustainability into supply chain models has become essential. So, a dual-channel supply chain (which includes both traditional and online channels) must adapt to sustainable practices to minimize waste, optimize resources, and reduce carbon footprints. On the other hand, the circular economy focuses on

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reducing waste, reusing materials, and recycling products, moving away from the traditional linear "take-make-dispose" model. Introducing a circular economic index in supply chain modeling ensures that businesses measure and improve their sustainability efforts, promoting closed-loop supply chains. Moreover, modern consumers are increasingly green-conscious and prefer products that are eco-friendly, ethically sourced, and sustainable. Integrating a green demand factor into the supply chain model helps businesses align with this shift and cater to environmentally responsible customers. Finally, while sustainability is important, price remains a key factor influencing consumer purchasing decisions. A dual-channel supply chain must consider price-sensitive demand, optimizing pricing strategies for both traditional and online markets while maintaining sustainability efforts. Motivated from the above mentioned factor, it is needed to develop a sustainable dual-channel supply chain model that integrates circular economy principles, green demand, and price sensitivity for modern businesses. It not only addresses environmental concerns but also enhances profitability, consumer trust, and long-term viability in a rapidly evolving market.

1.1. Literature review

The demand for everything needed for daily life has increased as the population has grown. As a result, numerous producing companies' production rates have been dramatically boosted. Together with the production, some of the pollutants, including CO_2, SO_2, and others, are discharged into the environment, posing a severe threat to human health and generating environmental contamination. People have learned about this predicament over time. Because of this, people are more conscious of the environment and their health. As a result of increased awareness, consumers are becoming more interested in green products, motivating the industry to make green goods. Cherian and Jacob [5] made an effort to understand what green marketing is and how consumers are aware of it. Ghosh and Shah [12], in addition to addressing the coordination concerns in their paper as green initiatives, tried to make the supply chain green by including environmentally sensitive consumer demand. By considering the green logistic performance index, Kim and Min [22] attempted to improve the supply chain's health. This young generation represents the majority of our nation. To prevent environmental chaos, young consumers' understanding of a healthy environment is crucial. Yadav and Pathak [50] examined young people's desire to buy environmentally friendly goods. A healthy supply chain can be achieved through green measures, and measuring carbon emissions [29] is one of them. A further green initiative is remanufacturing. Remanufacturing was taken into account by Giri *et al.* [15], which encourages the supply chain to become more environmentally friendly. Panja and Mondal [43] developed a model for a green supply chain by taking the greenness of products into account in the consumer demand function. Zhang *et al.* [53] generated two different product categories and marketed them to different consumers, types-one environmentally conscious and the other less so by way of a single shop. By developing a model, they were able to determine how many consumers were gravitational towards buying environmentally friendly goods. They also used the model to determine whether or not investing in green technologies would be more profitable. Ghosh *et al.* [13] observed how consumers are becoming increasingly aware of the need for greener products in today's life cycle for a healthy lifestyle. They worked hard to create a green product and developed their supply chain model using a consumer demand function based on the product's level of greenness. Wani and Mishra [48] viewed the manufacturer's choice to invest in the greening of products across the entire production process as crucial. Jamali and Rasti-Barzoki [19] in their model, also took into account green and non-green products to increase the sustainability of the supply chain. Sana [46] developed two types of demand functions, one for customers who care about the environment and another for those who don't. They invest in green technology to cut carbon emissions as well. Mondal and Giri [33] considered a supply chain with green products in demand for a produced good and also took remanufacturing into account to increase the supply chain's sustainability by controlling waste. Pal *et al.* [42] show their environmental awareness through considering green technology level of products and also apply cap and trade policy to reduce carbon emission. They consider recycling of end of life products along with production of new products. By lowering CO_2 emissions and PM (particulate matter) concentrations across the network, environmental goals are met in work of Kazancoglu *et al.* [20]. Using Big Data services, this article by Wu *et al.* [49] examines the issues of green innovation investment and coordination for a three-tiered closed-loop supply chain. Consciousness about the environment impels Haque *et al.* [17] to introduce the green-sensitive demand rate in their work [17].

The Circular Economic Index (CEI) alludes to the tactical, financial, and environmental benefits of recycling, reusing, reducing and recovering. The new tactic has employed a CEI as a supply chain decision-making instrument. To fully comprehend CEI, it must realize first the circular economy (CE). In today's world, numerous manufacturing firms have adopted the circular economy (CE) approach as a crucial sustainability strategy. The CEI is gaining more and more attention as a potential replacement for the traditional linear economic model that has a good influence on the environment while promoting economic progress. But what is CE? Many people are still unsure about the "circular economy" concept today. According to a study by Kirchherr *et al.* [23], by reducing, recycling, reusing, and recovering resources, a circular economy offers us access to an ecosystem that fosters social, economic, and environmental sustainability in both existing and emerging business models. In production processes, this 4R replaces the "end-of-life" concept. They formulated a coding framework for coding 114 definitions identified by them. Wastes and pollutants from a linear economy would be released back into the atmosphere, which is tremendously destructive to the environment. At the same time, the circular economy does not produce any trash. By proposing a circular system that preserves the value of goods, materials, and resources in the economy for as long as feasible, the CE seeks to do away with the take-make-dispose linear trend of manufacturing and utilization. Korhonen *et al.* [24] provided a valuable concept of CE from two perspectives and analyzed the upcoming issues that will arise for future researchers working on CE in making society more sustainable on both an economic and environmental level. The research work of Morseletto [36] assessed the circular economy's targets and looked at which targets make switching from a linear to a circular economy easier. To thoroughly investigate the phenomena using strict and repeatable research criteria, the paper of Merli *et al.* [30] employed a systematic approach to examining the body of CE literature. The purpose of study of Lewandowski [25] is to classify the attributes of the CE by the structure of the business model. There are two different types of researchers: one who looks into how the CE concept will affect plans and designs, and the other who looks into how the performance of the CE may be gauged. For this, some literature on CE indicators. This (Maio and Rem [9]) study's proposed CEI as,

$$\text{CEI} = \frac{\text{Material value recycled from EOL (end of life) product(s)}}{\text{Material value needed for (re-)producing EOL products(s)}}\cdot$$

The primary index-based environmental sustainability evaluation approaches used today were examined by Elia *et al.* [10] regarding how well they work to determine a system's circularity. This paper highlights the significant and crucial steps in evaluating a CE strategy along with a systematic technique for selecting the appropriate methodology. In the work of Moraga *et al.* [35], the bulk of the indicators, as shown by the framework's portrayal, emphasize the preservation of resources through recycling. The framework graphic of this work advises employing a group of indicators instead of one indicator to assess CE. Rabta [44] developed an EOQ inventory model, with the circular economic index as a primary decision tool influencing the demand rate and unit gross profit. Wani and Mishra [48] and Thomas and Mishra [47] both developed a circular economic model in their papers, with the CEI as a primary decision tool influencing both the demand rate and unit gross profit. The results in the work of Khorshidvand *et al.* [21] demonstrate that encouraging circular economy activities through cooperation can benefit both the environment and the economy. According to Sahadevan and Mishra [45] findings, the application of the circularity concept increases supply chain profitability while preventing pollution.

Given the current state of technology development, ordinary people may readily access all communication devices (including mobile phones, laptops, etc.) as well as internet services. So, in addition to the conventional single retail channel, many production managers are considering implementing a direct channel through online communication. Modern technology has given people a chance to do marketing activities at home. Online marketing offers the opportunity to see several items together in less time. Currently, there are two categories of people: those drawn to online marketing and those considering the offline market. Manufacturers preferred to have two types of supply chains, one direct internet channel, and the other traditional retail channel, in order to better meet consumer expectations in this cutthroat industry. Gangshu (George) Cai [3] created four situations, two of which contained dual channels. One is made up of a retail channel and a direct channel, while the other is made up of two retail channels. They made an effort to compare these four possibilities. Dan *et al.* [7] developed an analytical framework to investigate pricing strategies in two distinct channels-traditional retail and direct online and they also attempted to determine the influence of retail services on these pricing strategies using two Stackelberg game theoretic models. Chen and Bell [4] created a two-channel supply chain by segmenting the market into two, one with a complete return policy to customers and the other. Matsuo and Ohmura [37] reevaluated of Chen and Bell [4] research by applying the risk aversion effect. Giri *et al.* [14] put in place two dual supply chains: one is forward dual supply, consisting of traditional retail and direct online channels, and the other is reverse dual supply, where returned goods are collected through a collector in one channel and directly collected by an e-tail channel in another. Li *et al.* [26] developed a dual supply chain model, one with a traditional retail channel and the other with an e-tail channel. They looked into which channel was ideal for implementing return policies. Later, Mondal *et al.* [34] similarly created two dual supply chains as in Giri *et al.* [14]. However, in their model, retailers collected returned instead of collectors by retailers as opposed to collectors through traditional offline channels. In the research work of Batarfi *et al.* [2] the manufacturer used traditional retail channels to sell customized goods (*i.e.*, make-to-order) and online retail channels to sell standard goods (*i.e.*, make-to-stock) to increase its earnings. Dey and Giri [8] created a concept in which two channels carried out recycling. Recycling was done in two ways: by the recycler in one channel and the manufacturer in the other. Yang *et al.* [51] created a game-theoretical model with dual channel illumination and online customer reviews. Due to the existence of two different sorts of clients, one of whom is price-sensitive and the other is quality-sensitive, there were two different quality levels in each of the dual channels [55]. In this work Zhang *et al.* [54] developed a dynamic pricing algorithm for solving a two-stage dual supply chain model. One e-commerce and one traditional retail channel made up each stage. Recycling in dual channels exists in the work of Huang and Liang [18]. They created three models-one collected by a collector, one *via* online mode, and the final one combining both. Mandal and Pal [28] construct different game theoretical strategies for assessing the behavior of closed-loop dual channel (direct online channel and traditional off-line retail channel), which deals with fresh, green, and refurbished products. Through dual-channel (direct online channel and traditional off-line retail channel) marketing for green products, Barman *et al.* [1] seek to evaluate the most efficient pricing tactics with and without government subsidy policies. Pal *et al.* [41] in their work optimize prices, green level of products and collection effort level with dual channel strategy both in selling and collection of products. Pal [39] investigates the optimal pricing, green level of products and buy-back price strategies under dual channel environment in a closed-loop supply chain model. Guin *et al.* [16] consider online direct channel and offline channel through retailer as a selling strategy of their green goods. Pal [38] consider online offline selling strategy with stochastic demand functions. Pal and Sarkar [40] show how different pricing strategies of supply chain members in dual channel selling environment affect the optimal decisions of the members under their different power strategies. The work of Li *et al.* [27] demonstrates how green investments can reach pareto equilibrium in various scenarios using a dual-channel structure. This study of Das *et al.* [6] examines the combination of product movement across traditional retail and online channels. According to the computational findings in the work of Feng *et al.* [11], the cost-sharing contract can raise the dual-channel supply chain's overall profit.

1.2. Research gaps

The literature review highlights a significant research gap and indicates that there has been no prior effort to implement a dual-channel supply chain model with a manufacturer and online-offline retailer with price, green, and CEI sensitive demand. It would be more beneficial to investigate all the genuine issues like the impact of online and offline retail prices, CEI and green degree on customer demand, both economical and environmental awareness in terms of investment for increasing greenness and CEI of product, one manufacturer and online-offline retailer business situation and so on under one umbrella. This study aims to bridge the research gaps and obtain higher supply chain profits by increasing products' green degree and CEI in online and offline retail

Table 1. Comparison between current and existing works based on different considerations.

Author(s) (Year)	Demand rate dependent on product's		Green investment	CEI investment	Multiple retailers	Dual supply chain	Online retail channel	Off-line retail channel	CEI dependent price
	Green level	CEI							
Dan *et al.* [7]						√		√	
Ghosh and Shah [12]	√		√					√	
Manna *et al.* [29]					√			√	
Jamali and Rasti-Barzoki [19]	√		√		√	√		√	
Giri *et al.* [15]	√		√					√	
Li *et al.* [26]						√		√	
Mondal *et al.* [34]	√		√			√		√	
Panja and Mondal [43]	√							√	
Rabta [44]		√							√
Zhang *et al.* [53]	√		√					√	
Ghosh *et al.* [13]	√		√		√			√	
Zhang *et al.* [55]	√		√		√	√		√	
Zhang *et al.* [54]						√		√	
Giri and Mondal [15]	√		√		√			√	
Wani and Mishra [48]		√	√						√
Huang and Liang [18]						√			
Thomas and Mishra [47]		√	√						√
Mondal and Giri [31]	√		√					√	
Barman *et al.* [1]	√		√			√		√	
Mandal and Pal [28]	√		√			√		√	
This work	√	√	√	√	√	√	√	√	√

environments, promoting economic and environmental sustainability. Table 1 presents a comparative analysis of this study with relevant literature.

Table 1 provides the summary of the relevant literature as follows:

– While existing studies have analyzed the impact of CEI on demand and pricing, the role of technological investments in enhancing CEI and the combined influence of CEI and greenness on consumer demand has been understudied. Also, the maximization of individual supply chain member profits has not been a primary focus in previous research on CEI.
– Most works consider products' greenness, prompting investment in green technologies. Product recyclability (CEI) is a critical factor for sustainability and waste reduction, but it's often neglected in their research on consumer demand and technological investment. If it is possible to produce products with high recyclability levels, then it can contribute to a more sustainable future by streamlining waste management procedures.

1.3. Contributions

In this work, a circular economic green product based on a dual-channel supply chain model is developed. The concepts of online and offline marketing with different selling prices are included in this supply chain model. In the proposed supply chain model, single manufacturer and retailers (online and offline) are considered. Here, retailers' selling price, product's green level and circular economic index sensitive market demand is assumed for formulating the model. Also, the retailers' selling price is dependent on the product's circular economic index. The main objective of this study is to determine the optimal values of the product's circular economic index and green level, wholesale prices of the manufacturer, retailers' selling prices that maximize the profits of the integrated supply chain system as well as manufacturer and retailers (online and offline). For this, based on the manufacturer and retailers' profits, four maximization problems are constructed. Among them three maximization problems are solved by the Stackelberg game approach in decentralized model where the manufacturer and retailers are considered as leader and followers respectively. Then, to verify the feasibility of the model, a numerical example is carried out. Finally, sensitivity analyses are performed by increasing and decreasing the parameters' values of the model to get a effective conclusion. Our primary objective is to identify the existing research gaps and to explore the following research questions:

– What are the most effective strategies for manufacturer and retailers to meet the diverse needs of price-conscious, green-conscious, and CEI-conscious consumers across both online and offline channels?
– Which parameters are most influential in driving increased profitability for manufacturer and retailers?
– Which parameters contribute to increased green levels and CEI in products?
– How are consumer demands affected in online and offline retail channels?

This study offers the following insights: Firstly, our research focuses on developing strategies to enhance products' reusability (CEI) while maintaining their green attributes. Secondly, the manufacturer makes technological investments to increase the greenness and CEI of products. Thirdly, online and offline retailers exist here, *i.e.*, it is investigated the interplay between online and offline retail channels in satisfying consumer demand. Finally, retail prices exhibit a linear relationship with product reusability (CEI).

The rest of this work is constructed as follows: Some notations and assumptions, with some considerations, are given in Section 2 for the formulation of used models. In Section 3, the centralized model and Stackelberg game theoretic decentralized model is developed for performing the optimizations of profit functions with some propositions. Section 4 validates the formulated models of Section 3 numerically. The numerical results are elaborated in this section. After validating the models numerically, in Section 5, how changes in critical parameters' values affect the key decisions of supply chain members is analyzed. Section 6 discusses how the results obtained from constructed models in this work are helpful for production managers through economic and environmental sustainability. Section 7 concludes this work with some limitations and gives the future direction of this research.

2. Notations and assumptions

The following notations and assumptions have been used for developing the proposed model:

2.1. Notations

D_o	:	Online customers' demand (unit)
D_{off}	:	Offline customers' demand (unit)
P_o	:	Selling price of online retailer (Rs./unit)
P_{off}	:	Selling price of off-line retailer (Rs./unit)
c	:	Unit production cost (Rs./unit)
W_o	:	Wholesale price of the manufacturer to the online retailer (decision variable) (Rs./unit)
W_{off}	:	Wholesale price of manufacturer to offline retailer (decision variable) (Rs./unit)
ϕ_1	:	Initial product price of a product of online retailer (decision variable) (Rs./unit)
ψ_1	:	Initial product price of a product of offline retailer (decision variable) (Rs./unit)
θ	:	Green level of product (decision variable)
V	:	Circular Economic Index (CEI) of product (decision variable)
a	:	Basic market demand of online channel (unit)
a_1	:	Basic market demand of off-line channel (unit)
α_0	:	Selling self-price sensitivity parameter of online retailer
α_1	:	Selling self-price sensitivity parameter of offline retailer

β_0	:	Selling cross-price sensitivity parameter of online retailer
β_1	:	Selling cross-price sensitivity parameter of offline retailer
γ_0	:	Green level sensitivity parameter of online retailer
γ_1	:	Green level sensitivity parameter of offline retailer
λ_0	:	Circular economic index sensitivity parameter of online retailer
λ_1	:	Circular economic index sensitivity parameter of offline retailer
ϕ_2	:	Effective parameter of online retailer price w.r.t. CEI
ψ_2	:	Effective parameter of offline retailer price w.r.t. CEI
I	:	Investment cost parameter of manufacturer w.r.t. green degree (Rs.)
μ	:	Investment cost parameter of manufacturer w.r.t. CEI (Rs.)
$(.)^{\text{cs}}$	:	Centralized scenario
$(.)^{\text{ds}}$	:	Decentralized scenario
$\Pi_M(.)$	:	Manufacturer's profit
$\Pi_R^{(o)}(.)$	:	Online retailer's profit
$\Pi_R^{(\text{off})}(.)$	:	Off-line retailer's profit
$\Pi_{sc}(.)$	:	Whole supply chain profit

2.2. Assumptions

In this subsection, the most fundamental assumptions of the problem are summarized as follows:

(i) This economical and environmentally sustainable supply chain with a circularity index contains one manufacturer and two retailers. One is an online retailer who buys products from the manufacturer through an online channel, and sells them through online. Another is an offline retailer that traditionally buys products from the manufacturer through off-line mode, and sells them through offline mode [52].

(ii) Due to global warming, pollution is getting worse every day, which becomes a severe threat to the environment and human health. So, environmentally and health-conscious people are ready to pay more for green products. So, the manufacturer produces green products [32].

