

FORMAL SOLUTION FOR THE GENERALIZED MUTUAL EXCLUSION CONSTRAINTS PROBLEM IN A CLASS OF TIMED PETRI NETS

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Abstract. In this paper, we propose an analytical solution to the problem of Generalized Mutual Exclusion Constraints (GMECs) in a Network of Timed Event Graphs (NTEGs). The contribution of the paper lies in the development of a method to design control laws that satisfy these constraints across different paths in the graphs. Using Min-Plus dioid algebra, we employ algebraic techniques to express GMECs as weighted inequalities. The solution involves translating the constraints into Min-Plus linear equations that describe the behavior of NTEGs. We provide sufficient conditions for the existence of causal control laws, considering both the initial marking of the NTEGs and the parameters of the GMECs. To demonstrate the practical application of the approach, we include a case study that illustrates the effectiveness of the proposed control strategy.

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1. INTRODUCTION

Discrete Event Systems (DESs) can be found in a wide range applications, including, chemical treatments [1], flexible manufacturing workshops [2], and transportation networks. These systems are dynamic structures with discrete state spaces [3], whose proper control is essential to ensure safe and efficient operation. Indeed, the complexity of DESs imposes several restrictions (real time, state constraints, etc.) that must be addressed to improve the performance of production systems. Failure to consider these constraints can lead to undesirable situations, such as deadlocks or resource conflicts. To avoid such undesirable states, some specifications must be included in the synthesis of control laws.

Numerous notable papers in the literature have attempted to tackle the control problem under various DES constraints. Authors such as [4] concentrated on meeting time restrictions in sensitive areas by computing control laws using Max-Plus formalisms. Others exploited the concept of (A, B) -invariance [5] to create a feedback controller for timed Petri Nets (PNs).

Several research have handled capacity specifications that are defined by marking constraints or forbidden states [6]. These constraints provide a logical framework to describe the concurrent use of finite resources in manufacturing systems (machines, robots, etc.) or transportation networks (stations, paths, etc.).

Keywords. Discrete event systems, networks of timed event graphs, Min-Plus algebra, generalized mutual exclusion constraints, control laws.

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This study builds on the combined use of Min-Plus algebra and Networks of Timed Event Graphs (NTEGs). Such paradigms have been used to solve other control problems including tracking and disturbance rejection [7], just-in-time control [8]. In this context, we focus on the control synthesis problem for Generalized Mutual Exclusion Constraints (GMECs), which are specifications expressed as linear constraints that restrict a weighted sum of tokens in a set of places and/or paths. By associating weighting coefficients with place markings, GMECs reduce the number of control actions or additional resources required. Researchers in [9] used simple linear algebraic notions as they have resorted to the theory of regions when dealing with the problem of the automated manufacturing systems supervisory control. We also highlight the recent approach [10] where authors deployed labeled PNs to build monitor functions that solve the control problem of DESs subject to deterministic GMECs. Works such as [11] used the P -invariant property to create monitors based on place invariants. For fully observable and controllable systems, this technique offers a very easy and optimal solution.

Despite the importance of the time factor in modeling DESs, multiple methodologies do not explicitly account for durations. For instance, Atli *et al.* [12] addressed forbidden state problems in timed place marked graphs *via* supervisory control. Recent research [13] emphasized the control problem of time PNs with uncontrollable transitions to design supervisions that prevent the occurrence of forbidden states by enforcing GMECs. This strategy would decrease the system performance by resulting in deadlock states. Afterwards, Li *et al.* [14] expanded the previous work by examining the use of GMECs and deadlock freeness. Furthermore, Li *et al.* [15] suggested a novel study for enforcing GMECs on time PNs, where each constraint is associated with a restricted time interval. However, applying these techniques to more complex, real-world systems can result in combinatorial explosion.

The approach developed in this work incorporates the time parameter and avoids combinatorial state explosion. While prior studies [13–15] focused on enforcing GMECs in arbitrary timed PNs, this paper focuses on NTEGs, a subclass of timed PNs connected with dioid algebra. These models depict synchronization, delays, and parallel events but do not handle conflicts or resource sharing. TEGs are advantageous because their initial behavior can be described using linear equations in dioid algebra, enabling the development of general Max-Plus and Min-Plus linear system theories. In [16], authors were devised to enforce marking constraints in partially observable TEGs. Reference [17] extended this to feedback control ensuring marking constraints including disturbances, and Bouazza *et al.* [18] expanded it to partially observable timed event graphs.

Yet, these methods primarily dealt with specific marking constraints. In a related context, Rajah *et al.* [19] explored more general constraints known as Generalized Marking Constraints (GMCs), associating coefficients to places or paths in fully observable TEGs. However, this technique considered only maximum tokens enforced on places connected by paths. The study in [20] analyzed NTEGs under Mutual Exclusion Constraints (MECs) where controlled places had different paths. Recent work [21] generalized this to GMECs, while [22] incorporated partially observable NTEGs. Nevertheless, these purviews are limited because constraints are enforced only on individual places.

Extending [21], the present work develops control laws that guarantee compliance with GMECs applied to paths of fully observable NTEGs. Constrained paths may vary in length, meaning the number of places in each restricted path differs between TEGs. Previous research primarily focused on enforcing constraints on individual NTEG places, which limits practical applicability. We generalize these results to synthesize state feedback control strategies for enforcing GMECs on sets of paths, which is particularly useful for complex systems with numerous GMECs.

The significant scientific contributions of the current study are:

- (1) A mathematical approach based on Min-Plus algebra to formalize and solve the GMECs problem.
- (2) Design of observable state feedback to ensure compliance with generalized mutual exclusion constraints.
- (3) Graphical representation of the control laws using control places connected to the initial NTEGs to prevent the violation of GMECs.
- (4) Analytical modeling of NTEGs dynamic behavior and translation of GMEC specifications into Min-Plus equations and inequalities.
- (5) Application to a realistic manufacturing workshop with generalized mutual exclusion constraint is applied.

Compared to the literature, the current control strategies offer the following advantages:

- (i) They are comprehensive and avoid combinatorial state explosion.
- (ii) They do not require specialized hardware or extensive computational resources to compute control laws.
- (iii) They are feasible to implement and program without significant memory requirements.

The remainder of the paper is organized as follows: Section 2 reviews TEG concepts and Min-Plus properties. Section 3 formulates the GMECs problem. Section 4 presents the new control mechanism. Section 5 illustrates the theoretical results through a case study. Section 6 concludes with discussion and future work prospects.

2. PRELIMINARIES

2.1. Min-Plus basics

A dioid is a set D with an internal composition operation (\oplus) that is associative, commutative and has a neutral element, denoted by ε so: $\forall a \in D, a \oplus \varepsilon = \varepsilon \oplus a = a$, with the neutral element $\varepsilon = +\infty$.

The second operation (\otimes) is associative, right and left distributive over (\oplus) . It has a neutral element, defined by e with $e = 0$, and an absorbing element supplied by ε i.e.: $\forall a \in D, a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$. It should be observed that (\oplus) is idempotent, that is: $\forall a \in D, a \oplus a = a$. If the second operation (\otimes) is commutative, a dioid is said to be commutative.

A complete dioid is a set where (\otimes) is right and left distributive over infinite sums, which means: $\forall b \in D, \forall A \subseteq D : b \otimes (\oplus_{a \in A} a) = \oplus_{a \in A} (b \otimes a)$ and $(\oplus_{a \in A} a) \otimes b = \oplus_{a \in A} (a \otimes b)$. In the present paper, we explore the Min-Plus commutative dioid, generally known as Min-Plus algebra and given by $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \oplus, \otimes)$. In this situation, the law (\oplus) corresponds to the (min) application, i.e. $\forall a, b \in \mathbb{R}_{\min} : a \oplus b = \min(a, b)$. The second law (\otimes) is analogous to the (classical addition), namely: $\forall a, b \in \mathbb{R}_{\min} : a \otimes b = a + b$.

