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ON FULLY INTUITIONISTIC FUZZY LINEAR FRACTIONAL PROGRAMMING *VIA* NOVEL PARAMETERIZED MEMBERSHIP FUNCTIONS WITH APPLICATION IN PORTFOLIO OPTIMIZATION

Abhishek Chauhan, Sumati Mahajan* and Shubhpreet Kaur

Abstract. Fractional programming is an extensively used technique to simultaneously deal with two conflicting objectives, by optimizing their ratio. This paper focuses on fully intuitionistic fuzzy fractional programming to address real-world problems. Firstly, the intuitionistic fuzzy model is converted into a crisp multi-objective problem with fractional objectives. Subsequently, a novel intuitionistic fuzzy programming approach is proposed, offering an insightful method for selecting least acceptable values. The existing literature on intuitionistic fuzzy programming employs linear, exponential, hyperbolic, or parabolic membership and non-membership functions. This article presents that these functions result in a highly restrictive feasibility region. Thus, a family of parameterized membership and non-membership functions is introduced, which overcomes the limitations of conventionally used functions and effectively captures both the acceptance as well as rejection degrees. In this article, theoretical foundations are established for the proposed technique through several theorems which have been constructed and proved. Additionally, the proposed technique is illustrated through a numerical example and applied to a real-life portfolio optimization problem, demonstrating its effectiveness. Finally, a comparative analysis with prevalent studies is also conducted to emphasize the versatile nature of the proposed functions.

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1. Introduction

A standard linear programming problem (LPP) is a mathematical problem comprising a linear objective function and constraints. The motive is to optimize the objective and obtain a solution while satisfying some constraints or restrictions imposed on the model. LPPs have widespread applications in real-world scenarios like profit maximization, cost minimization, production optimization, transportation problems, diet planning problems, etc. A linear fractional programming problem (LFPP) is a kind of LPP that optimizes the ratio of two linear functions. These LFPPs find a wide range of applications in production planning, nutrition planning as well as staff management problems by optimizing ratios like production/wastage, patient/doctor, profit/cost, machine run time/operating cost, etc.

Keywords. Intuitionistic fuzzy approach, linear fractional programming problem, parametric membership functions, portfolio optimization.

Department of Mathematics, Punjab Engineering College (Deemed to be University), Chandigarh 160012, India.
*Corresponding author: sumatimahajan@pec.edu.in

As a first, Isbell and Marlow [24] formulated LFPPs and solved them using linear programming. In this direction, Charnes and Cooper [10] proposed an ingenious technique to solve LFPPs by using the variable transformation method to reduce the fractional problem into an LPP. Next, Gilmore and Gomory [22] used the gradient of the objective function to develop a modified simplex method for such problems. In a similar fashion, Martos *et al.* [35] developed a detailed method using a simpler gradient form. Following the development made by Charnes and Cooper, Jagannathan [25] explored parametric convex programming problems linking with fractional programming problems and later extended the findings to non-linear programming problems, with particular emphasis on continuous criteria functions. In due course, significant work on LFPPs was done by Swarup [50], Dinkelbach [19], Bitran and Novaes [7], Pal and Basu [42], etc. Afterwards, Craven and Mond [14] as well as Schaible [45] investigated the duality aspect of the LFPPs. Recently, Rizk-Allah *et al.* [44] studied fractional programming problems by suggesting a chaotic crow search algorithm. As a deflection, Arya *et al.* [4] developed a new branch-bound technique for multi-objective LFPPs using the weak duality theorem. In recent times, Jiao and Li [26] studied min–max LFPPs and proposed a novel algorithm using an image space branch-and-bound scheme.

LFPPs are a great tool to overcome a plethora of optimization problems. However, in many real-world scenarios, the parameters involved in the LFPPs are not always precise, rather have some ambiguity in the data. To cater to these real-world problems, it is important to incorporate uncertainty in some form, while modeling the problem. Among several ways to deal with this imprecision and uncertainty, one of the predominant techniques is to incorporate the fuzzy logic introduced by Zadeh [55]. In fuzzy set theory, Zadeh extended the notion of classical sets by allowing non-binary values of the membership function, which is the generalization of the characteristic function of classical sets. Consequently, in an uncertain environment, the widely applicable Charnes and Cooper's method cannot be applied directly to LFPPs.

Over time, different researchers have examined various approaches to handle LFPPs involving data uncertainty. The interval-valued parameters for generalized LFPPs were considered by Borza *et al.* [9] and Hladík [23]. However, Nayak and Ojha [40] studied an iterative ϵ-constraint method for LFPPs by converting the fractional objective into a non-fractional parametric function. Under a fuzzy setting, Toksarı [51] developed a Taylor series approximation-based approach for fuzzy LFPPs using linear membership functions. Subsequently, Mehra *et al.* [36] obtained the optimal value of LFPP by defining an (α, β) acceptable solution. Afterwards, Pop and Stancu-Minasian [43] worked on the LFPPs having crisp variables and triangular fuzzy coefficients. Later, for such LFPPs, Borza and Rambely [8] as well as Das and Edalatpanah [15] used fuzzy arithmetic operations to obtain an equivalent tri-objective model to solve the problem. In this direction, Agarwal *et al.* [1] investigated the optimality condition for multi-objective linear fractional programming problems (MOLFPPs).

Deviating from standard practice, Veeramani and Sumathi [52], Das *et al.* [17], Das and Mandal [16] along with Pandian and Jayalakshmi [41] studied fully fuzzy LFPPs, where all the variables and parameters are taken as triangular fuzzy numbers (TFNs) and proposed a new component-wise optimization technique. Subsequently, Das *et al.* [18] proposed a new ordering for TFNs and then used it to solve LFPPs. Next, the duality approach for solving fuzzy linear fractional transportation problems was examined by Liu [29]. Stanojević [49] developed an approach based on the extension principle for full fuzzy MOLFPPs. Afterwards, Bhatia *et al.* [6] suggested a new method, *viz.*, the mehar approach, that yields an exact fuzzy optimal solution to an existing fully fuzzy linear fractional transportation problem. The α-cut of fuzzy numbers was employed to solve fully fuzzy LFPPs numerically by Chinnadurai and Muthukumar [13]. The shortcomings of their study were highlighted and rectified in the work accomplished by Ebrahimnejad *et al.* [20] and Chauhan *et al.* [12]. In a significant development, Mahajan *et al.* [32] studied quadratic fractional programming problems with multiple objectives employing the α-cut of fuzzy numbers and the fuzzy programming approach using variation of parameters. In a recent study, Mishra *et al.* [37] solved non-linear fuzzy fractional programming problems by using the α-cut of fuzzy numbers to convert the equivalent non-fractional version of the problem into an interval-valued problem.

Extending the notion of fuzzy logic, Atanassov [5] presented the concept of intuitionistic fuzzy (IF) sets, where a non-membership function is also considered alongside the membership function, subject to the condition that their sum should not exceed unity. Since IF theory is the generalization of fuzzy logic, the study of LFPPs by means of IF sets is far more efficient and pragmatic. In the IF environment, Singh and Yadav [47] as well as Sharma *et al.* [46] used a fuzzy programming approach employing only the membership functions to solve LFPPs with triangular IF parameters. In another direction, Amer [3] explored non-linear fuzzy fractional programming problems using an interactive algorithm. Afterwards, El Sayed and Abo-Sinna [21] employed the ranking function of IF sets to convert a fully IF fractional programming problem with multiple objectives into an equivalent crisp model and then solved it by using IF programming for linear, hyperbolic and parabolic membership and non-membership functions. Under similar settings, following Mahajan and Gupta [30], Malik and Gupta [34] incorporated the optimistic, pessimistic and mixed-view points in the solution procedure. However, Moges *et al.* [39] used (α, β)-cuts of IF sets to convert fully IF bi-level multi-objective LFPPs into a crisp version and proposed a new compensatory IF mathematical method for such problems. In a recent study, Kara *et al.* [27] employed a fuzzy bisection algorithm for intuitionistic fuzzy linear fractional programming problems (IFLFPPs) with IF parameters. Subsequently, Yuvashri and Saraswathi [54] considered parameters as pentagonal IF numbers and proposed a ranking function to convert IF model into a crisp problem.

Nowadays, fuzzy as well as IFLFPPs find numerous applications in different fields, *viz.* business management, production planning, supplier selection, route optimization, etc. In one of their studies, Yang *et al.* [53] studied the agricultural structure planning problem. The said problem was modelled as a fuzzy LFPP and a superiority-inferiority measure based technique was employed to solve the problem. In recent times, Agarwal *et al.* [2] developed the KKT conditions for LFPPs having crisp variables and fuzzy parameters. Furthermore, an application in the share market was solved using the proposed optimality conditions. In an IF environment, Malik and Gupta [33] have proposed a method that amalgamates change of variables, goal programming and membership function technique to solve a multi-objective optimization problem with fractional objectives having IF variables and parameters. Subsequently, Moges *et al.* [38] considered an application in the problem of land allocation for agricultural use by suggesting a new method for IF multi-objective LFPPs.

1.1. Research gaps and motivation

The literature survey indicates that LFPPs are significantly used to model and solve real-world problems, especially in fuzzy as well as IF settings. In this article, the following research gaps and challenges are addressed:

(1) The existing studies are widespread, but there are no consistent and flexible techniques that are applicable to LFPPs having all the variables and parameters as IF numbers. Additionally, the studies where fully IFLFPPs are considered do not completely adhere to the IF aspect of the problem. For instance, in Singh and Yadav [47] and Sharma *et al.* [46], fuzzy programming is used without considering the IF environment. Moreover, the least acceptable values are fixed as zero for all the objectives, which may not be realistic. Similarly, in Malik and Gupta [33] goal-setting ignores the non-membership aspect, despite the inherent IF environment.
(2) In the studies [27, 46, 47, 54], either parameters or variables are taken as IF numbers while fully IFLFPPs are seldomly studied due to high complexity.
(3) In a study by El Sayed and Abo-Sinna [21] the proposed parabolic membership and non-membership functions are found to be incorrect as their sum exceeds unity, thereby violating the fundamental nature of IF numbers.
(4) In a recent study, Chauhan and Mahajan [11] proposed a novel generalized IF programming for non-linear multi-objective programming problems (MOPPs) having IF parameters and crisp non-linear variables. In their approach, the accuracy function of IF numbers is employed to transform the IF problem into a crisp one. However, IFLFPPs inherently involve ratios, thereby limiting the applicability of accuracy functions for defuzzification and hence their proposed technique.
(5) In IF programming, the least acceptance degree among the membership functions (α) and the largest rejection degree (λ) between the non-membership functions are evaluated and a restriction is imposed *i.e.* the largest rejection degree should not exceed the least acceptance degree $(\alpha \geq \lambda)$. This results in a highly rigid feasibility region of the problem as presented in Section 4.2. Therefore, in most cases, to cater to

TABLE 1. Features of the related approaches in LFPPs.

Reference	Uncertainty environment	Variables	Parameters	Approach	Nature of membership and non-membership functions
Nayak and Ojha [40]	Crisp	Crisp	Crisp	Iterative ϵ-constraint technique for linear parametric functions	Linear
Nayak and Ojha [40] Toksarı [51], Stanojević [48]	Crisp	Crisp	Crisp	Optimization of approximated membership functions using Taylor series	Linear
Pop and Stancu-Minasian [43]	Fuzzy	TFNs	TFNs	Multi-objective optimization using Pop and Stancu-Minasian [43] Kerre's method [28]	–
Borza and Rambely [8]	Fuzzy	Crisp	TFNs	α-cut and max–min based approach	Linear
Das and Edalatpanah [15]	Fuzzy	Crisp	TFNs	Ranking of TFNs	–
Das *et al.* [17]	Fuzzy	TFNs	TFNs	Component wise optimization	–
Chauhan *et al.* [12]	Fuzzy	TFNs	TFNs	α-cut-based numerical approach	–
Mahajan *et al.* [32]	Fuzzy	Crisp	fuzzy	Fuzzy programming	Linear
Singh and Yadav [47]	IF	Crisp	Triangular IF numbers (TIFNs)	Fuzzy programming	Linear
Sharma *et al.* [46]	IF	Crisp	TIFNs	Fuzzy programming	Linear, hyperbolic and parabolic
El Sayed and Abo-Sinna [21]	IF	Trapezoidal IF numbers (TrIFNs)	TrIFNs	IF programming	Linear, hyperbolic and parabolic
Malik and Gupta [33]	IF	TIFNs	TIFNs	Goal programming using membership functions	Linear and exponential
Kara *et al.* [27]	IF	Crisp	TIFNs	Integerated fuzzy bisection agorithm and IF programming using membership functions	Linear, exponential and hyperbolic
Yuvashri and Saraswathi [54]	IF	Crisp	Pentagonal IF numbers	Accuracy function and lingprog	–
Proposed approach	IF	TrIFNs	TrIFNs	IF programming	Parametrized family of membership and non-membership functions

the IF nature of the problem, the non-membership aspect cannot be directly incorporated into the model without affecting the problem's feasibility. Additionally, If infeasibility arises then presently there are no means to deal with it.

It is observed that, the problem is yet solvable if $\alpha \geq \lambda$ restriction is removed. However, this is undesirable as ideally a solution with high acceptance and small rejection degree is sought. Thus, there is a need to explore means to manipulate the problem's feasibility region by altering the membership and non-membership functions in a meaningful and acceptable way. These gaps motivate us to develop a novel, consistent technique for fully IFLFPPs that addresses the pertinent IF environment by proposing novel membership and non-membership functions to enhance the feasibility region of the problem while catering to the decision maker's preferences. Table 1 highlights the features of related works in fuzzy/IFLFPPs.

1.2. Main contributions

The main contributions made through this article are listed below:

(1) A novel parameterized family of membership and non-membership functions is developed to overcome the infeasibility issue faced by related studies.
(2) The above-mentioned functions are used to develop a novel IF programming approach for a fully IFLFPP by recasting it as a MOPP with fractional objectives.

