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GENERATING COMPROMISE SOLUTION OF BI-LEVEL MULTI-OBJECTIVE INTUITIONISTIC FUZZY FRACTIONAL PROGRAMMING PROBLEM

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Abstract. The real world optimization problems in hierarchical decision making systems, often encounter multiple functions in fractional forms with uncertain parameters. To tackle such situation of uncertainty, this paper proposes a novel methodology to find a compromise solution of a bi-level multi-objective linear fractional programming problem which is designed in a fuzzy environment with its parameters expressed as intuitionistic triangular fuzzy numbers. Based on the concept of intuitionistic fuzzy (α, β)-cuts and some theoretical aspects, the bi-level intuitionistic fuzzy model is formulated into an equivalent bi-level optimization with multiple interval valued fractional functions. The method proposed by Chakraborty and Gupta, is utilized to compute the individual compromise solution of each interval valued fractional objective function. Subsequently, the upper and lower level compromise solutions are computed to ascertain the aspiration values of the multiple interval valued fractional functions and the decision variables controlled at the upper level. Goal programming approach using a proposed modified linearization technique for fractional functions, is implemented to derive the compromise solution of the bi-level fuzzy optimization model. An existing numerical example, a practical problem in production sector are solved and the comparative discussion on result analysis is incorporated to demonstrate the feasibility and efficiency of the proposed approach.

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1. Introduction

Numerous real-world problems in different hierarchical decision making systems can be represented as bi-level programming problems (BLPP) [20,21] in mathematical form, which comprises two decision makers(DMs) allocated at two different levels *i.e.* upper and lower, in order of their decision making capacities. Upper-level DM (ULDM) and lower-level DM (LLDM) perform as the leader and follower of the BLPP respectively where both DMs have different goals and each controls individually a set of decision variables. The decision-making process is initiated by the leader and forwarded to the follower who determines own plan following the leader's decision. The goals attained by the leader may be impacted by the follower's strategy and also the actions of the leader may influence the follower's strategy. Therefore, a compromise solution is required which avoids the

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situation of decision deadlock and becomes suitable for both level DMS with overall benefit and growth of the system.

Fractional programming [48] is a nonlinear optimization problem which incorporates its objective function expressed as the ratio of functions to apropriately fit the formulation of various real-world into optimization models. Multi-objective linear fractional programming problem (MOLFPP) [16,48] contains multiple conflicted and inter-related objective functions which exist in fractional form of linear functions. MOLFPP has lots of applications in various fields like business, industries, management, economics, engineering, health care etc., which are according to several relative ratios of physical and economical quantities to be simultaneously optimized. Some commonly found examples of fractional objectives are, profit/cost, output/employee, production/time, inventory/sale, etc. In such situations, the parameters associated with the objective functions as well as constraints are all expected to be precisely known, but practically these parameters are not always exact because of several reasons like errors in measurement, fluctuations in market conditions, unpredictable state of climate etc. These circumstances can be effectively formulated into optimization models by utilizing a fuzzy environment to tackle the uncertainty. An intuitionistic fuzzy set(IFS) [7] is the extension of a fuzzy set and its degrees of membership, non-membership functions are considered with a condition that these are not always the complement of each other whereas the fuzzy set involves only the degree of membership. In order to appropriately formulate certain practical problems with uncertainty into fuzzy optimization models, intuitionistic fuzzy environment can be suitably applied. Several solution methodologies are developed in literature for solving various forms of linear fractional programming problems (LFPP). The variable transformation approach was developed by Charnes and Cooper [15] to solve LFPP. Schaible [44] discussed the duality concept and proposed a solution algorithm of LFPP. Guzel and Sivri [24] proposed a method of solution for multi-objective LFPP using Taylor series approach. Chakraborty and Gupta [14] proposed an algorithm to solve MOLFPP on the basis of a new linearization technique whereas Toksari [50] designed a solution methodology for MOLFPP with linearization of the fractional objectives on the basis of Taylor's series expansion. Bhati and Singh [11] used branch and bound computational method to solve MOLFPP. Pal *et al.* [41] developed a linearization technique for fractional membership functions and proposed a methodology of MOLFPP by using fuzzy goal programming method. Das *et al.* [18] designed a method for solving fully fuzzy LFPP using variable transformation method and multi-objective linear programming problem (MOLPP). Ammar and Khalifa [6] proposed a method of solution for MOLFPP with (γ, δ) fuzzy interval valued parameters based on ranking function, transformation method and fuzzy programming. Mishra [36] developed a method to solve bi-level LFPP (BL-LFPP) using the weighting sum approach. Calvete and Gale [13] proposed an approach to find a global optimal solution of a bi-level LPP/LFPP having linear and linear fractional objective functions at the first and second level respectively with its feasible region as a polyhedron. Hashem *et al.* [40] proposed a model of bi-level multi-objective optimization using fuzzy goal programming to measure the performance of a decision making unit comprising fractional objective functions at both level.

The notion and idea of intuitionistic fuzzy set (IFS) was initially proposed by Atanassov [7] in 1986. Sujeet and Shiv [46] proposed a technique to address LFPP in an intuitionistic environment of fuzziness whereas Bharti *et al.* [9] proposed a solution approach for intuitionistic MOLPP by constructing membership and non-membership functions of maximization and minimization type respectively. Bharti and Singh [10] developed an algorithm to solve interval valued intuitionistic fuzzy MOLPP assuming uncertain degrees of membership and non-membership. Fathy [23] developed a solution approach for an uncertain LPP in which all the parameters comprising decision variables are stated in terms of interval-valued intuitionistic fuzzy numbers. Sahoo *et al.* [43] derived a solution methodology for MOLFPP which involves all its parameters designed as pentagonal intuitionistic fuzzy numbers (IFN) using its equivalent conversion into a crisp MOLPP based on an accuracy function. Yuvashri and Saraswathi [52] also developed a solution approach to MOLFPP in intuitionistic fuzzy environment of pentagonal fuzzy numbers by deriving its equivalent crisp form. Several ranking methods for IFNs were proposed in literature by many researchers. Deng *et al.* [27] proposed a ratio ranking method for triangular IFNs. Satyajit and Debashree [17] developed a centroid-based ranking method for trapezoidal IFNs and added its application in a multicriteria optimization problem. Biswas and De [12] proposed an efficient ranking method for intuitionistic fuzzy bi-level LPP using defuzzification of IFNs and probability density functions for membership, nonmembership functions. Li [26] developed a ranking method of triangular IFNs based on the concept of ratio between value index and ambiguity index, also discussed its application to a multi attribute decision-making problem. Alessa [5] developed a solution procedure to BLFPP by introducing membership and nonmembership functions for the uncertainty of decision makers in an intuitionistic fuzzy environment. Maiti and Ray [31] proposed a ranking function approach for bi-level programming of Stackelberg game in an intuitionistic fuzzy environment. Aggarwal and Gupta [3] proposed a method to find an intuitionistic fuzzy efficient solution of a bilevel multi-objective LPP (BL-MOLPP) with its technological coefficients and resources as IFNs. Singh *et al.* [47] proposed an application to the production planning problems based on BL-MOLPP in intuitionistic fuzzy optimization and developed its solution algorithm based on TOPSIS method. Mollalign *et al.* [39] developed a methodology using two phase fuzzy goal programming to derive a compromise solution of a multi-level MOLPP with intuitionistic fuzzy parameters where its equivalent crisp form was obtained applying accuracy function. Moges *et al.* [38] proposed a method to obtain a compensatory solution of decentralized bi-level MOLFPP with intuitionistic trapezoidal fuzzy parameters where single DM remains at upper level but multiple DMs remain at the lower level. Moges *et al.* [37] also developed a method of solution for MOLFPP with triangular intuitionistic fuzzy numbers using a two-phase approach and weighted goal programming method. Roy and Maiti [42] developed some mathematical models to solve BLPP in Stackelberg game with two different cases under type-2 fuzzy environment. Maiti and Roy [29] proposed a computational algorithm to solve interval-valued and multi-choice BLPP for the Stackelberg game in intuitionistic fuzzy environment. Maiti and Roy [28] also developed a solution approach for multi-choice stochastic BLPP where, the parameters associated with the objective functions and constraints are multi-choice types and follow normal distribution respectively. Maiti *et al.* [30] proposed a model to improve better cooperative games in Gaussian type-2 fuzzy environment by implementing both parametric programming and fuzzy optimization. Das *et al.* [19] proposed a multi-objective optimization approach based on the concepts of a new ranking approach and epsilon-constraint method to solve two stage solid logistic network problem with some parameters designed as triangular type-2 neutrosophic fuzzy numbers. Malik and Gupta [32] developed a method to solve fully fuzzy MOLFPP with intuitionistic fuzzy parameters using variable transformation technique and weighted goal programming approach. Ahmad *et al.* [4] proposed an algorithm to solve multi-level MOLFPP using the neutrosophic fuzzy concepts where rough intervals are considered as the coefficients in the objective functions. Adhami *et al.* [2] addressed a multilevel supplier selection problem with fuzzy valued supply, demand and constraints based on the neutrosophic approach. Arora *et al.* [1] developed a method to solve a Transshipment problem where the cost coefficients are considered as asymmetric pentagonal fuzzy numbers. Barman *et al.* [8] proposed a decision making method based on multi-objective optimization to address and derive the solution of a practical problem related to tomato crops. Mardanya and Roy [33] proposed an approach of solution for a problem of multi-objective and multi-item solid transportation in trapezoidal fuzzy environment.