Considering a matrix dioid, we represent the set of $n \times m$ matrices for $n, m \in \mathbb{N}$ over \mathbb{R}_{\min} by $\mathbb{R}_{\min}^{n \times m}$ and the following operations are stated, $\forall A, B \in \mathbb{R}_{\min}^{n \times m}, \forall E \in \mathbb{R}_{\min}^{m \times p}$:

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}, \quad \forall i = 1, \dots, n \quad \forall j = 1, \dots, m \quad (1)$$

$$(A \otimes E)_{ij} = \oplus_{k=1}^m A_{ik} \otimes E_{kj} \quad \forall i = 1, \dots, n \quad \forall k = 1, \dots, p. \quad (2)$$

Theorem 1. Reference [23] demonstrated that the implicit equation $X = A \otimes X \oplus B$, defined on a complete dioid D , has as a solution $X = A^* \otimes B$. Take note that $A^* = \oplus_{i \in \mathbb{N}} A^i$, where A^* is the Kleene star of A and $A^i = A^{i-1} \otimes A$.

2.2. Timed event graphs

This section outlines the fundamentals of PNs and TEGs. We note that PN is a 4-tuple $Q = (P, T, A, m_0)$, where P and T are respectively the finite sets of places and transitions, $A \subseteq (P \times T) \cup (T \times P)$ is a finite set of arcs relying places to transitions and *vice versa* and m_0 is a vector whose i th component represents the initial marking of the place $p_i \in P$. A timed PN is defined as a couple $V = (Q, Tempo)$, where *tempo* is the delay associated with places and/or transitions.

A TEG is a type of PN in which each place has exactly one upstream and downstream transitions. These models aim to represent synchronization, delays and parallel occurrences, but not conflicts or resource sharing. A time delay τ_{ij} and a number of tokens m_{ij} create a place p_{ij} that links the transition $t_j \in T$ to t_i .

To derive the dynamic behavior of a given TEG, each transition t_i is coupled with a counter function $\theta_i(t)$. In fact, this counter represents the cumulated number of transition t_i firings at instant t . $u(t)$ denotes the counter function of the control transition t_u , such that $u(t) \in \mathbb{R}_{\min}^m$ is a vector in which m defines the number of control transitions without any further input transitions.

Therefore, the dynamic behavior of a TEG is expressed by the equation below in Min-Plus algebra:

$$\theta(t) = \oplus_{\tau=0}^{\tau_{\max}} (A_{\tau} \otimes \theta(t - \tau) \oplus B_{\tau} \otimes u(t - \tau)). \quad (3)$$

$\theta(t) \in \bar{\mathbb{R}}_{\min}^n$ is a vector where n is the number of transitions with at least one upstream place. $A_\tau \in \bar{\mathbb{R}}_{\min}^{n \times n}$ is the matrix whose terms are defined by $A_{\tau,ij}$ with entries $A_{\tau,ij}$ equal to e if the place p_{ij} is not marked with tokens, m_{ij} the number of the initial marking of the place p_{ij} , if this place exists, and ε else. $B_\tau \in \bar{\mathbb{R}}_{\min}^{n \times m}$ relates to the initial marking of the exit places of control transitions.

The TEG behavior is expressed implicitly in (3). We get the following equation to represent the explicit form:

$$\theta(t) = \oplus_{\tau > 0} (A_0^* \otimes A_\tau \otimes \theta(t - \tau) \oplus A_0^* \otimes B_\tau \otimes u(t - \tau)). \quad (4)$$

A_0^* is the Kleene star of A_0 (mentioned in Thm. 1).

2.3. Networks of timed event graphs

We consider a NTEGs [22] made up of a fixed number (L) of separated TEGs, with each TEG is indexed by (TEG^{*l*}), where $l \in \{1, \dots, L\}$. Figure 1 depicts a simple example of a NTEGs where places are temporized and identified with weighting coefficients. We shall use the following notations and terminology in this paper:

- P^l represents the finite set of places in each TEG and T^l designs the set of transitions.
- The control transition t_u^l manage the timing and sequencing of events in the TEGs, ensuring that transitions fire at the correct times and in the correct order.
- In NTEGs, a constrained path denoted by ρ^l , refers to a sequence of events (places) that are restricted by certain conditions or constraints. Noting that ρ^l connects $t_{j_1}^l$ to $t_{i_w}^l$. Both transitions indicate the path upstream and downstream transitions, respectively. W^l denotes the number of places belonging to ρ^l .
- We designate α^l the path that relates the control transition t_u^l to the upstream transition $t_{j_1}^l$ of the constrained path ρ^l with τ_{α^l} denoting the time delay of α^l .
- Since we are talking about GMECs, each constrained place belonging to the path ρ^l is associated with a weighting coefficient equal or greater than 1. This coefficient is designed by μ_w^l .

2.4. Networks of timed event graphs linear Min-Plus equations

The present paper investigates NTEGs composed of multiple independent TEGs represented by (TEG^{*l*}), where $l \in \{1, \dots, L\}$. Thus, the explicit equation of the dynamic behavior is given by:

$$\theta^l(t) = \oplus_{\tau_l > 0} \left(A_0^{l*} \otimes A_{\tau_l} \otimes \theta^l(t - \tau_l) \oplus A_0^{l*} \otimes B_{\tau_l} \otimes u^l(t - \tau_l) \right). \quad (5)$$

A_0^{l*} is the Kleene star of A_0^l .

Similarly to the case of typical linear systems, if the temporizations in the TEG are commensurable to a single time delay, (5) can be translated to state space form. As a result, the delay of each place must not exceed 1. To do this, we extend the initial TEG by timing all places to 0 or 1 and inserting n_{1_l} intermediate transitions. These added transitions are correlated with counters that make up the components of a vector $\bar{\theta}^l(t) \in \bar{\mathbb{R}}_{\min}^{n_{1_l}}$, with $x^l(t)$ representing the resulting extended state vector:

$$x^l(t) = \begin{bmatrix} \theta^l(t) \\ \bar{\theta}^l(t) \end{bmatrix}. \quad (6)$$

Consequently, the state equation can be written formally as the following:

$$x^l(t) = A^l \otimes x^l(t - 1) \oplus B^l \otimes u^l(t). \quad (7)$$

With:

$$A^l = A_0^{l*} \otimes A_1^l, \quad (7a)$$

and

$$B^l = A_0^{l*} \otimes B_0^l. \quad (7b)$$

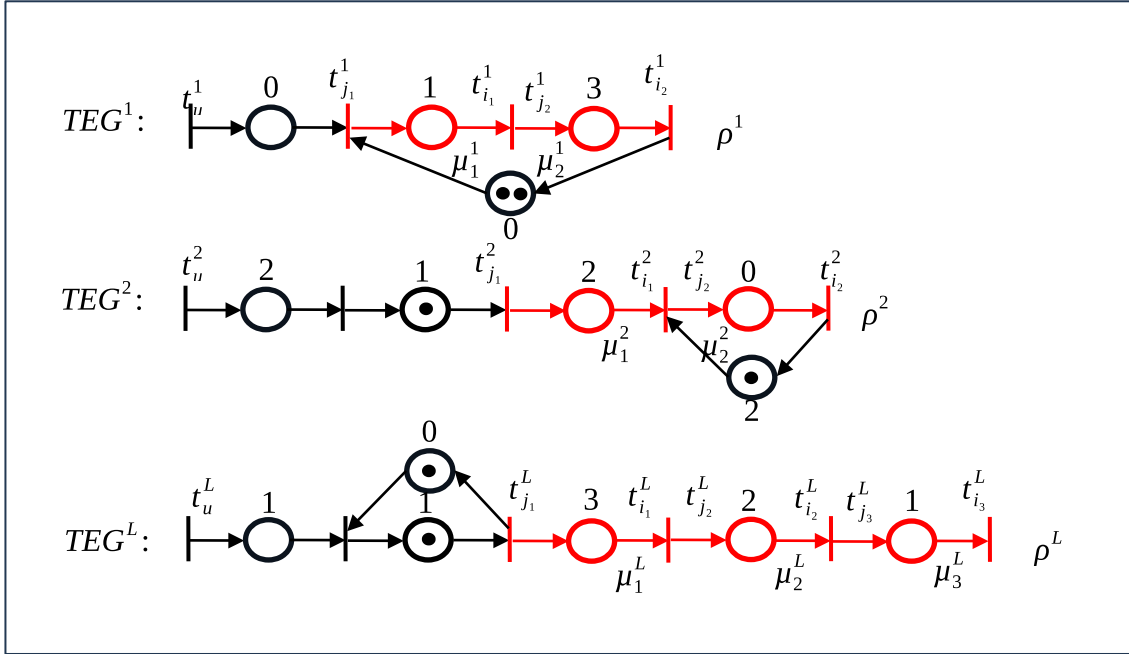


FIGURE 1. An example of a NTEGs.