(3) The IF programming approach for MOPPs is carried out by proposing the individual least acceptable value to be taken as the corresponding minimum value of the objective function.
(4) Several theorems are stated as well as proved to support the proposed IF programming approach for the suggested membership and non-membership functions.
(5) A numerical illustration and a real-world portfolio optimization problem, are solved using the proposed technique by framing the problems as IFLFPPs.
(6) A thorough comparative analysis is carried out among the related studies and the proposed approach to emphasize that the proposed approach outperforms them and is broadly applicable.
(7) Finally, another comparison is made to highlight the efficacy and applicability of the suggested membership and non-membership functions over some frequently used ones.

The remaining article is organized as follows: The preliminary definitions and notations are mentioned in Section 2. In Section 3, Charnes and Cooper's technique to solve general LFPP is discussed. The relation between the solution of a fully IFLFPP and the equivalent MOPP is established in Section 4. Afterwards, the IF programming along with motivation to develop new membership and non-membership functions is presented. In Section 5, a novel parameterized family of membership and non-membership functions is proposed to develop IF programming with theoretical support. Next, in Section 6, an illustrative example is solved using the proposed approach. Section 7 is dedicated to a real-world application alongside managerial insight and comparative analysis. To sum up, the conclusion and future scope are stated in Section 8.

2. Preliminaries & definitions

Some definitions and notations used throughout this article are stated below.

Definition 2.1 ([56])**.** Let $\mathbb{X}$ be a universal set. Then a fuzzy set $\tilde{F} = \{(x, \mu_{\tilde{F}}(x)) |\ x \in \mathbb{X}\}$ in $\mathbb{X}$ can be defined using its membership function $\mu_{\tilde{F}}$ which maps elements of $\mathbb{X}$ to the set $[0,1]$ ($\mu_{\tilde{F}} : \mathbb{X} \to [0,1]$). For some $x \in \mathbb{X}$, $\mu_{\tilde{F}}(x) \in [0,1]$ is the membership degree of x in $\tilde{F}$.

Definition 2.2 ([11])**.** An IF set in the universal set $\mathbb{X}$ is the collection of ordered triplets:

$$\tilde{F}^I = \{(x, \mu_{\tilde{F}^I}(x), \nu_{\tilde{F}^I}(x)) |\ \mu_{\tilde{F}^I}(x) + \nu_{\tilde{F}^I}(x) \leq 1, \forall x \in \mathbb{X}\}$$

where $\mu_{\tilde{F}^I}, \nu_{\tilde{F}^I}$ are the membership and non-membership function of $\tilde{F}^I$, respectively. For some fixed $x \in \mathbb{X}$, $\mu_{\tilde{F}^I}(x)$ is the membership degree and $\nu_{\tilde{F}^I}(x)$ is the non-membership degree of x in $\tilde{F}^I$.

Remark 2.1. $\mu_{\tilde{F}^I}(x)$ and $\nu_{\tilde{F}^I}(x)$ are also called the acceptance and rejection degree of x to be an element of $\tilde{F}^I$.

Definition 2.3 ([31])**.** A TrIFN $\tilde{G}^I = \{(r, \mu_{\tilde{G}^I}(r), \nu_{\tilde{G}^I}(r)) | \forall r \in \mathbb{R}\}$ is an IF set in $\mathbb{R}$ where

$$\mu_{\tilde{G}^I}(r) = \begin{cases} \frac{r-g_1}{g_2-g_1}, & \text{if } g_1 < r < g_2, \\ 1, & \text{if } g_2 \leq r \leq g_3, \\ \frac{g_4-r}{g_4-g_3}, & \text{if } g_3 < r < g_4, \\ 0, & \text{otherwise} \end{cases} \text{ and } \nu_{\tilde{G}^I}(r) = \begin{cases} \frac{g_2'-r}{g_2'-g_1'}, & \text{if } g_1' < r < g_2', \\ 0, & \text{if } g_2' \leq r \leq g_3', \\ \frac{r-g_3'}{g_4'-g_3'}, & \text{if } g_3' < r < g_4', \\ 1, & \text{otherwise,} \end{cases}$$

such that $g_1' \leq g_1 \leq g_2' \leq g_2 \leq g_3 \leq g_3' \leq g_4 \leq g_4'$. Figure 1 represents a TrIFN $\tilde{G}^I$ denoted as $\tilde{G}^I = (g_1, g_2, g_3, g_4;\ g_1', g_2', g_3', g_4')$. Throughout this paper, for simplicity's sake, the parameters and variables are considered to be the special case of TrIFNs where $g_2' = g_2$ and $g_3' = g_3$.

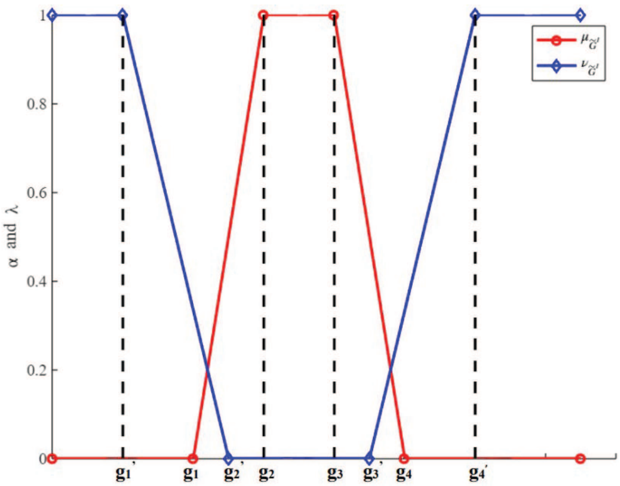

FIGURE 1. TrIFN $\tilde{G}^I = (g_1, g_2, g_3, g_4;\ {g_1}', g_2, g_3, {g_4}')$.

Remark 2.2 ([31])**.** A TrIFN $\tilde{G}^I = (g_1, g_2, g_3, g_4;\ {g_1}', g_2, g_3, {g_4}')$ is said to be non-negative ($\tilde{G}^I \succeq \tilde{0}^I$) *iff* ${g_1}' \geq 0$ and positive ($\tilde{G}^I \succ \tilde{0}^I$) *iff* ${g_1}' > 0$, where $\tilde{0}^I = (0, 0, 0, 0;\ 0, 0, 0, 0)$.

Definition 2.4 ([31])**.** Arithmetic operations on TrIFNs:
Let $\tilde{G}^{1I} = (g_1^1, g_2^1, g_3^1, g_4^1;\ {g_1^1}', g_2^1, g_3^1, {g_4^1}')$ and $\tilde{G}^{2I} = (g_1^2, g_2^2, g_3^2, g_4^2;\ {g_1^2}', g_2^2, g_3^2, {g_4^2}')$ be two TrIFNs then the arithmetic operations are defined as follows:

(i) Addition:

$$\tilde{G}^{1I} \oplus \tilde{G}^{2I} = \left(g_1^1 + g_1^2, g_2^1 + g_2^2, g_3^1 + g_3^2, g_4^1 + g_4^2;\ {g_1^1}' + {g_1^2}', g_2^1 + g_2^2, g_3^1 + g_3^2, {g_4^1}' + {g_4^2}'\right).$$

(ii) Scalar multiplication:
  (a) If $k \geq 0$ then $k\tilde{G}^{1I} = (kg_1^1, kg_2^1, kg_3^1, kg_4^1;\ k{g_1^1}', kg_2^1, kg_3^1, k{g_4^1}')$.
  (b) If $k < 0$ then $k\tilde{G}^{1I} = (kg_4^1, kg_3^1, kg_2^1, kg_1^1;\ k{g_4^1}', kg_3^1, kg_2^1, k{g_1^1}')$.

(iii) Subtraction:

$$\tilde{G}^{1I} \ominus \tilde{G}^{2I} = \left(g_1^1 - g_4^2, g_2^1 - g_3^2, g_3^1 - g_2^2, g_4^1 - g_1^2;\ {g_1^1}' - {g_4^2}', g_2^1 - g_3^2, g_3^1 - g_2^2, {g_4^1}' - {g_1^2}'\right).$$

(iv) Multiplication:

$$\begin{aligned}
&\tilde{G}^{1I} \otimes \tilde{G}^{2I} = (g_1, g_2, g_3, g_4;\ {g_1}', g_2, g_3, {g_4}') \text{ where}\\
&g_1 = \min\{g_1^1 g_1^2, g_1^1 g_4^2, g_4^1 g_1^2, g_4^1 g_4^2\},\\
&g_4 = \max\{g_1^1 g_1^2, g_1^1 g_4^2, g_4^1 g_1^2, g_4^1 g_4^2\},\\
&{g_1}' = \min\{{g_1^1}'{g_1^2}', {g_1^1}'{g_4^2}', {g_4^1}'{g_1^2}', {g_4^1}'{g_4^2}'\},\\
&{g_4}' = \max\left\{{g_1^1}'{g_1^2}', {g_1^1}'{g_4^2}', {g_4^1}'{g_1^2}', {g_4^1}'{g_4^2}'\right\},\\
&g_2 = \min\{g_2^1 g_2^2, g_2^1 g_3^2, g_3^1 g_2^2, g_3^1 g_3^2\},\\
&g_3 = \max\{g_2^1 g_2^2, g_2^1 g_3^2, g_3^1 g_2^2, g_3^1 g_3^2\}.
\end{aligned}$$

(v) Division:

$$\tilde{G}^{1I} \oslash \tilde{G}^{2I} = \left(h_1, h_2, h_3, h_4;\ {h_1}', h_2, h_3, {h_4}'\right) \text{ where}$$

$$h_1 = \min\{g_1^1/g_1^2, g_1^1/g_4^2, g_4^1/g_1^2, g_4^1/g_4^2\},$$
$$h_4 = \max\{g_1^1/g_1^2, g_1^1/g_4^2, g_4^1/g_1^2, g_4^1/g_4^2\},$$
$$h_1{}' = \min\{g_1^{1'}/g_1^{2'}, g_1^{1'}/g_4^{2'}, g_4^{1'}/g_1^{2'}, g_4^{1'}/g_4^{2'}\},$$
$$h_4{}' = \max\{g_1^{1'}/g_1^{2'}, g_1^{1'}/g_4^{2'}, g_4^{1'}/g_1^{2'}, g_4^{1'}/g_4^{2'}\},$$
$$h_2 = \min\{g_2^1/g_2^2, g_2^1/g_3^2, g_3^1/g_2^2, g_3^1/g_3^2\},$$
$$h_3 = \max\{g_2^1/g_2^2, g_2^1/g_3^2, g_3^1/g_2^2, g_3^1/g_3^2\}, \text{ provided that either } g_1^2 > 0 \text{ or } g_4^{2'} < 0.$$

Remark 2.3. If $\tilde{G}^{1I} \succeq \tilde{0}^I$ and $\tilde{G}^{2I} \succ \tilde{0}^I$ then

(i) $\tilde{G}^{1I} \otimes \tilde{G}^{2I} = (g_1^1 g_1^2, g_2^1 g_2^2, g_3^1 g_3^2, g_4^1 g_4^2;\ g_1^{1'} g_1^{2'}, g_2^1 g_2^2, g_3^1 g_3^2, g_4^{1'} g_4^{2'})$.
(ii) $\tilde{G}^{1I} \oslash \tilde{G}^{2I} = (g_1^1/g_4^2, g_2^1/g_3^2, g_3^1/g_2^2, g_4^1/g_1^2;\ g_1^{1'}/g_4^{2'}, g_2^1/g_3^2, g_3^1/g_2^2, g_4^{1'}/g_1^{2'})$.

Definition 2.5 ([31])**.** The ordering between two TrIFNs $\tilde{G}^{1I} = (g_1^1, g_2^1, g_3^1, g_4^1;\ g_1^{1'}, g_2^1, g_3^1, g_4^{1'})$ and $\tilde{G}^{2I} = (g_1^2, g_2^2, g_3^2, g_4^2;\ g_1^{2'}, g_2^2, g_3^2, g_4^{2'})$ is defined as:

(i) $\tilde{G}^{1I} \succ \tilde{G}^{2I}$ *iff* $g_1^{1'} > g_1^{2'},\ g_1^1 > g_1^2,\ g_2^1 > g_2^2,\ g_3^1 > g_3^2,\ g_4^1 > g_4^2,\ g_4^{1'} > g_4^{2'}$.
(ii) $\tilde{G}^{1I} \succeq \tilde{G}^{2I}$ *iff* $g_1^{1'} \geq g_1^{2'},\ g_1^1 \geq g_1^2,\ g_2^1 \geq g_2^2,\ g_3^1 \geq g_3^2,\ g_4^1 \geq g_4^2,\ g_4^{1'} \geq g_4^{2'}$.
(iii) $\tilde{G}^{1I} \approx \tilde{G}^{2I}$ *iff* $\tilde{G}^{1I} \succeq \tilde{G}^{2I}$ and $\tilde{G}^{2I} \succeq \tilde{G}^{1I}$ *iff* $g_1^{1'} = g_1^{2'},\ g_1^1 = g_1^2,\ g_2^1 = g_2^2,\ g_3^1 = g_3^2,\ g_4^1 = g_4^2,\ g_4^{1'} = g_4^{2'}$.

3. Linear fractional programming problem

In this section, Charnes and Cooper [10] method is discussed for a general LFPP which is as follows:

$$\left.\begin{array}{l} \text{Max } Z(x) = \frac{N(x)}{D(x)} = \frac{\sum_{j=1}^n p_j x_j + q}{\sum_{j=1}^n r_j x_j + s}, \\ \text{subject to} \\ \sum_{j=1}^n a_{ij} x_j \leq b_i \text{ for } i = 1, 2, \cdots, m, \\ x_j \geq 0 \text{ for } j = 1, 2, \cdots, n. \end{array}\right\} \text{(LFP)}$$

Let $S = \{x = (x_1, x_2, \ldots, x_n)\}$ be the set of all feasible solutions of (LFP). For some values of x, $D(x) = 0$ may occur. To overcome this, we fix either $D(x) > 0$ or $D(x) < 0$. Conventionally, we assume $D(x) > 0$.