Intuitionistic fuzzy numbers are extensions of standard fuzzy numbers which tackle the uncertainty more accurately. In this paper, a novel solution methodology is developed for bi-level multi-objective intuitionistic fuzzy LFPP where all the parameters exist in a special form of triangular intuitionistic fuzzy numbers. The methodology proposed here implements the concept of intuitionistic fuzzy (α, β)-cuts for deriving an equivalent interval valued bi-level model. It also utilizes the relationship between interval valued and bi-objective optimization, goal programming, method of Chakraborty and Gupta [14], a modified linearization technique for fractional functions to derive the compromise solution of the bi-level intuitionistic fuzzy fractional problem. The existing methods in literature mostly deal with deterministic forms of bi-level LPP, bi-level FPP with single or multiple objectives at each level, otherwise bi-level LPP, MOLPP in classical or intuitionistic fuzzy environment but the proposed solution approach attempts to solve and produce a compromise solution of a bi-level MOLFPP in an intuitionistic fuzzy environment of advanced triangular IFNs. To the best of author's belief and knowledge, almost no or very rare solution methodologies may be established in literature to address this particular form of bi-level intuitionistic fuzzy MOLFPP. Numerous problems in reality comprise imprecise information about the sources or parameters which enhances the level of difficulty in decision making context. Appropriate concepts of fuzzy logic can be implemented to mathematically address the impreciseness. Since, multiple interrelated and conflicted functions in fractional form with uncertain parameters sometimes need simultaneous optimization at different hierarchical levels of decision making problems, the proposed methodology is expected to be effectively utilized in decision making of hierarchical organizations where the problems can be mathematically modeled in form of BL-MOLFPP with intuitionistic fuzzy parameters.

This paper is structured as: followed by an introduction and brief description about the contribution of the work in Sections 1 and 2 covers the fundamental concepts related to intuitionistic fuzzy numbers, conversion of TIFNs to interval valued forms using fuzzy (α,β)-cuts, interval-valued optimization and its equivalent bi-objective optimization, linearization of fuzzy fractional membership functions and a solution approach for MOLFPP. Section 3 includes the mathematical formulation of the bi-level multi-objective triangular intuitionistic fuzzy LFPP whereas Section 4 incorporates the main work *i.e.*, the proposed solution methodology to determine its compromise solution. Further, an algorithm and a flowchart are also included in Section 4 to illustrate the proposed solution approach. Section 5 contains one existing numerical example, a practical problem in production planning as a real-world application, their solutions using the proposed methodology and a detailed comparative discussion on analysis of the results. Finally, concluding discussions are incorporated in Section 6.

2. Preliminaries

Definition 1 ([7])**.** Let U be the universal set then an intuitionistic fuzzy set (IFS) $\widetilde{F}^I$ in U is defined as a set of ordered triplets of the element x associated with the values of its degree of membership and degree of non membership. The mathematical expression of IFS can be represented as follows.

$$\widetilde{F}^I = \{\langle x, \mu_{\widetilde{F}^I}(x), \nu_{\widetilde{F}^I}(x)\rangle : x \in U\},$$

where, $\mu_{\widetilde{F}^I(x)}$ is the degree of membership and $\nu_{\widetilde{F}^I}(x)$ is the degree of non-membership of the element $x \in U$. $\mu_{\widetilde{F}^I} : U \to [0,1]$ *i.e.*, $\mu_{\widetilde{F}^I}(x) \in [0,1]$ and $\nu_{\widetilde{F}^I} : U \to [0,1]$ *i.e.*, $\nu_{\widetilde{F}^I}(x) \in [0,1]$. $0 \le \mu_{\widetilde{F}^I}(x) + \nu_{\widetilde{F}^I}(x) \le 1\ \forall\ x \in U$. $\pi_{\widetilde{F}^I}(x) = 1 - \mu_{\widetilde{F}^I}(x) - \nu_{\widetilde{F}^I}(x)$ represents the degree of non-determinacy (or uncertainty) $\forall\ x\ \in$ U in the IFS $\widetilde{F}^I$.

Definition 2 ([45])**.** An intuitionistic fuzzy set $\widetilde{F}^I$ is said to be an intuitionistic fuzzy number (IFN) if it possess the following properties.

(i) $\widetilde{F}^I$ is a subset of the real line $\mathbb{R}$.
(ii) $\widetilde{F}^I$ is normal *i.e.*, there exists at least one $x \in \mathbb{R}$ such that $\mu_{\widetilde{F}^I}(x) = 1$ ($\nu_{\widetilde{F}^I}(x) = 0$).
(iii) The associated membership function $\mu_{\widetilde{F}^I}(x)$ is convex *i.e.*, $\forall\ x, y \in \mathbb{R}$ and $\lambda \in [0,1]$, $\mu_{\widetilde{F}^I}(\lambda x + (1-\lambda)y) \ge \min\{\mu_{\widetilde{F}^I}(x), \mu_{\widetilde{F}^I}(y)\}$.
(iv) The associated non-membership function $\nu_{\widetilde{F}^I}(x)$ is concave *i.e.*, $\forall\ x, y \in \mathbb{R}$ and $\lambda \in [0,1]$, $\nu_{\widetilde{F}^I}(\lambda x + (1-\lambda)y) \le \max\{\nu_{\widetilde{F}^I}(x), \nu_{\widetilde{F}^I}(y)\}$.

Definition 3 ([45])**.** A triangular intuitionistic fuzzy number (TIFN) $\tilde{T}^I = \langle \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3; \mathfrak{w}, \mathfrak{u}\rangle$ is a special IFS on $\mathbb{R}$, having its membership and non-membership functions defined in the following forms.

$$\mu_{\widetilde{T}^I}(x) = \begin{cases} \frac{(x-\mathfrak{a}_1+\mathfrak{a}_2)\mathfrak{w}}{\mathfrak{a}_2}, & \mathfrak{a}_1 - \mathfrak{a}_2 \le x < \mathfrak{a}_1 \\ \frac{(\mathfrak{a}_1+\mathfrak{a}_3-x)\mathfrak{w}}{\mathfrak{a}_3}, & \mathfrak{a}_1 \le x \le \mathfrak{a}_1 + \mathfrak{a}_3 \\ 0, & \text{otherwise} \end{cases},$$

$$\nu_{\widetilde{T}^I}(x) = \begin{cases} \frac{[(\mathfrak{a}_1-x)+\mathfrak{u}(x-\mathfrak{a}_1+\mathfrak{a}_2)]}{\mathfrak{a}_2}, & \mathfrak{a}_1 - \mathfrak{a}_2 \le x < \mathfrak{a}_1 \\ \frac{[(x-\mathfrak{a}_1)+\mathfrak{u}(\mathfrak{a}_1+\mathfrak{a}_3-x)]}{\mathfrak{a}_3}, & \mathfrak{a}_1 \le x \le \mathfrak{a}_1 + \mathfrak{a}_3 \\ 1, & \text{otherwise} \end{cases}$$

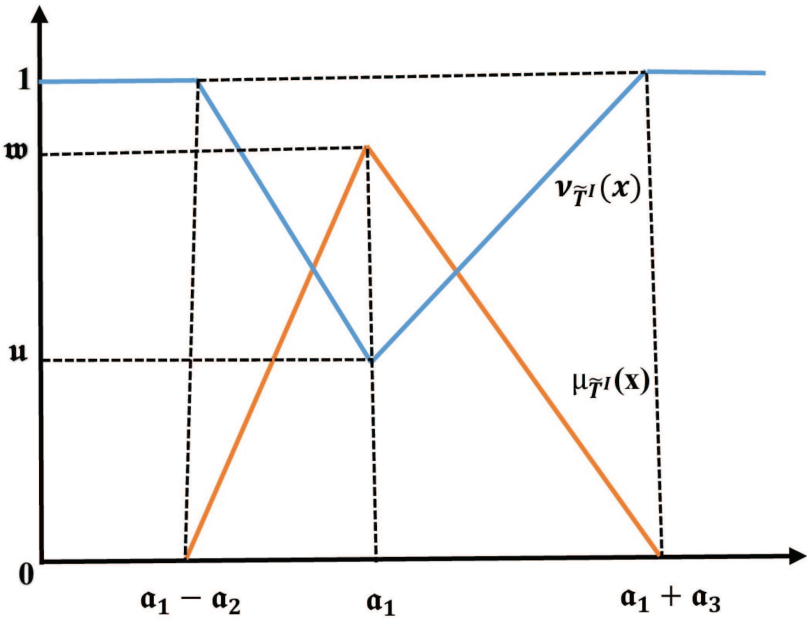

FIGURE 1. Triangular intuitionistic fuzzy number.

where, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ are not necessarily same, called as the left and right spreads respectively, $\mathfrak{a}_1$ is the mean value with the maximum value of degree of membership $\mathfrak{w}$ and the minimum value of degree of non-membership $\mathfrak{u}$ which satisfy the conditions $0 \leq \mathfrak{w} \leq 1$, $0 \leq \mathfrak{u} \leq 1$, $0 \leq \mathfrak{w}+\mathfrak{u} \leq 1$. The following Figure 1 illustrates the membership function $\mu_{\widetilde{T}^I}(x)$ and also the non-membership function $\nu_{\widetilde{T}^I}(x)$ of the TIFN $\widetilde{T}^I$.