In equation (7), we make τ_l substitutions with ($\tau_l \geq 1$), which is easily demonstrated by recurrence to yield the expression below:

$$x^l(t) = A^{\tau_l} \otimes x^l(t - \tau_l) \oplus [\oplus_{k_l=0}^{\tau_l-1} A^{k_l} \otimes B^l \otimes u^l(t - k_l)]. \quad (8)$$

3. PROBLEM STATEMENT

Mutual exclusion constraints (MECs) or the so-called (unweighted GMECs) are employed to avoid shared system resources from being utilized concurrently. A MEC is a specification that limits the sum of tokens included in some places. We explored GMECs [24] thoroughly from the standpoint of MECs in general. It is a condition which restricts the weighted sum of markings in certain TEGs places or paths. This sort of constraint allows the translation of mutual exclusion into the use of common resources shared by multiple processes. This can be accomplished by implementing a set of linear weighted inequalities. The relevant constraint is then expressed as below:

$$\sum_{l=1}^L M_{\rho}^l(t) \leq b. \quad (9)$$

L designates the number of TEGs in a NTEGs. ρ^l identifies the constrained path that contains W^l places and each place of them is indexed by ω ranging from 1 to W^l . b is the number of tokens not to be exceeded in the path ρ^l and $M_{\rho}^l(t)$ defines the marking of ρ^l at an instant t , with:

$$M_{\rho}^l(t) = \sum_{\omega=1}^{W^l} \mu_{\omega}^l \times M_{i_{\omega}j_{\omega}}^l(t). \quad (10)$$

μ_ω^l is the weighting coefficient associated with each place $p_{i_\omega j_\omega}^l$ belonging to the path ρ^l and $M_{i_\omega j_\omega}^l(t)$ is the marking of each constrained place existing in ρ^l , with:

$$M_{i_\omega j_\omega}^l(t) = x_{j_\omega}^l(t) - x_{i_\omega}^l(t) + M_{0_{i_\omega j_\omega}}^l. \quad (11)$$

$t_{j_\omega}^l$ and $t_{i_\omega}^l$ which are respectively the input and output transitions of each place $p_{i_\omega j_\omega}^l$ are associated with the function counters $x_{j_\omega}^l(t)$ and $x_{i_\omega}^l(t)$. $M_{0_{i_\omega j_\omega}}^l$ is the initial marking of $p_{i_\omega j_\omega}^l$.

We replace now ($M_{i_\omega j_\omega}^l(t)$) in equation (10) with its expression in (11) to obtain:

$$M_\rho^l(t) = \sum_{\omega=1}^{W^l} \mu_\omega^l \times \left(x_{j_\omega}^l(t) - x_{i_\omega}^l(t) + M_{0_{i_\omega j_\omega}}^l \right). \quad (12)$$

Subsequently, to simplify and develop the expression, we expand the terms in the sum in equation (12). This leads to the detailed form in equation (13), where the terms are separated and grouped to clarify the contribution of each component in the summation:

$$M_\rho^l(t) = \sum_{\omega=1}^{W^l} \mu_\omega^l \times x_{j_\omega}^l(t) - \sum_{\omega=1}^{W^l} \mu_\omega^l \times x_{i_\omega}^l(t) + \sum_{\omega=1}^{W^l} \mu_\omega^l \times M_{0_{i_\omega j_\omega}}^l. \quad (13)$$

In this step, we introduce the fact that for all $l \in \{1, \dots, L\}$ and $\forall \omega \in \{2, \dots, W^l\}$, the following equality holds:

$$x_{j_\omega}^l(t) = x_{i_{\omega-1}}^l(t). \quad (14)$$

The equation (14) is illustrated with a simple example in Figure 1. Using this, we can rewrite the expression for $M_\rho^l(t)$ as shown in equation (15):

$$M_\rho^l(t) = \mu_1^l \times x_{j_1}^l(t) + \sum_{\omega=2}^{W^l} \mu_\omega^l \times x_{j_\omega}^l(t) - \sum_{\omega=2}^{W^l} \mu_{(\omega-1)}^l \times x_{i_{(\omega-1)}}^l(t) - \mu_{W^l}^l \times x_{i_{W^l}}^l(t) + M_{o_\rho}^l. \quad (15)$$

With:

$$M_{o_\rho}^l = \sum_{\omega=1}^{W^l} \mu_\omega^l \times M_{0_{i_\omega j_\omega}}^l. \quad (15a)$$

Then, equation (15) is simplified to the following form, which shows a clearer relationship between the terms:

$$M_\rho^l(t) = \mu_1^l \times x_{j_1}^l(t) + \sum_{\omega=2}^{W^l} \mu_\omega^l \times x_{j_\omega}^l(t) - \sum_{\omega=2}^{W^l} \mu_{(\omega-1)}^l \times x_{j_\omega}^l(t) - \mu_{W^l}^l \times x_{i_{W^l}}^l(t) + M_{o_\rho}^l. \quad (16)$$

Finally, we simplify the expression (16) even further by recognizing the difference between successive μ_ω^l terms, leading to the final form of the equation:

$$M_\rho^l(t) = \mu_1^l \times x_{j_1}^l(t) + \sum_{\omega=2}^{W^l} \left| \mu_\omega^l - \mu_{(\omega-1)}^l \right| \times x_{j_\omega}^l(t) - \mu_{W^l}^l \times x_{i_{W^l}}^l(t) + M_{o_\rho}^l. \quad (17)$$

$x_{j_1}^l(t)$, $x_{j_\omega}^l(t)$ and $x_{i_{W^l}}^l(t)$ respectively denote the counter functions related to the input, intermediaries and output transitions of the path ρ^l . μ_1^l , μ_ω^l and $\mu_{W^l}^l$ are the coefficients respectively associated with the first, intermediates and last places in ρ^l .

After that, equation (17) is integrated into (9):

$$\sum_{l=1}^L \left(\mu_1^l \times x_{j_1}^l(t) + \sum_{\omega=2}^{W^l} \left| \mu_\omega^l - \mu_{(\omega-1)}^l \right| \times x_{j_\omega}^l(t) - \mu_W^l \times x_{i_W}^l(t) + M_{o_\rho}^l \right) \leq b. \quad (18)$$

Next, we move the third and fourth terms ($\sum_{l=1}^L \mu_W^l \times x_{i_W}^l(t)$) and ($\sum_{l=1}^L M_{o_\rho}^l$) of inequality (18) to the other side, we get:

$$\sum_{l=1}^L \mu_1^l \times x_{j_1}^l(t) + \sum_{l=1}^L \left(\sum_{\omega=2}^{W^l} \left| \mu_\omega^l - \mu_{(\omega-1)}^l \right| \times x_{j_\omega}^l(t) \right) \leq \left(b - \sum_{l=1}^L M_{o_\rho}^l \right) + \sum_{l=1}^L \mu_W^l \times x_{i_W}^l(t). \quad (19)$$

The inequality (19) can be transformed into the following expression using Min-Plus algebra:

$$\otimes_{l=1}^L \left[\left(x_{j_1}^l(t) \right)^{\otimes \mu_1^l} \otimes \left(\otimes_{\omega=2}^{W^l} \left(x_{j_\omega}^l(t) \right)^{\otimes \left| \mu_\omega^l - \mu_{(\omega-1)}^l \right|} \right) \right] \leq \otimes_{l=1}^L \left[\left(b - M_{o_\rho}^l \right) \otimes \left(x_{i_W}^l(t) \right)^{\otimes \mu_W^l} \right]. \quad (20)$$

Remark 1. The classical multiplication (\times) corresponds to a power in Min-Plus algebra, *i.e.* $\forall h \in \mathbb{Z}$ and $\forall J \in \mathbb{R}_{\min} : J^{\otimes h} = h \times J$. In Min-Plus algebra, we add the symbol (\otimes) to the power to distinguish between the multiplication and the power [21].

4. CALCULATION OF CONTROL LAWS

We provide an algebraic solution to the control problem of DESs subject to GMECs. The idea is to synthesize feedback control laws to ensure that the related specifications are met. We start with a single constraint and then we extend the technique to satisfy a set of GMECs. Yet, some prerequisites must be satisfied to demonstrate the efficiency of this method.