(iii) Increasing technologies and vast population growth cause massive waste production. For environmental and economic sustainability, waste minimization has more value. Consideration of the circular economic index (CEI) in the supply chain has a significant impact on waste minimization. The CEI measures how much the original product is reusable or recyclable. The CEI of product is considered here according to DiMaio and Rem [9], Rabta [44], Wani and Mishra [48], Thomas and Mishra [47].

(iv) With minimizing waste by increasing CEI, an increment in profits by raising selling prices is very much needed. The environmentally conscious people are ready to invest more in circular economic indexed products, too. So, the selling prices of both online and offline retail products depend on their initial prices and CEI linearly. The unit product price of an online retailer is given by $P_o(\phi_1, V) = \phi_1 + \phi_2 V$, and the unit product price of an offline retailer is given by $P_{\text{off}}(\psi_1, V) = \psi_1 + \psi_2 V$.

(v) As environmental and economic-conscious people always want to buy green and circular economic indexed products, the consumer demand increases with the increasing green level and CEI of products. In this regard, it is assumed that both the online and offline demand functions depend on the green level and CEI of products. The online and offline market demand is dependent linearly on retailers' selling price, product's green level, and circular economic index. The mathematical form of the online and offline market demands of the product is given by

$$D_o(\phi_1, \psi_1, \theta, V) = a - \alpha_0 P_o + \beta_0 P_{\text{off}} + \gamma_0\theta + \lambda_0 V$$
$$D_{\text{off}}(\phi_1, \psi_1, \theta, V) = a_1 - \alpha_1 P_{\text{off}} + \beta_1 P_o + \gamma_1\theta + \lambda_1 V$$

respectively, where $a > 0$, $a_1 > 0$, $\alpha_i > \beta_i > 0$, $\gamma_i > 0$, $\lambda_i > 0$ for $i = 0, 1$. For positivity of customer demands, it is assumed that $a - \alpha_0 P_o > 0$ and $a_1 - \alpha_1 P_{\text{off}} > 0$.

3. Problem description and optimization

Here, a dual channel green supply chain model is considered in which the manufacturer produces green products and then sells them to the retailers through online and offline retail channel. The manufacturer decides the product's green level (θ). The retailer in an online psychic purchases goods from the manufacturer and then sells them to people online customer. In an offline psychic, products are delivered to the retailer and sold to customers offline. Here, the manufacturer produces circular economic products. By this consideration, the manufacturer tries to make the supply chain economically and environmentally more sustainable. The manufacturer decides what should be the CEI such that wastage is minimized and profit is maximized with the increasing demand of environmentally conscious consumers. The online retailer sells the products to consumers with online price (P_o), which is a function of the initial online selling price (ϕ_1), and CEI (V). The offline retailer sells the products to consumers with off-line price (P_{off}), which is a function of the initial offline selling price (ψ_1) and CEI (V). To optimize the objective operations of the supply chain, two models are formulated: centralized (cs) and decentralized (ds). In the centralized scenario, the profit functions of the manufacturer and both channel retailers are integrated. Then it is optimized by the concavity of the Hessian matrix w.r.t. the given decision variables. In a decentralized scenario, a Stackelberg competition occurs between the manufacturer and both channel retailers. In the Stackelberg competition, the manufacturer acts as a leader, and retailers are followers. In the Stackelberg game theoretic model, the offline retailer optimizes his/her decision first, then the online retailer optimizes his/her profit function by using the optimal decision variable of the off-line retailer. After getting the best decisions from both retailers, the manufacturer optimizes his/her profit function by applying these optimal decision variables of both retailers. The schematic representation of the whole supply chain is shown in Figure 1.

Based on the assumptions and model description, the profits of the manufacturer, online retailer, offline retailer and integrated system are respectively given by

$$\begin{aligned}
\Pi_M(W_o, W_{\text{off}}, \theta, V) &= (W_o - c)D_o + (W_{\text{off}} - c)D_{\text{off}} - I\frac{\theta^2}{2} - \mu\frac{V^2}{2} \\
\Pi_R^{(o)}(\phi_1) &= (P_o - W_o)D_o \\
\Pi_R^{(\text{off})}(\psi_1) &= (P_{\text{off}} - W_{\text{off}})D_{\text{off}} \\
\Pi_{sc}(\phi_1, \psi_1, \theta, V) &= (P_o - c)D_o + (P_{\text{off}} - c)D_{\text{off}} - I\frac{\theta^2}{2} - \mu\frac{V^2}{2}\cdot
\end{aligned}$$

3.1. Optimization for centralized policy

In this model manufacturer, online and offline retailers are three members of the supply chain system. In the centralized policy, it is considered that all the members are making decisions based on an online retail selling price, an offline retail selling price, a green level of product, and a circular economic index jointly in order to maximize the integrated supply chain profit. So, the maximization problem in the centralized policy corresponding to the integrated supply chain profit is given by

$$\begin{aligned}
&\text{Maximize } \Pi_{sc}(\phi_1, \psi_1, \theta, V) \qquad (1)\\
&\text{subject to } \phi_1 > 0,\ \psi_1 > 0,\ 0 < \theta < 1,\ 0 < V < 1,
\end{aligned}$$

$$\begin{aligned}
\text{where, } \Pi_{sc}(\phi_1, \psi_1, \theta, V) = {} & (\phi_1 + \phi_2 V - c)(a - \alpha_0\phi_1 - \alpha_0\phi_2 V + \beta_0\psi_1 + \beta_0\psi_2 V + \gamma_0\theta + \lambda_0 V) \\
& + (\psi_1 + \psi_2 V - c)(a_1 - \alpha_1\psi_1 - \alpha_1\psi_2 V + \beta_1\phi_1 + \beta_1\phi_2 V + \gamma_1\theta + \lambda_1 V) \\
& - I\frac{\theta^2}{2} - \mu\frac{V^2}{2}\cdot \qquad (2)
\end{aligned}$$

Proposition 1. *The integrated supply chain profit function,* $\Pi_{sc}(\phi_1, \psi_1, \theta, V)$ *will be concave w.r.t.* ϕ_1*,* ψ_1*,* θ *and* V *if it satisfies the following conditions:*

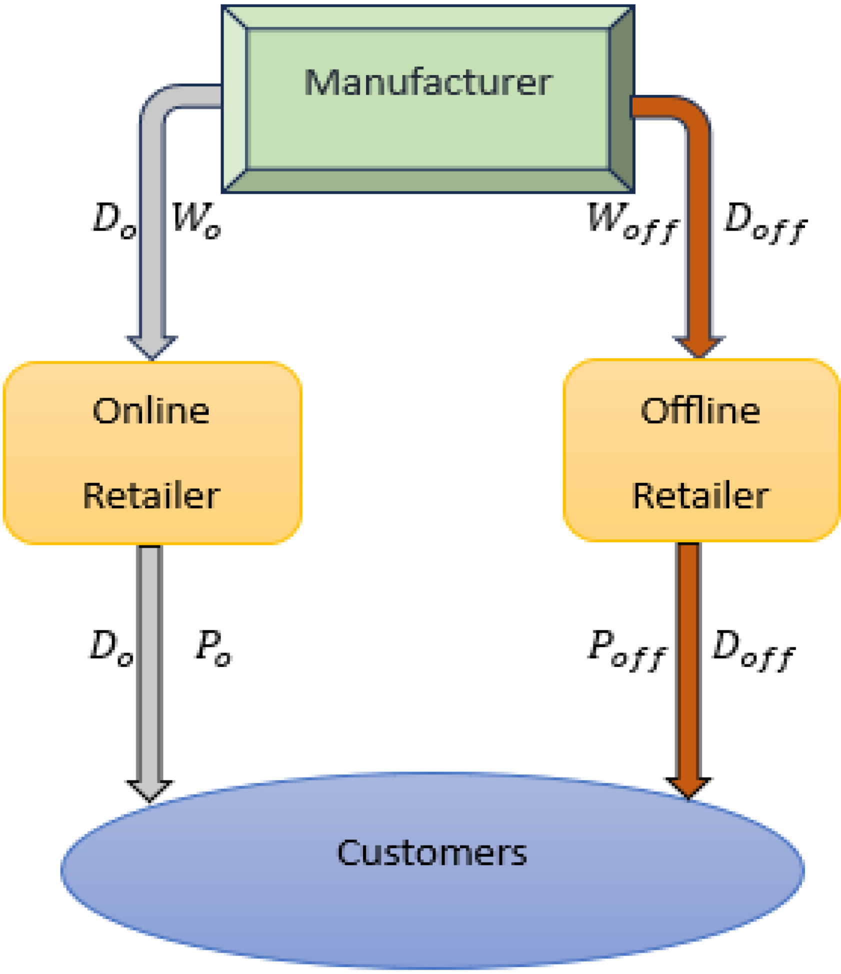

FIGURE 1. Schematic representation of supply chain model.

(i) $4\alpha_0\alpha_1 > (\beta_0+\beta_1)^2$,

(ii) $I > \dfrac{2\alpha_0\gamma_1^2 + 2\gamma_0\gamma_1(\beta_0+\beta_1) + 2\alpha_1\gamma_0^2}{4\alpha_0\alpha_1 - (\beta_0+\beta_1)^2}$,

(iii) $\mu > \dfrac{2\gamma_0\gamma_1\lambda_0\lambda_1 + 2\alpha_1 I\lambda_0^2 - \lambda_1(\gamma_0^2\lambda_1 - 2I((\beta_0+\beta_1)\lambda_0 + \alpha_0\lambda_1)) - \gamma_1^2\lambda_0^2}{4\alpha_0\alpha_1 I - (\beta_0+\beta_1)^2 I - 2\gamma_1^2\alpha_0 - 2\gamma_0\gamma_1(\beta_0+\beta_1) - 2\alpha_1\gamma_0^2}$.

Proof. The Hessian matrix associated with the whole supply chain profit function $\Pi_{sc}(\phi_1,\psi_1,\theta,V)$ is given by

$$H_1 = \begin{pmatrix} \frac{\partial^2\Pi_{sc}}{\partial\phi_1^2} & \frac{\partial^2\Pi_{sc}}{\partial\psi_1\partial\phi_1} & \frac{\partial^2\Pi_{sc}}{\partial\theta\partial\phi_1} & \frac{\partial^2\Pi_{sc}}{\partial V\partial\phi_1} \\ \frac{\partial^2\Pi_{sc}}{\partial\phi_1\partial\psi_1} & \frac{\partial^2\Pi_{sc}}{\partial\psi_1^2} & \frac{\partial^2\Pi_{sc}}{\partial\theta\partial\psi_1} & \frac{\partial^2\Pi_{sc}}{\partial V\partial\psi_1} \\ \frac{\partial^2\Pi_{sc}}{\partial\phi_1\partial\theta} & \frac{\partial^2\Pi_{sc}}{\partial\psi_1\partial\theta} & \frac{\partial^2\Pi_{sc}}{\partial\theta^2} & \frac{\partial^2\Pi_{sc}}{\partial V\partial\theta} \\ \frac{\partial^2\Pi_{sc}}{\partial\phi_1\partial V} & \frac{\partial^2\Pi_{sc}}{\partial\psi_1\partial V} & \frac{\partial^2\Pi_{sc}}{\partial\theta\partial V} & \frac{\partial^2\Pi_{sc}}{\partial V^2} \end{pmatrix} = \begin{pmatrix} -2\alpha_0 & \beta_0+\beta_1 & \gamma_0 & -2\alpha_0\phi_2+\beta_0\psi_2+\lambda_0+\beta_1\psi_2 \\ \beta_0+\beta_1 & -2\alpha_1 & \gamma_1 & \beta_0\phi_2-2\alpha_1\psi_2+\beta_1\phi_2+\lambda_1 \\ \gamma_0 & \gamma_1 & -I & \gamma_0\phi_2+\gamma_1\psi_2 \\ A_1 & A_2 & \gamma_0\phi_2+\gamma_1\psi_2 & -2\alpha_0\phi_2^2+2\beta_0\phi_2\psi_2+A_3 \end{pmatrix}$$

□

(For expressions of derivatives see Appendix A.1)

$$\begin{aligned}\text{where, } A_1 &= -2\alpha_0\phi_2 + \beta_0\psi_2 + \lambda_0 + \beta_1\psi_2,\ A_2 = \beta_0\phi_2 - 2\alpha_1\psi_2 + \beta_1\phi_2 + \lambda_1\\ A_3 &= 2\lambda_0\phi_2 - 2\alpha_1\psi_2^2 + 2\beta_1\phi_2\psi_2 + 2\lambda_1\psi_2 - \mu.\end{aligned}$$

From the Hessian matrix, it is seen that the principal minor is $|h_{11}| = -2\alpha_0 < 0$ *as* $\alpha_0 > 0$ *by assumption. The second order principal minor is*

$$|h_{12}| = \begin{vmatrix} -2\alpha_0 & \beta_0+\beta_1 \\ \beta_0+\beta_1 & -2\alpha_1 \end{vmatrix} = 4\alpha_0\alpha_1 - (\beta_0+\beta_1)^2.$$

The third order principal minor is

$$|h_{13}| = \begin{vmatrix} -2\alpha_0 & \beta_0+\beta_1 & \gamma_0 \\ \beta_0+\beta_1 & -2\alpha_1 & \gamma_1 \\ \gamma_0 & \gamma_1 & -I \end{vmatrix} = I\left((\beta_0+\beta_1)^2 - 4\alpha_0\alpha_1\right) + \left(2\alpha_0\gamma_1^2 + 2\gamma_0\gamma_1(\beta_0+\beta_1) + 2\alpha_1\gamma_0^2\right).$$

The fourth order principal minor is

$$\begin{aligned}|h_{14}| &= \begin{vmatrix} -2\alpha_0 & \beta_0+\beta_1 & \gamma_0 & -2\alpha_0\phi_2+\beta_0\psi_2+\lambda_0+\beta_1\psi_2 \\ \beta_0+\beta_1 & -2\alpha_1 & \gamma_1 & \beta_0\phi_2-2\alpha_1\psi_2+\beta_1\phi_2+\lambda_1 \\ \gamma_0 & \gamma_1 & -I & \gamma_0\phi_2+\gamma_1\psi_2 \\ A_1 & A_2 & \gamma_0\phi_2+\gamma_1\psi_2 & -2\alpha_0\phi_2^2+2\beta_0\phi_2\psi_2+A_3 \end{vmatrix}\\ &= \mu(4\alpha_0\alpha_1 I - (\beta_0+\beta_1)^2 I - 2\gamma_1^2\alpha_0 - 2\gamma_0\gamma_1(\beta_0+\beta_1) - 2\alpha_1\gamma_0^2) - (2\gamma_0\gamma_1\lambda_0\lambda_1 + 2\alpha_1 I\lambda_0^2 - \lambda_1(\gamma_0^2\lambda_1\\ &\quad - 2I((\beta_0+\beta_1)\lambda_0 + \alpha_0\lambda_1)) - \gamma_1^2\lambda_0^2).\end{aligned}$$

Now, the Hessian matrix will be negative definite if $|h_{11}| < 0$, $|h_{12}| > 0$, $|h_{13}| < 0$ *and* $|h_{14}| > 0$. *Here,* $|h_{11}| = -2\alpha_0 < 0$, *since* $\alpha_0 > 0$.
Again, $|h_{12}| > 0 \Rightarrow 4\alpha_0\alpha_1 > (\beta_0+\beta_1)^2$ *and* $|h_{13}| < 0 \Rightarrow I > \dfrac{2\alpha_0\gamma_1^2 + 2\gamma_0\gamma_1(\beta_0+\beta_1) + 2\alpha_1\gamma_0^2}{4\alpha_0\alpha_1 - (\beta_0+\beta_1)^2}$.

Finally,

$$|h_{14}| > 0 \Rightarrow \mu > \frac{2\gamma_0\gamma_1\lambda_0\lambda_1 + 2\alpha_1 I\lambda_0^2 - \lambda_1(\gamma_0^2\lambda_1 - 2I((\beta_0+\beta_1)\lambda_0 + \alpha_0\lambda_1)) - \gamma_1^2\lambda_0^2}{4\alpha_0\alpha_1 I - (\beta_0+\beta_1)^2 I - 2\gamma_1^2\alpha_0 - 2\gamma_0\gamma_1(\beta_0+\beta_1) - 2\alpha_1\gamma_0^2}.$$

Since the Hessian matrix corresponding to the integrated supply chain profit function is negative definite, so, it is proved that for the given conditions in Proposition 1, the integrated supply chain profit function is concave w.r.t. the decision variables ϕ_1, ψ_1, θ *and* V.