3.1. Charnes and Cooper [10] technique

Charnes and Cooper [10] presented a new method to deal with LFPPs by transforming the problem into an LPP by substituting $x = y/t$ where $y = (y_1, y_2, \ldots, y_n)$ and $t > 0$ is a real number. When $N(x) \geq 0$ and $D(x) > 0$ then the transformed equivalent LPP by using the technique given in [10] is as follows:

$$\left.\begin{array}{ll} \text{Max } tN(y/t) & = \sum_{j=1}^n p_j y_j + qt \\ \text{subject to} & \sum_{j=1}^n a_{ij} y_j - b_i t \leq 0 \text{ for } i = 1, 2, \cdots, m, \\ & \sum_{j=1}^n r_j y_j + st = 1,\ t > 0,\ y_j \geq 0 \text{ for } j = 1, 2, \cdots, n. \end{array}\right\} \text{(LFPP-1)}$$

Remark 3.1 ([10])**.** If (y^*, t^*) is an optimal solution of (LFPP-1) then the optimal solution of (LFP) (x^*) is obtained as $x^* = y^*/t^*$.

4. Fully intuitionistic fuzzy linear fractional programming

In this section, we formulate a fully IFLFPP having all the parameters and variables as TrIFNs. A fully IFLFPP is as follows:

$$\left.\begin{array}{ll}\text{Max } \tilde{Z}^I(x) & = \frac{\sum_{j=1}^n \tilde{p}_j^I \otimes \tilde{x}_j^I \oplus \tilde{q}^I}{\sum_{j=1}^n \tilde{r}_j^I \otimes \tilde{x}_j^I \oplus \tilde{s}^I} \text{ where } \sum_{j=1}^n \tilde{r}_j^I \otimes \tilde{x}_j^I \oplus \tilde{s}^I \succ \tilde{0}^I \\ \text{subject to} & \sum_{j=1}^n \tilde{a}_{ij}^I \otimes \tilde{x}_j^I \preceq \tilde{b}_i^I \text{ for } i = 1, 2, \cdots, m, \\ & \tilde{x}_j^I \succeq \tilde{0}^I \text{ for } j = 1, 2, \cdots, n. \end{array}\right\} \text{(IFLFP)}$$

In this article, all the variables and parameters are assumed to be non-negative TrIFNs. The (IFLFP) model can be rewritten as:

$$\left.\begin{array}{ll}\text{Max } \tilde{Z}^I(x) = & \dfrac{\begin{array}{c}\sum_{j=1}^n (p_{j1}, p_{j2}, p_{j3}, p_{j4};\ p'_{j1}, p_{j2}, p_{j3}, p'_{j4}) \otimes (x_{j1}, x_{j2}, x_{j3}, x_{j4};\ x'_{j1}, x_{j2}, x_{j3}, x'_{j4}) \\ \oplus (q_1, q_2, q_3, q_4;\ q'_1, q_2, q_3, q'_4)\end{array}}{\begin{array}{c}\sum_{j=1}^n (r_{j1}, r_{j2}, r_{j3}, r_{j4};\ r'_{j1}, r_{j2}, r_{j3}, r'_{j4}) \otimes (x_{j1}, x_{j2}, x_{j3}, x_{j4};\ x'_{j1}, x_{j2}, x_{j3}, x'_{j4}) \\ \oplus (s_1, s_2, s_3, s_4;\ s'_1, s_2, s_3, s'_4)\end{array}} \\ \text{where} & \left\{\begin{array}{c}\sum_{j=1}^n (r_{j1}, r_{j2}, r_{j3}, r_{j4};\ r'_{j1}, r_{j2}, r_{j3}, r'_{j4}) \otimes \\ (x_{j1}, x_{j2}, x_{j3}, x_{j4};\ x'_{j1}, x_{j2}, x_{j3}, x'_{j4}) \oplus (s_1, s_2, s_3, s_4;\ s'_1, s_2, s_3, s'_4)\end{array}\right\} \succ \tilde{0}^I, \\ \text{subject to} & \left\{\begin{array}{c}\sum_{j=1}^n (a_{ij_1}, a_{ij_2}, a_{ij_3}, a_{ij_4};\ a'_{ij_1}, a_{ij_2}, a_{ij_3}, a'_{ij_4}) \otimes \\ (x_{j1}, x_{j2}, x_{j3}, x_{j4};\ x'_{j1}, x_{j2}, x_{j3}, x'_{j4})\end{array}\right\} \preceq (b_{i1}, b_{i2}, b_{i3}, b_{i4};\ b'_{i1}, b_{i2}, b_{i3}, b'_{i4}), \\ & \text{for } i = 1, 2, \cdots, m, \\ & (x_{j1}, x_{j2}, x_{j3}, x_{j4};\ x'_{j1}, x_{j2}, x_{j3}, x'_{j4}) \succeq \tilde{0}^I \text{ for } j = 1, 2, \cdots, n. \end{array}\right\} \text{(IFLFP-1)}$$

Such fully IFLFPP cannot be solved easily, therefore the IF problem is solved by converting into an equivalent crisp version. Hence, using the arithmetic operations defined for TrIFNs and Remark 2.3, (IFLFP-1) can be reduced into the following model:

$$\left.\begin{array}{l}\text{Max } \tilde{Z}^I(x) = \big(Z_1(x), Z_2(x), Z_3(x), Z_4(x);\ Z_5(x), Z_2(x), Z_3(x), Z_6(x)\big), \\ \text{subject to} \\ \sum_{j=1}^n a_{ij_1} x_{j1} \le b_{i1},\ \sum_{j=1}^n a_{ij_2} x_{j2} \le b_{i2},\ \sum_{j=1}^n a_{ij_3} x_{j3} \le b_{i3} \\ \sum_{j=1}^n a_{ij_4} x_{j4} \le b_{i4},\ \sum_{j=1}^n a'_{ij_1} x'_{j1} \le b'_{i1},\ \sum_{j=1}^n a'_{ij_4} x'_{j4} \le b'_{i4} \text{ for } i = 1, 2, \ldots, m, \\ x'_{j1} \ge 0,\ x_{j1} - x'_{j1} \ge 0,\ x_{j2} - x_{j1} \ge 0,\ x_{j3} - x_{j2} \ge 0, \\ x_{j4} - x_{j3} \ge 0,\ x'_{j4} - x_{j4} \ge 0,\ \text{for } j = 1, 2, \ldots, n. \end{array}\right\} \text{(IFLFP-2)}$$

As all the coefficients and variables are non-negative TrIFNs, therefore, we get

$$Z_1(x) = \frac{\sum_{j=1}^n p_{j1} x_{j1} + q_1}{\sum_{j=1}^n r_{j4} x_{j4} + s_4},\ Z_2(x) = \frac{\sum_{j=1}^n p_{j2} x_{j2} + q_2}{\sum_{j=1}^n r_{j3} x_{j3} + s_3},\ Z_3(x) = \frac{\sum_{j=1}^n p_{j3} x_{j3} + q_3}{\sum_{j=1}^n r_{j2} x_{j2} + s_2},$$
$$Z_4(x) = \frac{\sum_{j=1}^n p_{j4} x_{j4} + q_4}{\sum_{j=1}^n r_{j1} x_{j1} + s_1},\ Z_5(x) = \frac{\sum_{j=1}^n p'_{j1} x'_{j1} + q'_1}{\sum_{j=1}^n r'_{j4} x'_{j4} + s'_4} \text{ and } Z_6(x) = \frac{\sum_{j=1}^n p'_{j4} x'_{j4} + q'_4}{\sum_{j=1}^n r'_{j1} x'_{j1} + s'_1}$$

with $\sum_{j=1}^n r_{ju} x_{ju} + s_u > 0$ for $u = 1, \ldots, 4$ and $\sum_{j=1}^n r'_{jv} x'_{jv} + s'_v > 0$ for $v = 1$ and 4.

We propose that (IFLFP-2) can be modeled as the following equivalent crisp MOPP:

$$\left.\begin{array}{l}\text{Max } Z(x) = \{Z_1(x), Z_2(x), \ldots, Z_6(x)\}, \\ \text{subject to the constraints of (IFLFP-2).}\end{array}\right\} \text{(MOP)}$$

In such MOPPs, we cannot always find an optimal solution that gives the optimal value for all the objective functions, thus we find an efficient or weakly efficient solution $\bar{x}$ as defined ahead.

Definition 4.1. A feasible solution $\bar{x}$ of (MOP) is said to be an efficient solution if there does not exist any other feasible solution $\hat{x}$ such that $Z_k(\bar{x}) \leq Z_k(\hat{x})$ for $k = 1, 2, \ldots, 6$ with strict inequality for at least one k. Additionally, it is said to be weakly efficient if the strict inequality holds $\forall k$. Clearly, it can be observed that every efficient solution is a weakly efficient solution, but the reverse side implication need not be true.

Theorem 4.1. *A weakly efficient solution of (MOP) is an optimal solution of (IFLFP-2).*

Proof. Let x be a weakly efficient solution of (MOP), therefore, it is also a feasible solution of (IFLFP-2). For instance, let us assume that x is not an optimal solution of (IFLFP-2) *i.e.* there exists some other feasible solution y such that:

$$\tilde{Z}^I(x) \preceq \tilde{Z}^I(y),$$

where

$$\tilde{Z}^I(x) = \big(Z_1(x), Z_2(x), Z_3(x), Z_4(x);\ Z_5(x), Z_2(x), Z_3(x), Z_6(x)\big) \text{ and}$$
$$\tilde{Z}^I(y) = \big(Z_1(y), Z_2(y), Z_3(y), Z_4(y);\ Z_5(y), Z_2(y), Z_3(y), Z_6(y)\big),$$

such that $Z_k(x) < Z_k(y)$, for at least one k. Since y is also a feasible solution of (MOP), therefore, x cannot be an efficient solution. This contradicts that x is a weakly efficient solution of (MOP) by Definiton 4.1. This amounts to saying that our assumption is wrong that x is not an optimal solution of (IFLFP-2). Hence the result. □

Remark 4.1. As an efficient solution x is also a weakly efficient solution of (MOP) by Definition 4.1, it follows that x is an optimal solution of (IFLFP-2).

Using Theorem 4.1 and Remark 4.1, to solve (IFLFP-2) it is sufficient to solve (MOP) for an efficient/weakly efficient solution.

4.1. Intuitionistic fuzzy programming

We propose to solve (MOP) using IF programming approach for MOPPs with non-negative objectives. The multi-objective model is first converted into 6 single objective sub-problems where the kth sub-problem is:

$$\text{Max } Z_k(x) \text{ subject to the constraints of (MOP).} \qquad (k\text{-MOP})$$

For $k = 1, 2, \ldots, 6$.

Let x_k^* be the optimal solution of (k-MOP) and $Z_k^* = Z_k(x_k^*)$ be the optimal value of the kth objective function. If $x_{k1}^* = x_{k2}^*$ for every distinct pair of $k1,\ k2 \in \{1, 2, \ldots, 6\}$ then x_k^* is an optimal solution of (MOP) $\forall k$. If the solutions are not all the same, then there is a need to obtain an efficient/weakly efficient solution. Such a solution can be obtained using the IF programming approach, where membership $(\mu_k\big(Z_k(x)\big))$ and non-membership $(\nu_k\big(Z_k(x)\big))$ functions are associated with each of the k objective functions such that

$$\mu_k\big(Z_k(x)\big) + \nu_k\big(Z_k(x)\big) \leq 1.$$

The main idea behind the IF programming approach is to maximize the smallest acceptance degree and minimize the largest rejection degree among the membership and non-membership functions, respectively. Let $\alpha = \min_{1\leq k\leq 6}\{\mu_k\big(Z_k(x)\big)\}$ and $\lambda = \max_{1\leq k\leq 6}\{\nu_k\big(Z_k(x)\big)\}$. Thus, the IF programming model is formulated as follows:

$$\left.\begin{aligned}&\text{Max } \alpha-\lambda\\&\text{subject to}\\&\mu_k\big(Z_k(x)\big)\geq\alpha,\\&\nu_k\big(Z_k(x)\big)\leq\lambda,\ \text{ for } k=1,2,\ldots,6,\\&\alpha\geq 0,\ \lambda\geq 0,\\&\alpha\geq\lambda,\ \alpha+\lambda\leq 1 \text{ and the constraints of (MOP).}\end{aligned}\right\}\text{(IFP)}$$

The membership and non-membership functions for kth objective function are defined using the most acceptable value (Z_k^U) and the least acceptable value (Z_k^L) of the objective function, such that

$$Z_k^L \leq Z_k(x) \leq Z_k^U.$$

The most and least acceptable values are predetermined, in this study, for the kth objective function, the most acceptable value is taken as the optimal value of (k-MOP) *i.e.* $Z_k^U = Z_k^*$. However, we propose the least acceptable value Z_k^L to be taken as the minimum value of the kth objective function in the set of feasible solutions *i.e.* $Z_k^L = \min\{Z_k(x)|x \text{ is a feasible solution of (MOP)}\}$. One clear and prominent advantage of selecting $Z_k^{U'}s$ and $Z_k^{L'}s$ in such a fashion is that all the possible values of objectives in the given solution space are covered. This provides flexibility in obtaining an efficient solution to a multi-objective problem by incorporating the largest possible range of values.

The membership and non-membership functions are selected in such a manner that

$$\mu_k\big(Z_k(x)\big) = 1,\ \nu_k\big(Z_k(x)\big) = 0 \text{ whenever } Z_k(x) \geq Z_k^U$$

and

$$\mu_k\big(Z_k(x)\big) = 0,\ \nu_k\big(Z_k(x)\big) = 1 \text{ for } Z_k(x) \leq Z_k^L.$$

Additionally, when $Z_k^L \leq Z_k(x) \leq Z_k^U$, these functions offer flexibility in effectively reflecting decision-maker's choices.