Taking ($\phi_l = \tau_l$) in equation (8) of Section 2.4, it yields:

$$x_i^l(t) = \left[\oplus_{r=1}^{n_l} A_{ir}^{\phi_l} \otimes x_r^l(t - \phi_l) \right] \oplus \left[\oplus_{k_l=0}^{\phi_l-1} (A^{k_l} \otimes B^l)_i \otimes u^l(t - k_l) \right] \quad (21)$$

where $\phi_l \geq 1$ and $A_{ir}^{\phi_l}$ is the i th row of the matrix A^{ϕ_l} . This allows us to express the equation (21) in a more compact matrix form:

$$x_i^l(t) = \left[A^{\phi_l}(i, :) \otimes x^l(t - \phi_l) \right] \oplus \left[\oplus_{k_l=0}^{\phi_l-1} (A^{k_l} \otimes B^l)_i \otimes u^l(t - k_l) \right]. \quad (22)$$

A^{ϕ_l} and A^{k_l} are respectively the state matrix of the TEG ^{l} to the power ϕ_l and k_l . B^l corresponds to the initial marking of the exit places of the source transitions t_u^l .

We are mainly interested in the counter associated with the output transition $t_{i_W}^l$ of the constrained path ρ^l :

$$x_{i_W}^l(t) = \left[A^{\phi_l}(i_W, :) \otimes x^l(t - \phi_l) \right] \oplus \left[\oplus_{k_l=0}^{\phi_l-1} (A^{k_l} \otimes B^l)_{i_W} \otimes u^l(t - k_l) \right]. \quad (23)$$

We classically multiply (power in Min-Plus algebra) equation (23) by (μ_W^l) :

$$\left(x_{i_W}^l(t) \right)^{\otimes \mu_W^l} = \left[\left(A^{\phi_l}(i_W, :) \otimes x^l(t - \phi_l) \right) \oplus \left(\oplus_{k_l=0}^{\phi_l-1} (A^{k_l} \otimes B^l)_{i_W} \otimes u^l(t - k_l) \right) \right]^{\otimes \mu_W^l}. \quad (24)$$

We add the sum ($\otimes_{l=1}^L$) in (24):

$$\otimes_{l=1}^L \left(x_{i_W}^l(t) \right)^{\otimes \mu_W^l} = \otimes_{l=1}^L \left[\left(A^{\phi_l}(i_W, :) \otimes x^l(t - \phi_l) \right) \oplus \left(\oplus_{k_l=0}^{\phi_l-1} (A^{k_l} \otimes B^l)_{i_W} \otimes u^l(t - k_l) \right) \right]^{\otimes \mu_W^l}. \quad (25)$$

Assumption 1. For each TEG, we suppose that there is at least one path α^l connecting the control transition t_u^l to the input transition $t_{j_1}^l$ of the restricted path ρ^l . This condition ensures that there is a connection between the control transition and the input transition of the restricted path. This connectivity is crucial for appropriately defining the control laws. In TEG networks, only the transitions t_u^l are controllable and can be acted upon to guarantee compliance with constraints. In general, the manufacturing systems are often cyclical, and their models are closely related. In these practical contexts, we often find paths linking controllable inputs to system states.

The path time delay is denoted by τ_{α_l} . This definition results in:

$$x_{j_1}^l(t) \leq (A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \otimes u^l(t - \tau_{\alpha_l}). \quad (26)$$

$A^{\tau_{\alpha_l}}$ is the state matrix of the TEG^{*l*} to the power τ_{α_l} .

We apply the Min-Plus power operation by multiplying Inequality (26) by (μ_1^l) , leading to (27):

$$(x_{j_1}^l(t))^{\otimes \mu_1^l} \leq \left[(A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \otimes u^l(t - \tau_{\alpha_l}) \right]^{\otimes \mu_1^l}. \quad (27)$$

Extending this operation to all L elements of $(x_{j_1}^l(t))^{\otimes \mu_1^l}$, we obtain the product of terms involving $(x_{j_1}^l(t))^{\otimes \mu_1^l}$, resulting in the inequality (28):

$$\otimes_{l=1}^L (x_{j_1}^l(t))^{\otimes \mu_1^l} \leq \otimes_{l=1}^L \left[(A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \otimes u^l(t - \tau_{\alpha_l}) \right]^{\otimes \mu_1^l}. \quad (28)$$

According to what regards intermediate transitions, we have the following expression:

$$\otimes_{\omega=2}^{W^l} (x_{j_\omega}^l(t))^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \leq \otimes_{\omega=2}^{W^l} \left[(A^{\tau_{\alpha_{l\omega}}} \otimes B^l)_{j_\omega} \otimes u^l(t - \tau_{\alpha_{l\omega}}) \right]^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|}. \quad (29)$$

Since we know that $u^l(t)$ is an increasing function, i.e. $u^l(t) \geq u^l(t-1) \geq \dots \geq u^l(t-n_l)$, we can deduce that $\forall \omega \in [2, W^l]$, $u^l(t - \tau_{\alpha_{l\omega}}) \leq u^l(t - \tau_{\alpha_l})$. $\tau_{\alpha_{l\omega}}$ designates the delay of the path connecting the control transition t_u^l to the transition $t_{j_\omega}^l$ with $(\tau_{\alpha_{l\omega}} \geq \tau_{\alpha_l})$. The expression (30) shows the inequality with Min-Plus algebra, involving a product over W^l elements for each ω :

$$\left(\otimes_{\omega=2}^{W^l} (x_{j_\omega}^l(t))^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \leq \otimes_{\omega=2}^{W^l} \left[(A^{\tau_{\alpha_{l\omega}}} \otimes B^l)_{j_\omega} \otimes u^l(t - \tau_{\alpha_l}) \right]^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|}. \quad (30)$$

In what concerns L elements, we extend the formulation (30), by taking a product over both l and ω elements.

$$\otimes_{l=1}^L \left(\otimes_{\omega=2}^{W^l} (x_{j_\omega}^l(t))^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \leq \otimes_{l=1}^L \left[\otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_{l\omega}}} \otimes B^l)_{j_\omega} \otimes u^l(t - \tau_{\alpha_l}) \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right]. \quad (31)$$

Combining inequalities (28) and (31), we perform the classical sum (Min-Plus multiplication) to get:

$$\begin{aligned} \otimes_{l=1}^L \left[(x_{j_1}^l(t))^{\otimes \mu_1^l} \otimes \left(\otimes_{\omega=2}^{W^l} (x_{j_\omega}^l(t))^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \right] &\leq \otimes_{l=1}^L \left[\left((A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \right. \right. \\ &\left. \left. \otimes u^l(t - \tau_{\alpha_l}) \right)^{\otimes \mu_1^l} \otimes \left(\otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_{l\omega}}} \otimes B^l)_{j_\omega} \otimes u^l(t - \tau_{\alpha_l}) \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \right]. \end{aligned} \quad (32)$$

Simplifying (32) leads to the expression (33), which expresses the result in a more compact form:

$$\begin{aligned} \otimes_{l=1}^L \left[(x_{j_1}^l(t))^{\otimes \mu_1^l} \otimes \left(\otimes_{\omega=2}^{W^l} (x_{j_\omega}^l(t))^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \right] &\leq \otimes_{l=1}^L \left[\left((A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \right)^{\otimes \mu_1^l} \right. \\ &\left. \otimes \left(\otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_{l\omega}}} \otimes B^l)_{j_\omega} \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \otimes (u^l(t - \tau_{\alpha_l}))^{\otimes \mu_W^l} \right]. \end{aligned} \quad (33)$$

4.1. Paths under one generalized mutual exclusion constraint

We start the process by considering NTEGs subject to one GMEC. The constraint is imposed on some paths containing a determined number of places.

Theorem 2. *A NTEGs made up of L TEGs and subject to a single constraint of the form (9) imposed on paths represented by ρ^l , provides a feedback controller expressed by the following formulation:*

$$\otimes_{l=1}^L u^l(t) \leq \otimes_{l=1}^L F^l \otimes x^l(t-1) \quad (34)$$

with:

$$F^l = \max \left(0, \left[C_l^{\otimes \frac{1}{\mu_W^l}} \right] \otimes A^{\phi_l}(i_W, :) \right), \quad (35)$$

and

$$C_l = b - M_{0_\rho}^l - \left((A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \right)^{\otimes \mu_1^l} - \otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_l \omega}} \otimes B^l)_{j_\omega} \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|}. \quad (36)$$

If the condition mentioned in (37) is met, such a controller exists:

$$\left((A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \right)^{\otimes \mu_1^l} \otimes \left(\otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_l \omega}} \otimes B^l)_{j_\omega} \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \leq (b - M_{0_\rho}^l) \otimes \left((A^{k_l} \otimes B^l)_{i_W} \right)^{\otimes \mu_W^l} \quad (37)$$

for $(k_l = 0, \dots, \tau_{\alpha_l})$.