Definition 4 ([45], (α,β)-cut of a TIFN)**.** The (α,β)-cut of a TIFN $\widetilde{T}^I = \langle \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3; \mathfrak{w}, \mathfrak{u}\rangle$ is a crisp subset of $\mathbb{R}$ that is defined as,
$\widetilde{T}^I_{\alpha,\beta} = \{x : \mu_{\widetilde{T}^I}(x) \geq \alpha, \nu_{\widetilde{T}^I}(x) \leq \beta\}$, where $0 \leq \alpha \leq \mathfrak{w}$, $\mathfrak{u} \leq \beta \leq 1$ and $0 \leq \alpha+\beta \leq 1$.

The α-cut of the TIFN $\widetilde{T}^I$ is $\widetilde{T}^I_\alpha = \{x : \mu_{\widetilde{T}^I}(x) \geq \alpha\}$, $0 \leq \alpha \leq \mathfrak{w}$, that can be represented in the interval valued form as, $\widetilde{T}^I_\alpha = [(\mathfrak{a}_1 - \mathfrak{a}_2) + \frac{\mathfrak{a}_2\alpha}{\mathfrak{w}}, (\mathfrak{a}_1 + \mathfrak{a}_3) - \frac{\mathfrak{a}_3\alpha}{\mathfrak{w}}] = [\widetilde{T}_L(\alpha), \widetilde{T}_R(\alpha)]$.

The β-cut of the TIFN $\widetilde{T}^I$ is $\widetilde{T}^I_\beta = \{x : \nu_{\widetilde{T}^I}(x) \leq \beta\}$, $\mathfrak{u} \leq \beta \leq 1$, that can be represented in the interval valued form as, $\widetilde{T}^I_\beta = [(\mathfrak{a}_1 - \mathfrak{a}_2) + \frac{(1-\beta)\mathfrak{a}_2}{(1-\mathfrak{u})}, (\mathfrak{a}_1 + \mathfrak{a}_3) - \frac{(1-\beta)\mathfrak{a}_3}{(1-\mathfrak{u})}] = [\widetilde{T}_L(\beta), \widetilde{T}_R(\beta)]$.

Theorem 1 ([25])**.** *For any* $\alpha \in [0,\mathfrak{w}]$, $\beta \in [\mathfrak{u},1]$ *and* $0 \leq \alpha+\beta \leq 1$, $\widetilde{T}^I_{\alpha,\beta} = \widetilde{T}^I_\alpha \cap \widetilde{T}^I_\beta = [\widetilde{T}_L, \widetilde{T}_R]$, *where,* $\widetilde{T}_L = \max\{\widetilde{T}_L(\alpha), \widetilde{T}_L(\beta)\}$ *and* $\widetilde{T}_R = \min\{\widetilde{T}_R(\alpha), \widetilde{T}_R(\beta)\}$.

Definition 5 (Arithmetic operations on TIFN, [45])**.** Let $\widetilde{\mathcal{A}}^I = \langle \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3; \mathfrak{w}_\mathfrak{a}, \mathfrak{u}_\mathfrak{a}\rangle$ and $\widetilde{\mathcal{B}}^I = \langle \mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3; \mathfrak{w}_\mathfrak{b}, \mathfrak{u}_\mathfrak{b}\rangle$ be two TIFNs on $\mathbb{R}$ and $\lambda \in \mathbb{R}$ then, the following arithmetic operations are defined on the TIFNs.

(i) $\widetilde{\mathcal{A}}^I \oplus \widetilde{\mathcal{B}}^I = \langle \mathfrak{a}_1+\mathfrak{b}_1, \mathfrak{a}_2+\mathfrak{b}_2, \mathfrak{a}_3+\mathfrak{b}_3; \min\{\mathfrak{w}_\mathfrak{a}, \mathfrak{w}_\mathfrak{b}\}, \max\{\mathfrak{u}_\mathfrak{a}, \mathfrak{u}_\mathfrak{b}\}\rangle$.
(ii) $\widetilde{\mathcal{A}}^I \ominus \widetilde{\mathcal{B}}^I = \langle \mathfrak{a}_1-\mathfrak{b}_1, \mathfrak{a}_2+\mathfrak{b}_3, \mathfrak{a}_3+\mathfrak{b}_2; \min\{\mathfrak{w}_\mathfrak{a}, \mathfrak{w}_\mathfrak{b}\}, \max\{\mathfrak{u}_\mathfrak{a}, \mathfrak{u}_\mathfrak{b}\}\rangle$.

$$\text{(iii)}\ \widetilde{\mathcal{A}}^I \otimes \widetilde{\mathcal{B}}^I = \begin{cases} \langle \mathfrak{a}_1\mathfrak{b}_1, \mathfrak{a}_1\mathfrak{b}_2 + \mathfrak{b}_1\mathfrak{a}_2 - \mathfrak{a}_2\mathfrak{b}_2, \mathfrak{a}_1\mathfrak{b}_3 + \mathfrak{b}_1\mathfrak{a}_3 + \mathfrak{a}_3\mathfrak{b}_3; \\ \min\{\mathfrak{w}_\mathfrak{a}, \mathfrak{w}_\mathfrak{b}\}, \max\{\mathfrak{u}_\mathfrak{a}, \mathfrak{u}_\mathfrak{b}\}\rangle, \quad \widetilde{\mathcal{A}}^I > 0, \widetilde{\mathcal{B}}^I > 0 \\ \\ \langle \mathfrak{a}_1\mathfrak{b}_1, -\mathfrak{a}_1\mathfrak{b}_3 + \mathfrak{b}_1\mathfrak{a}_2 + \mathfrak{a}_2\mathfrak{b}_3, -\mathfrak{a}_1\mathfrak{b}_2 + \mathfrak{b}_1\mathfrak{a}_3 - \mathfrak{a}_3\mathfrak{b}_2; \\ \min\{\mathfrak{w}_\mathfrak{a}, \mathfrak{w}_\mathfrak{b}\}, \max\{\mathfrak{u}_\mathfrak{a}, \mathfrak{u}_\mathfrak{b}\}\rangle, \quad \widetilde{\mathcal{A}}^I < 0, \widetilde{\mathcal{B}}^I > 0 \\ \\ \langle \mathfrak{a}_1\mathfrak{b}_1, -\mathfrak{a}_1\mathfrak{b}_3 - \mathfrak{b}_1\mathfrak{a}_3 - \mathfrak{a}_3\mathfrak{b}_3, -\mathfrak{a}_1\mathfrak{b}_2 - \mathfrak{b}_1\mathfrak{a}_2 + \mathfrak{a}_2\mathfrak{b}_2; \\ \min\{\mathfrak{w}_\mathfrak{a}, \mathfrak{w}_\mathfrak{b}\}, \max\{\mathfrak{u}_\mathfrak{a}, \mathfrak{u}_\mathfrak{b}\}\rangle, \quad \widetilde{\mathcal{A}}^I < 0, \widetilde{\mathcal{B}}^I < 0. \end{cases}$$

$$\text{(iv)}\ \frac{\widetilde{\mathcal{A}}^I}{\widetilde{\mathcal{B}}^I} = \begin{cases} \langle \frac{\mathfrak{a}_1}{\mathfrak{b}_1}, \frac{\mathfrak{a}_1\mathfrak{b}_3 + \mathfrak{b}_1\mathfrak{a}_2 - \mathfrak{a}_2\mathfrak{b}_3}{\mathfrak{b}_1^2}, \frac{\mathfrak{a}_1\mathfrak{b}_2 + \mathfrak{b}_1\mathfrak{a}_3 + \mathfrak{a}_3\mathfrak{b}_2}{\mathfrak{b}_1^2}; \\ \min\{\mathfrak{w}_\mathfrak{a}, \mathfrak{w}_\mathfrak{b}\}, \max\{\mathfrak{u}_\mathfrak{a}, \mathfrak{u}_\mathfrak{b}\}\rangle, \quad \widetilde{\mathcal{A}}^I > 0, \widetilde{\mathcal{B}}^I > 0 \\ \\ \langle \frac{\mathfrak{a}_1}{\mathfrak{b}_1}, \frac{\mathfrak{a}_1\mathfrak{b}_3 - \mathfrak{b}_1\mathfrak{a}_3 + \mathfrak{a}_3\mathfrak{b}_3}{\mathfrak{b}_1^2}, \frac{\mathfrak{a}_1\mathfrak{b}_2 - \mathfrak{b}_1\mathfrak{a}_2 - \mathfrak{a}_2\mathfrak{b}_2}{\mathfrak{b}_1^2}; \\ \min\{\mathfrak{w}_\mathfrak{a}, \mathfrak{w}_\mathfrak{b}\}, \max\{\mathfrak{u}_\mathfrak{a}, \mathfrak{u}_\mathfrak{b}\}\rangle, \quad \widetilde{\mathcal{A}}^I < 0, \widetilde{\mathcal{B}}^I > 0 \\ \\ \langle \frac{\mathfrak{a}_1}{\mathfrak{b}_1}, \frac{-\mathfrak{a}_1\mathfrak{b}_2 - \mathfrak{b}_1\mathfrak{a}_3 - \mathfrak{a}_3\mathfrak{b}_2}{\mathfrak{b}_1^2}, \frac{-\mathfrak{a}_1\mathfrak{b}_3 - \mathfrak{b}_1\mathfrak{a}_2 + \mathfrak{a}_2\mathfrak{b}_3}{\mathfrak{b}_1^2}; \\ \min\{\mathfrak{w}_\mathfrak{a}, \mathfrak{w}_\mathfrak{b}\}, \max\{\mathfrak{u}_\mathfrak{a}, \mathfrak{u}_\mathfrak{b}\}\rangle, \quad \widetilde{\mathcal{A}}^I < 0, \widetilde{\mathcal{B}}^I < 0. \end{cases}$$