4.2. Motivation to develop new functions

According to the constraints of (IFP), a feasible solution exists in the region where $\mu_k\big(Z_k(x)\big) \geq \nu_k\big(Z_k(x)\big)$, for the scope of the present study this region is referred to as the feasible/feasibility region. In literature, mainly linear, exponential, hyperbolic, etc. classes of membership and non-membership functions are studied. For such functions, the feasible region is represented in Figure 2 by the region shaded in blue colour. Such functions provide little flexibility in terms of incorporating linear or non-linear change in acceptance or rejection degree. As indicated in Figure 2, the feasibility region cannot be altered irrespective of the nature of membership and non-membership functions. Due to this reason, in some cases, infeasibility arises when such membership and non-membership functions are used to solve the (IFP) model.

It is observed that the infeasible problem can be converted into a feasible problem by increasing the feasibility region. This can be carried out by incorporating tolerance in the least or most acceptable values, but there is no specific way of assigning such tolerance values without employing some additional parameters. In this direction, we propose a new class of parameterized membership and non-membership functions, in the next section. The proposed membership and non-membership functions are so devised that the said region can be increased by gradually increasing the value of the associate parameter t. It can be seen in Figure 3, that as the value of t increases, the feasible region also keeps on increasing. We claim that such kind of functions can be effectively used to solve IF programming problems.

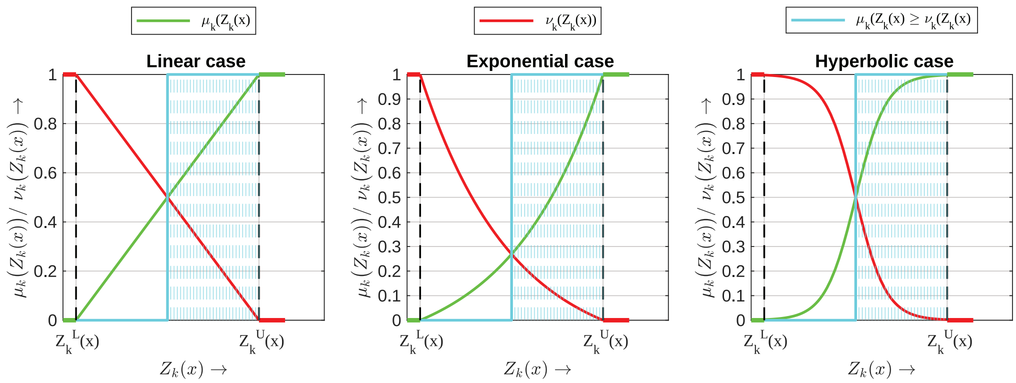

FIGURE 2. Existing linear as well as non-linear membership and non-membership functions.

5. Proposed membership and non-membership functions

Using a parameter $t \geq 1$, we define the parameterized family of membership and non-membership functions as follows:

$$\mu_k\big(Z_k(x)\big) = \begin{cases} 1, & \text{if } Z_k(x) > Z_k^U, \\ \frac{\left(Z_k^U - Z_k^L\right)^t - \left(Z_k^U - Z_k(x)\right)^t}{\left(Z_k^U - Z_k^L\right)^t}, & \text{if } Z_k^L \leq Z_k(x) \leq Z_k^U, \\ 0, & \text{if } Z_k(x) < Z_k^L \end{cases}$$

and

$$\nu_k\big(Z_k(x)\big) = \begin{cases} 0, & \text{if } Z_k(x) > Z_k^U, \\ \left(\frac{Z_k^U - Z_k(x)}{Z_k^U - Z_k^L}\right)^t, & \text{if } Z_k^L \leq Z_k(x) \leq Z_k^U, \\ 1, & \text{if } Z_k(x) < Z_k^L. \end{cases}$$

Here, $\mu_k\big(Z_k(x)\big)$ and $\nu_k\big(Z_k(x)\big)$ are the membership and non-membership functions associated with the kth objective function. The parameter t is the shape parameter selected according to the preference of the decision-maker that depicts the acceptance/rejection degree for the value of objective between Z_k^L and Z_k^U.

Remark 5.1. As a special case, for $t = 1$, the proposed membership and non-membership functions reduce to linear membership and non-membership functions.

Figure 3 displays the different membership and non-membership function plots for different t selections. It is easy to see that, as the value of t increases, the membership and non-membership degrees increase and decrease, respectively. Furthermore, for a very large value of t, the plots depict that the decision maker has high acceptance and low rejection for the value of kth objective function lying between Z_k^L and Z_k^U. This indicates that the decision-maker tends to be more accepting (less rejecting) towards the objective function value that is farther from the most acceptable value. Observably, the value of t can be increased as much as possible to cater to the decision-maker's liking.

5.1. Proposed intuitionistic fuzzy programming

Fixing the value of the parameter $t \geq 1$ as desired by the decision-maker. The IF programming model (IFP) as proposed in Section 4.1 can be formulated using the proposed membership and non-membership functions

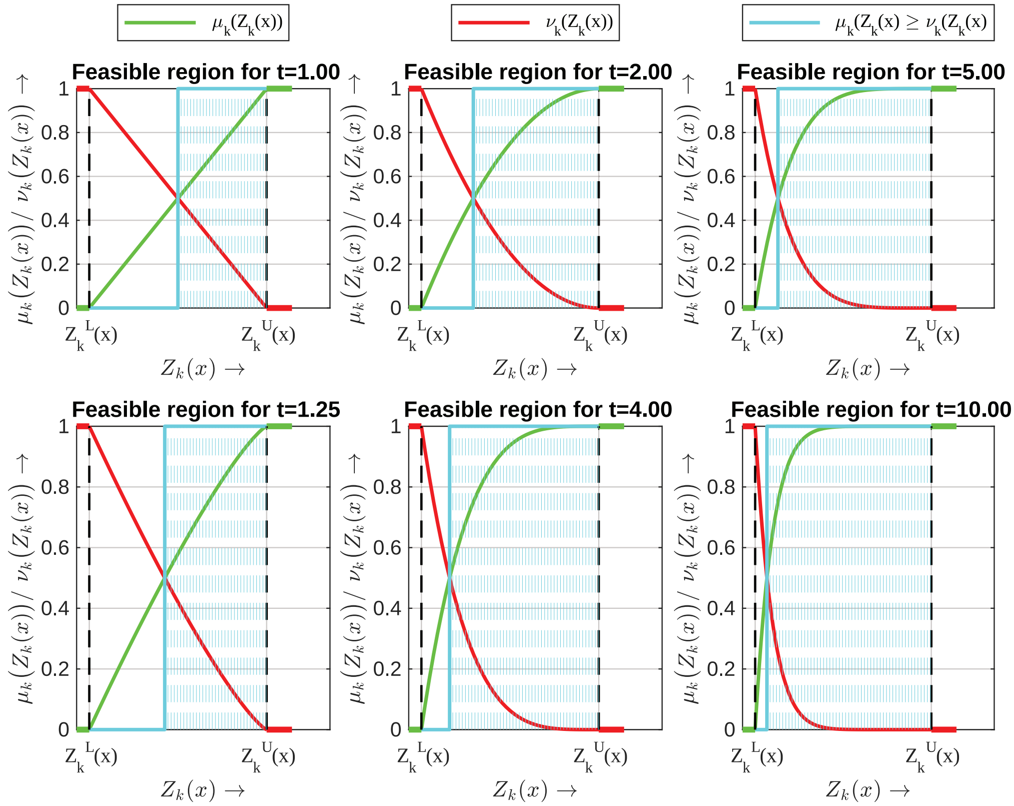

FIGURE 3. Increasing feasible region as t increases.

as follows:

$$\left.\begin{array}{l}\text{Max } \alpha - \lambda \\ \text{subject to} \\ Z_k(x) + (1-\alpha)^{1/t}\left({Z_k}^U - {Z_k}^L\right) \geq {Z_k}^U, \\ Z_k(x) + \lambda^{1/t}\left({Z_k}^U - {Z_k}^L\right) \geq {Z_k}^U, \ \text{ for } k = 1, 2, \ldots, 6, \\ \alpha \geq 0, \ \lambda \geq 0, \\ \alpha \geq \lambda, \ \alpha + \lambda \leq 1, \ \text{ and all the constraints of (IFLFP-2).}\end{array}\right\} \text{(IFP-}t\text{)}$$

Firstly $t = 1$ (linear functions) is selected, if the problem is feasible then there is no need to solve the problem for other values of t. However, if the problem is infeasible then the value of t can be increased until the model becomes feasible.

Remark 5.2. Since all the variables and coefficients are taken as non-negative TrIFNs, therefore $Z_k(x) \geq 0$ which implies ${Z_k}^L \geq 0$, $\forall k$.

Remark 5.3. The solution of (IFP-t), $x = \left\{x_{j1}, x_{j2}, x_{j3}, x_{j4}, x^{*'}_{j1}, x^{*'}_{j4} \mid j = 1, 2, \ldots, n\right\}$ can be rewritten in terms of TrIFN as $x = (\tilde{x}^I_1, \tilde{x}^I_2, \ldots, \tilde{x}^I_n)$ where $\tilde{x}^I_j = (x_{j1}, x_{j2}, x_{j3}, x_{j4};\ x^{'}_{j1}, x_{j2}, x_{j3}, x^{'}_{j4})$.

Theorem 5.1. *A unique optimal solution of (IFP-t) is an efficient solution of (IFLFP-2).*

Proof. Let $x^* = (\tilde{x}_1^{*I}, \tilde{x}_2^{*I}, \ldots, \tilde{x}_n^{*I})$ where $\tilde{x}_j^{*I} = (x_{j1}^*, x_{j2}^*, x_{j3}^*, x_{j4}^*;\ x_{j1}^{*'}, x_{j2}^*, x_{j3}^*, x_{j4}^{*'})$ for $j = 1, 2, \ldots, n$, be a unique optimal solution of (IFP-t). From the constraints of (IFP-t), we have

$$\begin{aligned}
&\sum_{j=1}^{n} a_{ij1} x_{j1}^* \le b_{i1},\ \sum_{j=1}^{n} a_{ij2} x_{j2}^* \le b_{i2},\ \sum_{j=1}^{n} a_{ij3} x_{j3}^* \le b_{i3},\\
&\sum_{j=1}^{n} a_{ij4} x_{j4}^* \le b_{i4},\ \sum_{j=1}^{n} a'_{ij1} x_{j1}^{*'} \le b'_{i1},\ \sum_{j=1}^{n} a'_{ij4} x_{j4}^{*'} \le b'_{i4} && \text{for } i = 1, 2, \ldots, m,\\
&x_{j1}^{*'} \ge 0\ x_{j1}^* - x_{j1}^{*'} \ge 0,\ x_{j2}^* - x_{j1}^* \ge 0,\ x_{j3}^* - x_{j2}^* \ge 0, x_{j4}^* - x_{j3}^* \ge 0,\\
&x_{j4}^{*'} - x_{j4}^* \ge 0, && \text{for } j = 1, 2, \ldots, n.
\end{aligned}$$

Observe that x^* also satisfies the constraints of (IFLFP-2). Therefore, x^* is also a feasible solution of (IFLFP-2).

Let $I_K = \{1, 2, \ldots, K\}$. If possible, let us assume that x^* is not an efficient solution of (IFLFP-2) *i.e.* there exists some other feasible solution y such that

$$Z_k(x^*) \le Z_k(y)\ \forall k \in I_K \text{ and the set } I_0 = \{k \in I_K | Z_k(x^*) < Z_k(y)\} \ne \emptyset. \tag{1}$$

Since, $Z_k^L \le Z_k(x) \le Z_k^U,\ \forall k \in I_K$ therefore, using (1) we obtain

$$\frac{Z_k^U - Z_k(x^*)}{Z_k^U - Z_k^L} \ge \frac{Z_k^U - Z_k(y)}{Z_k^U - Z_k^L}\ \forall k \in I_K \text{ and for some } k_0 \in I_0 \ne \emptyset,\ \frac{Z_{k_0}^U - Z_{k_0}(x^*)}{Z_{k_0}^U - Z_{k_0}^L} > \frac{Z_{k_0}^U - Z_{k_0}(y)}{Z_{k_0}^U - Z_{k_0}^L}. \tag{2}$$

For arbitrary $t \ge 1$, equation (2) yields

$$\left(\frac{Z_k^U - Z_k(x^*)}{Z_k^U - Z_k^L}\right)^t \ge \left(\frac{Z_k^U - Z_k(y)}{Z_k^U - Z_k^L}\right)^t \qquad \forall k \in I_K. \tag{3}$$

This implies

$$\nu_k\big(Z_k(x^*)\big) \ge \nu_k\big(Z_k(y)\big) \qquad \forall k. \tag{4}$$

From (3), it follows that

$$\frac{(Z_k^U - Z_k^L)^t - \big(Z_k^U - Z_k(x^*)\big)^t}{(Z_k^U - Z_k^L)^t} \le \frac{(Z_k^U - Z_k^L)^t - \big(Z_k^U - Z_k(y)\big)^t}{(Z_k^U - Z_k^L)^t} \qquad \forall k \in I_K. \tag{5}$$

Equivalently we have,

$$\mu_k\big(Z_k(x^*)\big) \le \mu_k\big(Z_k(y)\big) \qquad \forall k. \tag{6}$$

Let $\min\big\{\mu_k\big(Z_k(x)|k \in I_K\big\} = \alpha_x$ and $\max\big\{\nu_k\big(Z_k(x)|k \in I_K\big\} = \lambda_x$.

From (4) and (6) we get,

$$\alpha_{x^*} \le \alpha_y \text{ and } \lambda_{x^*} \ge \lambda y.$$

This gives that $\alpha_{x^*} - \lambda_{x^*} \le \alpha_y - \lambda_y$, which contradicts that x^* is a unique optimal solution to (IFP-t). Therefore, our assumption is wrong.