Proposition 2. *From Proposition 1, it is concluded that the whole supply chain profit function has a maximum value and the corresponding value is*

$$\begin{aligned}\Pi_{sc}^{\text{cs}}(\phi_1^{\text{cs}}, \psi_1^{\text{cs}}, \theta^{\text{cs}}, V^{\text{cs}}) &= (\phi_1^{\text{cs}} + \phi_2 V^{\text{cs}} - c)(a - \alpha_0\phi_1^{\text{cs}} - \alpha_0\phi_2 V^{\text{cs}} + \beta_0\psi_1^{\text{cs}} + \beta_0\psi_2 V^{\text{cs}} + \gamma_0\theta^{\text{cs}} + \lambda_0 V^{\text{cs}})\\ &\quad + (\psi_1^{\text{cs}} + \psi_2 V^{\text{cs}} - c)(a_1 - \alpha_1\psi_1^{\text{cs}} - \alpha_1\psi_2 V^{\text{cs}} + \beta_1\phi_1^{\text{cs}} + \beta_1\phi_2 V^{\text{cs}} + \gamma_1\theta^{\text{cs}} + \lambda_1 V^{\text{cs}})\\ &\quad - I\frac{\theta^{\text{cs}2}}{2} - \mu\frac{V^{\text{cs}2}}{2}\end{aligned}$$

where, $\phi_1^{\rm cs}$, $\psi_1^{\rm cs}$, $\theta^{\rm cs}$ *and* $V^{\rm cs}$ *are optimal values of* ϕ_1, ψ_1, θ *and* V *which are as follows:*

$$\phi_1^{\rm cs} = \frac{-a_1X_{11} + aX_{12} + c\left(X_{13} + X_{14}\lambda_1\phi_2 + \gamma_0\gamma_1X_{15} + (\gamma_1^2)X_{16} + \alpha_1X_{17}\right)}{Z}$$

$$\psi_1^{\rm cs} = \frac{-aX_{21} + a(X_{22})\psi_2 + c\left(X_{23} + X_{24}\psi_2 - \left(\gamma_1^2\right)\left(\left(\lambda_0^2\right) - 2\alpha_0\mu + \left(-\alpha_0 + \beta_0\right)\lambda_0\psi_2\right) + \gamma_0\gamma_1X_{25}\right) + a_1X_{26}}{Z}$$

$$\theta^{\rm cs} = \frac{\left(\left(a_1 + \left(-\alpha_1 + \beta_1\right)c\right)\lambda_0 - \left(a + \left(-\alpha_0 + \beta_0\right)c\right)\lambda_1\right)\left(\gamma_1\lambda_0 - \gamma_0\lambda_1\right) - \left(X_{31}\gamma_0 + X_{32}\gamma_1\right)\mu}{Z}$$

$$V^{\rm cs} = \frac{-\left(X_{41}\lambda_0\right) + X_{42}\lambda_1}{Z}\cdot$$

Proof. Since the integrated supply chain profit function Π_{sc} is concave w.r.t. the decision variables ϕ_1, ψ_1, θ, V so there exist the optimal values for which the integrated supply chain profit will be maximum. The necessary conditions of the maximization problem (1) are $\frac{\partial\Pi_{sc}}{\partial\phi_1} = 0$, $\frac{\partial\Pi_{sc}}{\partial\psi_1} = 0$, $\frac{\partial\Pi_{sc}}{\partial\theta} = 0$ and $\frac{\partial\Pi_{sc}}{\partial V} = 0$ which give the following equations.

$$-2\alpha_0\phi_1 + (\beta_0 + \beta_1)\psi_1 + (\beta_0\psi_2 - 2\alpha_0\phi_2 + \lambda_0 + \beta_1\psi_2)V + \gamma_0\theta - \beta_1c + \alpha_0c + a = 0 \tag{3}$$

$$(\beta_0 + \beta_1)\phi_1 - 2\alpha_1\psi_1 + (\beta_0\phi_2 - 2\alpha_1\psi_2 + \beta_1\phi_2 + \lambda_1)V + \gamma_1\theta + \alpha_1c - \beta_0c + a_1 = 0 \tag{4}$$

$$\gamma_0\phi_1 + \gamma_0\phi_2V - c\gamma_0 + \gamma_1\psi_1 + \gamma_1\psi_2V - \gamma_1c - I\theta = 0 \tag{5}$$

$$\begin{aligned} &a\phi_2 - 2\alpha_0\phi_1\phi_2 - 2\alpha_0\phi_2^2V + \alpha_0c\phi_2 + \beta_0\phi_2\psi_1 + \beta_0\phi_1\psi_2 + 2\beta_0\phi_2\psi_2V - \beta_0c\psi_2 \\ &+ \gamma_0\phi_2\theta + \lambda_0\phi_1 + 2\lambda_0\phi_2V - c\lambda_0 + a_1\psi_2 - 2\alpha_1\psi_1\psi_2 - 2\alpha_1\psi_2^2V + c\alpha_1\psi_2 + \beta_1\psi_2\phi_1 \\ &+ \beta_1\psi_1\phi_2 + 2\beta_1\psi_2\phi_2V - \beta_1c\phi_2 + \gamma_1\psi_2\theta + \lambda_1\psi_1 + 2\lambda_1\psi_2V - \lambda_1c - \mu V = 0. \end{aligned} \tag{6}$$

Solving (3)–(6), the optimal decision variables explicitly are given as

$$\phi_1^{\rm cs} = \frac{-a_1X_{11} + aX_{12} + c\left(X_{13} + X_{14}\lambda_1\phi_2 + \gamma_0\gamma_1X_{15} + \gamma_1^2X_{16} + \alpha_1X_{17}\right)}{Z}$$

$$\psi_1^{\rm cs} = \frac{-aX_{21} + aX_{22}\psi_2 + c\left[X_{23} + X_{24}\psi_2 - \gamma_1^2\left\{\lambda_0^2 - 2\alpha_0\mu + (\beta_0 - \alpha_0)\lambda_0\psi_2\right\} + \gamma_0\gamma_1X_{25}\right] + a_1X_{26}}{Z}$$

$$\theta^{\rm cs} = \frac{[\{a_1 + (\beta_1 - \alpha_1)c\}\lambda_0 - \{a + (\beta_0 - \alpha_0)c\}\lambda_1](\gamma_1\lambda_0 - \gamma_0\lambda_1) - (X_{31}\gamma_0 + X_{32}\gamma_1)\mu}{Z}$$

$$V^{\rm cs} = \frac{X_{42}\lambda_1 - X_{41}\lambda_0}{Z}\cdot$$

The mathematical expressions of X_{11}, X_{12}, X_{13}, X_{14}, X_{15}, X_{16}, X_{17}, Z, X_{21}, X_{22}, X_{23}, X_{24}, X_{25}, X_{26}, X_{31}, X_{32}, X_{41} and X_{42} are given in Appendix A.2.

Substituting the values of all the optimal decision variables from equations in (2), the optimal whole supply chain profit function is given as

$$\begin{aligned} \Pi_{sc}^{\rm cs}\left(\phi_1^{\rm cs}, \psi_1^{\rm cs}, \theta^{\rm cs}, V^{\rm cs}\right) = &\left(\phi_1^{\rm cs} + \phi_2V^{\rm cs} - c\right)\left(a - \alpha_0\phi_1^{\rm cs} - \alpha_0\phi_2V^{\rm cs} + \beta_0\psi_1^{\rm cs} + \beta_0\psi_2V^{\rm cs} + \gamma_0\theta^{\rm cs} + \lambda_0V^{\rm cs}\right) \\ &+ \left(\psi_1^{\rm cs} + \psi_2V^{\rm cs} - c\right)\left(a_1 - \alpha_1\psi_1^{\rm cs} - \alpha_1\psi_2V^{\rm cs} + \beta_1\phi_1^{\rm cs} + \beta_1\phi_2V^{\rm cs} + \gamma_1\theta^{\rm cs} + \lambda_1V^{\rm cs}\right) \\ &- I\frac{\theta^{\rm cs2}}{2} - \mu\frac{V^{\rm cs2}}{2}\cdot \end{aligned}$$

□

3.2. Decentralized policy

In decentralized policy, each supply chain member is willing to optimize self- profit function independently. Here, the manufacturer plays the role of Stackelberg leader, whereas online and off-line retailers act as the followers. A Stackelberg competition occurs between a manufacturer and retailers. In this policy, off-line retailer first optimizes his/her profit function. After getting the best reaction from off-line retailer, the online retailer optimizes his/her profit function. After accounting for the best reaction of online retailer and offline retailer manufacturer optimizes his/her profit function and computes the best values of his/her decision variables. So, the maximization problems in the decentralized policy corresponding to the profits of off-line and online retailers as well as manufacturer are given by

$$
\begin{aligned}
&\text{Maximize } \Pi_R^{(o)}(\phi_1)\\
&\text{subject to } \phi_1 > 0\\
&\text{Maximize } \Pi_R^{(\text{off})}(\psi_1)\\
&\text{subject to } \psi_1 > 0\\
&\text{Maximize } \Pi_M(W_o, W_{\text{off}}, \theta, V)\\
&\text{subject to } W_o > 0,\ W_{\text{off}} > 0,\ 0 < \theta < 1,\ 0 < V < 1
\end{aligned}
\tag{7}
$$

$$\text{where}, \Pi_R^{(o)}(\phi_1) = (\phi_1 + \phi_2 V - W_o)\{a - \alpha_0\phi_1 + \beta_0\psi_1 + \gamma_0\theta + (\lambda_0 - \alpha_0\phi_2 + \beta_0\psi_2)V\} \tag{8}$$

$$\Pi_R^{(\text{off})}(\psi_1) = (\psi_1 + \psi_2 V - W_{\text{off}})\{a_1 - \alpha_1\psi_1 + \beta_1\phi_1 + \gamma_1\theta + (\beta_1\phi_2 - \alpha_1\psi_2 + \lambda_1)V\} \tag{9}$$

$$
\begin{aligned}
\Pi_M(W_o, W_{\text{off}}, \theta, V) &= (W_o - c)(a - \alpha_0\phi_1 - \alpha_0\phi_2 V + \beta_0\psi_1 + \beta_0\psi_2 V + \gamma_0\theta + \lambda_0 V)\\
&+ (W_{\text{off}} - c)(a_1 - \alpha_1\psi_1 - \alpha_1\psi_2 V + \beta_1\phi_1 + \beta_1\phi_2 V + \gamma_1\theta + \lambda_1 V) - I\frac{\theta^2}{2} - \mu\frac{V^2}{2}\cdot
\end{aligned}
\tag{10}
$$

Proposition 3. *The profit function of off-line retailer is concave w.r.t. the decision variable ψ_1.*

Proof. The first and second order derivatives of profit function of off-line retailer from (9) w.r.t. ψ_1 are given by

$$
\begin{aligned}
\frac{\mathrm{d}\Pi_R^{(\text{off})}}{\mathrm{d}\psi_1} &= a_1 - 2\alpha_1\psi_1 - 2\alpha_1\psi_2 V + \alpha_1 W_{(\text{off})} + \beta_1\phi_1 + \beta_1\phi_2 V + \gamma_1\theta + \lambda_1 V\\
\frac{\mathrm{d}^2\Pi_R^{(\text{off})}}{\mathrm{d}\psi_1^2} &= -2\alpha_1 < 0 \quad \text{as } \alpha_1 > 0.
\end{aligned}
$$

Here, the second order derivative of the off-line retailer's profit function is negative. So, it is proved that the profit function of off-line retailer is concave w.r.t. the decision variable ψ_1. □

Proposition 4. *The profit function of online retailer is concave w.r.t. the decision variable ϕ_1 if $\alpha_0 > \frac{\beta_0\beta_1}{2\alpha_1}$.*

Proof. From $\frac{\mathrm{d}\Pi_R^{(\text{off})}}{\mathrm{d}\psi_1} = 0$, it is obtained that

$$\psi_1 = \frac{B}{2\alpha_1} \tag{11}$$

where $B = a_1 - 2\alpha_1\psi_2 V + \alpha_1 W_{\text{off}} + \beta_1\phi_1 + \beta_1\phi_2 V + \gamma_1\theta + \lambda_1 V$.

Putting the value of ψ_1 in equation (8), the profit function of online retailer becomes as follows

$$
\begin{aligned}
\Pi_R^{(o)}(\phi_1) = {} & a\phi_1 + a\phi_2 V - aW_o - \alpha_0\phi_1^2 - 2\alpha_0\phi_1\phi_2 V + \alpha_0 W_o\phi_1 - \alpha_0\phi_2^2 V^2 + \alpha_0\phi_2 V W_o + \beta_0\phi_1\frac{B}{2\alpha_1}\\
& + \beta_0\phi_2 V\frac{B}{2\alpha_1} - \beta_0 W_o\frac{B}{2\alpha_1} + \beta_0\psi_2 V\phi_1 + \beta_0\psi_2 V^2\phi_2 - \beta_0\psi_2 V W_o + \gamma_0\theta\phi_1 + \gamma_0\theta\phi_2 V\\
& - \gamma_0\theta W_o + \lambda_0 V\phi_1 + \lambda_0 V^2\phi_2 - \lambda_0 V W_o.
\end{aligned}
\tag{12}
$$

Now, the first order derivative of function given in (12) w.r.t. ϕ_1 is given by

$$\frac{\mathrm{d}\Pi_R^{(o)}}{\mathrm{d}\phi_1} = a - 2\alpha_0\phi_1 - 2\alpha_0\phi_2 V + \alpha_0 W_o + \frac{\beta_0\phi_2 V\beta_1}{2\alpha_1} - \frac{\beta_0 W_o\beta_1}{2\alpha_1} + \beta_0\psi_2 V + \gamma_0\theta + \lambda_0 V + \frac{\beta_0}{2\alpha_1}\left(a_1 - 2\alpha_1\psi_2 V + \alpha_1 W_{\text{off}} + 2\beta_1\phi_1 + \beta_1\phi_2 V + \gamma_1\theta + \lambda_1 V\right).$$

The second order derivative of function given in (12) w.r.t. ϕ_1 is given by

$$\frac{\mathrm{d}^2\Pi_R^{(o)}}{\mathrm{d}\phi_1^2} = -2\alpha_0 + \frac{\beta_0\beta_1}{\alpha_1}\cdot$$

The profit function of online retailer will be concave if $\frac{\mathrm{d}^2\Pi_R^{(o)}}{\mathrm{d}\phi_1^2} < 0$ *i.e.*, if $2\alpha_0 > \frac{\beta_0\beta_1}{\alpha_1}$.

Thus, it is proved that the profit function of online retailer is concave w.r.t. the decision variable ϕ_1 if $\alpha_0 > \frac{\beta_0\beta_1}{2\alpha_1}$. □

Proposition 5. *The profit function* $\Pi_M(W_o, W_{\text{off}}, \theta, V)$ *of the manufacturer is concave w.r.t. the decision variables* W_o*,* W_{off}*,* θ *and* V *if it satisfies the following conditions:*

(i) $-\alpha_0^2 Y_1^2 + \alpha_0 Z_1 - \frac{Z_2}{4\alpha_1^2} > 0$,

(ii) $I < \frac{(2\alpha_1(\gamma_0 - \alpha_0 Y_3) + \beta_0 Y_4)Z_5) - 2(\gamma_1 + \beta_1 Y_3 - \frac{Y_4}{2})Z_4}{2Z_3}$,

(iii) $\mu > \frac{2(\lambda_1 - \alpha_1\psi_2 + \beta_1(\phi_2 + Y_5) - \frac{Y_6}{2})Z_8) - 2(2\alpha_1(\lambda_0 + \beta_0\psi_2 - \alpha_0(\phi_2 + Y_5)) + \beta_0 Y_6)Z_7}{Z_6}$.

Proof. From $\frac{\mathrm{d}\Pi_R^{(o)}}{\mathrm{d}\phi_1} = 0$, the initial retail price (ϕ_1) of online retailer is derived as

$$\phi_1 = \frac{F}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} \tag{13}$$

where, $F = 2a\alpha_1 - 4\alpha_0\alpha_1\phi_2 V + 2\alpha_0\alpha_1 W_o + \beta_0 a_1 - 2\alpha_1\beta_0\psi_2 V + \beta_0\alpha_1 W_{\text{off}} + 2\beta_0\beta_1\phi_2 V + \beta_0\gamma_1\theta + \beta_0\lambda_1 V - \beta_0\beta_1 W_o + 2\alpha_1\beta_0\psi_2 V + 2\alpha_1\gamma_0\theta + 2\alpha_1\lambda_0 V$.

Putting the value of ϕ_1 as given in (13), from (11) the initial retail price (ψ_1) of off-line retailer is obtained as

$$\psi_1 = \frac{G}{2\alpha_1(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} \tag{14}$$

where, $G = 4\alpha_0\alpha_1 a_1 - 2\beta_0\beta_1 a_1 - 8\alpha_0(\alpha_1^2)\psi_2 V + 4\beta_0\beta_1\alpha_1\psi_2 V + 4\alpha_0(\alpha_1^2)W_{\text{off}} - 2\beta_0\beta_1\alpha_1 W_{\text{off}} + \beta_1 F + 4\alpha_0\alpha_1\beta_1\phi_2 V - 2\beta_0(\beta_1^2)\phi_2 V + 4\alpha_0\alpha_1\gamma_1\theta - 2\beta_0\beta_1\gamma_1\theta + 4\alpha_0\alpha_1\lambda_1 V - 2\beta_0\beta_1\lambda_1 V$.

Now, ϕ_1 and ψ_1 both are in terms of W_o, W_{off}, θ and V *i.e.*, in terms of decision variables of manufacturer as given in equations (13) and (14). Putting these values of ϕ_1 and ψ_1 in (10), the profit function of manufacturer becomes as follows:

$$\Pi_M(W_o, W_{\text{off}}, \theta, V) = (W_o - c)(a - \frac{\alpha_0 F}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} - \alpha_0\phi_2 V + \frac{\beta_0 G}{2\alpha_1(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} + \beta_0\psi_2 V + \gamma_0\theta + \lambda_0 V) + (W_{\text{off}} - c)(a_1 - \frac{G}{2(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} - \alpha_1\psi_2 V + \frac{\beta_1 F}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} + \beta_1\phi_2 V + \gamma_1\theta + \lambda_1 V) - I\frac{\theta^2}{2} - \mu\frac{V^2}{2}\cdot \tag{15}$$

The Hessian matrix associated with the manufacturer's profit function $\Pi_M(W_o, W_{\text{off}}, \theta, V)$ is given by

$$H_2 = \begin{pmatrix} \frac{\partial^2\Pi_M}{\partial W_o^2} & \frac{\partial^2\Pi_M}{\partial W_{\text{off}}\partial W_o} & \frac{\partial^2\Pi_M}{\partial\theta\partial W_o} & \frac{\partial^2\Pi_M}{\partial V\partial W_o} \\ \frac{\partial^2\Pi_M}{\partial W_o\partial W_{\text{off}}} & \frac{\partial^2\Pi_M}{\partial W_{\text{off}}^2} & \frac{\partial^2\Pi_M}{\partial\theta\partial W_{\text{off}}} & \frac{\partial^2\Pi_M}{\partial V\partial W_{\text{off}}} \\ \frac{\partial^2\Pi_M}{\partial W_o\partial\theta} & \frac{\partial^2\Pi_M}{\partial W_{\text{off}}\partial\theta} & \frac{\partial^2\Pi_M}{\partial\theta^2} & \frac{\partial^2\Pi_M}{\partial V\partial\theta} \\ \frac{\partial^2\Pi_M}{\partial W_o\partial V} & \frac{\partial^2\Pi_M}{\partial W_{\text{off}}\partial V} & \frac{\partial^2\Pi_M}{\partial\theta\partial V} & \frac{\partial^2\Pi_M}{\partial V^2} \end{pmatrix}$$
$$= \begin{pmatrix} -\alpha_0 + \frac{\beta_0\beta_1}{2\alpha_1} & -\alpha_0 Y_1 + \frac{\beta_0 Y_2}{2\alpha_1} & -\alpha_0 Y_3 + \frac{\beta_0 Y_4}{2\alpha_1} + \gamma_0 & Y_7 \\ -\alpha_0 Y_1 + \frac{\beta_0 Y_2}{2\alpha_1} & -Y_2 + 2\beta_1 Y_1 & -\frac{Y_4}{2} + \beta_1 Y_3 + \gamma_1 & Y_8 \\ -\alpha_0 Y_3 + \frac{\beta_0 Y_4}{2\alpha_1} + \gamma_0 & -\frac{Y_4}{2} + \beta_1 Y_3 + \gamma_1 & -I & 0 \\ Y_7 & Y_8 & 0 & -\mu \end{pmatrix}$$

(For expressions of derivatives see Appendix A.3).