$$\text{(v)}\ \lambda\widetilde{\mathcal{A}}^I = \begin{cases} \langle \lambda\mathfrak{a}_1, \lambda\mathfrak{a}_2, \lambda\mathfrak{a}_3; \mathfrak{w}_\mathfrak{a}, \mathfrak{u}_\mathfrak{a}\rangle & \lambda > 0 \\ \langle \lambda\mathfrak{a}_1, -\lambda\mathfrak{a}_3, -\lambda\mathfrak{a}_2; \mathfrak{w}_\mathfrak{a}, \mathfrak{u}_\mathfrak{a}\rangle & \lambda < 0. \end{cases}$$

$$\text{(vi)}\ (\widetilde{\mathcal{A}}^I)^{-1} = \langle \frac{1}{\mathfrak{a}_1}, \frac{\mathfrak{a}_3}{\mathfrak{a}_1^2 + \mathfrak{a}_1\mathfrak{a}_3}, \frac{\mathfrak{a}_2}{\mathfrak{a}_1^2 - \mathfrak{a}_1\mathfrak{a}_2}; \mathfrak{w}_\mathfrak{a}, \mathfrak{u}_\mathfrak{a}\rangle.$$

2.1. Bi-objective form of interval valued optimization

An optimization model $M_{\rm I}$ comprising interval-valued objective function, can have an equivalent expression of a bi-objective optimization model $M_{\rm II}$, Stanojevic *et al.* [49].

$$M_{\rm I} : \max_{x\in\Psi} \, [\mathcal{F}^L(x), \mathcal{F}^U(x)]. \tag{2.1}$$

$$M_{\rm II} : \max_{x\in\Psi} \, \{\mathcal{F}^L(x), \mathcal{F}^U(x)\}. \tag{2.2}$$

Definition 6 ([51])**.** Let, $[L_1, U_1]$ and $[L_2, U_2]$ be two real intervals, then $[L_1, U_1] \le [L_2, U_2]$ iff $L_1 \le U_1$ and $L_2 \le U_2$.

Definition 7 ([51])**.** $x^* \in \Psi$ is an efficient solution of $M_{\rm I}$ if $\nexists\ x \in \Psi$ such that $[\mathcal{F}^L(x^*), \mathcal{F}^U(x^*)] < [\mathcal{F}^L(x), \mathcal{F}^U(x)]$.

2.2. Linearization of fuzzy fractional membership function

Pal *et al.* [41] developed a process of linearization for the fuzzy membership functions associated with the fractional objective functions involved in a multi-objective optimization. Consider, the fuzzy membership function $\mu_{\tilde{\mathcal{F}}}(x)$ of a maximization type MOLFPP which comprises $\mathcal{F}(x)$ as one of its objective function having aspired(best) value $\mathfrak{u}$ and acceptable(worst) value $\mathfrak{l}$.

$$\mu_{\tilde{\mathcal{F}}}(x) = \frac{\mathcal{F}(x) - \mathfrak{l}}{\mathfrak{u} - \mathfrak{l}}, \ \text{where } \mathfrak{l} \le \mathcal{F}(x) = \frac{\mathfrak{c}x + \gamma}{\mathfrak{d}x + \delta} \le \mathfrak{u}. \tag{2.3}$$

$$\text{Fuzzy goal} : \frac{\mathcal{F}(x) - \mathfrak{l}}{\mathfrak{u} - \mathfrak{l}} - \mathfrak{d}^+ + \mathfrak{d}^- = 1 \tag{2.4}$$

where $\mathfrak{d}^+$, $\mathfrak{d}^-$ are the over and under deviations respectively from the aspiration value unity. The fuzzy fractional goal (2.4) can be further linearized on simplifying as follows.

$$\begin{aligned}
&\mathfrak{m}\mathcal{F}(x) - \mathfrak{m}\mathfrak{l} - \mathfrak{d}^+ + \mathfrak{d}^- = 1, \text{ where } \mathfrak{m} = \frac{1}{\mathfrak{u} - \mathfrak{l}} \\
&\mathfrak{m}\left(\frac{\mathfrak{c}x + \gamma}{\mathfrak{d}x + \delta}\right) - \mathfrak{d}^+ + \mathfrak{d}^- = 1 + \mathfrak{m}\mathfrak{l} \\
&\mathfrak{m}(\mathfrak{c}x + \gamma) - \mathfrak{d}^+(\mathfrak{d}x + \delta) + \mathfrak{d}^-(\mathfrak{d}x + \delta) = \mathfrak{m}'(\mathfrak{d}x + \delta), \text{ where } \mathfrak{m}' = 1 + \mathfrak{m}\mathfrak{l} \\
&(\mathfrak{m}\mathfrak{c} - \mathfrak{m}'\mathfrak{d})x - \mathfrak{d}^+(\mathfrak{d}x + \delta) + \mathfrak{d}^-(\mathfrak{d}x + \delta) = \mathfrak{m}'\delta - \mathfrak{m}\gamma \\
&\mathfrak{n}x - \mathfrak{D}^+ + \mathfrak{D}^- = \mathfrak{p} \\
&\text{where, } \mathfrak{n} = \mathfrak{m}\mathfrak{c} - \mathfrak{m}'\mathfrak{d}, \mathfrak{D}^+ = (\mathfrak{d}x + \delta)\mathfrak{d}^+, \ \mathfrak{D}^- = (\mathfrak{d}x + \delta)\mathfrak{d}^-, \ \ \mathfrak{p} = \mathfrak{m}'\delta - \mathfrak{m}\gamma.
\end{aligned} \tag{2.5}$$

As the considered LFPP remains in maximization form, it involves $\mathfrak{d}^- \leq 1$ to attain a nonzero degree of membership which generates $-\mathfrak{d}x + \mathfrak{D}^- \leq \delta$. A small change is added in our proposed method by eliminating over deviations since it shows the state of full achievement for a problem of maximization form. Thus, the goal (2.4) can be stated as,

$$\frac{\mathcal{F}(x) - \mathfrak{l}}{\mathfrak{u} - \mathfrak{l}} + \mathfrak{d}^- \geq 1. \tag{2.6}$$

The linearization of (2.6) generates the following two constraints, $\mathfrak{n}x + \mathfrak{D}^- \geq \mathfrak{p}$, and $-\mathfrak{d}x + \mathfrak{D}^- \leq \delta$, where, $\mathfrak{n} = \mathfrak{m}\mathfrak{c} - \mathfrak{m}'\mathfrak{d}$, $\mathfrak{D}^- = (\mathfrak{d}x + \delta)\mathfrak{d}^-$, $\mathfrak{p} = \mathfrak{m}'\delta - \mathfrak{m}\gamma$, $\mathfrak{m} = \frac{1}{\mathfrak{u}-\mathfrak{l}}$, $\mathfrak{m}' = 1 + \mathfrak{m}\mathfrak{l}$.

2.3. Method of solution for MOLFPP

Chakraborty and Gupta [14] developed the following solution approach to derive the compromise solution of a MOLFPP.

$$\text{MOLFPP}: \ \text{Max } Z_i(x) = \frac{N_i(x)}{D_i(x)} = \frac{\sum_{j=1}^{n} c_{ij}x_j + \alpha_i}{\sum_{j=1}^{n} d_{ij}x_j + \beta_i}, \qquad i = 1, 2 \dots k \tag{2.7}$$

subject to

$$\Omega_1 = \{Ax \leq b, x \geq 0\}$$

where, $x, c_i, d_i \in \mathbb{R}^n$, $\alpha_i, \beta_i \in \mathbb{R}$, b $\in \mathbb{R}^m$, A $\in \mathbb{R}^{m\times n}$ and $d_i x + \beta_i \geq 0 \ \forall \ x \in \Omega_1$. Consider the index set $I = \{i | \sum_{j=1}^{n} c_{ij}x_j + \alpha_i \geq 0\}$ and $I^c = \{i | \sum_{j=1}^{n} c_{ij}x_j + \alpha_i < 0\} \ \forall \ i \in \{1, 2, \dots, k\}$. Assume that, $\frac{1}{\sum_{j=1}^{n} d_{ij}x_j + \beta_i} \geq t \ \forall \ i \in I$ and $\frac{-1}{\sum_{j=1}^{n} c_{ij}x_j + \alpha_i} \geq t \ \forall \ i \in I^c$. Based on the transformation $y = tx (t > 0)$, the MOLFPP can be transformed into the following model.

$$\text{Max } \{tN_i(y/t), \text{ for } i \in I; tD_i(y/t), \text{ for } i \in I^c\}$$

subject to

$$\Omega_2 = \{tD_i(y/t) \leq 1, \text{ for } i \in I, \ -tN_i(y/t) \leq 1 \text{ for } i \in I^c, \ A(y/t) - b \leq 0, t, y \geq 0\}. \tag{2.8}$$

Defining fuzzy linear membership functions $\mu_i(tN_i(y/t))$ and $\mu_i(tD_i(y/t))$ and using Zimmermann's max-min operator method [53], the above model is formulated into the following single objective optimization.

$$\text{Max } \lambda$$

subject to

$$\mu_i(tN_i(y/t)) \geq \lambda \text{ for } i \in I, \ \mu_i(tD_i(y/t)) \geq \lambda \text{ for } i \in I^c, \ (y, t) \in \Omega_2. \tag{2.9}$$

3. Bi-level multi-objective intuitionistic fuzzy LFPP: problem formulation

Bi-level multi-objective optimization [21] frequently arises in decision making process of hierarchical organizations to adequately address a variety of practical problems. General fuzzy numbers are associated with only degree of membership whereas IFNs are associated with degrees of membership, non-membership and hesitancy. So, it provides a more flexible framework for dealing with the uncertainty and it can be appropriately used to address the informational ambiguity in the optimization problems. Consider the following bi-level multi-objective intuitionistic fuzzy LFPP (BL-MOIFLFPP) in a cooperative environment, which includes all its constants and coefficients as triangular intuitionistic fuzzy numbers.