Hence, x^* is an efficient solution of (IFLFP-2). □

Theorem 5.2. *An optimal solution of (IFP-t) is a weakly efficient solution of (IFLFP-2).*

Proof. Let $x^* = (\tilde{x}_1^{*I}, \tilde{x}_2^{*I}, \ldots, \tilde{x}_n^{*I})$ where $\tilde{x}_j^{*I} = (x_{j1}^*, x_{j2}^*, x_{j3}^*, x_{j4}^*;\ x_{j1}^{*'}, x_{j2}^*, x_{j3}^*, x_{j4}^{*'})$ for $j = 1, 2, \ldots, n$, be a optimal solution of (IFP-t). From the constraints of (IFP-t), we have

$$\begin{aligned}
&\sum_{j=1}^{n} a_{ij_1} x_{j1}^* \le b_{i1},\ \sum_{j=1}^{n} a_{ij_2} x_{j2}^* \le b_{i2},\ \sum_{j=1}^{n} a_{ij_3} x_{j3}^* \le b_{i3},\\
&\sum_{j=1}^{n} a_{ij_4} x_{j4}^* \le b_{i4},\ \sum_{j=1}^{n} a'_{ij_1} x_{j1}^{*'} \le b'_{i1},\ \sum_{j=1}^{n} a'_{ij_4} x_{j4}^{*'} \le b'_{i4} && \text{for } i = 1, 2, \ldots, m,\\
&x_{j1}^{*'} \ge 0\ x_{j1}^* - x_{j1}^{*'} \ge 0,\ x_{j2}^* - x_{j1}^* \ge 0,\ x_{j3}^* - x_{j2}^* \ge 0, x_{j4}^* - x_{j3}^* \ge 0,\\
&x_{j4}^{*'} - x_{j4}^* \ge 0, && \text{for } j = 1, 2, \ldots, n.
\end{aligned}$$

Observe that x^* also satisfies the constraints of (IFLFP-2). Therefore, x^* is also a feasible solution of (IFLFP-2).

If possible, let us assume that x^* is not a weakly efficient solution of (IFLFP-2) *i.e.* there exists some other feasible solution y such that $\forall k \in I_K,\ Z_k(x^*) < Z_k(y)$.

For arbitrary parameter $t \ge 1$, following the similar procedure as in Theorem 5.1, we have,

$$\left(\frac{Z_k^U - Z_k(x^*)}{Z_k^U - Z_k^L}\right)^t > \left(\frac{Z_k^U - Z_k(y)}{Z_k^U - Z_k^L}\right)^t \text{ and} \tag{7}$$

$$\frac{(Z_k^U - Z_k^L)^t - \left(Z_k^U - Z_k(x^*)\right)^t}{(Z_k^U - Z_k^L)^t} < \frac{(Z_k^U - Z_k^L)^t - \left(Z_k^U - Z_k(y)\right)^t}{(Z_k^U - Z_k^L)^t},\ \forall k \in I_K. \tag{8}$$

This amounts to saying

$$\nu_k\left(Z_k(x^*)\right) > \nu_k\left(Z_k(y)\right) \text{ and } \mu_k\left(Z_k(x^*)\right) < \mu_k\left(Z_k(y)\right), \qquad \forall k \in I_K. \tag{9}$$

Since $\min\left\{\mu_k\left(Z_k(x)|k \in I_K\right\} = \alpha_x$ and $\max\left\{\nu_k\left(Z_k(x)|k \in I_K\right\} = \lambda_x$.

Therefore, from (9) it follows that,

$$\alpha_{x^*} < \alpha_y \text{ and } \lambda_{x^*} > \lambda y.$$

This gives that $\alpha_{x^*} - \lambda_{x^*} < \alpha_y - \lambda_y$, which contradicts that x^* is an optimal solution to (IFP-t). Therefore, our assumption is wrong.

Hence, x^* is a weakly efficient solution of (IFLFP-2). □

Theorem 5.3. *A feasible solution of (IFP-t) is an efficient solution of (IFLFP-2) if it is a unique, weakly efficient solution.*

Proof. Let $x^* = (\tilde{x}_1^{*I}, \tilde{x}_2^{*I}, \ldots, \tilde{x}_n^{*I})$ where $\tilde{x}_j^{*I} = (x_{j1}^*, x_{j2}^*, x_{j3}^*, x_{j4}^*;\ x_{j1}^{*'}, x_{j2}^*, x_{j3}^*, x_{j4}^{*'})$ for $j = 1, 2, \ldots, n$, be a feasible solution of (IFP-t) and a unique weakly efficient solution of (IFLFP-2). To show that x^* is an efficient solution of (IFLFP-2), it suffices to show that it is a unique optimal solution of (IFP-t) by Theorem 5.1.

Let us assume that x^* is not a unique optimal solution of (IFP-t), hence there exists some alternate optimal solution. WLOG let y be the alternate optima. Using Theorem 5.2, we get y is also a weakly efficient solution of (IFLFP-2). Since, x^* is given to be a unique weakly efficient solution of (IFLFP-2), therefore, $x^* = y$. This contradicts our assumption that x^* is not a unique optimal solution of (IFP-t). Hence, the result. □

Remark 5.4. If $x^* = (\tilde{x}_1^{*I}, \tilde{x}_2^{*I}, \ldots, \tilde{x}_n^{*I})$ where $\tilde{x}_j^{*I} = (x_{j1}^*, x_{j2}^*, x_{j3}^*, x_{j4}^*;\ x_{j1}^{*'}, x_{j2}^*, x_{j3}^*, x_{j4}^{*'})$ for $j = 1, 2, \ldots, n$, is a feasible solution of (IFP-t) then, based on the results of Theorems 5.1, 5.2, 5.3 and Definition 4.1 the following implications can be drawn:

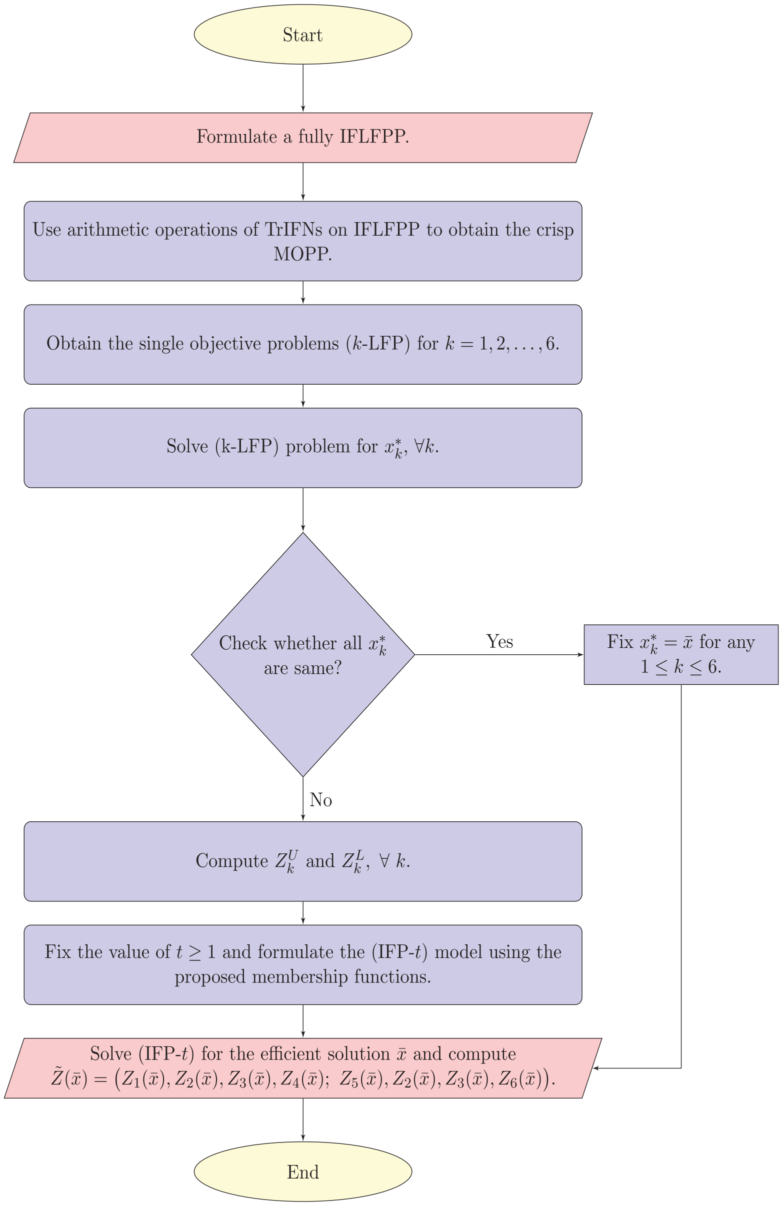

FIGURE 4. Flow chart of the proposed technique.

The steps involved in solving IFLFPP as (IFLFP) model using the above-mentioned technique are written below, along with the flow chart in Figure 4:

Step 1: Formulate a fully IFLFPP (IFLFP) using TrIFN.
Step 2: Convert the IF problem into an equivalent crisp MOPP (IFLFP-2) using the arithmetic operations of TrIFNs.

Step 3: Split the multi-objective problem into 6 sub-problems. For $k = 1, 2, \ldots, 6$, the kth sub-problem is stated as:

$$\text{Max } Z_k(x) \text{ subject to all the constraints of (IFLFP-2).} \qquad (k\text{-LFP})$$

Step 4: Solve (k-LFP) by using Charnes and Cooper [10] approach to obtain the optimal solution x_k^* and the optimal value Z_k^*, $\forall k$.

Step 5: Check whether $x_{k1}^* = x_{k2}^*$ for $k1,\ k2 = 1, 2, \ldots, 6$. If yes, fix $x_k^* = \bar{x}$ for any k and go to Step 8, else go to Step 6.

Step 6: Let S' be the set of all feasible solutions of (IFLFP-2). Evaluate $Z_k^U = Z_k^*$ and $Z_k^L = \min\{Z_k(x) | x \in S'\}$.

Step 7: Fix the value of the parameter $t \geq 1$ as desired by the decision-maker and formulate the IF programming model (IFP-t) as proposed in Section 5.1.

Step 8: Solve (IFP-t) to obtain a solution $\bar{x}$ and substitute in $\tilde{Z}^I(x)$ to find the IF value of (IFLFP) *i.e.* $\big(Z_1(\bar{x}), Z_2(\bar{x}), Z_3(\bar{x}), Z_4(\bar{x});\ Z_5(\bar{x}), Z_2(\bar{x}), Z_3(\bar{x}), Z_6(\bar{x})\big)$.

6. Illustrative example

In this section, a fully IFLFPP is solved using the proposed technique. The following example is worked out for illustration purposes using the steps mentioned in Section 5.1. All the mathematical models are solved using the software Lingo 19.0, the hardware is Intel(R) Core(TM) i7-8565U CPU @ 1.80 GHz, 1992 Mhz, 4 Core(s), 8 Logical Processor(s) and the operating system is Microsoft Windows 11 Pro.

Step 1: Consider the following IFLFPP:

$$\left.\begin{aligned}
&\text{Max } \tilde{Z}^I(x) = \tfrac{\tilde{p}_1^I \otimes \tilde{x}_1^I \oplus \tilde{p}_2^I \otimes \tilde{x}_2^I \oplus \tilde{q}^I}{\tilde{x}_1^I \oplus \tilde{x}_2^I \oplus \tilde{s}^I}\\
&\text{subject to}\\
&\tilde{x}_1^I \oplus \tilde{x}_2^I \preceq \tilde{b}_1^I,\\
&\tilde{a}_1^I \otimes \tilde{x}_1^I \oplus \tilde{a}_2^I \otimes \tilde{x}_2^I \preceq \tilde{b}_2^I,\\
&\tilde{x}_1^I \succeq \tilde{0}^I,\ \tilde{x}_2^I \succeq \tilde{0}^I,
\end{aligned}\right\} \text{(E1)}$$

where $\tilde{p}_1^I = (2, 3, 4, 5;\ 1, 3, 4, 6)$, $\tilde{p}_2^I = (2, 3, 5, 6;\ 1.5, 3, 5, 6.5)$, $\tilde{q}^I = (1.5, 3, 4, 5.5;\ 1, 3, 4, 5.5)$, $\tilde{s}^I = (3, 3.5, 3.5, 4;\ 2.5, 3.5, 3.5, 4.5)$, $\tilde{a}_1^I = (3, 3.5, 3.5, 4;\ 2, 3.5, 3.5, 5)$, $\tilde{a}_2^I = (8, 9, 10, 11;\ 8, 9, 10, 11.5)$, $\tilde{b}_1^I = (2.5, 3, 4, 4;\ 2, 3, 4, 5)$, $\tilde{b}_2^I = (9, 10, 10.5, 11;\ 8, 10, 10.5, 12)$ and the variables as $\tilde{x}_j^I = (\tilde{x}_{j1}, \tilde{x}_{j2}, \tilde{x}_{j3}, \tilde{x}_{j4}; \tilde{x}'_{j1}, \tilde{x}_{j2}, \tilde{x}_{j3}, \tilde{x}'_{j4})$ for $j = 1, 2$.

Clearly, we have $\tilde{x}_1^I \oplus \tilde{x}_2^I \oplus \tilde{s}^I \succ \tilde{0}^I$ and $\tilde{p}_1^I \otimes \tilde{x}_1^I \oplus \tilde{p}_2^I \otimes \tilde{x}_2^I \oplus \tilde{q}^I \succeq \tilde{0}^I$.