The mathematical expressions of Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7 and Y_8 are given in Appendix A.4.

The first and second order principal minors are $|h_{21}| = -\alpha_0 + \frac{\beta_0\beta_1}{2\alpha_1}$ and

$$|h_{22}| = \begin{vmatrix} -\alpha_0 + \frac{\beta_0\beta_1}{2\alpha_1} & -\alpha_0 Y_1 + \frac{\beta_0 Y_2}{2\alpha_1} \\ -\alpha_0 Y_1 + \frac{\beta_0 Y_2}{2\alpha_1} & -Y_2 + 2\beta_1 Y_1 \end{vmatrix} = -\alpha_0^2 Y_1^2 + \alpha_0 Z_1 - \frac{Z_2}{4\alpha_1^2} \text{ respectively.}$$

The third order principal minor is

$$|h_{23}| = \begin{vmatrix} -\alpha_0 + \frac{\beta_0\beta_1}{2\alpha_1} & -\alpha_0 Y_1 + \frac{\beta_0 Y_2}{2\alpha_1} & -\alpha_0 Y_3 + \frac{\beta_0 Y_4}{2\alpha_1} + \gamma_0 \\ -\alpha_0 Y_1 + \frac{\beta_0 Y_2}{2\alpha_1} & -Y_2 + 2\beta_1 Y_1 & -\frac{Y_4}{2} + \beta_1 Y_3 + \gamma_1 \\ -\alpha_0 Y_3 + \frac{\beta_0 Y_4}{2\alpha_1} + \gamma_0 & -\frac{Y_4}{2} + \beta_1 Y_3 + \gamma_1 & -I \end{vmatrix}$$
$$= \frac{1}{8\alpha_1^2}\left(2IZ_3 + 2\left(\gamma_1 + \beta_1 Y_3 - \frac{Y_4}{2}\right)Z_4 - \left(2\alpha_1\left(\gamma_0 - \alpha_0 Y_3\right) + \beta_0 Y_4\right)Z_5\right).$$

The fourth order principal minor is

$$|h_{24}| = |H_2|$$
$$= \begin{vmatrix} -\alpha_0 + \frac{\beta_0\beta_1}{2\alpha_1} & -\alpha_0 Y_1 + \frac{\beta_0 Y_2}{2\alpha_1} & -\alpha_0 Y_3 + \frac{\beta_0 Y_4}{2\alpha_1} + \gamma_0 & Y_7 \\ -\alpha_0 Y_1 + \frac{\beta_0 Y_2}{2\alpha_1} & -Y_2 + 2\beta_1 Y_1 & -\frac{Y_4}{2} + \beta_1 Y_3 + \gamma_1 & Y_8 \\ -\alpha_0 Y_3 + \frac{\beta_0 Y_4}{2\alpha_1} + \gamma_0 & -\frac{Y_4}{2} + \beta_1 Y_3 + \gamma_1 & -I & 0 \\ Y_7 & Y_8 & 0 & -\mu \end{vmatrix}$$
$$= \frac{1}{8\alpha_1^2}\left(\mu Z_6 + 2\left(2\alpha_1\left(\lambda_0 + \beta_0\psi_2 - \alpha_0\left(\phi_2 + Y_5\right)\right) + \beta_0 Y_6\right)Z_7 - 2\left(\lambda_1 - \alpha_1\psi_2 + \beta_1(\phi_2 + Y_5) - \frac{Y_6}{2}\right)Z_8\right).$$

The mathematical expressions of Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7 and Z_8 are given in Appendix A.4.

Now, the manufacturer's profit function will be concave if the Hessian matrix will be negative definite if $|h_{21}| < 0$, $|h_{22}| > 0$, $|h_{23}| < 0$ and $|h_{24}| > 0$.

Now, $|h_{21}| = -\alpha_0 + \frac{\beta_0\beta_1}{2\alpha_1}$ which is obviously less than 0 from Proposition 4.

Again, $|h_{22}| > 0 \Rightarrow -\alpha_0^2 Y_1^2 + \alpha_0 Z_1 - \frac{Z_2}{4\alpha_1^2} > 0$ and $|h_{23}| < 0 \Rightarrow I < \frac{(2\alpha_1(\gamma_0 - \alpha_0 Y_3) + \beta_0 Y_4)Z_5) - 2(\gamma_1 + \beta_1 Y_3 - \frac{Y_4}{2})Z_4}{2Z_3}$.

Also, $|h_{24}| > 0 \Rightarrow \mu > \frac{2(\lambda_1 - \alpha_1\psi_2 + \beta_1(\phi_2 + Y_5) - \frac{Y_6}{2})Z_8) - 2(2\alpha_1(\lambda_0 + \beta_0\psi_2 - \alpha_0(\phi_2 + Y_5)) + \beta_0 Y_6)Z_7}{Z_6}$.

Hence, it is proved that the manufacturer's profit function will be concave w.r.t. the decision variables W_o, W_{off}, θ and V if the given conditions in Proposition 5 hold. □

Proposition 6. *From Proposition 5 it is concluded that manufacturer's profit function has maximum value and the maximum value is*

$$\Pi_M^{\text{ds}}\left(W_o^{\text{ds}}, W_{\text{off}}^{\text{ds}}, \theta^{\text{ds}}, V^{\text{ds}}\right) = \left(W_o^{\text{ds}} - c\right)\left(a - \frac{\alpha_0 F^{\text{ds}}}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} - \alpha_0\phi_2 V^{\text{ds}} + \frac{\beta_0 G^{\text{ds}}}{2\alpha_1(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} + \beta_0\psi_2 V^{\text{ds}} + \gamma_0\theta^{\text{ds}} + \lambda_0 V^{\text{ds}}\right) + \left(W_{\text{off}}^{\text{ds}} - c\right)\left(a_1 - \frac{G^{\text{ds}}}{2(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} - \alpha_1\psi_2 V^{\text{ds}} + \frac{\beta_1 F^{\text{ds}}}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} + \beta_1\phi_2 V^{\text{ds}} + \gamma_1\theta^{\text{ds}} + \lambda_1 V^{\text{ds}}\right) - I\frac{{\theta^{\text{ds}}}^2}{2} - \mu\frac{{V^{\text{ds}}}^2}{2}$$

where, W_o^{ds}, $W_{\text{off}}^{\text{ds}}$, θ^{ds}, V^{ds} *are as follows:*

$$W_o^{\text{ds}} = \frac{\left(4\left(\alpha_0^2\right)\left(\alpha_1^2\right)c\left(-I\left(\lambda_1^2\right) - \left(\gamma_1^2\right)\mu + 4\alpha_1 I\mu\right) - \alpha_0\alpha_1 E_{11} + \beta_1 E_{12}\right)}{K_2}$$

$$W_{\text{off}}^{\text{ds}} = \frac{\left(2\left(\alpha_0^2\right)\alpha_1 E_{21} + \beta_1 E_{22} - \alpha_0 E_{23}\right)}{K_2}$$

$$\theta^{\text{ds}} = \frac{\left(a_1\alpha_1(2\alpha_0\alpha_1 - \beta_0\beta_1)\lambda_0(\gamma_1\lambda_0 - \gamma_0\lambda_1) + E_{31} + a_1 E_{32}\mu + \beta_1 E_{33}\mu + 2\left(\alpha_0^2\right)\left(\alpha_1^2\right)cE_{34} + \alpha_0\alpha_1 E_{35}\right)}{-K_2}$$

$$V^{\text{ds}} = \frac{\left(\alpha_1\beta_1 E_{41}\lambda_0 + a_1\alpha_1 E_{42}\lambda_0 + a_1 E_{43}\lambda_1 + \beta_0\beta_1 E_{44}\lambda_1 + \alpha_0\alpha_1 E_{45} + 2\left(\alpha_0^2\right)\left(\alpha_1^2\right)cE_{46}\right)}{-K_2}.$$

Proof. Since the manufacturer's profit function is concave w.r.t. the decision variables W_o, W_{off}, θ and V, so there exist the optimal values of W_o, W_{off}, θ and V for which the manufacturer's profit will be maximum. So, the necessary conditions of the maximization problem (7) are $\frac{\partial\Pi_M}{\partial W_o} = 0$, $\frac{\partial\Pi_M}{\partial W_{\text{off}}} = 0$, $\frac{\partial\Pi_M}{\partial\theta} = 0$ and $\frac{\partial\Pi_M}{\partial V} = 0$ which give the following equations.

$$a - \frac{\alpha_0 F}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} - \alpha_0\phi_2 V + \frac{\beta_0 G}{2\alpha_1(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} + \beta_0\psi_2 V + \gamma_0\theta + \lambda_0 V + (W_o - c)\left(-\frac{\alpha_0}{2} + \frac{\beta_0\beta_1}{4\alpha_1}\right) + (W_{\text{off}} - c)\left(\frac{\beta_1}{4}\right) = 0 \tag{16}$$

$$(W_o - c)\left(-\alpha_0 Y_1 + \frac{\beta_0 Y_2}{2\alpha_1}\right) + \left(a_1 - \frac{G}{2(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} - \alpha_1\psi_2 V + \frac{\beta_1 F}{4\alpha_0\alpha_1 - 2\beta_0\beta_1} + \beta_1\phi_2 V + \gamma_1\theta + \lambda_1 V\right) + (W_{\text{off}} - c)\left(-\frac{Y_2}{2} + \beta_1 Y_1\right) = 0 \tag{17}$$

$$(W_o - c)\left(-\alpha_0 Y_3 + \frac{\beta_0 Y_4}{2\alpha_1} + \gamma_0\right) + (W_{\text{off}} - c)\left(-\frac{Y_4}{2} + \beta_1 Y_3 + \gamma_1\right) - I\theta = 0 \tag{18}$$

$$(W_o - c)\left\{\frac{\beta_0 Y_6}{2\alpha_1} + \beta_0\psi_2 + \lambda_0 - \alpha_0(Y_5 + \phi_2)\right\} + (W_{\text{off}} - c)\left\{\beta_1(Y_5 + \phi_2) - \frac{Y_6}{2} - \alpha_1\psi_2 + \lambda_1\right\} - \mu V = 0. \tag{19}$$

Solving equations (16)–(19), the optimal decision variables of manufacturer are explicitly obtained as follows:

$$W_o^{\text{ds}} = \frac{\left(4\left(\alpha_0^2\right)\left(\alpha_1^2\right)c\left(-I\left(\lambda_1^2\right) - \left(\gamma_1^2\right)\mu + 4\alpha_1 I\mu\right) - \alpha_0\alpha_1 E_{11} + \beta_1 E_{12}\right)}{K_2}$$

$$W_{\text{off}}^{\text{ds}} = \frac{\left(2\left(\alpha_0^2\right)\alpha_1 E_{21} + \beta_1 E_{22} - \alpha_0 E_{23}\right)}{K_2}$$

$$\theta^{\text{ds}} = \frac{\left(a_1\alpha_1(2\alpha_0\alpha_1 - \beta_0\beta_1)\lambda_0(\gamma_1\lambda_0 - \gamma_0\lambda_1) + E_{31} + a_1 E_{32}\mu + \beta_1 E_{33}\mu + 2(\alpha_0^2)(\alpha_1^2)cE_{34} + \alpha_0\alpha_1 E_{35}\right)}{-K_2}$$

$$V^{\text{ds}} = \frac{\left(\alpha_1\beta_1 E_{41}\lambda_0 + a_1\alpha_1 E_{42}\lambda_0 + a_1 E_{43}\lambda_1 + \beta_0\beta_1 E_{44}\lambda_1 + \alpha_0\alpha_1 E_{45} + 2\left(\alpha_0^2\right)\left(\alpha_1^2\right)cE_{46}\right)}{-K_2}.$$

The mathematical expressions of E_{11}, E_{12}, K_2, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33}, E_{34}, E_{35}, E_{41}, E_{42}, E_{43}, E_{44}, E_{45} and E_{46} are given in Appendix A.5.

Putting these values of optimal decision variables $W_o^{\rm ds}$, $W_{\rm off}^{\rm ds}$, $\theta^{\rm ds}$ and $V^{\rm ds}$ in (15), the manufacturer's profit function is obtained as

$$\begin{aligned}\Pi_M^{\rm ds}\left(W_o^{\rm ds}, W_{\rm off}^{\rm ds}, \theta^{\rm ds}, V^{\rm ds}\right) = \left(W_o^{\rm ds} - c\right)\Bigg(a - \frac{\alpha_0 F^{\rm ds}}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} - \alpha_0\phi_2 V^{\rm ds} + \frac{\beta_0 G^{\rm ds}}{2\alpha_1(4\alpha_0\alpha_1 - 2\beta_0\beta_1)}\\ + \beta_0\psi_2 V^{\rm ds} + \gamma_0\theta^{\rm ds} + \lambda_0 V^{\rm ds}\Bigg) + \left(W_{\rm off}^{\rm ds} - c\right)\Bigg(a_1 - \frac{G^{\rm ds}}{2(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} - \alpha_1\psi_2 V^{\rm ds}\\ + \frac{\beta_1 F^{\rm ds}}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} + \beta_1\phi_2 V^{\rm ds} + \gamma_1\theta^{\rm ds} + \lambda_1 V^{\rm ds}\Bigg) - I\frac{{\theta^{\rm ds}}^2}{2} - \mu\frac{{V^{\rm ds}}^2}{2}\end{aligned}$$

where,

$$\begin{aligned}F^{\rm ds} &= 2a\alpha_1 - 4\alpha_0\alpha_1\phi_2V^{\rm ds} + 2\alpha_0\alpha_1W_o^{\rm ds} + \beta_0a_1 - 2\alpha_1\beta_0\psi_2V^{\rm ds} + \beta_0\alpha_1W_{\rm off}^{\rm ds} + 2\beta_0\beta_1\phi_2V^{\rm ds} + \beta_0\gamma_1\theta^{\rm ds}\\ &\quad + \beta_0\lambda_1V^{\rm ds} - \beta_0\beta_1W_o^{\rm ds} + 2\alpha_1\beta_0\psi_2V^{\rm ds} + 2\alpha_1\gamma_0\theta^{\rm ds} + 2\alpha_1\lambda_0V^{\rm ds}\\ G^{\rm ds} &= 4\alpha_0\alpha_1a_1 - 2\beta_0\beta_1a_1 - 8\alpha_0\left(\alpha_1^2\right)\psi_2V^{\rm ds} + 4\beta_0\beta_1\alpha_1\psi_2V^{\rm ds} + 4\alpha_0\left(\alpha_1^2\right)W_{\rm off}^{\rm ds} - 2\beta_0\beta_1\alpha_1W_{\rm off}^{\rm ds} + \beta_1F^{\rm ds}\\ &\quad + 4\alpha_0\alpha_1\beta_1\phi_2V^{\rm ds} - 2\beta_0\left(\beta_1^2\right)\phi_2V^{\rm ds} + 4\alpha_0\alpha_1\gamma_1\theta^{\rm ds} - 2\beta_0\beta_1\gamma_1\theta^{\rm ds} + 4\alpha_0\alpha_1\lambda_1V^{\rm ds} - 2\beta_0\beta_1\lambda_1V^{\rm ds}.\end{aligned}$$

□

Proposition 7. *From Proposition 4 it is concluded that profit function of an online retailer is maximum and the maximum value of this profit function is given by*

$$\begin{aligned}{\Pi_R^o}^{(ds)}(\phi_1^{\rm ds}) &= a\phi_1^{\rm ds} + a\phi_2V^{\rm ds} - aW_o^{\rm ds} - \alpha_0{\phi_1^{\rm ds}}^2 - 2\alpha_0\phi_1^{\rm ds}\phi_2V^{\rm ds} + \alpha_0W_o^{\rm ds}\phi_1^{\rm ds} - \alpha_0\phi_2^2{V^{\rm ds}}^2 + \alpha_0\phi_2V^{\rm ds}W_o^{\rm ds}\\ &\quad + \beta_0\phi_1^{\rm ds}\frac{B^{\rm ds}}{2\alpha_1} + \beta_0\phi_2V^{\rm ds}\frac{B^{\rm ds}}{2\alpha_1} - \beta_0W_o^{\rm ds}\frac{B^{\rm ds}}{2\alpha_1} + \beta_0\psi_2V^{\rm ds}\phi_1^{\rm ds} + \beta_0\psi_2{V^{\rm ds}}^2\phi_2\\ &\quad - \beta_0\psi_2V^{\rm ds}W_o^{\rm ds} + \gamma_0\theta^{\rm ds}\phi_1^{\rm ds} + \gamma_0\theta^{\rm ds}\phi_2V^{\rm ds} - \gamma_0\theta^{\rm ds}W_o^{\rm ds} + \lambda_0V^{\rm ds}\phi_1^{\rm ds} + \lambda_0{V^{\rm ds}}^2\phi_2 - \lambda_0V^{\rm ds}W_o^{\rm ds}\end{aligned}$$

where $\phi_1^{\rm ds} = \frac{(-2(\alpha_0^2)(\alpha_1^2)E_{51} + \alpha_0\alpha_1E_{52} + \beta_1E_{53})}{K_2}$.

Proof. Putting the values of optimal decision variables $W_o^{\rm ds}$, $W_{\rm off}^{\rm ds}$, $\theta^{\rm ds}$ and $V^{\rm ds}$ of manufacturer from Proposition 6 in (13) obtained in Proposition 5, the optimal decision variable of online retailer is obtained as

$$\phi_1^{\rm ds} = \frac{\left(-2\left(\alpha_0^2\right)\left(\alpha_1^2\right)E_{51} + \alpha_0\alpha_1E_{52} + \beta_1E_{53}\right)}{K_2}.$$

The mathematical expressions of E_{51}, E_{52} and E_{53} are given in Appendix A.6.