Model M_1: BL-MOIFLFPP

$$
\begin{aligned}
&\text{(ULDM)}: \max_{X_1} \mathcal{F}_{1i}(x) = \frac{\sum_{j=1}^{n} \tilde{r}^I_{1ij} x_j + \tilde{\gamma}^I_{1i}}{\sum_{j=1}^{n} \tilde{s}^I_{1ij} x_j + \tilde{\delta}^I_{1i}}, \qquad i = 1, 2, \ldots, m_1 \\
&\text{(LLDM)}: \max_{X_2} \mathcal{F}_{2i}(x) = \frac{\sum_{j=1}^{n} \tilde{r}^I_{2ij} x_j + \tilde{\gamma}^I_{2i}}{\sum_{j=1}^{n} \tilde{s}^I_{2ij} x_j + \tilde{\delta}^I_{2i}}, \qquad i = 1, 2, \ldots, m_2 \\
&\text{subject to} \\
&\tilde{\Delta}(x) = \left\{ \sum_{j=1}^{n} \tilde{\mathfrak{t}}^I_{jk} x_j \le \tilde{\mathfrak{p}}^I_k, x_j \ge 0, k = 1, 2, \ldots, m \right\} \\
&\qquad = \left\{ \tilde{T}^I_1 X_1 + \tilde{T}^I_2 X_2 \le \tilde{P}^I; X_1, X_2 \ge 0 \right\}
\end{aligned} \tag{3.1}
$$

where, $\mathcal{F}_{1i}(x) = \big(\mathcal{F}_{11}(x), \mathcal{F}_{12}(x), \ldots \mathcal{F}_{1m_1}(x)\big)$ and $\mathcal{F}_{2i}(x) = \big(\mathcal{F}_{21}(x), \mathcal{F}_{22}(x), \ldots \mathcal{F}_{2m_2}(x)\big)$ are the objective functions of ULDM and LLDM respectively, $x = (X_1, X_2) = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $X_1 \in \mathbb{R}^{n_1}, X_2 \in \mathbb{R}^{n_2}, (n_1 + n_2 = n)$ are controlled independently by the DMs at the upper level and lower level respectively. $\tilde{T}^I_1 \in \mathbb{R}^{m \times n_1}$ and $\tilde{T}^I_2 \in \mathbb{R}^{m \times n_2}$ are the coefficient matrices of X_1 and X_2 respectively consisting of TIFNs as the entries. $\tilde{r}^I_{lij}, \tilde{s}^I_{lij}, \tilde{\gamma}^I_{lij}, \tilde{\delta}^I_{lij}, \tilde{\mathfrak{t}}^I_{jk}, \tilde{\mathfrak{p}}^I_k$ are TIFNs defined on $\mathbb{R}^+$. It is assumed that, $\sum_{j=1}^{n} \tilde{s}^I_{lij} x_j + \tilde{\delta}^I_{li} > 0, l = 1, 2 \;\forall\; x \in \tilde{\Delta}(x)$.

The coefficients and constants present in the objective functions of each level and constraints, exist in form of TIFNs which are defined as follows.

$$
\begin{aligned}
\tilde{r}^I_{1ij} &= (\tilde{r}^{(1)I}_{1ij}, \tilde{r}^{(2)I}_{1ij}, \tilde{r}^{(3)I}_{1ij}; w_{r_{1i}}, u_{r_{1i}}),\ \tilde{r}^I_{2ij} = (\tilde{r}^{(1)I}_{2ij}, \tilde{r}^{(2)I}_{2ij}, \tilde{r}^{(3)I}_{2ij}; w_{r_{2i}}, u_{r_{2i}}), \\
\tilde{s}^I_{1ij} &= (\tilde{s}^{(1)I}_{1ij}, \tilde{s}^{(2)I}_{1ij}, \tilde{s}^{(3)I}_{1ij}; w_{s_{1i}}, u_{s_{1i}}),\ \tilde{s}^I_{2ij} = (\tilde{s}^{(1)I}_{2ij}, \tilde{s}^{(2)I}_{2ij}, \tilde{s}^{(3)I}_{2ij}; w_{s_{2i}}, u_{s_{2i}}), \\
\tilde{\gamma}^I_{1i} &= (\tilde{\gamma}^{(1)I}_{1i}, \tilde{\gamma}^{(2)I}_{1i}, \tilde{\gamma}^{(3)I}_{1i}; w_{\gamma 1i}, u_{\gamma 1i}),\ \tilde{\gamma}^I_{2i} = (\tilde{\gamma}^{(1)I}_{2i}, \tilde{\gamma}^{(2)I}_{2i}, \tilde{\gamma}^{(3)I}_{2i}; w_{\gamma 2i}, u_{\gamma 2i}), \\
\tilde{\delta}^I_{1i} &= (\tilde{\delta}^{(1)I}_{1i}, \tilde{\delta}^{(2)I}_{1i}, \tilde{\delta}^{(3)I}_{1i}; w_{\delta 1i}, u_{\delta 1i}),\ \tilde{\delta}^I_{2i} = (\tilde{\delta}^{(1)I}_{2i}, \tilde{\delta}^{(2)I}_{2i}, \tilde{\delta}^{(3)I}_{2i}; w_{\delta 2i}, u_{\delta 2i}), \\
\tilde{\mathfrak{t}}^I_{jk} &= (\tilde{\mathfrak{t}}^{(1)I}_{jk}, \tilde{\mathfrak{t}}^{(2)I}_{jk}, \tilde{\mathfrak{t}}^{(3)I}_{jk}; w_{\mathfrak{t}jk}, u_{\mathfrak{t}jk}),\ \tilde{\mathfrak{p}}^I_k = (\tilde{\mathfrak{p}}^{(1)I}_k, \tilde{\mathfrak{p}}^{(2)I}_k, \tilde{\mathfrak{p}}^{(3)I}_k; w_{\mathfrak{p}_k}, u_{\mathfrak{p}_k}).
\end{aligned} \tag{3.2}
$$

4. Proposed solution methodology of BL-MOIFLFPP

The concept of intuitionistic fuzzy (α, β)-cuts as discussed in Definition 4 and Theorem 1 *i.e.*, $\widetilde{T}^I_{\alpha,\beta} = \widetilde{T}^I_\alpha \cap \widetilde{T}^I_\beta$, is implemented in both the objective functions and constraints of the BL-MOIFLFPP (3.1). Consequently, the intuitionistic fuzzy parameters as described in (3.2) can be equivalently converted into interval valued forms for all the objective functions belonging to the upper, lower levels and also the constraints. At the upper and lower level, the mathematical expression for the objective functions can be defined as follows.

$$\mathcal{F}_{li}(x) = \frac{\sum_{j=1}^{n}\left\{\left[r_{lij}^{L^*}, r_{lij}^{U^*}\right] \cap \left[r_{lij}^{L^{**}}, r_{lij}^{U^{**}}\right]\right\}x_j + \left\{\left[\gamma_{li}^{L^*}, \gamma_{li}^{U^*}\right] \cap \left[\gamma_{li}^{L^{**}}, \gamma_{li}^{U^{**}}\right]\right\}}{\sum_{j=1}^{n}\left\{\left[s_{lij}^{L^*}, s_{lij}^{U^*}\right] \cap \left[s_{lij}^{L^{**}}, s_{lij}^{U^{**}}\right]\right\}x_j + \left\{\left[\delta_{li}^{L^*}, \delta_{li}^{U^*}\right] \cap \left[\delta_{li}^{L^{**}}, \delta_{li}^{U^{**}}\right]\right\}}$$