Step 2: Upon using the proposed algorithm, the model further reduces to an equivalent MOPP as follows:

$$\left.\begin{aligned}
&\text{Max } \tilde{Z}^I(x) = \big(Z_1(x), Z_2(x), Z_3(x), Z_4(x);\ Z_5(x), Z_2(x), Z_3(x), Z_6(x)\big)\\
&\text{subject to}\\
&x'_{11} + x'_{21} \leq 2,\ x_{11} + x_{21} \leq 2.5,\ x_{12} + x_{22} \leq 3,\\
&x_{12} + x_{23} \leq 4,\ x_{14} + x_{24} \leq 4,\ x'_{14} + x'_{24} \leq 5,\\
&2x'_{11} + 8x'_{21} \leq 8,\ 3x_{11} + 8x_{21} \leq 9,\ 3.5x_{12} + 9x_{22} \leq 10,\\
&3.5x_{12} + 10x_{23} \leq 10.5,\ 4x_{14} + 11x_{24} \leq 11,\ 5x'_{14} + 11.5x'_{24} \leq 12,\\
&x'_{j1} \geq 0,\ x_{j1} - x'_{j1} \geq 0,\ x_{j2} - x_{j1} \geq 0,\ x_{j3} - x_{j2} \geq 0,\\
&x_{j4} - x_{j3} \geq 0,\ x'_{j4} - x_{j4} \geq 0,\ \text{for } j = 1, 2
\end{aligned}\right\} \text{(E2)}$$

where $Z_1(x) = \frac{2x_{11}+2x_{21}+1.5}{x_{14}+x_{24}+4}$, $Z_2(x) = \frac{3x_{12}+3x_{22}+3}{x_{13}+x_{23}+3.5}$, $Z_3(x) = \frac{4x_{13}+5x_{23}+4}{x_{12}+x_{22}+3.5}$, $Z_4(x) = \frac{5x_{14}+6x_{24}+5.5}{x_{11}+x_{21}+3}$, $Z_5(x) = \frac{x'_{11}+1.5x'_{21}+1}{x'_{14}+x'_{24}+4.5}$ and $Z_6(x) = \frac{6x'_{14}+6.5x'_{24}+5.5}{x'_{11}+x'_{21}+2.5}$.

Since all the variables are non-negative, therefore it follows that $x_{14}+x_{24}+4>0$, $x_{13}+x_{23}+3.5>0$, $x_{12}+x_{22}+3.5>0$, $x_{11}+x_{21}+3>0$, $x'_{14}+x'_{24}+4.5>0$, $x'_{11}+x'_{21}+2.5>0$.

Step 3: (E2) is split into the following sub-problems, for $k = 1, 2, \ldots, 6$, kth sub-problem being:

$$\text{Max } Z_k(x) \text{ subject to all the constraints of (E2).} \tag{E2-k}$$

Step 4: Upon solving the sub-problems individually by using Charnes and Cooper's method we obtain the following optimal values

$$Z_1^* = 0.9843,\ Z_2^* = 1.728,\ Z_3^* = 3.885,\ Z_4^* = 5.83,\ Z_5^* = 0.4852,\ Z_6^* = 7.96.$$

Step 5 and 6: The solutions obtained are not all the same, therefore Z_k^L and Z_k^U values are evaluated. For $k = 1, 2, \ldots, 6$, fix $Z_k^U = Z_k^*$. The Z_k^L values obtained are

$$Z_1^L = 0.2343,\ Z_2^L = 0.5084,\ Z_3^L = 1.14286,\ Z_4^L = 1.833,\ Z_5^L = 0.1449,\ Z_6^L = 2.2.$$

Step 7: Using the proposed membership and non-membership functions for $t = 1$ (linear case), the IF programming model becomes:

$$\left.\begin{aligned}
&\text{Max } \alpha-\lambda\\
&\text{subject to}\\
&(2x_{11}+2x_{21})-(x_{14}+x_{24})\big(0.2343+0.75\alpha\big)-3\alpha+0.5628\geq 0,\\
&(2x_{11}+2x_{21})+(x_{14}+x_{24})\big(0.75\lambda-0.9843\big)+3\lambda-2.4372\geq 0,\\
&(3x_{12}+3x_{22})-(x_{13}+x_{23})\big(0.5084+1.2196\alpha\big)-4.2686\alpha+1.2206\geq 0,\\
&(3x_{12}+3x_{22})+(x_{13}+x_{23})\big(1.2196\lambda-1.728\big)+4.2686\lambda-3.048\geq 0,\\
&(4x_{13}+5x_{23})-(x_{12}+x_{22})\big(1.1428+2.7421\alpha\big)-9.597\alpha+0.0002\geq 0,\\
&(4x_{13}+5x_{23})+(x_{12}+x_{22})\big(2.7421\lambda-3.885\big)+9.597\lambda-9.5975\geq 0,\\
&(5x_{14}+6x_{24})-(x_{11}+x_{21})\big(1.833+3.997\alpha\big)-11.991\alpha+0.001\geq 0,\\
&(5x_{14}+6x_{24})+(x_{11}+x_{21})\big(3.997\lambda-5.83\big)+11.991\lambda-11.99\geq 0,\\
&(x'_{11}+1.5x'_{21})-(x'_{14}+x'_{24})\big(0.1449+0.3403\alpha\big)-1.531\alpha+0.348\geq 0,\\
&(x'_{11}+1.5x'_{21})+(x'_{14}+x'_{24})\big(0.3403\lambda-0.4852\big)+1.531\lambda-1.183\geq 0,\\
&(6x'_{14}+6.5x'_{24})-(x'_{11}+x'_{21})\big(2.2+5.76\alpha\big)-14.4\alpha\geq 0,\\
&(6x'_{14}+6.5x'_{24})+(x'_{11}+x'_{21})\big(5.76\lambda-7.96\big)+14.4\lambda-14.4\geq 0.\\
&\alpha\geq 0,\ \lambda\geq 0,\ \alpha\geq\lambda,\ \alpha+\lambda\leq 1,\ \text{and all the constraints of (E2).}
\end{aligned}\right\}\ \text{(E3)}$$

Step 8: On solving (E3), we obtain $\alpha = 0.5368$, $\lambda = 0.4631$ and $\alpha-\lambda = 0.086$. The solution obtained can be written as $\bar{x} = (\tilde{x}_1^I,\ \tilde{x}_2^I)$ where

$$\tilde{x}_1^I = (1.202, 1.418, 2.228, 2.4;\ 1.137, 1.418, 2.228, 2.236) \text{ and}$$
$$\tilde{x}_2^I = (0.071, 0.071, 0.071, 0.071;\ 0.071, 0.071, 0.071, 0.071).$$

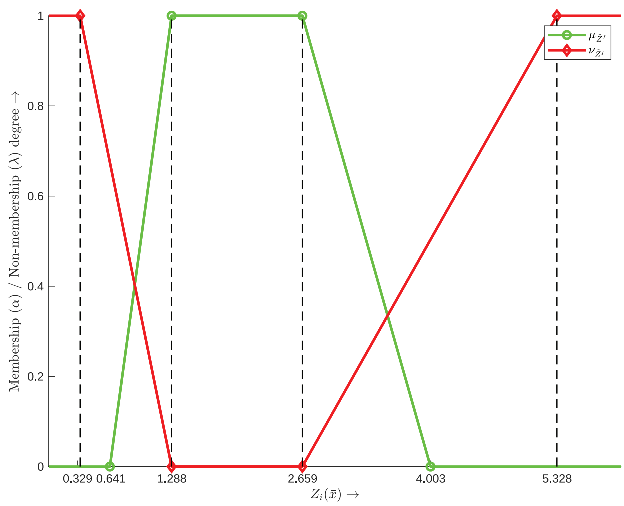

FIGURE 5. IF value of (IE) *i.e.* $\tilde{Z}^I(\bar{x})$.

Additionally, the corresponding value of the objectives comes out to be

$$Z_1(\bar{x}) = 0.641,\ Z_2(\bar{x}) = 1.288,\ Z_3(\bar{x}) = 2.659,\ Z_4(\bar{x}) = 4.003,\ Z_5(\bar{x}) = 0.329,\ Z_6(\bar{x}) = 5.328$$

and the associated TrIFN representation is presented graphically in Figure 5.

6.1. Validation of the proposed membership functions

In order to validate the proposed approach as well as the membership and non-membership functions, the same problem is solved for different values of t. Thus obtained results are presented in Table 2. Evidently, the solutions obtained for different values of t are all same, hence validating the proposed functions. Additionally, Figure 6 depicts the behaviour of the optimal value of (E3) for different values of t. One can observe from both Table 2 and Figure 6, that as the value of t increases the optimal value also increases. In addition, it can be seen that the corresponding value of α increases while λ decreases. This overlaps with the fact that for a larger value of t, the decision-maker has higher acceptance and lesser rejection towards the solution farther from the best solution.

7. Application in portfolio optimization

A pivotal discipline in the world of finance is portfolio optimization. This intricate process involves crafting an investment portfolio to maximize returns while managing risk. By meticulously analyzing the interplay of diverse assets, their historical performances and correlations, portfolio optimization aims to strike a harmonious balance between risk and reward. In this direction, a company plans to invest approximately \$100 K in two stocks: A and B. From the performance history of these stocks, the expected return and the associated risks are evaluated as presented in Table 3. Since the data obtained using the previous performance records lacks precision and accuracy, therefore TrIFNs are used instead of crisp ones.

The aim of the decision maker is to identify the fraction of the amount to be invested in both the stocks for which the expected return is maximum while the associated risk is minimum. Additionally, the company intends to allocate a minimum of approximately 25% ($\tilde{25}^I$%) of the total funds into individual stocks.

TABLE 2. Optimal value of (E3) for various values of t.

Parameter t	Optimal value	Solution $\left(\bar{x} = (\tilde{x}_1^I,\ \tilde{x}_2^I)\right)$
1	$\alpha = 0.543$ $\lambda = 0.457$ $\alpha - \lambda = 0.086$	$\tilde{x}_1^I = (1.202, 1.418, 2.228, 2.236;\ 1.137, 1.418, 2.228, 2.236)$ $\tilde{x}_2^I = (0.071, 0.071, 0.071, 0.071;\ 0, 0.071, 0.071, 0.071)$
1.5	$\alpha = 0.691$ $\lambda = 0.39$ $\alpha - \lambda = 0.382$	$\tilde{x}_1^I = (1.202, 1.418, 2.228, 2.236;\ 1.137, 1.418, 2.228, 2.236)$ $\tilde{x}_2^I = (0.071, 0.071, 0.071, 0.071;\ 0, 0.071, 0.071, 0.071)$
2	$\alpha = 0.791$ $\lambda = 0.209$ $\alpha - \lambda = 0.582$	$\tilde{x}_1^I = (1.202, 1.418, 2.228, 2.236;\ 1.137, 1.418, 2.228, 2.236)$ $\tilde{x}_2^I = (0.071, 0.071, 0.071, 0.071;\ 0, 0.071, 0.071, 0.071)$
2.5	$\alpha = 0.858$ $\lambda = 0.142$ $\alpha - \lambda = 0.715$	$\tilde{x}_1^I = (1.202, 1.418, 2.228, 2.236;\ 1.137, 1.418, 2.228, 2.236)$ $\tilde{x}_2^I = (0.071, 0.071, 0.071, 0.071;\ 0, 0.071, 0.071, 0.071)$
3	$\alpha = 0.905$ $\lambda = 0.095$ $\alpha - \lambda = 0.809$	$\tilde{x}_1^I = (1.202, 1.418, 2.228, 2.236;\ 1.137, 1.418, 2.228, 2.236)$ $\tilde{x}_2^I = (0.071, 0.071, 0.071, 0.071;\ 0, 0.071, 0.071, 0.071)$
5	$\alpha = 0.98$ $\lambda = 0.02$ $\alpha - \lambda = 0.96$	$\tilde{x}_1^I = (1.202, 1.418, 2.228, 2.236;\ 1.137, 1.418, 2.228, 2.236)$ $\tilde{x}_2^I = (0.071, 0.071, 0.071, 0.071;\ 0, 0.071, 0.071, 0.071)$
10	$\alpha = 0.9996$ $\lambda = 0.0003$ $\alpha - \lambda = 0.9992$	$\tilde{x}_1^I = (1.202, 1.418, 2.228, 2.236;\ 1.137, 1.418, 2.228, 2.236)$ $\tilde{x}_2^I = (0.071, 0.071, 0.071, 0.071;\ 0, 0.071, 0.071, 0.071)$

Let X and Y be the percentage of the amount allocated in Stock A and B, respectively. In realistic scenarios, uncertainty and ambiguity cannot be avoided, to address this problem the fraction of the amount to be invested in the stocks is considered as the following TrIFNs

$$\tilde{X}^I = (x_2, x_3, x_4, x_5;\ x_1, x_3, x_4, x_6) \text{ and } \tilde{Y}^I = (y_2, y_3, y_4, y_5;\ y_1, y_3, y_4, y_6).$$

To find the optimal amount to invest in both stocks, the present problem is modelled as an IFLFPP, where the objective is to maximize the ratio of expected return $(\tilde{15}^I \otimes \tilde{X}^I \oplus \tilde{17.5}^I \otimes \tilde{Y}^I)$ to the associated risk $(\tilde{8}^I \otimes \tilde{X}^I \oplus \tilde{12}^I \otimes \tilde{Y}^I)$. Thus the objective function is termed the expected return to risk ratio (ERRR) and the IFLFPP is as follows:

$$\left.\begin{array}{l} \text{Max } \tilde{Z}^I(x) = \frac{\tilde{15}^I \otimes \tilde{X}^I \oplus \tilde{17.5}^I \otimes \tilde{Y}^I}{\tilde{8}^I \otimes \tilde{X}^I \oplus \tilde{12}^I \otimes \tilde{Y}^I} \\ \text{subject to} \\ \tilde{X}^I \oplus \tilde{Y}^I \preceq \tilde{100}^I, \\ \tilde{25}^I \preceq \tilde{X}^I,\ \tilde{25}^I \preceq \tilde{Y}^I, \\ \tilde{X}^I \succeq \tilde{0}^I,\ \tilde{Y}^I \succeq \tilde{0}^I. \end{array}\right\} \text{(P1)}$$

The constraint $\tilde{X}^I \oplus \tilde{Y}^I \preceq \tilde{100}^I$ represents the availability restriction on the total amount of funds that can be used, while $\tilde{25}^I \preceq \tilde{X}^I$ and $\tilde{25}^I \preceq \tilde{Y}^I$ signify that at least 25% of the net amount is invested in both the stocks.