Proof. We respectively identify by $t_{j_1}^l$ and $t_{i_W}^l$ the input and output transitions of the constrained path ρ^l . W^l is the number of places corresponding to each ρ^l . Noting that each TEG is indexed by l , where l ranging from 1 to L . The GMEC, as previously stated, can be expressed in Min-Plus algebra by:

$$\otimes_{l=1}^L \left[(x_{j_1}^l(t))^{\otimes \mu_1^l} \otimes \left(\otimes_{\omega=2}^{W^l} (x_{j_\omega}^l(t))^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \right] \leq \otimes_{l=1}^L \left[(b - M_{0_\rho}^l) \otimes (x_{i_W}^l(t))^{\otimes \mu_W^l} \right]. \quad (38)$$

Next, we have the control laws described in Min-Plus algebra:

$$\begin{aligned} & \otimes_{l=1}^L \left[\left((A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \right)^{\otimes \mu_1^l} \otimes \left(\otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_l \omega}} \otimes B^l)_{j_\omega} \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \right. \\ & \left. \otimes (u^l(t - \tau_{\alpha_l}))^{\otimes \mu_W^l} \right] \leq \otimes_{l=1}^L \left[(b - M_{0_\rho}^l) \otimes (x_{i_W}^l(t))^{\otimes \mu_W^l} \right]. \end{aligned} \quad (39)$$

The combination of inequalities (33) and (39) indicates that the constraint (20) is met. Actually, if we choose a feedback satisfying the inequality (39) and the expression (33) describing the existence of a path α^l linking t_u^l to $t_{j_1}^l$ is always true, we can argue that the GMEC (20) is ensured by the transitivity relation. Then, in the inequality (39) we replace the counter function $x_{i_W}^l(t)^{\otimes \mu_W^l}$ with its expression provided in (24) to get the following expression:

$$\begin{aligned} & \otimes_{l=1}^L \left[\left((A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \right)^{\otimes \mu_1^l} \otimes \left(\otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_l \omega}} \otimes B^l)_{j_\omega} \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \otimes (u^l(t - \tau_{\alpha_l}))^{\otimes \mu_W^l} \right] \\ & \leq \otimes_{l=1}^L \left[(b - M_{0_\rho}^l) \otimes \left((A^{\phi_l}(i_W, :) \otimes x^l(t - \phi_l)) \oplus \left(\oplus_{k_l=0}^{\phi_l-1} (A^{k_l} \otimes B^l)_{i_W} \otimes u^l(t - k_l) \right) \right)^{\otimes \mu_W^l} \right]. \end{aligned} \quad (40)$$

We can prove that the expression (40) is equivalent to the inequalities system shown below (it is evident that if the minimum (\oplus) of two terms is greater than or equal to a value, then each of them is likewise greater than or equal to that value):

$$\begin{aligned} & \otimes_{l=1}^L \left[\left((A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \right)^{\otimes \mu_1^l} \otimes \left(\otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_{l\omega}}} \otimes B^l)_{j_\omega} \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \otimes (u^l(t - \tau_{\alpha_l}))^{\otimes \mu_W^l} \right] \\ & \leq \otimes_{l=1}^L \left[(b - M_{0\rho}^l) \otimes (A^{\phi_l}(i_W, \cdot) \otimes x^l(t - \phi_l))^{\otimes \mu_W^l} \right]. \end{aligned} \tag{41}$$

And

$$\begin{aligned} & \otimes_{l=1}^L \left[\left((A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \right)^{\otimes \mu_1^l} \otimes \left(\otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_{l\omega}}} \otimes B^l)_{j_\omega} \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \otimes (u^l(t - \tau_{\alpha_l}))^{\otimes \mu_W^l} \right] \\ & \leq \otimes_{l=1}^L \left[(b - M_{0\rho}^l) \otimes \left(\oplus_{k_l=0}^{\phi_l-1} (A^{k_l} \otimes B^l)_{i_W} \otimes u^l(t - k_l) \right)^{\otimes \mu_W^l} \right]. \end{aligned} \tag{42}$$

Checking that both inequalities (41) and (42) are simultaneously true is sufficient to verify the satisfaction of (39).

We move now the first and second elements of (41) to the other side:

$$\begin{aligned} & \otimes_{l=1}^L (u^l(t - \tau_{\alpha_l}))^{\otimes \mu_W^l} \leq \otimes_{l=1}^L \left[(b - M_{0\rho}^l - \left((A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \right)^{\otimes \mu_1^l} \right. \\ & \quad \left. - \otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_{l\omega}}} \otimes B^l)_{j_\omega} \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \otimes (A^{\phi_l}(i_W, \cdot) \otimes x^l(t - \phi_l))^{\otimes \mu_W^l} \right]. \end{aligned} \tag{43}$$

Next, the formulation (43) is simplified into (44):

$$\otimes_{l=1}^L (u^l(t - \tau_{\alpha_l}))^{\otimes \mu_W^l} \leq \otimes_{l=1}^L \left[C_l \otimes (A^{\phi_l}(i_W, \cdot) \otimes x^l(t - \phi_l))^{\otimes \mu_W^l} \right] \tag{44}$$

where C_l is defined in (45) as:

$$C_l = b - M_{0\rho}^l - \left((A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \right)^{\otimes \mu_1^l} - \otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_{l\omega}}} \otimes B^l)_{j_\omega} \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|}. \tag{45}$$

We are interested in developing control laws of the form: $\otimes_{l=1}^L u^l(t) = \otimes_{l=1}^L F^l \otimes x^l(t - 1)$. For this, we choose ($\phi_l = \tau_{\alpha_l} + 1$) and we classically multiply (44) by $(\frac{1}{\mu_W^l})$, which gives:

$$\otimes_{l=1}^L u^l(t) \leq \otimes_{l=1}^L \left(\left[C_l \otimes \frac{1}{\mu_W^l} \right] \otimes (A^{\phi_l}(i_W, \cdot) \otimes x^l(t - 1)) \right). \tag{46}$$

In what follows, the symbol $[a]$ represents the largest integer less or equal to a .

Referring to the expression (42), we derive the following formulation, which reflects the existence condition to be verified:

$$\left((A^{\tau_{\alpha_l}} \otimes B^l)_{j_1} \right)^{\otimes \mu_1^l} \otimes \left(\otimes_{\omega=2}^{W^l} \left((A^{\tau_{\alpha_{l\omega}}} \otimes B^l)_{j_\omega} \right)^{\otimes |\mu_\omega^l - \mu_{(\omega-1)}^l|} \right) \leq (b - M_{0\rho}^l) \otimes \left((A^{k_l} \otimes B^l)_{i_W} \right)^{\otimes \mu_W^l} \tag{47}$$

for $(k_l = 0, \dots, \tau_{\alpha_l})$. □

4.2. Comparative analysis and complexity

4.2.1. Approach positioning analysis

Compared to the present mathematical control technique, the methods advanced by Giua *et al.* [24, 25] did not integrate the time feature in the modelling of the system and in the computation of the control laws. The current method fully accounts for the temporal dimension, making it applicable to timed Petri nets with controllable input transitions while providing stronger performance guarantees.

Earlier methods [13–15, 26] may, under certain conditions, lead to blocking situations and exacerbate the state-space explosion problem. Studies such as [13, 15] concentrated on using GMECs in arbitrary TPNs, whereas [26] was restricted to one supervisor and deadlock-freeness on an underlying untimed Petri net system. Other approaches, such as those in [15, 27], relied on partially modified state class graphs or integer linear programming techniques. Nevertheless, these methods are based on worst-case analysis and may fail to guarantee a feasible solution under certain conditions, sometimes reducing system performance or even resulting in deadlocks. In contrast, our approach employs exhaustive algebraic techniques based on Min-Plus algebra, which ensure blocking-free analysis, integrate temporal information directly into the model and control specifications, and avoid combinatorial explosion of states.