Putting the optimal decision variables $\phi_1^{\rm ds}$, $W_o^{\rm ds}$, $W_{\rm off}^{\rm ds}$, $\theta^{\rm ds}$ and $V^{\rm ds}$ in (12) contained in the proof of Proposition 4, the optimal profit function of online retailer becomes

$$\begin{aligned}{\Pi_R^o}^{(\rm ds)}(\phi_1^{\rm ds}) &= a\phi_1^{\rm ds} + a\phi_2V^{\rm ds} - aW_o^{\rm ds} - \alpha_0{\phi_1^{\rm ds}}^2 - 2\alpha_0\phi_1^{\rm ds}\phi_2V^{\rm ds} + \alpha_0W_o^{\rm ds}\phi_1^{\rm ds} - \alpha_0\phi_2^2{V^{\rm ds}}^2 + \alpha_0\phi_2V^{\rm ds}W_o^{\rm ds}\\ &\quad + \beta_0\phi_1^{\rm ds}\frac{B^{\rm ds}}{2\alpha_1} + \beta_0\phi_2V^{\rm ds}\frac{B^{\rm ds}}{2\alpha_1} - \beta_0W_o^{\rm ds}\frac{B^{\rm ds}}{2\alpha_1} + \beta_0\psi_2V^{\rm ds}\phi_1^{\rm ds} + \beta_0\psi_2{V^{\rm ds}}^2\phi_2 - \beta_0\psi_2V^{\rm ds}W_o^{\rm ds}\\ &\quad + \gamma_0\theta^{\rm ds}\phi_1^{\rm ds} + \gamma_0\theta^{\rm ds}\phi_2V^{\rm ds} - \gamma_0\theta^{\rm ds}W_o^{\rm ds} + \lambda_0V^{\rm ds}\phi_1^{\rm ds} + \lambda_0{V^{\rm ds}}^2\phi_2 - \lambda_0V^{\rm ds}W_o^{\rm ds}\end{aligned}$$

where, $B^{\rm ds} = a_1 - 2\alpha_1\psi_2V^{\rm ds} + \alpha_1W_{\rm off}^{\rm ds} + \beta_1\phi_1^{\rm ds} + \beta_1\phi_2V^{\rm ds} + \gamma_1\theta^{\rm ds} + \lambda_1V^{\rm ds}$. □

Proposition 8. *From Proposition 3 it is concluded that profit function of an off-line retailer is maximum and the maximum value of this profit function is given by*

$$\Pi_R^{\text{off}\,(ds)}\left(\psi_1^{\text{ds}}\right) = \left(\psi_1^{\text{ds}} + \psi_2 V^{\text{ds}} - W_{\text{off}}^{\text{ds}}\right)\left(a_1 - \alpha_1\psi_1^{\text{ds}} - \alpha_1\psi_2 V^{\text{ds}} + \beta_1\phi_1^{\text{ds}} + \beta_1\phi_2 V^{\text{ds}} + \gamma_1\theta^{\text{ds}} + \lambda_1 V^{\text{ds}}\right)$$

where $\psi_1^{\text{ds}} = \frac{(2(\alpha_0^2)\alpha_1 E_{61} + \beta_1 E_{62} + \alpha_0 E_{63})}{K_2}$.

Proof. Putting the values of optimal decision variables W_o^{ds}, $W_{\text{off}}^{\text{ds}}$, θ^{ds}, V^{ds} and ϕ_1^{ds} of the manufacturer and the online retailer respectively in (14), which are contained in the proof of Proposition 5, the optimal decision variable of off-line retailer is obtained as

$$\psi_1^{\text{ds}} = \frac{\left(2\left(\alpha_0^2\right)\alpha_1 E_{61} + \beta_1 E_{62} + \alpha_0 E_{63}\right)}{K_2}.$$

The mathematical expressions of E_{61}, E_{62} and E_{63} are given in Appendix A.7.

Putting the optimal decision variables ϕ_1^{ds}, ψ_1^{ds}, $W_{\text{off}}^{\text{ds}}$, θ^{ds} and V^{ds} in (9), the optimal profit function of off-line retailer becomes

$$\Pi_R^{\text{off}\,(ds)}(\psi_1^{\text{ds}}) = \left(\psi_1^{\text{ds}} + \psi_2 V^{\text{ds}} - W_{\text{off}}^{\text{ds}}\right)\left(a_1 - \alpha_1\psi_1^{\text{ds}} - \alpha_1\psi_2 V^{\text{ds}} + \beta_1\phi_1^{\text{ds}} + \beta_1\phi_2 V^{\text{ds}} + \gamma_1\theta^{\text{ds}} + \lambda_1 V^{\text{ds}}\right).$$

. □

Proposition 9. *Demand in online and offline channels increases with the increase of* V *if* $\beta_0\psi_2 + \lambda_0 - \alpha_0\phi_2 > 0$ *and* $\beta_1\phi_2 + \lambda_1 - \alpha_1\psi_2 > 0$ *respectively.*

Proof. Demands of online retail channel and offline retail channel are given by

$$D_o = a - \alpha_0\phi_1 + \beta_0\psi_1 + \gamma_0\theta + (\beta_0\psi_2 + \lambda_0 - \alpha_0\phi_2)V$$
$$D_{\text{off}} = a_1 - \alpha_1\psi_1 + \beta_1\phi_1 + \gamma_1\theta + (\beta_1\phi_2 + \lambda_1 - \alpha_1\psi_2)V.$$

First order derivatives of D_o and D_{off} w.r.t. V are given by

$$\frac{\partial D_o}{\partial V} = \beta_0\psi_2 + \lambda_0 - \alpha_0\phi_2$$
$$\frac{\partial D_{\text{off}}}{\partial V} = \beta_1\phi_2 + \lambda_1 - \alpha_1\psi_2$$

D_o and D_{off} increase with increase of V if $\frac{\partial D_o}{\partial V} > 0$ and $\frac{\partial D_{\text{off}}}{\partial V} > 0$ respectively, *i.e.*, if $\beta_0\psi_2 + \lambda_0 - \alpha_0\phi_2 > 0$ and $\beta_1\phi_2 + \lambda_1 - \alpha_1\psi_2 > 0$ respectively. □

4. Numerical results

In this section, a thorough numerical analysis is presented to validate our proposed dual-channel supply chain model. A numerical example using the following set of parameter values is examined:

$a = 2000$, $a_1 = 1950$, $\alpha_0 = 5.8$, $\alpha_1 = 5.9$, $\beta_0 = 0.2$, $\beta_1 = 0.3$, $\gamma_0 = 28.5$, $\gamma_1 = 29$, $\lambda_0 = 56.5$, $\lambda_1 = 57$, $\phi_2 = 9.5$, $\psi_2 = 10$, $c = 240$, $I = 4300$, $\mu = 8200$. Using MATHEMATICA 11.3, the optimal values of all the decision variables and profit functions of each supply chain member and the whole supply chain profit in decentralized policy, and centralized policy respectively are obtained from these parametric values (Tab. 2).

In centralized policy, the optimal variables of the whole supply chain are obtained as $\phi_1^{\text{cs}} = 297.19$, $P_o^{\text{cs}} = 305.464$, $\psi_1^{\text{cs}} = 291.695$, $P_{\text{off}}^{\text{cs}} = 300.405$, $\theta^{\text{cs}} = 0.841271$, $V^{\text{cs}} = 0.870951$. The values of first order, second order, third order and fourth order principle minors are $-11.6 < 0$, $136.63 > 0$, $-567342 < 0$ and $4.31432 \times 10^9 > 0$ respectively, which implies profit function of the whole supply chain in centralized policy is maximized at these parametric values as the Hessian matrix of the profit function of the whole supply chain is negative definite at

TABLE 2. Optimum results for centralized and decentralized problems.

Proposed models	Members of supply chain	Objective functions and variables (dependent and independent)	Optimum values
Centralized policy	Manufacturer	Green level of product $\in \mathbf{R}[0,1]$	0.841271
		Circular economic index of product $\in \mathbf{R}[0,1]$	0.870951
	Online Retailer	Unit online initial retail price (in online channel), (in Rs.)	297.19
		Unit online retail price(in Rs.)	305.464
	Off-line Retailer	Unit off-line initial retail price (in off-line channel), (in Rs.)	291.695
		Unit off-line retail price(in Rs.)	300.405
		Total supply chain profit (in Rs.)	39774.9
Decentralized policy	Manufacturer	Green level of product $\in \mathbf{R}[0,1]$	0.406518
		Circular economic index of product $\in \mathbf{R}[0,1]$	0.420861
		Unit online wholesale price (in online channel) (in Rs.)	301.826
		Unit off-line wholesale price (in off-line channel) (in Rs.)	297.263
		Total profit (in Rs.)	19220.6
	Online Retailer	Unit online initial retail price (in online channel), (in Rs.)	328
		Unit online retail price (in Rs.)	331.999
		Total profit (in Rs.)	5275.43
	Off-line Retailer	Unit off-line initial retail price (in off-line channel) (in Rs.)	321.15
		Unit off-line retail price (in Rs.)	325.359
		Total profit (in Rs.)	4657.11

these parametric values. At these parametric values, the total supply chain profit is reached as $\Pi_{sc}(\phi_1^{\text{cs}}, \psi_1^{\text{cs}}, \theta^{\text{cs}}, V^{\text{cs}}) = 39774.9$. In decentralized policy, the decision variables of all supply chain members are obtained as $\phi_1^{\text{ds}} = 328$, $P_o^{\text{ds}} = 331.999$, $\psi_1^{\text{ds}} = 321.15$, $P_{\text{off}}^{\text{ds}} = 325.359$, $W_o^{\text{ds}} = 301.826$, $W_{\text{off}}^{\text{ds}} = 297.263$, $\theta^{\text{ds}} = 0.406518$, $V^{\text{ds}} = 0.420861$. At these parametric values, the second order derivative of online retailer's profit function w.r.t. the decision variable ϕ_1 is $-11.5898 < 0$, the second order derivative of off-line retailer's profit function w.r.t. the decision variable ψ_1 is $-11.8 < 0$ which implies that online retailer's and off-line retailer's profit functions are maximized at these parametric values. Also, at these parametric values, the value of first order, second order, third order and fourth order principle minors are $-5.79492 < 0$, $34.1725 > 0$, $-144399 < 0$ and $1.14146 \times 10^9 > 0$, respectively, which implies that the Hessian matrix of the profit function of the manufacturer is negative definite at these parametric values. So, the profit function of the manufacturer is also maximized at these parametric values.

The optimal outcomes for each model scenario are shown in Table 2. Supply chain members can optimize total profitability by using a collaborative decision-making strategy, as shown by numerical findings. By working together, they can provide end users with green and circular economic indexed products at a lesser cost than they could with a decentralized model, which will raise consumer satisfaction and boost sales. In the centralized strategy, a remarkable rise in the green level and CEI of products results in a tremendous increase of demand functions, raising integrated profitability. Furthermore, the retail pricing's linear relationship with CEI speeds up the rise in profitability as CEI rises in the centralized policy.

5. Sensitivity analyses and result discussion

Now, to study the influences of different parametric values on profit and other decision variables (dependent and independent both) of supply chain members in decentralized policy, a sensitivity analysis is performed through some tables and figures. It is desires to observe how the small changes in parameters' values have a huge impact on the profits of each supply chain member in a decentralized scenario.

The following observations are obtained from the Figures 2–17:

(i) a is the basic market potential of online retail channels. From Figure 5, it is observed that with the small increase of a green level and circular economic index (CEI) increases more. From Figure 4, it is seen that online demand increases more than offline demand with increase of a. Due to the higher increase rate of

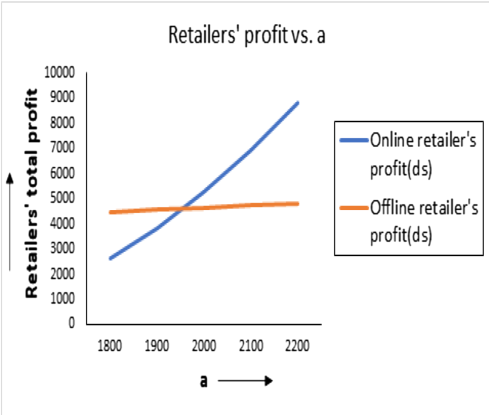

FIGURE 2. Retailers' profit *vs.* a in decentralized scenario (ds).

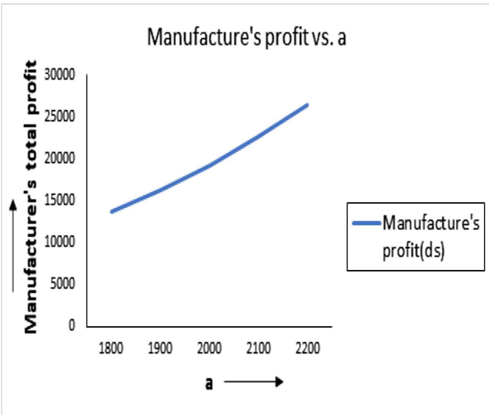

FIGURE 3. Manufacturer's profit *vs.* a in decentralized scenario (ds).

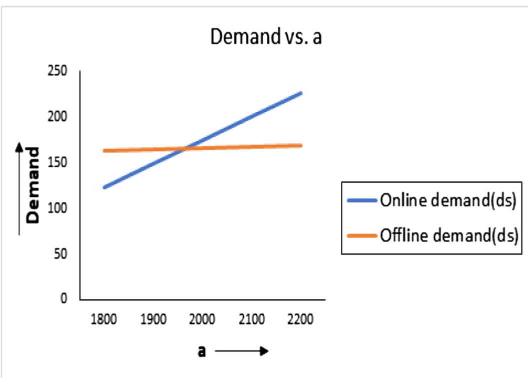

FIGURE 4. Demand *vs.* a in decentralized scenario (ds).

TABLE 3. Results of sensitivity analyses in decentralized policy for the changes of a, a_1, α_0 and α_1.

Parameter	Value	W_o^{ds}	$W_{\mathrm{off}}^{\mathrm{ds}}$	ϕ_1^{ds}	ψ_1^{ds}	P_o^{ds}	$P_{\mathrm{off}}^{\mathrm{ds}}$	θ^{ds}	V^{ds}	D_o	D_{off}	$\Pi_R^{o\,(ds)}$	$\Pi_R^{\mathrm{off}\,(ds)}$	Π_M^{ds}
	1800	284.048	295.953	301.986	320.005	305.348	323.544	0.342053	0.353893	123.432	162.787	2629.1	4491.43	13 780.3
	1900	292.937	296.608	314.993	320.578	318.673	324.451	0.374285	0.387377	149.138	164.274	3838.23	4573.9	16 277.8
a	2000	301.826	297.263	328	321.15	331.999	325.359	0.406518	0.420861	174.845	165.762	5275.43	4657.11	19 220.6
	2100	310.715	297.919	341.007	321.723	345.324	326.266	0.438751	0.454345	200.551	167.249	6940.7	4741.07	22 608.7
	2200	319.605	298.574	354.014	322.295	358.649	327.173	0.470983	0.487829	226.258	168.737	8834.04	4825.78	26 442.1
	1755	300.696	280.22	327.14	296.246	330.523	299.807	0.343747	0.356117	172.846	115.563	5155.53	2263.51	14 365
	1852.5	301.261	288.742	327.57	308.698	331.261	312.583	0.375133	0.388489	173.846	140.662	5215.31	3353.53	16 584.7
a_1	1950	301.826	297.263	328	321.15	331.999	325.359	0.406518	0.420861	174.845	165.762	5275.43	4657.11	19 220.6
	2047.5	302.392	305.785	328.431	333.602	332.736	338.134	0.437903	0.453233	175.844	190.861	5335.9	6174.24	22 272.5
	2145	302.957	314.307	328.861	346.054	333.474	350.91	0.469289	0.485605	176.843	215.961	5396.71	7904.92	25 740.6
	5.22	322.736	298.808	358.506	322.495	363.258	327.497	0.482921	0.500231	211.319	169.263	8563.11	4855.93	25 910.4
	5.51	311.713	297.994	342.424	321.786	346.779	326.37	0.442639	0.458384	193.036	167.417	6769.06	4750.6	22 269.6
α_0	5.8	301.826	297.263	328	321.15	331.999	325.359	0.406518	0.420861	174.845	165.762	5275.43	4657.11	19 220.6
	6.09	292.91	296.605	314.99	320.577	318.667	324.447	0.373945	0.387023	156.731	164.269	4036.95	4573.59	16 676.2
	6.38	284.827	296.008	303.196	320.057	306.582	323.62	0.344422	0.356353	138.684	162.915	3017	4498.53	14 565.6
	5.31	303.183	317.651	329.031	350.876	333.769	355.863	0.482035	0.498751	177.228	202.905	5420.77	7753.37	25 434.2
	5.605	302.468	306.903	328.488	335.205	332.836	339.782	0.44222	0.457685	175.972	184.286	5343.9	6059.1	22 042.5
α_1	5.9	301.826	297.263	328	321.15	331.999	325.359	0.406518	0.420861	174.845	165.762	5275.43	4657.11	19 220.6
	6.195	301.248	288.569	327.561	308.473	331.243	312.35	0.374322	0.387653	173.828	147.319	5214.06	3503.27	16 884.4
	6.49	300.724	280.688	327.162	296.981	330.559	300.557	0.345141	0.357554	172.907	128.945	5158.73	2561.91	14 965.8

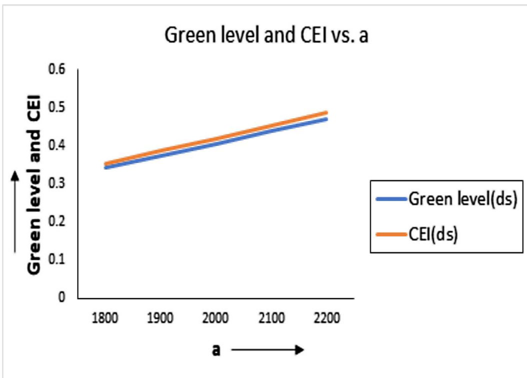

FIGURE 5. Green level and CEI *vs.* a in decentralized scenario (ds).

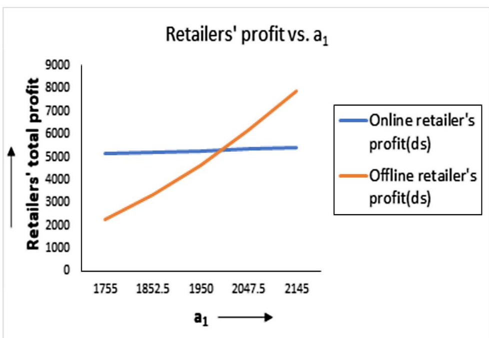

FIGURE 6. Retailers' profit *vs.* a_1 in decentralized scenario (ds).

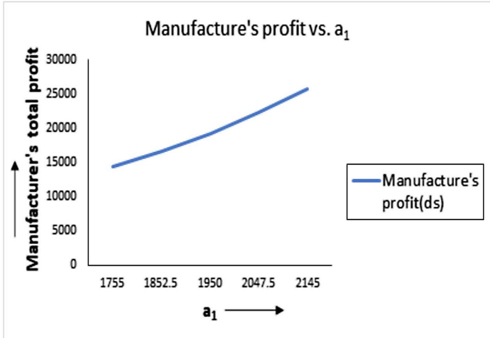

FIGURE 7. Manufacturer's profit *vs.* a_1 in decentralized scenario (ds).