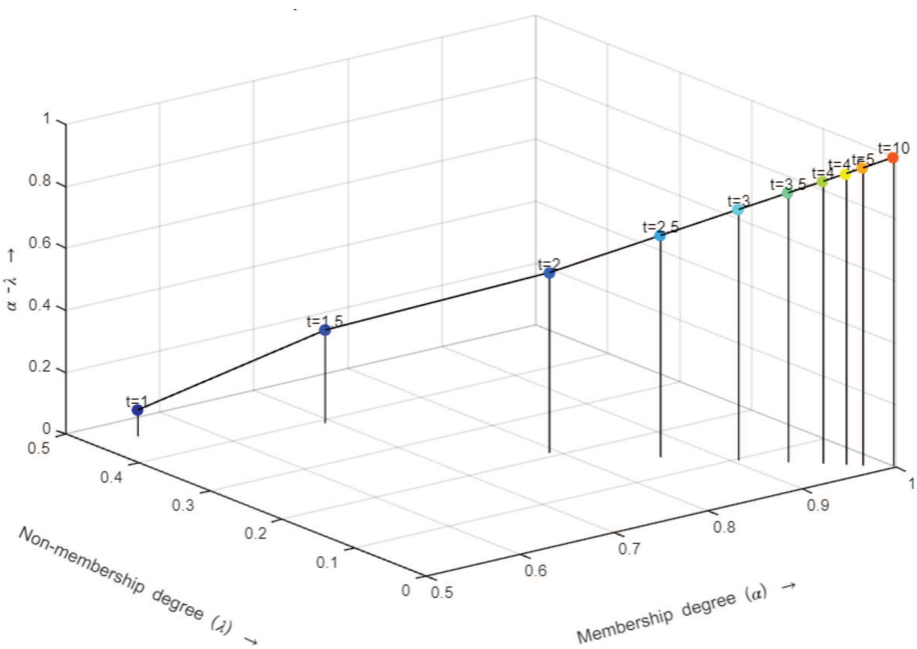

FIGURE 6. Increasing $\alpha - \lambda$ value for the different selections of t.

TABLE 3. The data associated with the stocks as TrIFNs.

Stock	Expected return (in %)	Risk percentage (in %)
Stock A	$\tilde{15}^I = (14, 14.5, 15.5, 16;\ 13, 14.5, 15.5, 17)$	$\tilde{8}^I = (7, 7.5, 8.5, 9;\ 6, 7.5, 8.5, 10)$
Stock B	$\tilde{17.5}^I = (16, 17, 18, 19;\ 15, 17, 18, 20)$	$\tilde{12}^I = (10, 11, 13, 14;\ 9, 11, 13, 15)$

To deal with underlying uncertainty and ambiguity, in some circumstances, the decision-maker can also go over budget as well as under the minimum allocation condition. Therefore to cater for this, we fix the following trapezoidal IF parameters *i.e.* $\tilde{100}^I = (95, 95, 100, 100;\ 90, 95, 100, 105)$ and $\tilde{25}^I = (23, 24, 26, 27;\ 22, 24, 26, 28)$.

Step 1: On substituting the parameters and variables as TrIFNs, the above-mentioned model can be recast as:

$$\left.\begin{aligned}
&\text{Max } \tilde{Z}^I(x) = \frac{\begin{array}{c}(14, 14.5, 15.5, 16;\ 13, 14.5, 15.5, 17) \otimes (x_2, x_3, x_4, x_5;\ x_1, x_3, x_4, x_6) \\ \oplus (16, 17, 18, 19;\ 15, 17, 18, 20) \otimes (y_2, y_3, y_4, y_5;\ y_1, y_3, y_4, y_6)\end{array}}{\begin{array}{c}(7, 7.5, 8.5, 9;\ 6, 7.5, 8.5, 10) \otimes (x_2, x_3, x_4, x_5;\ x_1, x_3, x_4, x_6) \\ \oplus (10, 11, 13, 14;\ 9, 11, 13, 15) \otimes (y_2, y_3, y_4, y_5;\ y_1, y_3, y_4, y_6)\end{array}} \\
&\text{subject to} \\
&(x_2, x_3, x_4, x_5;\ x_1, x_3, x_4, x_6) \oplus (y_2, y_3, y_4, y_5;\ y_1, y_3, y_4, y_6) \preceq (95, 95, 100, 100;\ 90, 95, 100, 105), \\
&(23, 24, 26, 27;\ 22, 24, 26, 28) \preceq (x_2, x_3, x_4, x_5;\ x_1, x_3, x_4, x_6), \\
&(23, 24, 26, 27;\ 22, 24, 26, 28) \preceq (y_2, y_3, y_4, y_5;\ y_1, y_3, y_4, y_6), \\
&(x_2, x_3, x_4, x_5;\ x_1, x_3, x_4, x_6) \succeq \tilde{0}^I,\ (y_2, y_3, y_4, y_5;\ y_1, y_3, y_4, y_6) \succeq \tilde{0}^I.
\end{aligned}\right\} \quad \text{(P2)}$$

Step 2: Upon using the proposed algorithm, the model further reduces into an MOPP as:

$$\left.\begin{aligned}
&\text{Max } \tilde{Z}^I(x) = \big(Z_2(x), Z_3(x), Z_4(x), Z_5(x);\ Z_1(x), Z_2(x), Z_3(x), Z_6(x)\big)\\
&\text{subject to}\\
&x_1 + y_1 \leq 90, x_2 + y_2 \leq 95, x_3 + y_3 \leq 95,\\
&x_4 + y_4 \leq 100, x_5 + y_5 \leq 100, x_6 + y_6 \leq 105,\\
&x_1 \geq 22, x_2 \geq 23, x_3 \geq 24, x_4 \geq 26, x_5 \geq 27, x_6 \geq 28,\\
&y_1 \geq 22, y_2 \geq 23, y_3 \geq 24, y_4 \geq 26, y_5 \geq 27, y_6 \geq 28,\\
&x_1 \geq 0,\ x_2 - x_1 \geq 0,\ x_3 - x_2 \geq 0,\ x_4 - x_3 \geq 0,\ x_5 - x_4 \geq 0,\ x_6 - x_5 \geq 0,\\
&y_1 \geq 0,\ y_2 - y_1 \geq 0,\ y_3 - y_2 \geq 0,\ y_4 - y_3 \geq 0,\ y_5 - y_4 \geq 0,\ y_6 - y_5 \geq 0.
\end{aligned}\right\}\text{(P3)}$$

where $Z_1(x) = \frac{13x_1+15y_1}{10x_6+15y_6}$, $Z_2(x) = \frac{14x_2+16y_2}{9x_5+14y_5}$, $Z_3(x) = \frac{14.5x_3+17y_3}{8.5x_4+13y_4}$, $Z_4(x) = \frac{15.5x_4+18y_4}{7.5x_3+11y_3}$, $Z_5(x) = \frac{16x_5+19y_5}{7x_2+10y_2}$, $Z_6(x) = \frac{17x_6+20y_6}{6x_1+9y_1}$.
Since all the variables are non-negative, it follows that $6x_1 + 9y_1 > 0,\ 7x_2 + 10y_2 > 0,\ 7.5x_3 + 11y_3 > 0,$ $8.5x_4 + 13y_4 > 0$, $9x_5 + 14y_5 > 0$, $10x_6 + 15y_6 > 0$.

Steps 3 and 4: Split the problem into 6 single objective sub-problems and solve the sub-problems individually by using Charnes and Cooper's method as presented in Section 3.1. The optimal values of the sub-problems are as follows:

$$Z_1^* = 1.179,\ Z_2^* = 1.398,\ Z_3^* = 1.560,\ Z_4^* = 3.902,\ Z_5^* = 4.652,\ Z_6^* = 6.109.$$

Steps 5 and 6: For $k = 1, 2, \ldots, 6$, fix $Z_k^U = Z_k^*$. Also, the Z_k^L values come out to be

$$Z_1^L = 0.429,\ Z_2^L = 0.546,\ Z_3^L = 0.642,\ Z_4^L = 1.724,\ Z_5^L = 1.984,\ Z_6^L = 2.364.$$

Step 7: Using the proposed membership and non-membership functions, for some $t \geq 1$, the IF programming model becomes:

$$\left.\begin{array}{ll}
\text{Max} & \alpha - \lambda \\
\text{subject to} & (1-\alpha)^{1/t}(7.5x_6 + 11.25y_6) + 13x_1 + 15y_1 \geq 11.79x_6 + 17.685y_6, \\
& (1-\alpha)^{1/t}(7.668x_5 + 11.928y_5) + 14x_2 + 16y_2 \geq 12.582x_5 + 19.572y_5, \\
& (1-\alpha)^{1/t}(7.803x_4 + 11.934y_4) + 14.5x_3 + 17y_3 \geq 13.26x_4 + 20.28y_4, \\
& (1-\alpha)^{1/t}(16.335x_3 + 23.958y_3) + 15.5x_4 + 18y_4 \geq 29.265x_3 + 42.922y_3, \\
& (1-\alpha)^{1/t}(18.676x_2 + 26.68y_2) + 16x_5 + 19y_5 \geq 32.564x_2 + 46.52y_2, \\
& (1-\alpha)^{1/t}(22.47x_1 + 33.705y_1) + 17x_6 + 20y_6 \geq 36.654x_1 + 54.981y_1, \\
& \lambda^{1/t}(7.5x_6 + 11.25y_6) + 13x_1 + 15y_1 \geq 11.79x_6 + 17.685y_6, \\
& \lambda^{1/t}(7.668x_5 + 11.928y_5) + 14x_2 + 16y_2 \geq 12.582x_5 + 19.572y_5, \\
& \lambda^{1/t}(7.803x_4 + 11.934y_4) + 14.5x_3 + 17y_3 \geq 13.26x_4 + 20.28y_4, \\
& \lambda^{1/t}(16.335x_3 + 23.958y_3) + 15.5x_4 + 18y_4 \geq 29.265x_3 + 42.922y_3, \\
& \lambda^{1/t}(18.676x_2 + 26.68y_2) + 16x_5 + 19y_5 \geq 32.564x_2 + 46.52y_2, \\
& \lambda^{1/t}(22.47x_1 + 33.705y_1) + 17x_6 + 20y_6 \geq 36.654x_1 + 54.981y_1, \\
& \alpha \geq 0,\ \lambda \geq 0,\ \alpha \geq \lambda,\ \alpha + \lambda \leq 1,\ \text{and all the constraints of (P3).}
\end{array}\right\} \text{(P4)}$$

Step 8: The solutions obtained upon solving (P4) for different selections of t are presented in Table 4. Additionally, the corresponding optimal value of the objective as well as the acceptance and rejection degree of the solution is presented.

7.1. Managerial insights

The decision maker can draw the following insights from Table 4 and Figures 7 and 8:

(1) It can be seen from Table 4 that the problem is infeasible for $t = 1$ and 1.15. However, as Table 4 illustrates, this problem becomes feasible only for some significantly bigger values of t. This is consistent with the idea that increasing the value of t increases the feasibility region.

(2) Table 4 displays the solution is identical for each of the selected t values. This shows that the technique is consistent as only the feasibility/infeasibility aspect of the problem is altered. Next, the (P4) model's optimal value against the α and λ values is shown in Figure 7. As the value of t increases, it can be seen that the related solution's acceptance (rejection) degree increases (decreases).

(3) The efficient solution obtained upon solving the feasible version of the problem using some suitable value of t is

$$\tilde{X}^I = (43.174, 43.898, 71.359, 72;\ 43.174, 43.898, 71.359, 75.145),$$
$$\tilde{Y}^I = (23, 24.51, 28, 28;\ 22, 24.51, 28, 28).$$

Consequently, the corresponding ERRR as a TrIFN comes out to be

$$\tilde{Z}^I(\tilde{X}^I, \tilde{Y}^I) = (0.935, 1.086, 2.688, 3.164;\ 0.761, 1.086, 2.688, 4.0203).$$

(4) The ERRR value is presented as a TrIFN graphically in Figure 8. In this figure, one can observe that the membership degree of $\tilde{Z}^I(\tilde{X}^I, \tilde{Y}^I)$ attains the maximum value over the range [1.09, 2.69], whereas the

TABLE 4. Fraction of the amount invested in stock A and B, respectively as TrIFNs.

Parameter t	Optimal value	Solution $\left(\bar{x} = (\tilde{X}^I,\ \tilde{Y}^I)\right)$
1	Infeasible	–
1.15	Infeasible	–
1.3	$\alpha = 0.532$ $\lambda = 0.468$ $\alpha - \lambda = 0.064$	$\tilde{X}^I = (43.174, 43.898, 71.359, 72;\ 43.174, 43.898, 71.359, 75.145)$ $\tilde{Y}^I = (23, 24.51, 28, 28;\ 22, 24.51, 28, 28)$
1.5	$\alpha = 0.584$ $\lambda = 0.417$ $\alpha - \lambda = 0.167$	$\tilde{X}^I = (43.174, 43.898, 71.359, 72;\ 43.174, 43.898, 71.359, 75.145)$ $\tilde{Y}^I = (23, 24.51, 28, 28;\ 22, 24.51, 28, 28)$
2	$\alpha = 0.689$ $\lambda = 0.311$ $\alpha - \lambda = 0.378$	$\tilde{X}^I = (43.174, 43.898, 71.359, 72;\ 43.174, 43.898, 71.359, 75.145)$ $\tilde{Y}^I = (23, 24.51, 28, 28;\ 22, 24.51, 28, 28)$
3	$\alpha = 0.827$ $\lambda = 0.173$ $\alpha - \lambda = 0.654$	$\tilde{X}^I = (43.174, 43.898, 71.359, 72;\ 43.174, 43.898, 71.359, 75.145)$ $\tilde{Y}^I = (23, 24.51, 28, 28;\ 22, 24.51, 28, 28)$
5	$\alpha = 0.946$ $\lambda = 0.054$ $\alpha - \lambda = 0.892$	$\tilde{X}^I = (43.174, 43.898, 71.359, 72;\ 43.174, 43.898, 71.359, 75.145)$ $\tilde{Y}^I = (23, 24.51, 28, 28;\ 22, 24.51, 28, 28)$
10	$\alpha = 0.997$ $\lambda = 0.029$ $\alpha - \lambda = 0.968$	$\tilde{X}^I = (43.174, 43.898, 71.359, 72;\ 43.174, 43.898, 71.359, 75.145)$ $\tilde{Y}^I = (23, 24.51, 28, 28;\ 22, 24.51, 28, 28)$

non-membership degree is minimum for these values of the objective function. This indicates that there is a very high possibility of the ERRR to be somewhere between [1.09, 2.69].