As opposed to our previous studies [19, 21, 22], which addressed only simple marking or mutual exclusion constraints on individual places, the present work considers a broader class of constraints, namely generalized mutual exclusion constraints (GMECs), imposed over paths rather than single places. This extension enables the approach to cover a wider range of application scenarios. Similarly, while earlier methods focusing on place-based constraints [17, 18] primarily dealt with capacity limitations applied to specific places, our framework extends beyond these formulations by handling generalized structural constraints that are not confined to individual places.

The main advantages and practical interest of these algebraic approaches are the following: (i) solving Min-Plus linear equations and calculating control laws does not require specific tools and heavy computing resources, (ii) the implementation and application of developed approaches are easy, and they do not require a lot of memory space, (iii) the adopted analytical method in this work does not suffer from the problem of the combinatorial explosion of states, as is the case for methods that use timed automata or other graphic formalisms of DESs, (iv) the various approaches that use Min-Plus models are analytical and exhaustive and can deal with complex applications and the temporal information is included in both the control specifications and the model and finally (v) this formal control methodology is exhaustive and admits a feasible solution if the sufficient conditions are satisfied.

4.2.2. Estimation of algorithmic complexity

In this section, we provide an estimation of the computational complexity of the proposed control synthesis algorithm for networks of timed event graphs (NTEGs) under generalized mutual exclusion constraints (GMECs). Understanding the algorithmic complexity is crucial for evaluating the scalability and efficiency of the method, especially when dealing with large-scale systems. We denote the following structural parameters of the NTEGs:

T : the number of transitions, P : the number of place paths involved in the constraints, C : the number of generalized mutual exclusion constraints. The proposed control synthesis algorithm proceeds through the following steps:

- (i) Computation of path-timing expressions: for each place path, a timing expression is constructed based on the firing delays of the involved transitions, using the Min-Plus algebra. Since each path may traverse up to T transitions, this step has a complexity of $O(P.T)$.
- (ii) Formulation of GMECs: each GMEC is translated into a Min-Plus inequality over the corresponding path timing expressions. Assuming each constraint involves a bounded number of paths, the total complexity of this step is $O(C.P)$.

- (iii) Solving the Min-Plus inequality system: the resulting system is modeled as a set of linear inequalities over the Min-Plus semiring. Solving this system typically involves computing the *Kleene star* of the associated Min-Plus matrix, which provides the least fixed-point solution. In the general case, the computational $O((P + T)^3)$, assuming dense matrix operations.

Therefore, the overall estimated computational complexity of the algorithm is:

$$O(P.T + C.P + (P + T)^3).$$

The current methodology provides an algebraic formula to compute control laws that ensure compliance with GMECs in TEG networks. The analytical expression of these computed supervisors is a function of the initial marking of the network, the computation of powers of Min-Plus matrices, and the parameters of all constraints. Consequently, the computational complexity of this method is inherently related to the size of the TEG and the order of the resulting matrices. While the core operations involve matrix computations in the Min-Plus algebra, handling large-scale systems may require additional optimizations.

It is important to note that, in the worst case, the complexity can grow exponentially with the size of the network, typically on the order of $O(2^n)$ where n is the number of places or transitions in the Petri net. This exponential growth arises from: (i) the combinatorial explosion in the number of reachable states, (ii) the NP-hard nature of solving the system under timing and mutual exclusion constraints and (iii) the added complexity of optimizing the control to meet these constraints. Thus, although the Min-Plus algebraic approach provides a structured and scalable framework, the problem remains computationally challenging for large-scale NTEGs.

4.3. The control synthesis algorithmic procedure

The feasibility of the computed control laws is guaranteed by Theorem 2. In the sequel, we created an algorithmic approach to demonstrate the applicability of mathematical methodologies even on complicated systems. The algorithmic below concerns the case of several GMECs enforced on paths of NTEGs.

Algorithm 1. NTEGs_control_laws_synthesis_under_GMECS.

Input: $L, b_s, A^l, B^l, M_{0\rho_s}^l, \mu_{\omega_s}^l, \mu_{1_s}^l, W_s^l, J_{1_s}^l, i_{W_s}^l$

Output: F_s^l

$\Phi_{l_s} := \tau_{\alpha_{l_s}} + 1$

$\mu_{1_s}^l \geq 1$ and $\mu_{\omega_s}^l \geq 1$

For $s \in \{1, \dots, S\}$ **do**

For $l \in \{1, \dots, L\}$ **do**

For $k_{l_s} \in \{1, \dots, \tau_{\alpha_{l_s}}\}$ **do**

For $\omega_s \in \{1, \dots, W_s^l\}$ **do**

If $((A^{\tau_{\alpha_{l_s}}} \otimes B^l)_{j_{1_s}})^{\otimes \mu_{1_s}^l} \otimes (\otimes_{\omega_s}^{W_s^l} ((A^{\tau_{\alpha_{l_s}}} \otimes B^l)_{j_{\omega_s}})^{\otimes |\mu_{\omega_s}^l - \mu_{(\omega_s-1)}^l|}) \leq (b_s - M_{0\rho_s}^l) \otimes ((A^{k_{l_s}} \otimes B^l)_{j_{W_s}})^{\otimes \mu_{W_s}^l}$ **and**

$b_s - M_{0\rho_s}^l - ((A^{\tau_{\alpha_{l_s}}} \otimes B^l)_{j_{1_s}})^{\otimes \mu_{1_s}^l} - \otimes_{\omega_s}^{W_s^l} ((A^{\tau_{\alpha_{l_s}}} \otimes B^l)_{j_{\omega_s}})^{\otimes |\mu_{\omega_s}^l - \mu_{(\omega_s-1)}^l|} \otimes A^{\Phi_{l_s}}(i_{W_s}, :) \geq 0$ **then**

print $F_s^l = b_s - M_{0\rho_s}^l - ((A^{\tau_{\alpha_{l_s}}} \otimes B^l)_{j_{1_s}})^{\otimes \mu_{1_s}^l} - \otimes_{\omega_s}^{W_s^l} ((A^{\tau_{\alpha_{l_s}}} \otimes B^l)_{j_{\omega_s}})^{\otimes |\mu_{\omega_s}^l - \mu_{(\omega_s-1)}^l|} \otimes A^{\Phi_{l_s}}(i_{W_s}, :)$

else

print failure;

return

end if

end for

end for

end for

end for

5. CASE STUDY: A FLEXIBLE MANUFACTURING CELL

We attempted to apply the current technique to a manufacturing system called “Heater Frame of GLX405” [28]. Figure 2 depicts two Production Lines (Pl₁ and Pl₂), where a robot R_1 transports the first type of material T_1 to the injection machine M_1 , which injects the material into its mold. After that, R_2 sends the injected pieces to a buffer B . Simultaneously, R_3 replaces a second type of raw material T_2 to a drilling machine M_2 . When M_2 finishes its operation, R_2 transfers the products to B . Both R_2 and B are the shared resources.

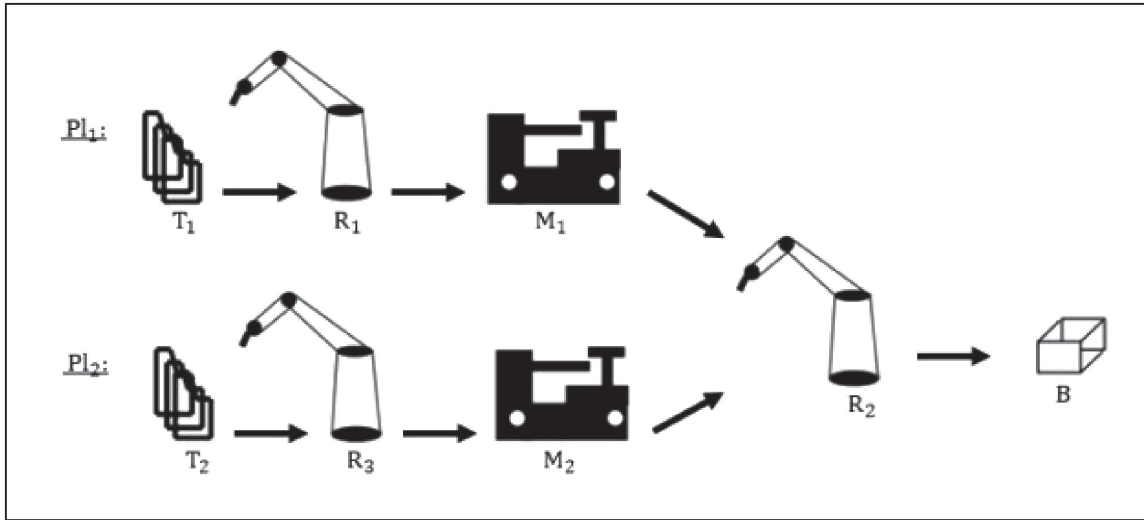


FIGURE 2. A flexible manufacturing cell [28].