where,

$$\begin{aligned}
r_{lij}^{L^*} &= \left(\tilde{r}_{lij}^{(1)I} - \tilde{r}_{lij}^{(2)I}\right) + \frac{\tilde{r}_{lij}^{(2)I}\alpha}{w_{r_{li}}},\ r_{lij}^{U^*} = \left(\tilde{r}_{lij}^{(1)I} + \tilde{r}_{lij}^{(3)I}\right) - \frac{\tilde{r}_{lij}^{(3)I}\alpha}{w_{r_{li}}},\\
r_{lij}^{L^{**}} &= \left(\tilde{r}_{lij}^{(1)I} - \tilde{r}_{lij}^{(2)I}\right) + \frac{(1-\beta)\tilde{r}_{lij}^{(2)I}}{(1-u_{r_{li}})},\ r_{lij}^{U^{**}} = \left(\tilde{r}_{lij}^{(1)I} + \tilde{r}_{lij}^{(3)I}\right) - \frac{(1-\beta)\tilde{r}_{lij}^{(3)I}}{(1-u_{r_{li}})},\\
\gamma_{li}^{L^*} &= \left(\tilde{\gamma}_{li}^{(1)I} - \tilde{\gamma}_{li}^{(2)I}\right) + \frac{\tilde{\gamma}_{li}^{(2)I}\alpha}{w_{li}},\ \gamma_{lj}^{U^*} = \left(\tilde{\gamma}_{li}^{(1)I} + \tilde{\gamma}_{li}^{(3)I}\right) - \frac{\tilde{\gamma}_{li}^{(3)I}\alpha}{w_{li}},\\
\gamma_{li}^{L^{**}} &= \left(\tilde{\gamma}_{li}^{(1)I} - \tilde{\gamma}_{li}^{(2)I}\right) + \frac{\tilde{\gamma}_{li}^{(2)I}\alpha}{w_{li}},\ \gamma_{li}^{U^{**}} = \left(\tilde{\gamma}_{li}^{(1)I} + \tilde{\gamma}_{li}^{(3)I}\right) - \frac{(1-\beta)\tilde{\gamma}_{li}^{(3)I}}{(1-u_{li})}\\
s_{lij}^{L^*} &= \left(\tilde{s}_{lij}^{(1)I} - \tilde{s}_{lij}^{(2)I}\right) + \frac{\tilde{s}_{lij}^{(2)I}\alpha}{w_{s_{li}}},\ s_{lij}^{U^*} = \left(\tilde{s}_{lij}^{(1)I} + \tilde{s}_{1ij}^{(3)I}\right) - \frac{\tilde{s}_{1ij}^{(3)I}\alpha}{w_{s_{li}}},\\
s_{lij}^{L^{**}} &= \left(\tilde{s}_{lij}^{(1)I} - \tilde{s}_{lij}^{(2)I}\right) + \frac{(1-\beta)\tilde{s}_{lij}^{(2)I}}{w_{s_{li}}},\ s_{lij}^{U^{**}} = \left(\tilde{s}_{lij}^{(1)I} + \tilde{s}_{lij}^{(3)I}\right) - \frac{(1-\beta)\tilde{s}_{lij}^{(3)I}}{(1-u_{s_{li}})},\\
\delta_{li}^{L^*} &= \left(\tilde{\delta}_{li}^{(1)I} - \tilde{\delta}_{li}^{(2)I}\right) + \frac{\tilde{\delta}_{li}^{(2)I}\alpha}{w_{li}},\ \delta_{li}^{U^*} = \left(\tilde{\delta}_{li}^{(1)I} + \tilde{\delta}_{li}^{(3)I}\right) - \frac{\tilde{\delta}_{li}^{(3)I}\alpha}{w_{li}}\\
\delta_{li}^{L^{**}} &= \left(\tilde{\delta}_{li}^{(1)I} - \tilde{\delta}_{li}^{(2)I}\right) + \frac{(1-\beta)\tilde{\delta}_{li}^{(2)I}}{(1-u_{li})},\ \delta_{li}^{U^{**}} = \left(\tilde{\delta}_{li}^{(1)I} + \tilde{\delta}_{li}^{(3)I}\right) - \frac{(1-\beta)\tilde{\delta}_{li}^{(3)I}}{(1-u_{li})}.
\end{aligned} \tag{4.1}$$

Based on Theorem 1, the coefficients and constants involved in the objective functions of (4.1) can be converted into single intervals. Suppose that,

$$\begin{aligned}
&\left\{\left[r_{lij}^{L^*}, r_{lij}^{U^*}\right] \cap \left[r_{lij}^{L^{**}}, r_{lij}^{U^{**}}\right]\right\} = [r_{lij}^L, r_{lij}^U],\ \left\{\left[\gamma_{li}^{L^*}, \gamma_{li}^{U^*}\right] \cap \left[\gamma_{li}^{L^{**}}, \gamma_{li}^{U^{**}}\right]\right\} = [\gamma_{li}^L, \gamma_{li}^U]\\
&\left\{\left[s_{lij}^{L^*}, s_{lij}^{U^*}\right] \cap \left[s_{lij}^{L^{**}}, s_{lij}^{U^{**}}\right]\right\} = [s_{lij}^L, s_{lij}^U],\ \left\{\left[\delta_{li}^{L^*}, \delta_{li}^{U^*}\right] \cap \left[\delta_{li}^{L^{**}}, \delta_{li}^{U^{**}}\right]\right\} = [\delta_{li}^L, \delta_{li}^U].
\end{aligned} \tag{4.2}$$

On substituting the single interval expressions of (4.2), the mathematical formulation for the objective functions belonging to both the upper level and lower level as explained in (4.1), can be stated as follows.

$$\begin{aligned}
\mathcal{F}_{li}(x) &= \frac{\sum_{j=1}^{n}[r_{lij}^L, r_{lij}^U]x_j + [\gamma_{li}^L, \gamma_{li}^U]}{\sum_{j=1}^{n}[s_{lij}^L, s_{lij}^U]x_j + [\delta_{li}^L, \delta_{li}^U]} = \frac{\left[\sum_{j=1}^{n} r_{lij}^L x_j + \gamma_{li}^L, \sum_{j=1}^{n} r_{lij}^U x_j + \gamma_{li}^U\right]}{\left[\sum_{j=1}^{n} s_{lij}^L x_j + \delta_{li}^L, \sum_{j=1}^{n} s_{lij}^U x_j + \delta_{li}^U\right]}\\
&= \left[\frac{\sum_{j=1}^{n} r_{lij}^L x_j + \gamma_{li}^L}{\sum_{j=1}^{n} s_{lij}^U x_j + \delta_{li}^U}, \frac{\sum_{j=1}^{n} r_{lij}^U x_j + \gamma_{li}^U}{\sum_{j=1}^{n} s_{lij}^L x_j + \delta_{li}^L}\right] = \left[\mathcal{F}_{li}^L(x), \mathcal{F}_{li}^U(x)\right],\ i = 1,2,3,\ldots,m_l.
\end{aligned} \tag{4.3}$$

At the upper and lower level of (3.1), the objective functions can be mathematically expressed as,

$$\mathcal{F}_{li}(x) = \begin{cases} \left[\mathcal{F}^L_{1i}(x), \mathcal{F}^U_{1i}(x)\right], & i = 1,2,3,\ldots,m_1 \\ \text{for } l = 1(\text{Upper level}) \\ \left[\mathcal{F}^L_{2i}(x), \mathcal{F}^U_{2i}(x)\right], & i = 1,2,3,\ldots,m_2 \\ \text{for } l = 2(\text{Lower level}). \end{cases} \tag{4.4}$$

Subsequently, the constraints can be simplified in the following forms using the concept of intuitionistic fuzzy (α,β)-cuts over the coefficients and constants involved in the constraints (3.1).

$$\begin{aligned} \sum_{j=1}^{n} \Bigg\{ & \left[(\tilde{\mathfrak{t}}^{(1)I}_{jk} - \tilde{\mathfrak{t}}^{(2)I}_{jk}) + \frac{\tilde{\mathfrak{t}}^{(2)I}_{jk}\alpha}{w_{\mathfrak{t}}}, (\tilde{\mathfrak{t}}^{(1)I}_{jk} + \tilde{\mathfrak{t}}^{(3)I}_{jk}) - \frac{\tilde{\mathfrak{t}}^{(3)I}_{jk}\alpha}{w_{\mathfrak{t}}}\right] \\ & \cap \left[(\tilde{\mathfrak{t}}^{(1)I}_{jk} - \tilde{\mathfrak{t}}^{(2)I}_{jk}) + \frac{(1-\beta)\tilde{\mathfrak{t}}^{(2)I}_{jk}}{(1-u_{\mathfrak{t}})}, (\tilde{\mathfrak{t}}^{(1)I}_{jk} + \tilde{\mathfrak{t}}^{(3)I}_{jk}) - \frac{(1-\beta)\tilde{\mathfrak{t}}^{(3)I}_{jk}}{(1-u_{\mathfrak{t}})}\right]\Bigg\} x_j \\ & \leq \Bigg\{\left[(\tilde{\mathfrak{p}}^{(1)I}_{k} - \tilde{\mathfrak{p}}^{(2)I}_{k}) + \frac{\tilde{\mathfrak{p}}^{(2)I}_{k}\alpha}{w_{\mathfrak{p}}}, (\tilde{\mathfrak{p}}^{(1)I}_{k} + \tilde{\mathfrak{p}}^{(3)I}_{k}) - \frac{\tilde{\mathfrak{p}}^{(3)I}_{k}\alpha}{w_{\mathfrak{p}}}\right] \\ & \cap \left[(\tilde{\mathfrak{p}}^{(1)I}_{k} - \tilde{\mathfrak{p}}^{(2)I}_{k}) + \frac{(1-\beta)\tilde{\mathfrak{p}}^{(2)I}_{k}}{(1-u_{\mathfrak{p}})}, (\tilde{\mathfrak{p}}^{(1)I}_{k} + \tilde{\mathfrak{p}}^{(3)I}_{k}) - \frac{(1-\beta)\tilde{\mathfrak{p}}^{(3)I}_{k}}{(1-u_{\mathfrak{p}})}\right]\Bigg\}. \end{aligned} \tag{4.5}$$