(5) The solutions $\tilde{X}$ and $\tilde{Y}$ are the fraction of the total amount to be invested in Stock A and Stock B, respectively. Therefore, the amount which the decision maker should invest in Stock A is

$$\$(43174, 43898, 71359, 72000;\ 43174, 43898, 71359, 75145)$$

and in Stock B is

$$\$(23000, 24510, 28000, 28000;\ 22000, 24510, 28000, 28000).$$

7.2. Comparative analysis

The above-mentioned portfolio optimization problem is solved using various different approaches and then the results are compared with the solution obtained using the proposed approach. To compare the approaches, we compute the distance of the IF solution from the a selected best solution. For some solution, let trapezoidal IF value of the objective be $\tilde{Z}^I = \left(Z_2(x), Z_3(x), Z_4(x), Z_5(x);\ \ Z_1(x), Z_2(x), Z_3(x), Z_6(x)\right)$ and $\tilde{Z}^I_B = \left(Z^U_2(x), Z^U_3(x), Z^U_4(x), Z^U_5(x);\ \ Z^U_1(x), Z^U_2(x), Z^U_3(x), Z^U_6(x)\right)$ be the best solution. Then,

$$D^+_{(\tilde{Z}^I_B,\ \tilde{Z}^I)} = ||\tilde{Z}^I_B - \tilde{Z}^I||_2 = \sqrt{\sum_{i=1}^{6} |Z^U_i(x) - Z_i(x)|^2}$$

is the distance between $\tilde{Z}^I_B$ and $\tilde{Z}^I$. If we fix $\tilde{Z}^I = \tilde{Z}^I_B$ then we obtain $D^+_{(\tilde{Z}^I_B,\ \tilde{Z}^I)} = 0$, making it the ideal case. Hence, the smaller value of $D^+_{(\tilde{Z}^I_B,\ \tilde{Z}^I)}$ indicates that the solution is close to the best solution. Similarly, let

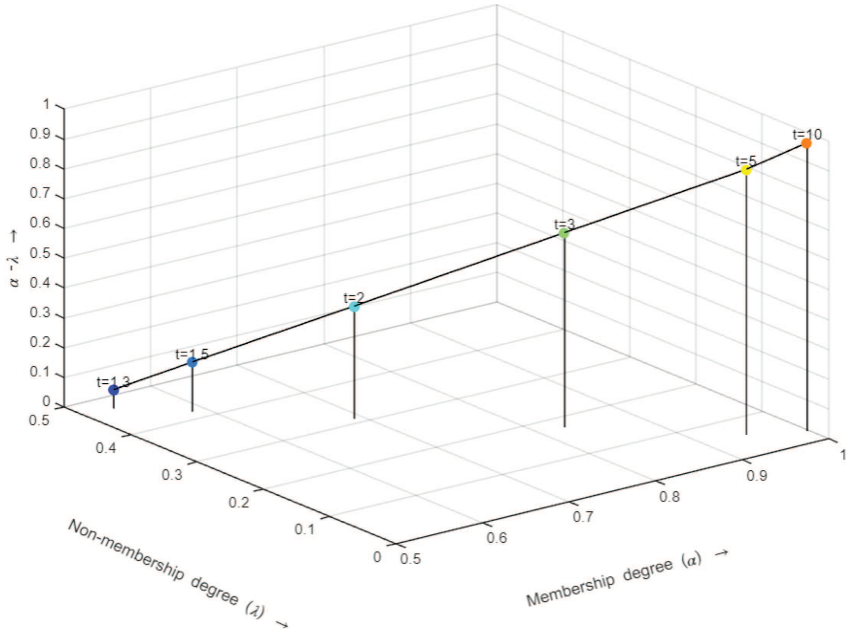

FIGURE 7. Increasing $\alpha - \lambda$ value for the portfolio optimization problem as the parameter t increases.

$\tilde{Z}_W^I = \big(Z_2^L(x), Z_3^L(x), Z_4^L(x), Z_5^L(x);\ Z_1^L(x), Z_2^L(x), Z_3^L(x), Z_6^L(x)\big)$ be the worst/least desired solution and

$$D^-_{(\tilde{Z}_W^I,\ \tilde{Z}^I)} = ||\tilde{Z}_W^I - \tilde{Z}^I||_2 = \sqrt{\sum_{i=1}^{6} |Z_i^L(x) - Z_i(x)|^2}$$

be the distance between $\tilde{Z}_W^I$ and $\tilde{Z}^I$. In this case, a larger D^- value indicate greater distance from the least desired solution and making it more desirable. Individually, D^+ as well as D^- are sufficient for comparison purposes. However, we use the following relativistic comparison factor that amalgamates both D^+ and D^- thus providing a relative form of comparison:

$$D^*_{((\tilde{Z}_B^I,\ \tilde{Z}_W^I),\ \tilde{Z}^I)} = \frac{D^-_{(\tilde{Z}_W^I,\ \tilde{Z}^I)}}{D^+_{(\tilde{Z}_B^I,\ \tilde{Z}^I)} + D^-_{(\tilde{Z}_W^I,\ \tilde{Z}^I)}}.$$

It is easy to check that the value of D^* lies in $[0,1]$. The solution for which D^* value is closer to 1 is highly desirable as in that case D^+ value comes out to be closer to 0, whereas for the solution when D^* is closer to 0 indicates that the solution is farther from the best solution. This way, we can efficiently compare the solutions obtained using different approaches.
Evidently, from Table 5 the D^+, D^- and D^* values obtained are closer to their ideal/desirable values in the case of the proposed approach. Additionally, these values are significantly better using the proposed approach.

Further, to illustrate the efficiency and applicability of the presented technique the (P4) model is solved using different membership and non-membership functions from the literature. The results are presented in Table 6 which shows that the problem is infeasible if the membership and non-membership functions are taken as linear, exponential or hyperbolic. In contrast, using the proposed membership and non-membership degree, the problem can be converted to a feasible one for a suitably large value of t. Furthermore, the comparison of D^* values indicates that we get an improved solution using the proposed functions.

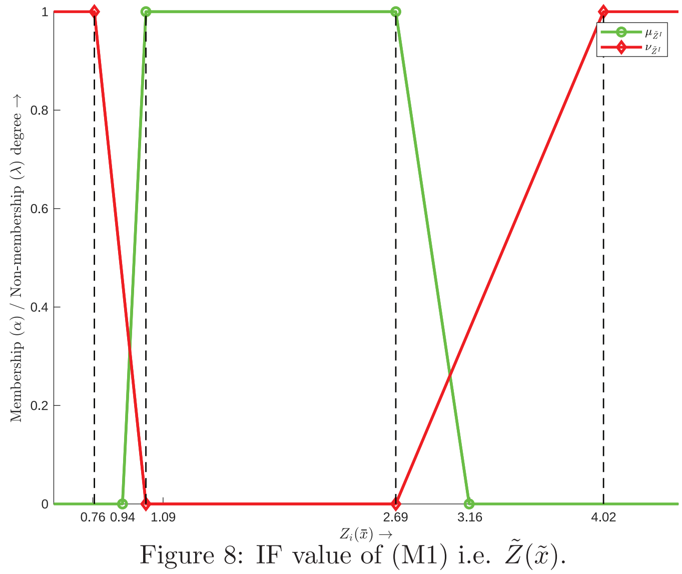

FIGURE 8. IF value of (M1) *i.e.* $\tilde{Z}(\tilde{x})$.

TABLE 5. Comparison of some prominent studies for the presented portfolio optimization problem.

Reference	Solution $\left(\bar{x} = (\tilde{X}^I,\ \tilde{Y}^I)\right)$ and ERRR as a TrIFN $\left(\tilde{Z}(\bar{x})\right)$	D^+	D^-	D^*	Rank
Malik and Gupta [34]	$\tilde{X}^I = (72.5, 72.5, 72.5, 72.5;\ 72.5, 72.5, 72.5, 72.5)$ $\tilde{Y}^I = (25, 25, 25, 25;\ 25, 25, 25, 25)$ $\tilde{Z}^I(\bar{x}) = (1.412, 1.5684, 1.922, 2.158;\ 1.1977, 1.5684, 1.922, 2.625)$	4.72	1.53	0.25	2
Singh and Yadav [47]	$\tilde{X}^I = (62.76, 62.9, 63.142, 63.142;\ 62, 62.9, 63.142, 63.142)$ $\tilde{Y}^I = (28, 28, 28, 28;\ 28, 28, 28, 28)$ $\tilde{Z}^I(\bar{x}) = (1.38, 1.54, 1.9, 2.14;\ 1.17, 1.54, 1.9, 2.62)$	4.74	1.48	0.24	3
Malik and Gupta [33]	$\tilde{X}^I = (27.99, 27.99, 28, 28;\ 27.99, 27.99, 28, 28)$ $\tilde{Y}^I = (28, 28, 28, 28;\ 28, 28, 28, 28)$ $\tilde{Z}^I(\bar{x}) = (1.3043, 1.465, 1.8108, 2.0588;\ 1.1199, 1.465, 18108, 2.4667)$	4.94	1.32	0.21	4
Proposed approach	$\tilde{X}^I = (43.174, 43.898, 71.359, 72;\ 43.174, 43.898, 71.359, 75.145)$ $\tilde{Y}^I = (23, 24.51, 28, 28;\ 22, 24.51, 28, 28)$ $\tilde{Z}^I(\bar{x}) = (0.935, 1.086, 2.688, 3.164;\ 0.761, 1.086, 2.688, 4.0203)$	**2.94**	**2.35**	**0.44**	1

8. CONCLUSION AND FUTURE SCOPE

In this paper, Charnes and Cooper [10] technique and IF programming approach for MOPPs are discussed to solve fully IFLFPPs. This article highlights a novel approach for IFLFPPs where all the coefficients and parameters are taken as TrIFNs. Initially, the problem is transformed into an equivalent model with multiple objective functions using the arithmetic operations defined in Section 2. Next, the optimal solution for each objective function is found. Then, the IF programming approach is used to obtain the efficient solution by

TABLE 6. Solution comparison for different membership and non-membership functions.

Reference	Nature of membership and non-membership function	Parameter (same for all membership/non-membership functions)	ERRR as TrIFN $(\tilde{Z}^I(x))$	D^+	D^-	D^*	Rank
Mahajan and Gupta [30], Mahajan and Gupta [31]	Linear	–	Infeasible	–	–	–	–
	Exponential	–	Infeasible	–	–	–	–
	Hyperbolic	–	Infeasible	–	–	–	–
Malik and Gupta [33]	Linear	–	Infeasible	–	–	–	–
	Exponential	–	Infeasible	–	–	–	–
Mahajan and Gupta [30], Malik and Gupta [33]	Linear - Optimistic case	Tolerance value = 10	$(1.056, 1.403, 2.138, 2.838; 0.728, 1.403, 2.138, 4.15)$	3.255	1.859	0.364	2
	Exponential - Optimistic case	Tolerance value = 9	$(1.054, 1.402, 2.138, 2.833; 0.735, 1.402, 2.138, 4.13)$	3.267	1.843	0.361	3
Malik and Gupta [33], Mahajan and Gupta [30]	Linear - Pessimistic case	–	Infeasible	–	–	–	–
	Exponential - Pessimistic case	–	Infeasible	–	–	–	–
Mahajan and Gupta [30]	Hyperbolic - Optimistic case	Tolerance value = 8.5	$(1.055, 1.402, 2.137, 2.83; 0.737, 1.402, 2.138, 4.12)$	3.274	1.834	0.359	4
	Hyperbolic - Pessimistic case	–	Infeasible	–	–	–	–
El Sayed and Abo-Sinna [21]	Linear	–	Infeasible	–	–	–	–
	Exponential	–	Infeasible	–	–	–	–
	Parabolic	–	Infeasible	–	–	–	–
ine Proposed approach	Non-linear	$t \geq 1.3$	$(0.935, 1.086, 2.688, 3.164; 0.761, 1.086, 2.688, 4.0203)$	**2.94**	**2.35**	**0.44**	1

proposing the least acceptable value to be taken as $\min\{Z_k(x)\}$ subject to the constraints of the original problem, for $k \in \{1, 2, \ldots, K\}$.

Furthermore, a novel parameterized family of membership and non-membership functions is proposed, highlighting its salient features over existing ones in Section 5. Next, several theorems are established to show the validity of the solutions obtained. Later, a numerical example is solved to illustrate the proposed approach. Afterwards, a real-world portfolio optimization problem is framed and solved using the proposed IF programming. Later, Tables 5 and 6 are used to present detailed comparative analysis with existing studies as well as some predominantly used membership and non-membership functions. The results indicate that the proposed study outperforms existing techniques.

The MOPP considered in this paper has all the objectives of maximization type. In future, the development of novel membership and non-membership functions with mixed kinds of objectives would be worthwhile. Additionally, an optimistic/pessimistic approach can be explored for such problems. Furthermore, a similar approach can be developed for type-2 or interval-valued IF sets. In another direction, the concept of (α, β)-cut can also be incorporated to examine the outcomes.

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Conflicts of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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