The system is modeled by NTEGs (Fig. 3) composed of two-timed event graphs and subject to one GMEC, where t_u^1 and t_u^2 are the control transitions and the constraint is imposed on two paths ρ^1 and ρ^2 . Indeed, the input and output transitions of the path ρ^1 are respectively determined by t_2^1 and t_4^1 . The input and output transitions of the second constrained path ρ^2 are defined by t_2^2 and t_4^2 .

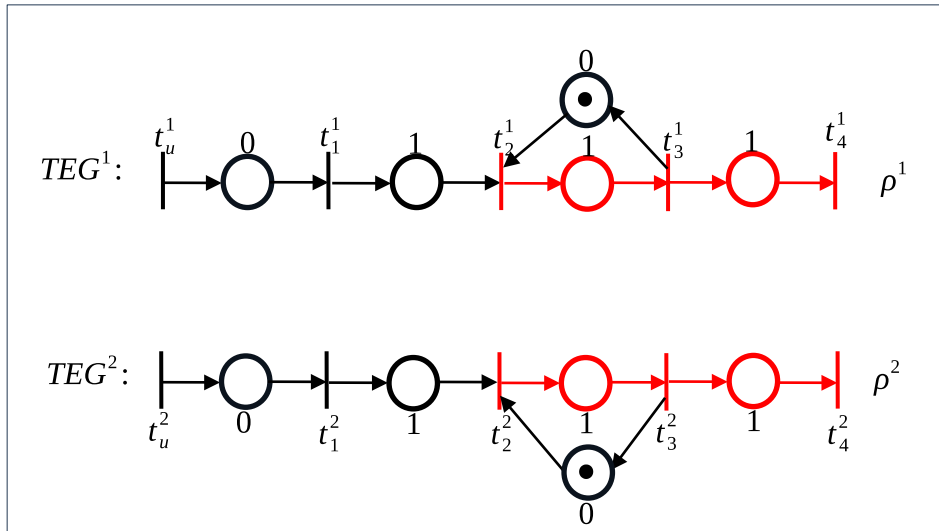


FIGURE 3. The manufacturing cell NTEGS model.

TABLE 1. Places description.

Notation	Description
p_{1u}^1	R_1 transports T_1 to the machine
p_{21}^1	M_1 injects T_1
p_{32}^1	R_2 is ready to carry out its mission
p_{43}^1	The fabricated item is transferred to buffer B
p_{1u}^2	R_3 transports T_2 to the machine
p_{21}^2	M_2 drills T_2
p_{32}^2	R_2 is prepared to complete its mission
p_{43}^2	The product is transferred to buffer B

The Table 1 illustrates the interpretation and description of the restricted places:

At an instant t , the number of tokens in both constrained paths exceeds three tokens. Consequently, we are discussing mutual exclusion problem, which can be phrased as follows:

$$M_\rho^1(t) + M_\rho^2(t) \leq 3 \quad (48)$$

and which is also equal to:

$$[M_{32}^1(t) + 2 \times M_{43}^1(t)] + [2 \times M_{32}^2(t) + M_{43}^2(t)] \leq 3. \quad (49)$$

In this example, we suppose that: $L = 2$, $S = 1$, $\mu_1^1 = 1$, $\mu_2^1 = 2$, $\mu_1^2 = 2$, $\mu_2^2 = 1$, $j_1^1 = 2$, $i_1^1 = 3$, $j_2^1 = 3$, $i_2^1 = 4$, $j_1^2 = 2$, $i_1^2 = 3$, $j_2^2 = 3$, $i_2^2 = 4$, $W^1 = 2$, $W^2 = 2$ and $b = 3$.

Next, we respectively attach a counter function with each transition of both TEGs ($\text{TEG}^1, \text{TEG}^2$), where $\theta_i^1(t)$ such that $i^1 \in \{1, 2, 3, 4\}$ and $\theta_i^2(t)$ with $i^2 \in \{1, 2, 3, 4\}$. Starting with the first TEG, we obtain:

$$\begin{cases} \theta_1^1(t) = e \otimes u^1(t) \\ \theta_2^1(t) = e \otimes \theta_1^1(t-1) \oplus 1 \otimes \theta_3^1(t) \\ \theta_3^1(t) = e \otimes \theta_2^1(t-1) \\ \theta_4^1(t) = e \otimes \theta_3^1(t-1). \end{cases} \quad (50)$$

Similarly, the dynamical behavior of the second TEG is described by:

$$\begin{cases} \theta_1^2(t) = e \otimes u^2(t) \\ \theta_2^2(t) = e \otimes \theta_1^2(t-1) \oplus 1 \otimes \theta_3^2(t) \\ \theta_3^2(t) = e \otimes \theta_2^2(t-1) \\ \theta_4^2(t) = e \otimes \theta_3^2(t-1). \end{cases} \quad (51)$$

The implicit matrix of $\theta^1(t)$ and $\theta^2(t)$ are given by the following representations:

$$\theta^1(t) = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \otimes \theta^1(t) \oplus \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \end{bmatrix} \otimes \theta^1(t-1) \oplus \begin{bmatrix} e \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix} \otimes u^1(t). \quad (52)$$

And

$$\theta^2(t) = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \otimes \theta^2(t) \oplus \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \end{bmatrix} \otimes \theta^2(t-1) \oplus \begin{bmatrix} e \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix} \otimes u^2(t). \quad (53)$$

The explicit forms of (52) and (53) are:

$$x^1(t) = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ e & 1 & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \end{bmatrix} \otimes x^1(t-1) \oplus \begin{bmatrix} e \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix} \otimes u^1(t). \tag{54}$$

And

$$x^2(t) = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ e & 1 & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \end{bmatrix} \otimes x^2(t-1) \oplus \begin{bmatrix} e \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix} \otimes u^2(t). \tag{55}$$

In TEG^1 , we have $M_{0_p}^1 = 0$ and the time delay of the path α^1 relating t_u^1 to t_2^1 is denoted as $\tau_{\alpha_1} = 1$. This results in $\phi_1 = 2$. Consequently, the sufficient condition is satisfied for k_1 ranging from 0 to 1: $(A^1 \otimes B^1)_2 \otimes (A^2 \otimes B^1)_3 \leq 3 \otimes ((A^{k_1} \otimes B^1)_4)^{\otimes 2}$ with $(A^1 \otimes B^1)_2 = e$, $(A^2 \otimes B^1)_3 = e$ and $(A^{k_1} \otimes B^1)_4 = [\varepsilon \ \varepsilon]$.

For TEG^2 we have $M_{0_p}^2 = 0$ and $\tau_{\alpha_2} = 1$, such that α^2 is the path that connects t_u^2 to t_2^2 . This gives $\phi_2 = 2$. The condition is verified for each k_2 going from 0 to 1: $((A^1 \otimes B^2)_2)^{\otimes 2} \otimes (A^2 \otimes B^2)_3 \leq 3 \otimes (A^{k_2} \otimes B^2)_4$ with $(A^1 \otimes B^2)_2 = e$, $(A^2 \otimes B^2)_3 = e$ and $(A^{k_2} \otimes B^2)_4 = [\varepsilon \ \varepsilon]$.

The control law is therefore expressed as below: $u^1(t) \otimes u^2(t) = 1 \otimes x_2^1(t-1) \otimes 1 \otimes x_2^2(t-1)$.