According to Theorem 1, minimum of the intervals in (4.5) can be further expressed in form of intervals. Suppose that,

$$\begin{aligned} & \Bigg\{\left[(\tilde{\mathfrak{t}}^{(1)I}_{jk} - \tilde{\mathfrak{t}}^{(2)I}_{jk}) + \frac{\tilde{\mathfrak{t}}^{(2)I}_{jk}\alpha}{w_{\mathfrak{t}}}, (\tilde{\mathfrak{t}}^{(1)I}_{jk} + \tilde{\mathfrak{t}}^{(3)I}_{jk}) - \frac{\tilde{\mathfrak{t}}^{(3)I}_{jk}\alpha}{w_{\mathfrak{t}}}\right] \\ & \quad \cap \left[(\tilde{\mathfrak{t}}^{(1)I}_{jk} - \tilde{\mathfrak{t}}^{(2)I}_{jk}) + \frac{(1-\beta)\tilde{\mathfrak{t}}^{(2)I}_{jk}}{(1-u_{\mathfrak{t}})}, (\tilde{\mathfrak{t}}^{(1)I}_{jk} + \tilde{\mathfrak{t}}^{(3)I}_{jk}) - \frac{(1-\beta)\tilde{\mathfrak{t}}^{(3)I}_{jk}}{(1-u_{\mathfrak{t}})}\right]\Bigg\} \\ & \quad = [\mathfrak{t}^L_{jk}, \mathfrak{t}^U_{jk}], \\ & \Bigg\{\left[(\tilde{p}^{(1)I}_{k} - \tilde{p}^{(2)I}_{k}) + \frac{\tilde{p}^{(2)I}_{k}\alpha}{w_p}, (\tilde{p}^{(1)I}_{k} + \tilde{p}^{(3)I}_{k}) - \frac{\tilde{p}^{(3)I}_{k}\alpha}{w_p}\right] \\ & \quad \cap \left[(\tilde{p}^{(1)I}_{k} - \tilde{p}^{(2)I}_{k}) + \frac{(1-\beta)\tilde{p}^{(2)I}_{k}}{(1-u_p)}, (\tilde{p}^{(1)I}_{k} + \tilde{p}^{(3)I}_{k}) - \frac{(1-\beta)\tilde{p}^{(3)I}_{k}}{(1-u_p)}\right]\Bigg\} \\ & \quad = [p^L_k, p^U_k]. \end{aligned} \tag{4.6}$$

The constraints of (4.5) can be changed into the following forms using the interval valued expressions considered in (4.6).

$$\begin{aligned} & \sum_{j=1}^{n} \left[\mathfrak{t}^L_{jk}, \mathfrak{t}^U_{jk}\right] x_j \leq \left[\mathfrak{p}^L_k, \mathfrak{p}^U_k\right], && x_j \geq 0, \quad k = 1,2,\ldots,m, \ j = 1,2,\ldots,n \\ & i.e., \ \left[\sum_{j=1}^{n} \mathfrak{t}^L_{jk} x_j, \sum_{j=1}^{n} \mathfrak{t}^U_{jk} x_j\right] \leq \left[\mathfrak{p}^L_k, \mathfrak{p}^U_k\right], && x_j \geq 0, \ k = 1,2,\ldots,m, \ j = 1,2,\ldots,n. \end{aligned} \tag{4.7}$$

The constraints of (4.7) can be further formulated into the following linear inqualities [34,51].

$$\sum_{j=1}^{n} \mathfrak{t}_{jk}^{L} x_j \leq \mathfrak{p}_k^L, \quad \sum_{j=1}^{n} \mathfrak{t}_{jk}^{U} x_j \leq \mathfrak{p}_k^U, \qquad k = 1, 2, \ldots, m. \tag{4.8}$$

4.1. Models developed to solve BL-MOIFLFPP

The BL-MOIFLFPP ($\mathbf{M_1}$) in (3.1) can be now formulated as an interval valued BL-MOLFPP with its constraints expressed in form of linear inequalities as described in the following model ($\mathbf{M_2}$).

Model M_2: Interval valued BL-MOLFPP

$$\begin{aligned}
&(\text{ULDM}): \max_{X_1} \mathcal{F}_{1i}(x) = [\mathcal{F}_{1i}^L(x), \mathcal{F}_{1i}^U(x)] = \left[\frac{\sum_{j=1}^n r_{1ij}^L x_j + \gamma_{1i}^L}{\sum_{j=1}^n s_{1ij}^U x_j + \delta_{1i}^U}, \frac{\sum_{j=1}^n r_{1ij}^U x_j + \gamma_{1i}^U}{\sum_{j=1}^n s_{1ij}^L x_j + \delta_{1i}^L}\right], \qquad i = 1,2,3\ldots m_1\\
&(\text{LLDM}): \max_{X_2} \mathcal{F}_{2i}(x) = [\mathcal{F}_{2i}^L(x), \mathcal{F}_{2i}^U(x)] = \left[\frac{\sum_{j=1}^n r_{2ij}^L x_j + \gamma_{2i}^L}{\sum_{j=1}^n s_{2ij}^U x_j + \delta_{2i}^U}, \frac{\sum_{j=1}^n r_{2ij}^U x_j + \gamma_{2i}^U}{\sum_{j=1}^n s_{2ij}^L x_j + \delta_{2i}^L}\right], \qquad i = 1,2,3\ldots m_2\\
&\quad \text{subject to}\\
&\quad \sum_{j=1}^{n} \mathfrak{t}_{jk}^{L} x_j \leq \mathfrak{p}_k^L, \quad \sum_{j=1}^{n} \mathfrak{t}_{jk}^{U} x_j \leq \mathfrak{p}_k^U, \; x_j \geq 0, \qquad k = 1,2,3\ldots m.
\end{aligned} \tag{4.9}$$

Using the concept as described in Section 2.1, each interval valued fractional objective function at both upper and lower levels of the model ($\mathbf{M_2}$) in (4.9) can be individually considered as a bi-objective optimization problem as follows,

$$\begin{aligned}
&\max \; \mathcal{F}_{li}(x) = \{\mathcal{F}_{li}^L(x), \mathcal{F}_{li}^U(x)\} = \left\{\frac{\sum_{j=1}^n r_{lij}^L x_j + \gamma_{li}^L}{\sum_{j=1}^n s_{lij}^U x_j + \delta_{li}^U}, \frac{\sum_{j=1}^n r_{lij}^U x_j + \gamma_{li}^U}{\sum_{j=1}^n s_{lij}^L x_j + \delta_{li}^L}\right\}\\
&\text{subject to}\\
&\quad \sum_{j=1}^{n} \mathfrak{t}_{jk}^{L} x_j \leq \mathfrak{p}_k^L, \quad \sum_{j=1}^{n} \mathfrak{t}_{jk}^{U} x_j \leq \mathfrak{p}_k^U, \; x_j \geq 0, \qquad k = 1,2,\ldots,m.
\end{aligned} \tag{4.10}$$

where,

$$\begin{cases} l = 1 \text{ and } i = 1,2,\ldots,m_1 \text{ at Upper level}\\ l = 2 \text{ and } i = 1,2,\ldots,m_2 \text{ at Lower level}\end{cases}$$

Determine the individual optimal solutions separately for each objective function $\mathcal{F}_{li}(x)$ on solving the problem (4.10) for each l and each i, using the proposed method of Chakraborty and Gupta [14], Section 2.3. Let, $\hat{X}_{li}$ be the individual optimal solutions of $\mathcal{F}_{li}(x)$ respectively at each level. The aspiration values of the interval valued objective functions $[\mathcal{F}_{li}^L(x), \mathcal{F}_{li}^U(x)]$ are computed as $[\hat{\mathcal{F}}_{li}^L, \hat{\mathcal{F}}_{li}^U] = [\mathcal{F}_{li}^L(\hat{X}_{li}), \mathcal{F}_{li}^U(\hat{X}_{li})]$ for each $i = 1, 2, \ldots, m_1$ and $i = 1, 2, ..., m_2$ at level $l = 1$ and $l = 2$ respectively. The goals [35] for the interval valued objective functions at both the levels are designed as follows;

$$[\mathcal{F}_{li}^L(x),\ \mathcal{F}_{li}^U(x)] + [\mathfrak{d}_{li}^{L^-},\ \mathfrak{d}_{li}^{U^-}] - [\mathfrak{d}_{li}^{L^+},\ \mathfrak{d}_{li}^{U^+}] = [\hat{\mathcal{F}}_{li}^L, \hat{\mathcal{F}}_{li}^U], \quad i = 1,2,\ldots m_l \text{ for } l = 1,2$$

where, $[\mathfrak{d}_{li}^{L^-},\ \mathfrak{d}_{li}^{U^-}]$ and $[\mathfrak{d}_{li}^{L^+},\ \mathfrak{d}_{li}^{U^+}]$ are the interval valued under and over deviations from the aspiration values.