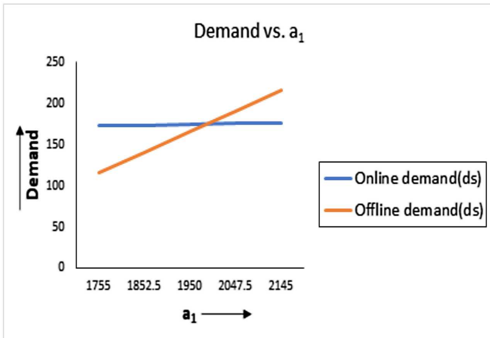

FIGURE 8. Demand *vs.* a_1 in decentralized scenario (ds).

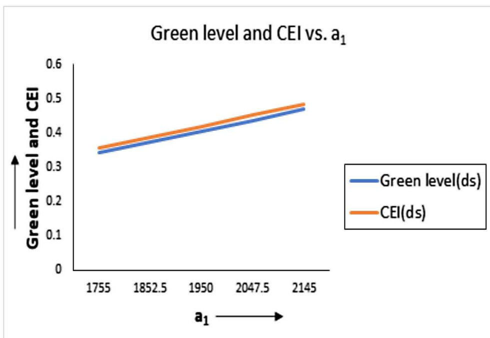

FIGURE 9. Green level and CEI *vs.* a_1 in decentralized scenario (ds).

green level and CEI offline demand also increases but the increasing rate is very low. As online demand and offline demand both increase, although the increasing rate is different, the online and offline retailers both increase their initial unit retail prices (Tab. 3). The manufacturer also increases his/her online and offline wholesale prices. As a result, from Figures 2 and 3, it is seen that with the increasing demand, the profit of each supply chain member increases. In this case, the online retailer's profit increases more than the offline retailer's profit due to increasing online demand more than offline demand w.r.t. a.

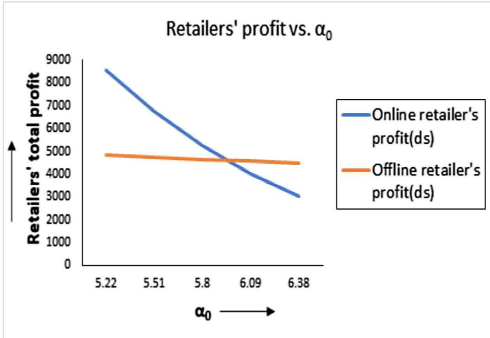

FIGURE 10. Retailers' profit *vs.* α_0 in decentralized scenario (ds).

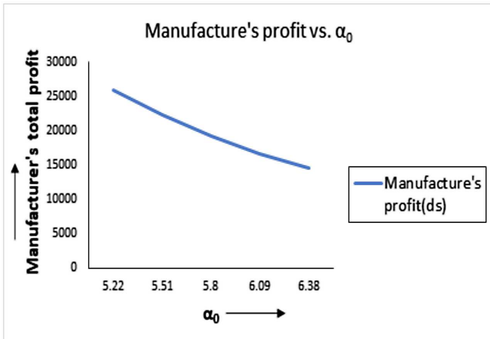

FIGURE 11. Manufacturer's profit *vs.* α_0 in decentralized scenario (ds).

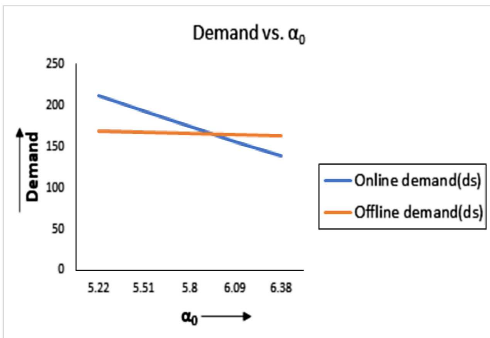

FIGURE 12. Demand *vs.* α_0 in decentralized scenario (ds).

(ii) A similar effect can be seen with offline basic market demand a_1 as a. Here, with a small increase of a_1 offline demand increases more than online demand. The reason behind the increasing demand is the higher rate of green level and CEI with the small increase of a_1 (Fig. 9). Due to more increase in offline demand than online demand, offline retailer increases his/her initial unit retail price more than online retailer. So, with increasing offline demand more than online demand, the profit of offline retailer also increases more

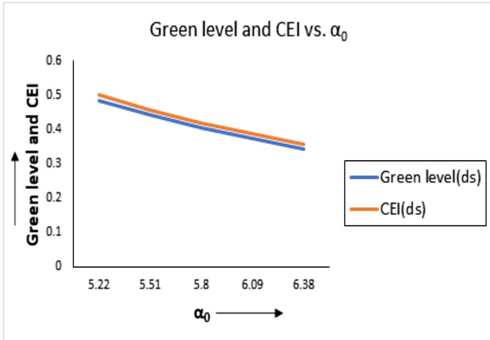

FIGURE 13. Green level and CEI *vs.* α_0 in decentralized scenario (ds).

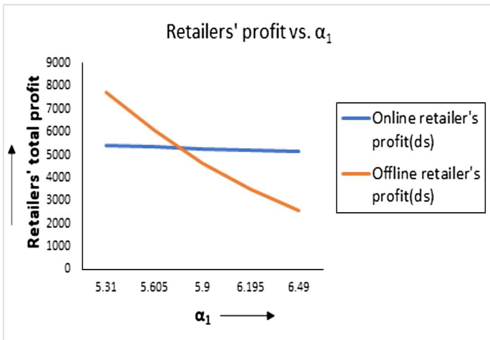

FIGURE 14. Retailers' profit *vs.* α_1 in decentralized scenario (ds).

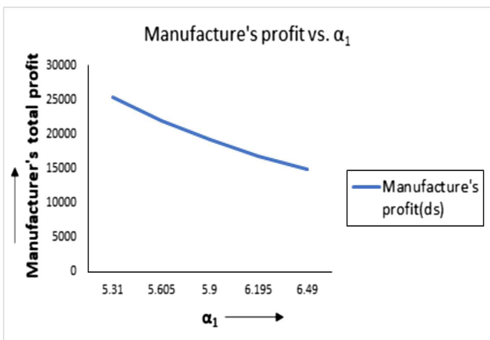

FIGURE 15. Manufacturer's profit *vs.* α_1 in decentralized scenario (ds).

than online retailer (Fig. 6). Manufacturer's profit also increases due to the increase of both online and offline demand along with the increasing wholesale prices (Tab. 3).

(iii) α_0 is the self-price sensitivity parameter of online retail channels. From Figure 13, it is seen that the small increase of α_0 causes a greater decrease in the green level and CEI. Demand in both channels also decreases with small increase of α_0 but the decreasing rate in online channels is higher than in offline channels (Fig. 12). As demand decreases, both retailers decrease their initial retail prices (Tab. 3). Manufacturer

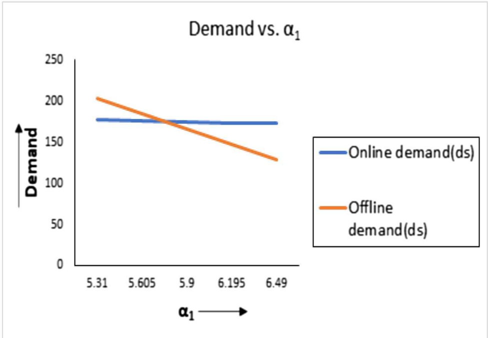

FIGURE 16. Demand *vs.* α_1 in decentralized scenario (ds).

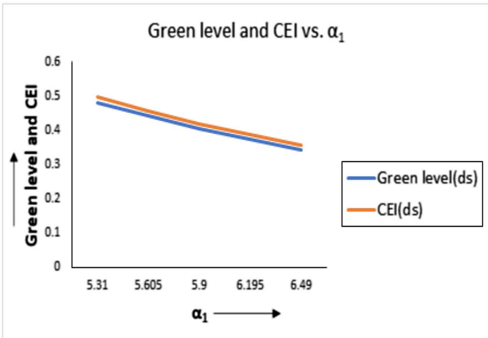

FIGURE 17. Green level and CEI *vs.* α_1 in decentralized scenario (ds).

also decreases his/her wholesale prices in both channels (Tab. 3). As a result of Figures 10 and 11, it is noticed that the profit of each member decreases with the small increase of α_0. But the online retailer's profit decreases more than the offline retailer's profit.

(iv) A similar effect can be observed for offline self-price sensitivity parameter α_1 as α_0. But in this case, the offline retailer's profit decreases more than the online retailer's profit as offline demand decreases more than online demand. These effects can be seen in Figures 14 and 16.

6. Managerial insights and practical applications

To reduce environmental pollution and save natural resources of raw-material, green and circular manufacturing based dual-channel supply chain model is developed and analyzed different aspects on industry and environment which are given below.

(i) By increasing online and offline basic market demand parameter, the production manager can increase the profit of each individual member with enhanced green level and CEI of products.

(ii) By enhancing the greenness and CEI of manufactured goods, supply chain managers can improve their profitability. The green level and circular economic index vary more with a minor change in the self-price sensitivity parameters. By altering these parametric values, the production managers can boost their profits while improving the sustainability of the environment. Manufacturing managers may focus on certain sensitivity characteristics to increase profits and improve environmental sustainability.

(iii) By raising the CEI of produced products, the industrial managers can reduce wastages, which will have a positive effect on creating a healthy environment, and also can extend consumer awareness about the CEI of the products by producing more products with a higher CEI.
(iv) The presence of both traditional offline retail channels and online retail channels in this engaged life cycle aids supply chain members and production managers in surviving in a competitive market with healthy profit margins. Keeping this in mind, it is incorporated online and offline retail channels in our work. In this study, it is found that customer demand for the online retail channel is higher than that for the offline channel. So, the managers can increase their profits by raising customer demands in the online retail channel by investing in it for higher customer satisfaction.

Thus, by controlling the green level of the items as well as the CEI of the products, this work instructs managers on how to improve their profitability while simultaneously improving the environment's health and sustainability.

A sustainable dual-channel supply chain model incorporating circular economy, green demand, and price sensitivity is transforming industries by reducing waste, optimizing costs, and catering to eco-conscious and budget-conscious consumers. Businesses can benefit by enhancing customer loyalty, regulatory compliance, and long-term profitability. Some real-life examples are given as follows where the concept of the developed model can be implemented:

(i) Businesses like Amazon, Walmart, and IKEA operate both online and offline sales channels. Also, they can implement circular economy practices by encouraging product returns, refurbishments, and resale through both channels.
(ii) Companies like Dell, HP, and Lenovo integrate circular economy by offering refurbished products through both online and off-line retail channels.
(iii) Green and price-sensitive demand is addressed by offering eco-friendly product alternatives and dynamic pricing strategies based on customer preferences. Moreover, consumers who are price-sensitive but also want eco-friendly choices can opt for certified refurbished laptops and smartphones at lower prices.

7. Conclusions and future research

Looking at the present attraction towards online marketing and growing pollution due to huge technological advancements, this study depicts a dual channel supply chain model with green and CEI based demand functions. This work constitutes the manufacturer, the online retailer and the offline retailer. Optimal prices, green level and CEI of products are found out after optimizing the profit functions of the online and offline retailers,the manufacturer and the whole supply chain through centralized and decentralized policies. It is abundantly evident from the numerical results that the centralized policy is preferable to the decentralized policy, raising the products' green level and circular economic index to have better profit with a healthier environment. Sensitivity analysis reveals that basic market potential parameters a and a_1 are more sensitive to profit, consumers' demand, green level and CEI of product. The self-price sensitivity parameters α_0 and α_1 also be seen highly sensible to the aforementioned. It is seen that as a increases, total online demand increases more than offline demand, and vice versa for a_1. It is also observed that with increasing values of self price sensitivity parameter (α_0) in the online channel , total online demand declines more than offline demand and vice versa for the offline channel self-price sensitivity parameter(α_1). It is concluded that a and a_1 are more effective parameters to increase the members' individual profits and also to enhance the green level and CEI of products.

A number of noteworthy observations from this study are highlighted as follows:

– In today's industrialized world, waste management through the reuse of used products is also required for a sustainable environment, as the greening of products is insufficient to safeguard the environment. To increase environmental sustainability, this study considers the CEI along with the greenness of manufactured goods. This indicates that our research aims to create healthy environments that promote the well-being of living things.

– Our research demonstrates that centralized policy outperform decentralized policy in terms of improved environmental performance and increased revenues at a lower cost.
– Meeting consumer demands online and offline enhances the profits for each supply chain member with customer satisfaction.

Our research has limits even though it contributes to the creation of healthy surroundings. This study takes into account variable-dependent deterministic demand functions, but to make this work more believable, stochastic demand might be created. It might be feasible to promote healthy competition among the retailers by utilizing an additional game-theoretical strategy. It is possible to think of a contract as a coordinated mechanism that enhances the profit of each individual member. However, complexity prevents us from producing such a well-coordinated model in our job.

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Conflicts of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Appendix A.

A.1.

The first order partial derivatives of $\Pi_{sc}(\phi_1, \psi_1, \theta, V)$ as given in (2) w.r.t. ϕ_1, ψ_1, θ and V are

$$\frac{\partial \Pi_{sc}}{\partial \phi_1} = a - 2\alpha_0\phi_1 - 2\alpha_0\phi_2 V + \alpha_0 c + \beta_0\psi_1 + \beta_0\psi_2 V + \gamma_0\theta + \lambda_0 V + \beta_1\psi_1 + \beta_1\psi_2 V - \beta_1 c$$

$$\frac{\partial \Pi_{sc}}{\partial \psi_1} = \beta_0\phi_1 + \beta_0\phi_2 V - \beta_0 c + a_1 - 2\alpha_1\psi_1 - 2\alpha_1\psi_2 V + \alpha_1 c + \beta_1\phi_1 + \beta_1\phi_2 V + \gamma_1\theta + \lambda_1 V$$

$$\begin{aligned}\frac{\partial\Pi_{sc}}{\partial\theta} &= \gamma_0\phi_1 + \gamma_0\phi_2 V - c\gamma_0 + \gamma_1\psi_1 + \gamma_1\psi_2 V - \gamma_1 c - I\theta\\ \frac{\partial\Pi_{sc}}{\partial V} &= a\phi_2 - 2\alpha_0\phi_1\phi_2 - 2\alpha_0\phi_2^2 V + \alpha_0 c\phi_2 + \beta_0\phi_2\psi_1 + \beta_0\phi_1\psi_2 + 2\beta_0\phi_2\psi_2 V\\ &\quad - \beta_0 c\psi_2 + \gamma_0\phi_2\theta + \lambda_0\phi_1 + 2\lambda_0\phi_2 V - c\lambda_0 + a_1\psi_2 - 2\alpha_1\psi_1\psi_2 - 2\alpha_1\psi_2^2 V\\ &\quad + c\alpha_1\psi_2 + \beta_1\psi_2\phi_1 + \beta_1\psi_1\phi_2 + 2\beta_1\psi_2\phi_2 V - \beta_1 c\phi_2 + \gamma_1\psi_2\theta + \lambda_1\psi_1 + 2\lambda_1\psi_2 V - \lambda_1 c - \mu V.\end{aligned}$$

The second order partial derivatives w.r.t. the decision variables ϕ_1, ψ_1, θ and V are given by

$$\begin{aligned}&\frac{\partial^2\Pi_{sc}}{\partial\phi_1^2} = -2\alpha_0,\ \frac{\partial^2\Pi_{sc}}{\partial\psi_1\partial\phi_1} = \beta_0 + \beta_1,\ \frac{\partial^2\Pi_{sc}}{\partial\theta\partial\phi_1} = \gamma_0,\\ &\frac{\partial^2\Pi_{sc}}{\partial V\partial\phi_1} = -2\alpha_0\phi_2 + \beta_0\psi_2 + \lambda_0 + \beta_1\psi_2,\ \frac{\partial^2\Pi_{sc}}{\partial\psi_1^2} = -2\alpha_1,\\ &\frac{\partial^2\Pi_{sc}}{\partial\theta\partial\psi_1} = \gamma_1,\ \frac{\partial^2\Pi_{sc}}{\partial V\partial\psi_1} = \beta_0\phi_2 - 2\alpha_1\psi_2 + \beta_1\phi_2 + \lambda_1,\ \frac{\partial^2\Pi_{sc}}{\partial\theta^2} = -I,\ \frac{\partial^2\Pi_{sc}}{\partial V\partial\theta} = \gamma_0\phi_2 + \gamma_1\psi_2,\\ &\frac{\partial^2\Pi_{sc}}{\partial V^2} = -2\alpha_0\phi_2^2 + 2\beta_0\phi_2\psi_2 + 2\lambda_0\phi_2 - 2\alpha_1\psi_2^2 + 2\beta_1\phi_2\psi_2 + 2\lambda_1\psi_2 - \mu.\end{aligned}$$

A.2.