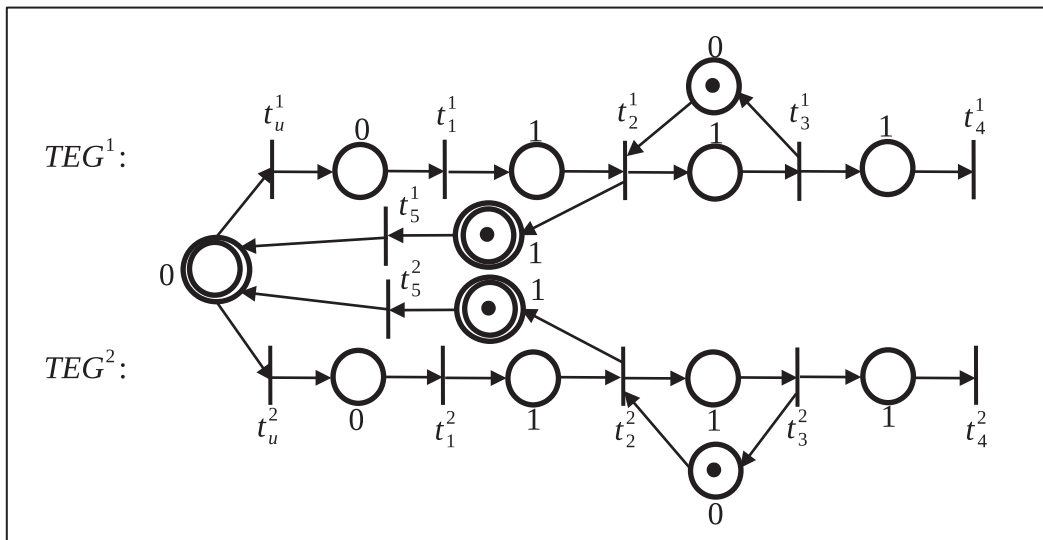


FIGURE 4. Petri net model of the controlled flexible production cell.

Remark 2. The resultant controllers can be characterized by timed and marked places related to the initial graph, which can be a software installed into the system or timed contactors (Fig. 4). We can also witness the efficiency of the current mathematical approach in Figure 5, which represents the reachability graph of the controlled Petri net. In addition, implementing the suggested solutions is simple and does not necessitate a huge memory space. In the short term, we intend to implement these procedures in the **MATLAB** software. The validity hypotheses of the proposed framework and practical scope are outlined below:

- (i) The system must be modeled as a network of timed event graph (a subclass of timed Petri nets), where places represent delays and transitions represent events.

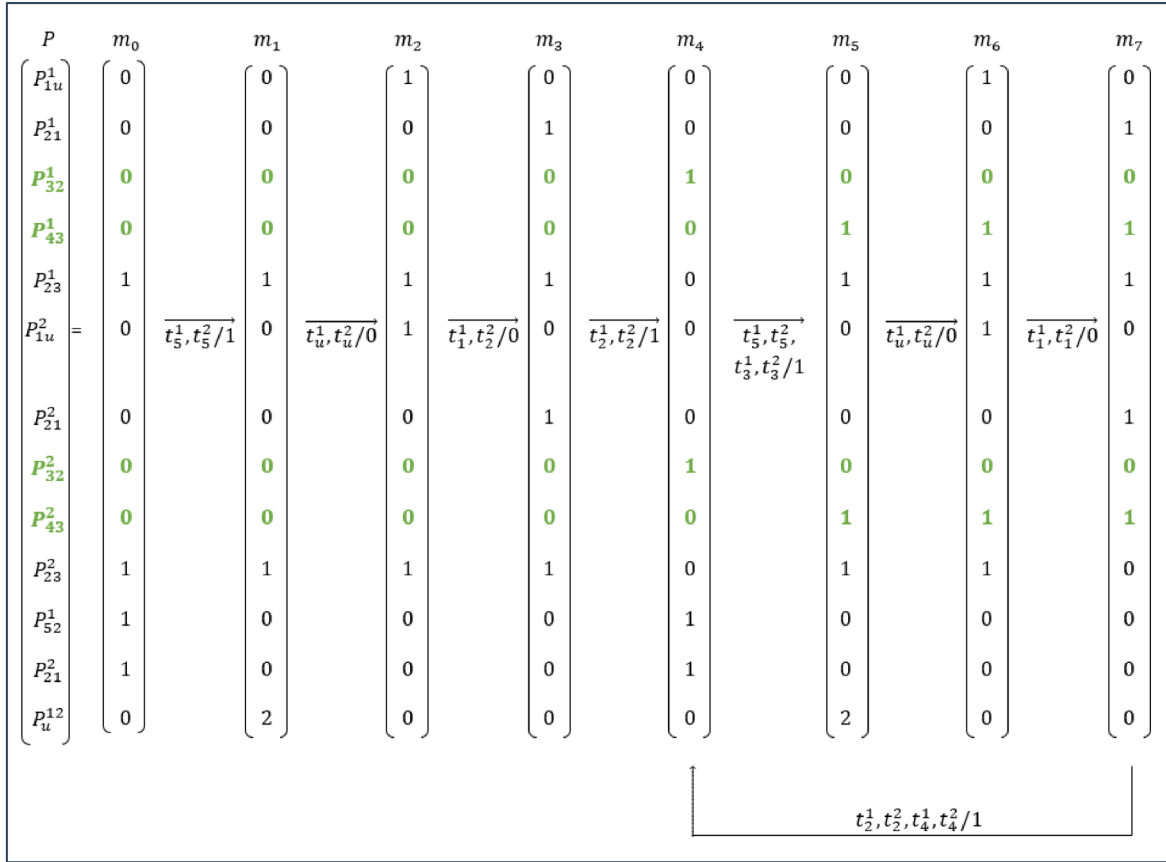


FIGURE 5. Reachability graph of the controlled Petri net.

- (ii) The timing behavior of the system is assumed to be deterministic and bounded, which ensures that the Min-Plus algebraic modeling is appropriate.
- (iii) The GMECs must be formulated over identifiable and structurally relevant place paths.
- (iv) The structure of the Petri net must allow for the derivation of well-defined timing paths, and must not involve conflicts or nondeterministic branching, which would violate the assumptions of linearity in the Min-Plus setting.

While the framework is not universally applicable to all classes of Petri nets, it is particularly well-suited to synchronous and cyclic systems found in manufacturing, transportation, and embedded real-time control systems, where such structural and timing conditions naturally hold.

6. CONCLUSION

This paper presented an innovative and effective algebraic method to address the control problem of discrete event systems (DESs) under capacity constraints, using Networked Timed Event Graphs (NTEGs) and Min-Plus algebra. The main contributions included modeling the behavior of DESs with linear equations in Min-Plus algebra and expressed Generalized Mutual Exclusion Constraints (GMECs) as weighted inequalities in Min-Plus algebra. A synthesis method for feedback control laws ensuring compliance with GMECs on specific NTEG paths was developed. Furthermore, sufficient conditions were derived to guarantee the existence of such control

laws. The proposed method efficiency was demonstrated through its application to a flexible manufacturing system. For future studies, we will focus on partially observable NTEGs and establishing sufficient and necessary conditions for control laws. It is also interesting to discuss the robustness and optimality of the computed control laws.

DATA AVAILABILITY STATEMENT

No new data/codes were created or analyzed in this study.

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APPENDIX A. TABLE OF NOTATIONS

Notations	Description
P^l	Finite set of places in each TEG^l
T^l	Finite set of transitions in each TEG^l
ρ^l	Constrained path in each TEG^l
t_u^l	Control transition of each TEG^l
$t_{j_1}^l$	Upstream transition of the constrained path ρ^l
$t_{i_W}^l$	Downstream transition of the constrained path ρ^l
L	Number of TEGs
α^l	Path connecting t_u^l to $t_{j_1}^l$
τ_{α^l}	Time delay of the path α^l
W^l	Number of places in ρ^l
M_ρ^l	Marking of the path ρ^l
$M_{i_\omega j_\omega}^l$	Marking of the place $p_{i_\omega j_\omega}^l$
$M_{o_{i_\omega j_\omega}}^l$	Initial marking of the place $p_{i_\omega j_\omega}^l$
$p_{i_\omega j_\omega}^l$	Places existing in ρ^l
$t_{i_\omega}^l$	Output transition of the place $p_{i_\omega j_\omega}^l$
$t_{j_\omega}^l$	Input transition of the place $p_{i_\omega j_\omega}^l$
μ_1^l	Coefficient associated with the first place in ρ^l
μ_ω^l	Coefficient associated with the intermediate places in ρ^l
μ_W^l	Coefficient associated with the last place in ρ^l
S^l	Number of constraints in each TEG^l