The mathematical expressions of X_{11}, X_{12}, X_{13}, X_{14}, X_{15}, X_{16}, X_{17}, Z, X_{21}, X_{22}, X_{23}, X_{24}, X_{25}, X_{26}, X_{31}, X_{32}, X_{41} and X_{42} which are used in Proposition 2, are as follows:

$$\begin{aligned}X_{11} &= I\{(\beta_0+\beta_1)\mu - 2\alpha_0\lambda_1\phi_2 + \lambda_0(\lambda_1 - (\beta_0+\beta_1)\phi_2)\} + \gamma_0\{\gamma_0\lambda_1\phi_2 + \gamma_1(\mu - \lambda_0\phi_2)\}\\ X_{12} &= I\lambda_1(\lambda_1 + (\beta_0+\beta_1)\phi_2) - 2\alpha_1 I(\mu - \lambda_0\phi_2) + \gamma_1(\gamma_0\lambda_1\phi_2 + \gamma_1(\mu - \lambda_0\phi_2))\\ X_{13} &= 2\beta_0 I\lambda_0\lambda_1 + \beta_1 I\lambda_0\lambda_1 - (\gamma_0^2)(\lambda_1^2) + \alpha_0 I(\lambda_1^2) + \beta_0 I(\lambda_1^2) + (\beta_0^2)I\mu + \beta_0\beta_1 I\mu + \beta_1(\beta_0+\beta_1)I\lambda_0\phi_2\\ X_{14} &= -\beta_1(\gamma_0^2) + \beta_0(-\alpha_0+\beta_0)I + (\alpha_0+\beta_0)\beta_1 I\\ X_{15} &= 2\lambda_0\lambda_1 + (\alpha_1 + 2\beta_0 + \beta_1)\mu + (-\alpha_1+\beta_1)\lambda_0\phi_2 + (-\alpha_0+\beta_0)\lambda_1\phi_2\\ X_{16} &= (\alpha_0+\beta_0)\mu - \lambda_0(\lambda_0 + (-\alpha_0+\beta_0)\phi_2)\\ X_{17} &= (\gamma_0^2)(2\mu + \lambda_1\phi_2) + I(\lambda_0(2\lambda_0+\lambda_1) + (-2\alpha_0 - \beta_0 + \beta_1)\mu - ((-\beta_0+\beta_1)\lambda_0 + 2\alpha_0(\lambda_0+\lambda_1))\phi_2)\\ Z &= \lambda_1(-(\gamma_0^2)\lambda_1 + 2I((\beta_0+\beta_1)\lambda_0 + \alpha_0\lambda_1)) + ((\beta_0+\beta_1)^2)I\mu - (\gamma_1^2)((\lambda_0^2) - 2\alpha_0\mu)\\ &\quad + 2\gamma_0\gamma_1(\lambda_0\lambda_1 + (\beta_0+\beta_1)\mu) + 2\alpha_1(I(\lambda_0^2) + (\gamma_0^2)\mu - 2\alpha_0 I\mu)\\ X_{21} &= I\lambda_0\lambda_1 + \gamma_0\gamma_1\mu + (\beta_0+\beta_1)I\mu\\ X_{22} &= -(\gamma_1^2)\lambda_0 + 2\alpha_1 I\lambda_0 + \gamma_0\gamma_1\lambda_1 + (\beta_0+\beta_1)I\lambda_1\\ X_{23} &= \alpha_1 I(\lambda_0^2) + \beta_1 I(\lambda_0^2) + \alpha_0 I\lambda_0\lambda_1 + \beta_0 I\lambda_0\lambda_1 + 2\beta_1 I\lambda_0\lambda_1 - (\gamma_0^2)(\lambda_1^2) + 2\alpha_0 I(\lambda_1^2) + \alpha_1(\gamma_0^2)\mu + \beta_1(\gamma_0^2)\mu\\ &\quad - 2\alpha_0\alpha_1 I\mu + \alpha_0\beta_0 I\mu - \alpha_0\beta_1 I\mu + \beta_0\beta_1 I\mu + (\beta_1^2)I\mu\\ X_{24} &= \alpha_1(\beta_0-\beta_1)I\lambda_0 + \beta_1(\beta_0+\beta_1)I\lambda_0 + (\alpha_1-\beta_1)(\gamma_0^2)\lambda_1 + \beta_0(\beta_0+\beta_1)I\lambda_1 - \alpha_0 I((\beta_0-\beta_1)\lambda_1\\ &\quad + 2\alpha_1(\lambda_0+\lambda_1))\\ X_{25} &= 2\lambda_0\lambda_1 + (\alpha_0 + \beta_0 + 2\beta_1)\mu + (-\alpha_1+\beta_1)\lambda_0\psi_2 + (-\alpha_0+\beta_0)\lambda_1\psi_2\\ X_{26} &= I((\lambda_0^2) - 2\alpha_0\mu + (\beta_0+\beta_1)\lambda_0\psi_2 + 2\alpha_0\lambda_1\psi_2) + \gamma_0(\gamma_1\lambda_0\psi_2 + \gamma_0(\mu - \lambda_1\psi_2))\\ X_{31} &= 2a\alpha_1 + a_1(\beta_0+\beta_1) - 2\alpha_0\alpha_1 c + \alpha_1(\beta_0-\beta_1)c + \beta_1(\beta_0+\beta_1)c\\ X_{32} &= 2a_1\alpha_0 + a(\beta_0+\beta_1) + \alpha_0(-2\alpha_1 - \beta_0 + \beta_1)c + \beta_0(\beta_0+\beta_1)c\\ X_{41} &= \gamma_1((a_1 + (-\alpha_1+\beta_1)c)\gamma_0 - (a + (-\alpha_0+\beta_0)c)\gamma_1) + (2a\alpha_1 + a_1(\beta_0+\beta_1) - 2\alpha_0\alpha_1 c\\ &\quad + \alpha_1(\beta_0-\beta_1)c + \beta_1(\beta_0+\beta_1)c)I\\ X_{42} &= \gamma_0((a_1 + (-\alpha_1+\beta_1)c)\gamma_0 - (a + (-\alpha_0+\beta_0)c)\gamma_1) - (2a_1\alpha_0 + a(\beta_0+\beta_1) + \alpha_0(-2\alpha_1\\ &\quad - \beta_0 + \beta_1)c + \beta_0(\beta_0+\beta_1)c)I.\end{aligned}$$

A.3.

From (15), the first order partial derivatives of $\Pi_M(W_o, W_{\text{off}}, \theta, V)$ w.r.t. the decision variables W_o, W_{off}, θ and V are calculated as followings:

$$\begin{aligned}
\frac{\partial \Pi_M}{\partial W_o} &= a - \frac{\alpha_0 F}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} - \alpha_0\phi_2 V + \frac{\beta_0 G}{2\alpha_1(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} + \beta_0\psi_2 V + \gamma_0\theta + \lambda_0 V \\
&\quad + (W_o - c)\left(-\frac{\alpha_0}{2} + \frac{\beta_0\beta_1}{4\alpha_1}\right) + (W_{\text{off}} - c)\left(\frac{\beta_1}{4}\right) \\
\frac{\partial \Pi_M}{\partial W_{\text{off}}} &= (W_o - c)\left(-\alpha_0 Y_1 + \frac{\beta_0 Y_2}{2\alpha_1}\right) + \left(a_1 - \frac{G}{2(4\alpha_0\alpha_1 - 2\beta_0\beta_1)} - \alpha_1\psi_2 V + \frac{\beta_1 F}{4\alpha_0\alpha_1 - 2\beta_0\beta_1} + \beta_1\phi_2 V\right. \\
&\quad \left. + \gamma_1\theta + \lambda_1 V\right) + (W_{\text{off}} - c)\left(-\frac{Y_2}{2} + \beta_1 Y_1\right) \\
\frac{\partial \Pi_M}{\partial \theta} &= (W_o - c)\left(-\alpha_0 Y_3 + \frac{\beta_0 Y_4}{2\alpha_1} + \gamma_0\right) + (W_{\text{off}} - c)\left(-\frac{Y_4}{2} + \beta_1 Y_3 + \gamma_1\right) - I\theta \\
\frac{\partial \Pi_M}{\partial V} &= (W_o - c)\left(-\alpha_0 Y_5 - \alpha_0\phi_2 + \frac{\beta_0 Y_6}{2\alpha_1} + \beta_0\psi_2 + \lambda_0\right) + (W_{\text{off}} - c)\left(-\frac{Y_6}{2} - \alpha_1\psi_2 + \beta_1 Y_5 + \beta_1\phi_2 + \lambda_1\right) \\
&\quad - \mu V.
\end{aligned}$$

The second order partial derivatives of $\Pi_M(W_o, W_{\text{off}}, \theta, V)$ w.r.t. the decision variables W_o, W_{off}, θ and V are as followings:

$$\begin{aligned}
&\frac{\partial^2 \Pi_M}{\partial W_o^2} = -\alpha_0 + \frac{\beta_0\beta_1}{2\alpha_1},\ \frac{\partial^2 \Pi_M}{\partial W_{\text{off}}\partial W_o} = -\alpha_0 Y_1 + \frac{\beta_0 Y_2}{2\alpha_1},\ \frac{\partial^2 \Pi_M}{\partial \theta\partial W_o} = -\alpha_0 Y_3 + \frac{\beta_0 Y_4}{2\alpha_1} + \gamma_0, \\
&\frac{\partial^2 \Pi_M}{\partial V\partial W_o} = -\alpha_0 Y_5 - \alpha_0\phi_2 + \frac{\beta_0 Y_6}{2\alpha_1} + \beta_0\psi_2 + \lambda_0,\ \frac{\partial^2 \Pi_M}{\partial W_{\text{off}}^2} = -Y_2 + 2\beta_1 Y_1,\ \frac{\partial^2 \Pi_M}{\partial \theta\partial W_{\text{off}}} = -\frac{Y_4}{2} + \beta_1 Y_3 + \gamma_1, \\
&\frac{\partial^2 \Pi_M}{\partial V\partial W_{\text{off}}} = -\frac{Y_6}{2} - \alpha_1\psi_2 + \beta_1 Y_5 + \beta_1\phi_2 + \lambda_1,\ \frac{\partial^2 \Pi_M}{\partial \theta^2} = -I,\ \frac{\partial^2 \Pi_M}{\partial V\partial \theta} = 0,\ \frac{\partial^2 \Pi_M}{\partial V^2} = -\mu.
\end{aligned}$$

A.4.

The mathematical expressions of Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7 and Y_8 which are used in Proposition 5, are as follows: $Y_1 = \frac{\beta_0\alpha_1}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)}$, $Y_2 = \frac{(4\alpha_0(\alpha_1^2) - \beta_0\beta_1\alpha_1)}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)}$, $Y_3 = \frac{(\beta_0\gamma_1 + 2\alpha_1\gamma_0)}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)}$, $Y_4 = \frac{(2\beta_1\alpha_1\gamma_0 + 4\alpha_0\alpha_1\gamma_1 - \beta_0\beta_1\gamma_1)}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)}$,

$Y_5 = \frac{(-4\alpha_0\alpha_1\phi_2 + 2\beta_0\beta_1\phi_2 + \beta_0\lambda_1 + 2\alpha_1\lambda_0)}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)}$, $Y_6 = \frac{(-8\alpha_0(\alpha_1^2)\psi_2 + 4\beta_0\beta_1\alpha_1\psi_2 + 4\alpha_0\alpha_1\lambda_1 - \beta_0\beta_1\lambda_1 + 2\alpha_1\beta_1\lambda_0)}{(4\alpha_0\alpha_1 - 2\beta_0\beta_1)}$,

$Y_7 = -\alpha_0 Y_5 - \alpha_0\phi_2 + \frac{\beta_0 Y_6}{2\alpha_1} + \beta_0\psi_2 + \lambda_0$, $Y_8 = -\frac{Y_6}{2} - \alpha_1\psi_2 + \beta_1 Y_5 + \beta_1\phi_2 + \lambda_1$.

The mathematical expressions of Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7 and Z_8 used in Proposition 5, are as follows:

$$\begin{aligned}
Z_1 &= -2\beta_1 Y_1 + Y_2 + \frac{\beta_0 Y_1 Y_2}{\alpha_1} \\
Z_2 &= \beta_0(\beta_0 Y_2^2 + 2\alpha_1\beta_1(-2\beta_1 Y_1 + Y_2)) \\
Z_3 &= 4\alpha_1 Y_1\left(2\alpha_0\alpha_1\beta_1 - \beta_0\left(\beta_1^2\right) + \left(\alpha_0^2\right)\alpha_1 Y_1\right) + 2\alpha_1\left(\beta_0\beta_1 - 2\alpha_0\left(\alpha_1 + \beta_0 Y_1\right)\right) Y_2 + \left(\beta_0^2\right)\left(Y_2^2\right) \\
Z_4 &= \alpha_1(2\alpha_0\alpha_1 - \beta_0\beta_1)(2\gamma_1 + 2\beta_1 Y_3 - Y_4) + (2\alpha_0\alpha_1 Y_1 - \beta_0 Y_2)(-2\alpha_1\gamma_0 + 2\alpha_0\alpha_1 Y_3 - \beta_0 Y_4) \\
Z_5 &= \alpha_1\left(8\beta_1\gamma_0 Y_1 + 4\alpha_0\gamma_1 Y_1 - 4\gamma_0 Y_2 - 4\alpha_0\beta_1 Y_1 Y_3 + 4\alpha_0 Y_2 Y_3 - 2\alpha_0 Y_1 Y_4\right) - \beta_0\big(2Y_2\left(\gamma_1 + \beta_1 Y_3\right) \\
&\quad + \left(-4\beta_1 Y_1 + Y_2\right) Y_4\big) \\
Z_6 &= -2I\left(4\alpha_1 Y_1\left(2\alpha_0\alpha_1\beta_1 - \beta_0\left(\beta_1^2\right) + \left(\alpha_0^2\right)\alpha_1 Y_1\right) + 2\alpha_1\left(\beta_0\beta_1 - 2\alpha_0\left(\alpha_1 + \beta_0 Y_1\right)\right) Y_2 + \left(\beta_0^2\right)\left(Y_2^2\right)\right) \\
&\quad - 2\left(\gamma_1 + \beta_1 Y_3 - \frac{Y_4}{2}\right)\big(\alpha_1\left(2\alpha_0\alpha_1 - \beta_0\beta_1\right)\left(2\gamma_1 + 2\beta_1 Y_3 - Y_4\right) + \left(2\alpha_0\alpha_1 Y_1 - \beta_0 Y_2\right)\big(-2\alpha_1\gamma_0 \\
&\quad + 2\alpha_0\alpha_1 Y_3 - \beta_0 Y_4\big)\big) + \left(2\alpha_1\left(\gamma_0 - \alpha_0 Y_3\right) + \beta_0 Y_4\right)\left(\alpha_1(8\beta_1\gamma_0 Y_1 + 4\alpha_0\gamma_1 Y_1 - 4\gamma_0 Y_2 - 4\alpha_0\beta_1 Y_1 Y_3\right. \\
&\quad \left. + 4\alpha_0 Y_2 Y_3 - 2\alpha_0 Y_1 Y_4) - \beta_0(2Y_2(\gamma_1 + \beta_1 Y_3) + (-4\beta_1 Y_1 + Y_2)Y_4)\right) \\
Z_7 &= I\Bigg(\left(2\alpha_0\alpha_1 Y_1 - \beta_0 Y_2\right)\left(\lambda_1 - \alpha_1\psi_2 + \beta_1(\phi_2 + Y_5) - \frac{Y_6}{2}\right) + (2\beta_1 Y_1 - Y_2)(2\alpha_1(\lambda_0 + \beta_0\psi_2
\end{aligned}$$

$$-\alpha_0(\phi_2+Y_5))+\beta_0Y_6)\Big)-(\gamma_1+\beta_1Y_3-\frac{Y_4}{2})((2\alpha_1(\gamma_0-\alpha_0Y_3)+\beta_0Y_4)\left(\lambda_1-\alpha_1\psi_2+\beta_1(\phi_2+Y_5)-\frac{Y_6}{2}\right)$$
$$-\frac{1}{2}\left(2\gamma_1+2\beta_1Y_3-Y_4\right)\left(2\alpha_1\left(\lambda_0+\beta_0\psi_2-\alpha_0\left(\phi_2+Y_5\right)\right)+\beta_0Y_6\right))$$
$$Z_8=I\Big(2\alpha_1(2\alpha_0\alpha_1-\beta_0\beta_1)\left(\lambda_1-\alpha_1\psi_2+\beta_1(\phi_2+Y_5)-\frac{Y_6}{2}\right)+(2\alpha_0\alpha_1Y_1-\beta_0Y_2)(-2\alpha_1(\lambda_0+\beta_0\psi_2$$
$$-\alpha_0(\phi_2+Y_5))-\beta_0Y_6)\Big)-(2\alpha_1(\gamma_0-\alpha_0Y_3)+\beta_0Y_4)\Big((2\alpha_1(\gamma_0-\alpha_0Y_3)+\beta_0Y_4)\Big(\lambda_1-\alpha_1\psi_2$$
$$+\beta_1(\phi_2+Y_5)-\frac{Y6}{2}\Big)-\frac{1}{2}(2\gamma_1+2\beta_1Y_3-Y_4)(2\alpha_1(\lambda_0+\beta_0\psi_2-\alpha_0(\phi_2+Y_5))+\beta_0Y_6)\Big).$$

A.5.

The mathematical expressions of E_{11}, E_{12}, K_2, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33}, E_{34}, E_{35}, E_{41}, E_{42}, E_{43}, E_{44}, E_{45} and E_{46} which are used in Proposition 6, are as follows.

$$E_{11}=\beta_0(4\beta_0-\beta_1)c(I(\lambda_1^2)+(\gamma_1^2)\mu)+4(\alpha_1^2)(cI\lambda_0(2\lambda_0+\lambda_1)-4aI\mu+c(\gamma_0(2\gamma_0+\gamma_1)+(-\beta_0+\beta_1)I)\mu)$$
$$+2\alpha_1(2a(I(\lambda_1^2)+(\gamma_1^2)\mu)-2a_1(I\lambda_0\lambda_1+\gamma_0\gamma_1\mu+(3\beta_0+\beta_1)I\mu)+c(\lambda_1(3\beta_1I\lambda_0-(\gamma_0^2)\lambda_1+2\beta_0I(3\lambda_0$$
$$+\lambda_1))+\beta_0(\beta_0+11\beta_1)I\mu-(\gamma_1^2)((\lambda_0^2)-2\beta_0\mu)+\gamma_0\gamma_1(2\lambda_0\lambda_1+3(2\beta_0+\beta_1)\mu)))$$
$$E_{12}=-2(\alpha_1^3)c(I(\lambda_0^2)+(\gamma_0^2)\mu)+2(\beta_0^3)c(I(\lambda_1^2)+(\gamma_1^2)\mu)+(\alpha_1^2)(I\lambda_0(2(a_1+2\beta_0c-\beta_1c)\lambda_0-(2a+\beta_0c)\lambda_1)$$
$$+(2(a_1+2\beta_0c-\beta_1c)(\gamma_0^2)-(2a+\beta_0c)\gamma_0\gamma_1+2a(-5\beta_0+\beta_1)I+3\beta_0(-\beta_0+\beta_1)cI)\mu)+\alpha_1\beta_0(a(I(\lambda_1^2)$$
$$+(\gamma_1^2)\mu)-a_1(I\lambda_0\lambda_1+\gamma_0\gamma_1\mu+(7\beta_0+\beta_1)I\mu)+c(\lambda_1(\beta_1I\lambda_0-(\gamma_0^2)\lambda_1+\beta_0I(6\lambda_0+\lambda_1))$$
$$+\beta_0(\beta_0+7\beta_1)I\mu+(\gamma_1^2)(-(\lambda_0^2)+\beta_0\mu)+\gamma_0\gamma_1(2\lambda_0\lambda_1+(6\beta_0+\beta_1)\mu)))$$
$$K_2=(8(\alpha_0^2)(\alpha_1^2)(-I(\lambda_1^2)-(\gamma_1^2)\mu+4\alpha_1I\mu)+\beta_1(4(\alpha_1^2)(\beta_0-\beta_1)(I(\lambda_0^2)+(\gamma_0^2)\mu)+2(\beta_0^3)(I(\lambda_1^2)+(\gamma_1^2)\mu)$$
$$+\alpha_1\beta_0(-(\gamma_1^2)(\lambda_0^2)+\lambda_1(2(3\beta_0+\beta_1)I\lambda_0-(\gamma_0^2)\lambda_1)+((\beta_0^2)+14\beta_0\beta_1+(\beta_1^2))I\mu+2\gamma_0\gamma_1(\lambda_0\lambda_1+3\beta_0\mu$$
$$+\beta_1\mu)))-2\alpha_0\alpha_1(4(\alpha_1^2)(I(\lambda_0^2)+(\gamma_0^2)\mu)+\beta_0(2\beta_0-\beta_1)(I(\lambda_1^2)+(\gamma_1^2)\mu)+\alpha_1(-(\gamma_1^2)(\lambda_0^2)$$
$$+\lambda_1(6(\beta_0+\beta_1)I\lambda_0-(\gamma_0^2)\lambda_1)+((\beta_0^2)+22\beta_0\beta_1+(\beta_1^2))I\mu+2\gamma_0\gamma_1(\lambda_0\lambda_1+3(\beta_0+\beta_1)\mu))))$$
$$E_{21}=-cI\lambda_1(\beta_0\lambda_1+2\alpha_1(\lambda_0+2\lambda_1))+(-c\gamma_1(\beta_0\gamma_1+2\alpha_1(\gamma_0+2\gamma_1))+2\alpha_1(4a_1+(4\alpha_1-\beta_0+\beta_1)c)I)\mu$$
$$E_{22}=2(\alpha_1^2)(\beta_0-2\beta_1)c(I(\lambda_0^2)+(\gamma_0^2)\mu)+(\beta_0^2)(I\lambda_1(a_1\lambda_0+\beta_1c\lambda_0-a\lambda_1+\beta_0c\lambda_1)+\gamma_1(a_1\gamma_0+\beta_1c\gamma_0-a\gamma_1$$
$$+\beta_0c\gamma_1)\mu+(-\beta_0+\beta_1)(a_1+\beta_1c)I\mu)+\alpha_1\beta_0(2a_1(I(\lambda_0^2)+(\gamma_0^2)\mu)-2a(I\lambda_0\lambda_1+\gamma_0\gamma_1\mu+(\beta_0+3\beta_1)I\mu)$$
$$+c(-(\gamma_1^2)(\lambda_0^2)+2\beta_1I(\lambda_0^2)+2\gamma_0\gamma_1\lambda_0\lambda_1+3\beta_0I\lambda_0\lambda_1+2\beta_1I\lambda_0\lambda_1-(\gamma_0^2)(\lambda_1^2)+(3\beta_0\gamma_0\gamma_1+2\beta_1\gamma_0(\gamma_0+\gamma_1)$$
$$+\beta_1(7\beta_0+\beta_1)I)\mu))$$
$$E_{23}=4(\alpha_1^3)c(I(\lambda_0^2)+(\gamma_0^2)\mu)-(\beta_0^2)\beta_1c(I(\lambda_1^2)+(\gamma_1^2)\mu)+2(\alpha_1^2)(2a_1(I(\lambda_0^2)+(\gamma_0^2)\mu)-2a(I\lambda_0\lambda_1+\gamma_0\gamma_1\mu$$
$$+(\beta_0+3\beta_1)I\mu)+c(-(\gamma_1^2)(\lambda_0^2)+2\beta_1I(\lambda_0^2)+2\gamma_0\gamma_1\lambda_0\lambda_1+3\beta_0I\lambda_0\lambda_1+6\beta_1I\lambda_0\lambda_1-(\gamma_0^2)(\lambda_1^2)+(3\beta_0\gamma_0\gamma_1$$
$$+2\beta_1\gamma_0(\gamma_0+3\gamma_1)+\beta_1(11\beta_0+\beta_1)I)\mu))+2\alpha_1\beta_0(a_1\gamma_0\gamma_1\mu+a_1I(\lambda_0\lambda_1-\beta_0\mu+5\beta_1\mu)$$
$$-a(I(\lambda_1^2)+(\gamma_1^2)\mu)+(\beta_0-\beta_1)c((\gamma_1^2)\mu+I((\lambda_1^2)-2\beta_1\mu)))$$
$$E_{31}=\alpha_1\beta_0\beta_1((\alpha_1-\beta_1)c\lambda_0+(a+\beta_0c)\lambda_1)(\gamma_1\lambda_0-\gamma_0\lambda_1)$$
$$E_{32}=\alpha_1(-6\alpha_0\alpha_1(\beta_0+\beta_1)+\beta_0\beta_1(3\beta_0+\beta_1))\gamma_0+2(-2\alpha_0\alpha_1(2\alpha_0\alpha_1+(\beta_0^2))+\beta_0(\alpha_0\alpha_1+(\beta_0^2))\beta_1)\gamma_1$$
$$E_{33}=\alpha_1(4a\alpha_1(\beta_0-\beta_1)+\beta_0(\alpha_1\beta_0-5\alpha_1\beta_1+3\beta_0\beta_1+(\beta_1^2))c)\gamma_0+\beta_0(a\alpha_1(3\beta_0+\beta_1)$$
$$+\beta_0(2\beta_0\beta_1+\alpha_1(\beta_0+\beta_1))c)\gamma_1$$
$$E_{34}=\lambda_1(\gamma_1\lambda_0-\gamma_0\lambda_1)+(3\beta_0-\beta_1)\gamma_1\mu+4\alpha_1(\gamma_0+\gamma_1)\mu$$
$$E_{35}=-(2\alpha_1(\alpha_1-\beta_1)c\lambda_0+2a\alpha_1\lambda_1+\beta_0(2\alpha_1+\beta_1)c\lambda_1)(\gamma_1\lambda_0-\gamma_0\lambda_1)-(2\alpha_1(\alpha_1\beta_0-3\alpha_1\beta_1+5\beta_0\beta_1+(\beta_1^2))c\gamma_0$$
$$+\beta_0(2\alpha_1\beta_0+8\alpha_1\beta_1+7\beta_0\beta_1-(\beta_1^2))c\gamma_1+2a\alpha_1(4\alpha_1\gamma_0+3(\beta_0+\beta_1)\gamma_1))\mu$$
$$E_{41}=-\beta_0\gamma_1((\alpha_1-\beta_1)c\gamma_0+(a+\beta_0c)\gamma_1)+(4a\alpha_1(\beta_0-\beta_1)+\beta_0(\alpha_1(\beta_0-5\beta_1)+\beta_1(3\beta_0+\beta_1))c)I$$
$$E_{42}=-2\alpha_0\alpha_1(\gamma_0\gamma_1+3(\beta_0+\beta_1)I)+\beta_0\beta_1(\gamma_0\gamma_1+(3\beta_0+\beta_1)I)$$
$$E_{43}=\alpha_1(2\alpha_0\alpha_1-\beta_0\beta_1)(\gamma_0^2)-4\alpha_0\alpha_1(2\alpha_0\alpha_1+(\beta_0^2))I+2\beta_0(\alpha_0\alpha_1+(\beta_0^2))\beta_1I$$
$$E_{44}=\alpha_1\gamma_0((\alpha_1-\beta_1)c\gamma_0+(a+\beta_0c)\gamma_1)+(a\alpha_1(3\beta_0+\beta_1)+\beta_0(2\beta_0\beta_1+\alpha_1(\beta_0+\beta_1))c)I$$
$$E_{45}=(\gamma_1(2\alpha_1(\alpha_1-\beta_1)c\gamma_0+2a\alpha_1\gamma_1+\beta_0(2\alpha_1+\beta_1)c\gamma_1)-2\alpha_1(4a\alpha_1+\alpha_1(\beta_0-3\beta_1)c+\beta_1(5\beta_0+\beta_1)c)I)\lambda_0$$

$$
\begin{aligned}
&- (\gamma_0(2\alpha_1(\alpha_1-\beta_1)c\gamma_0+2a\alpha_1\gamma_1+\beta_0(2\alpha_1+\beta_1)c\gamma_1)+(6a\alpha_1(\beta_0+\beta_1)+\beta_0((7\beta_0-\beta_1)\beta_1\\
&+2\alpha_1(\beta_0+4\beta_1))c)I)\lambda_1\\
E_{46} &= -(\gamma_1^2)\lambda_0+\gamma_0\gamma_1\lambda_1+(3\beta_0-\beta_1)I\lambda_1+4\alpha_1 I(\lambda_0+\lambda_1)
\end{aligned}
$$

A.6.

The mathematical expressions of E_{51}, E_{52} and E_{53} which are used in Proposition 7, are as follows:

$$
\begin{aligned}
E_{51} &= 4a_1 I\lambda_1\phi_2+c(I\lambda_1(\lambda_1+(-4\alpha_1-3\beta_0+\beta_1)\phi_2)-4\alpha_1 I(\mu+\lambda_0\phi_2)+\gamma_1(-\gamma_0\lambda_1\phi_2+\gamma_1(\mu+\lambda_0\phi_2)))\\
E_{52} &= -\beta_0(4\beta_0-\beta_1)c(I(\lambda_1^2)+(\gamma_1^2)\mu)+\beta_0(\beta_1 c(\gamma_1^2)\lambda_0-4a_1\beta_0 I\lambda_1+\beta_1(-c\gamma_0\gamma_1+2a_1 I+(-7\beta_0+\beta_1)cI)\lambda_1)\phi_2\\
&- 2(\alpha_1^2)\big(4aI(-3\mu+\lambda_0\phi_2)+c\big(\big(-3\beta_0+\beta_1\big)I\mu+I\lambda_0(4\lambda_0+3\lambda_1+\beta_0\phi_2-3\beta_1\phi_2)+\gamma_0\big(4\gamma_0\mu+3\gamma_1\mu\\
&- \gamma_1\lambda_0\phi_2+\gamma_0\lambda_1\phi_2\big)\big)\big)-2\alpha_1(-a_1(3\gamma_0\gamma_1\mu+(9\beta_0+\beta_1)I\mu+\gamma_0(-\gamma_1\lambda_0+\gamma_0\lambda_1)\phi_2+3I\lambda_0(\lambda_1-(\beta_0\\
&+\beta_1)\phi_2))+a(3I\lambda_1(\lambda_1+(\beta_0+\beta_1)\phi_2)+\gamma_1(3\gamma_1\mu-\gamma_1\lambda_0\phi_2+\gamma_0\lambda_1\phi_2))+c((\beta_1 I\lambda_0-(\gamma_0^2)\lambda_1)(\lambda_1+\beta_1\phi_2)\\
&-(\gamma_1^2)((\lambda_0^2)-3\beta_0\mu+\beta_0\lambda_0\phi_2)+(\beta_0^2)I(\mu+\lambda_1\phi_2)+\gamma_0\gamma_1(2\lambda_0\lambda_1+6\beta_0\mu+\beta_1\mu+\beta_1\lambda_0\phi_2+\beta_0\lambda_1\phi_2)\\
&+\beta_0 I(3\lambda_1(2\lambda_0+\lambda_1)+5\beta_1\mu+\beta_1(5\lambda_0+4\lambda_1)\phi_2)))\\
E_{53} &= -4(\alpha_1^3)c(I(\lambda_0^2)+(\gamma_0^2)\mu)+2(\beta_0^3)(c(\gamma_1^2)\mu+a_1 I\lambda_1\phi_2+cI\lambda_1(\lambda_1+\beta_1\phi 2))+\alpha_1\beta_0(-a_1(\gamma_0\gamma_1\mu\\
&+(11\beta_0+\beta_1)I\mu+\gamma_0(-\gamma_1\lambda_0+\gamma_0\lambda_1)\phi_2+I\lambda_0(\lambda_1-(3\beta_0+\beta_1)\phi_2))+c((\beta_1 I\lambda_0-(\gamma_0^2)\lambda_1)(\lambda_1+\beta_1\phi_2)\\
&-(\gamma_1^2)((\lambda_0^2)-\beta_0\mu+\beta_0\lambda_0\phi_2)+(\beta_0^2)I(\mu+\lambda_1\phi_2)+\gamma_0\gamma_1(2\lambda_0\lambda_1+6\beta_0\mu+\beta_1\mu+\beta_1\lambda_0\phi_2+\beta_0\lambda_1\phi_2)\\
&+\beta_0 I(6\lambda_0\lambda_1+(\lambda_1^2)+3\beta_1\mu+\beta_1(3\lambda_0+\lambda_1)\phi_2))+a(I\lambda_1(\lambda_1+(3\beta_0+\beta_1)\phi_2)+\gamma_1(\gamma_0\lambda_1\phi_2\\
&+\gamma_1(\mu-\lambda_0\phi_2))))+(\alpha_1^2)(4a_1(I(\lambda_0^2)+(\gamma_0^2)\mu)+(\beta_0^2)cI(-5\mu+\lambda_0\phi_2)-4a(\gamma_0\gamma_1\mu+I\lambda_0(\lambda_1+\beta_1\phi_2))\\
&+\beta_0(4aI(-4\mu+\lambda_0\phi_2)+c(\beta_1 I\mu+I\lambda_0(4\lambda_0-3\lambda_1-5\beta_1\phi_2)+\gamma_0(4\gamma_0\mu-3\gamma_1\mu-\gamma_1\lambda_0\phi_2\\
&+\gamma_0\lambda_1\phi_2)))).
\end{aligned}
$$

A.7.

The mathematical expressions of E_{61}, E_{62} and E_{63} which are used in Proposition 8, are as follows:

$$
\begin{aligned}
E_{61} &= -2\beta_0 c(I(\lambda_1^2)+(\gamma_1^2)\mu)+4(\alpha_1^2)cI(\mu+(\lambda_0+\lambda_1)\psi_2)-\alpha_1(4a_1 I(-3\mu+\lambda_1\psi_2)+c((\beta_0-3\beta_1)I\mu\\
&+I\lambda_1(3\lambda_0+4\lambda_1-3\beta_0\psi_2+\beta_1\psi_2)+\gamma_1(3\gamma_0\mu+4\gamma_1\mu+\gamma_1\lambda_0\psi_2-\gamma_0\lambda_1\psi_2)))\\
E_{62} &= (\alpha_1^2)((\beta_0-4\beta_1)c(I(\lambda_0^2)+(\gamma_0^2)\mu)+(-\beta_0 c\gamma_0\gamma_1+4a(\beta_0-\beta_1)I+\beta_0(\beta_0-5\beta_1)cI)\lambda_0\psi_2+\beta_0 c\gamma_0^2\lambda_1\psi_2)\\
&+2(\beta_0^2)(a_1\gamma_0\gamma_1\mu+\beta_1 c\gamma_0\gamma_1\mu+(\beta_1^2)cI\mu-a(I(\lambda_1^2)+(\gamma_1^2)\mu)+\beta_1 cI\lambda_1(\lambda_0+\beta_0\psi_2)+a_1 I(\lambda_0\lambda_1+\beta_1\mu\\
&+\beta_0\lambda_1\psi_2))+\alpha_1\beta_0(-a(3I\lambda_0\lambda_1+3\gamma_0\gamma_1\mu+(\beta_0+11\beta_1)I\mu)+a(-(\gamma_1^2)\lambda_0+\gamma_0\gamma_1\lambda_1+(3\beta_0+\beta_1)I\lambda_1)\psi_2\\
&+a_1(I\lambda_0(3\lambda_0+(3\beta_0+\beta_1)\psi_2)+\gamma_0(3\gamma_0\mu+\gamma_1\lambda_0\psi_2-\gamma_0\lambda_1\psi_2))+c(-(\gamma_0^2)(\lambda_1^2)-(\gamma_1^2)\lambda_0(\lambda_0+\beta_0\psi_2)\\
&+\beta_0 I\lambda_1(\lambda_0+\beta_0\psi_2)+(\beta_1^2)I(\mu+\lambda_0\psi_2)+\beta_1(\gamma_0^2)(3\mu-\lambda_1\psi_2)+\gamma_0\gamma_1(2\lambda_0\lambda_1+\beta_0\mu+2\beta_1\mu+\beta_1\lambda_0\psi_2\\
&+\beta_0\lambda_1\psi_2)+\beta_1 I(3(\lambda_0^2)+2\lambda_0\lambda_1+\beta_0\mu+\beta_0(3\lambda_0+\lambda_1)\psi_2)))\\
E_{63} &= 2(\beta_0^2)\beta_1 c(I(\lambda_1^2)+(\gamma_1^2)\mu)-2(\alpha_1^3)(4aI\lambda_0\psi_2+cI\lambda_0(\lambda_0+(\beta_0-3\beta_1)\psi_2)+c\gamma_0(-\gamma_1\lambda_0\psi_2+\gamma_0(\mu+\lambda_1\psi_2)))\\
&-2(\alpha_1^2)(-a(3I\lambda_0\lambda_1+3\gamma_0\gamma_1\mu+(\beta_0+9\beta_1)I\mu)+a(-(\gamma_1^2)\lambda_0+\gamma_0\gamma_1\lambda_1+3(\beta_0+\beta_1)I\lambda_1)\psi_2\\
&+a_1(3I\lambda_0(\lambda_0+(\beta_0+\beta_1)\psi_2)+\gamma_0(3\gamma_0\mu+\gamma_1\lambda_0\psi_2-\gamma_0\lambda_1\psi_2))+c(-(\gamma_0^2)(\lambda_1^2)-(\gamma_1^2)\lambda_0(\lambda_0+\beta_0\psi_2)\\
&+\beta_0 I\lambda_1(\lambda_0+\beta_0\psi_2)+(\beta_1^2)I(\mu+\lambda_0\psi_2)+\beta_1(\gamma_0^2)(3\mu-\lambda_1\psi_2)+\gamma_0\gamma_1(2\lambda_0\lambda_1+\beta_0\mu+6\beta_1\mu+\beta_1\lambda_0\psi_2\\
&+\beta_0\lambda_1\psi_2)+\beta_1 I(3\lambda_0(\lambda_0+2\lambda_1)+4\beta_0\mu+\beta_0(5\lambda_0+4\lambda_1)\psi_2)))+\alpha_1\beta_0(4a(I(\lambda_1^2)+(\gamma_1^2)\mu)\\
&+(\beta_1^2)cI(-7\mu+\lambda_1\psi_2)-2a_1(2\gamma_0\gamma_1\mu+I(2\lambda_0\lambda_1+9\beta_1\mu+2\beta_0\lambda_1\psi_2\\
&-\beta_1\lambda_1\psi_2))+\beta_1 c(\beta_0 I\mu+I\lambda_1(-\lambda_0+2\lambda_1-7\beta_0\psi_2)+\gamma_1(\gamma_1(2\mu+\lambda_0\psi_2)-\gamma_0(\mu+\lambda_1\psi_2)))).
\end{aligned}
$$