To determine the compromise solutions at the upper and lower level, the following model $M_l^{(\mathrm{I})}$ is formulated separately at each level ($l = 1, 2$) using goal programming which eliminates over deviations and minimizes only the under deviations.

Model $M_l^{(\mathrm{I})}, (l = 1, 2)$:

$$
\begin{aligned}
&\min \left[\sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-},\ \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-}\right] \\
&\text{subject to} \\
&\left[\mathcal{F}_{li}^{L}(x),\ \mathcal{F}_{li}^{U}(x)\right] + \left[\mathfrak{d}_{li}^{L^-},\ \mathfrak{d}_{li}^{U^-}\right] \geq \left[\hat{\mathcal{F}}_{li}^{L}, \hat{\mathcal{F}}_{li}^{U}\right], \qquad i = 1, 2, \ldots, m_l \\
&\sum_{j=1}^{n} \mathfrak{t}_{jk}^{L} x_j \leq \mathfrak{p}_k^{L},\ \sum_{j=1}^{n} \mathfrak{t}_{jk}^{U} x_j \leq \mathfrak{p}_k^{U},\ x_j \geq 0, \\
&\mathfrak{d}_{li}^{L^-},\ \mathfrak{d}_{li}^{U^-} \geq 0, \qquad k = 1, 2, \ldots, m.
\end{aligned} \tag{4.11}
$$

The model (4.11) can be equivalently formulated as follows.

Model $M_l^{(\mathrm{II})}, (l = 1, 2)$:

$$
\begin{aligned}
&\min \left(\sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-} + \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-}\right) \\
&\text{subject to} \\
&\mathcal{F}_{li}^{L}(x) + \mathfrak{d}_{li}^{L^-} \geq \hat{\mathcal{F}}_{li}^{L},\ \mathcal{F}_{li}^{U}(x) + \mathfrak{d}_{li}^{U^-} \geq \hat{\mathcal{F}}_{li}^{U}, \qquad i = 1, 2, \ldots, m_l \\
&\sum_{j=1}^{n} \mathfrak{t}_{jk}^{L} x_j \leq \mathfrak{p}_k^{L},\ \sum_{j=1}^{n} \mathfrak{t}_{jk}^{U} x_j \leq \mathfrak{p}_k^{U},\ x_j \geq 0, \\
&\mathfrak{d}_{li}^{L^-}, \mathfrak{d}_{li}^{U^-} \geq 0, \qquad k = 1, 2, \ldots, m.
\end{aligned} \tag{4.12}
$$

Theorem 2. *An optimal solution $(x^*, \mathfrak{d}^*)$ of $M_l^{(\mathrm{II})}$ is an efficient solution of $M_l^{(\mathrm{I})}$.*

Proof. Consider, $(x^*, \mathfrak{d}^*)$ is an optimal solution of $M_l^{(\mathrm{II})}$.

i.e., $\mathcal{F}_{li}^{L}(x^*) + \mathfrak{d}_{li}^{L^-*} \geq \hat{\mathcal{F}}_{li}^{L},\ \mathcal{F}_{li}^{U}(x^*) + \mathfrak{d}_{li}^{U^-*} \geq \hat{\mathcal{F}}_{li}^{U}, \quad i = 1, 2, \ldots, m_l$

i.e., $[\mathcal{F}_{li}^{L}(x^*),\ \mathcal{F}_{li}^{U}(x^*)] + [\mathfrak{d}_{li}^{L^-*},\ \mathfrak{d}_{li}^{U^-*}] \geq [\hat{\mathcal{F}}_{li}^{L}, \hat{\mathcal{F}}_{li}^{U}], \quad i = 1, 2, \ldots m_l$ based on the concept of intervals in Section 2.1.

i.e., $(x^*, \mathfrak{d}^*)$ is a feasible solution of $M_l^{(\mathrm{I})}$.

Suppose that, $(x^*, \mathfrak{d}^*)$ is not an efficient solution of $M_l^{(\mathrm{I})}$.

i.e., $\exists$ a feasible solution $(x, \mathfrak{d})$ of $M_l^{(\mathrm{I})}$ such that,

$$
\left[\sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-},\ \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-}\right] < \left[\sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-*},\ \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-*}\right]
$$

$$i.e., \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-} < \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-*} \text{ and } \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-} < \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-*},$$
$$\text{or} \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-} \leq \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-*} \text{ and } \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-} < \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-*},$$
$$\text{or} \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-} < \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-*} \text{ and } \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-} \leq \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-*}$$
$$i.e., \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-} + \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-} < \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{L^-*} + \sum_{i=1}^{m_l} \mathfrak{d}_{li}^{U^-*}$$

which leads to the contradiction that $(x^*, \mathfrak{d}^*)$ is an optimal solution of $M_l^{(\mathrm{II})}$.

Therefore, $(x^*, \mathfrak{d}^*)$ is an efficient solution of $M_l^{(\mathrm{I})}$. □

Linearizing fractional constraints of $M_l^{(\mathrm{II})}$:

To linearize the functions $\mathcal{F}_{li}^L(x)$ and $\mathcal{F}_{li}^U(x)$ $(i = 1, 2, \ldots, m_l)$ of (4.12) which exist in linear fractional forms, the modified linearization method is used as discussed in Section 3, which can be derived as follows.

$$\mathcal{F}_{li}^L(x) + \mathfrak{d}_{li}^{L^-} \geq \hat{\mathcal{F}}_{li}^L, \; i.e., \; \frac{\sum_{j=1}^n r_{lij}^L x_j + \gamma_{li}^L}{\sum_{j=1}^n s_{lij}^U x_j + \delta_{li}^U} + \mathfrak{d}_{li}^{L^-} \geq \hat{\mathcal{F}}_{li}^L$$
$$\left(\sum_{j=1}^n r_{lij}^L x_j + \gamma_{li}^L\right) + \left(\sum_{j=1}^n s_{lij}^U x_j + \delta_{li}^U\right)\mathfrak{d}_{li}^{L^-} \geq \hat{\mathcal{F}}_{li}^L\left(\sum_{j=1}^n s_{lij}^U x_j + \delta_{li}^U\right)$$
$$\left(\sum_{j=1}^n r_{lij}^L x_j + \gamma_{li}^L\right) - \left(\hat{\mathcal{F}}_{li}^L \sum_{j=1}^n s_{lij}^U x_j\right) + \left(\sum_{j=1}^n s_{lij}^U x_j + \delta_{li}^U\right)\mathfrak{d}_{li}^{L^-} \geq \delta_{li}^U \hat{\mathcal{F}}_{li}^L$$
$$\sum_{j=1}^n (r_{lij}^L - \hat{\mathcal{F}}_{li}^L s_{lij}^U) x_j + \left(\sum_{j=1}^n s_{lij}^U x_j + \delta_{li}^U\right)\mathfrak{d}_{li}^{L^-} \geq \delta_{li}^U \hat{\mathcal{F}}_{li}^L - \gamma_{li}^L$$
$$\bar{\mathcal{F}}_{li}^L(x) + \mathfrak{D}_{li}^{L^-} \geq \hat{\mathcal{F}}_{li}^{L^{(1)}}$$
$$\text{where, } \mathfrak{D}_{li}^{L^-} = \left(\sum_{j=1}^n s_{lij}^U x_j + \delta_{li}^U\right)\mathfrak{d}_{li}^{L^-}, \; \bar{\mathcal{F}}_{li}^L(x) = \sum_{j=1}^n (r_{lij}^L - \hat{\mathcal{F}}_{li}^L s_{lij}^U) x_j \text{ and } \hat{\mathcal{F}}_{li}^{L^{(1)}} = \delta_{li}^U \hat{\mathcal{F}}_{li}^L - \gamma_{li}^L \tag{4.13}$$
$$\text{but, } \mathfrak{d}_{li}^{L^-} \leq \hat{\mathcal{F}}_{li}^L, \; i.e., \; \frac{\mathfrak{D}_{li}^{L^-}}{\left(\sum_{j=1}^n s_{lij}^U x_j + \delta_{li}^U\right)} \leq \hat{\mathcal{F}}_{li}^L$$
$$\mathfrak{D}_{li}^{L^-} \leq \hat{\mathcal{F}}_{li}^L\left(\sum_{j=1}^n s_{lij}^U x_j + \delta_{li}^U\right), \; i.e., \; -\hat{\mathcal{F}}_{li}^L\left(\sum_{j=1}^n s_{lij}^U x_j\right) + \mathfrak{D}_{li}^{L^-} \leq \hat{\mathcal{F}}_{li}^L \delta_{li}^U$$
$$\underline{\mathcal{F}}_{li}^L(x) + \mathfrak{D}_{li}^{L^-} \leq \hat{\mathcal{F}}_{li}^{L^{(2)}},$$
$$\text{where, } \underline{\mathcal{F}}_{li}^L(x) = -\hat{\mathcal{F}}_{li}^L\left(\sum_{j=1}^n s_{lij}^U x_j\right) \text{ and } \hat{\mathcal{F}}_{li}^{L^{(2)}} = \hat{\mathcal{F}}_{li}^L \delta_{li}^U.$$

So, the inequality constraint with fractional function $\mathcal{F}_{li}^L(x)$ can be linearized into the following two inequalities,

$$\mathcal{F}_{li}^L(x) + \mathfrak{d}_{li}^{L^-} \geq \hat{\mathcal{F}}_{li}^L \text{ implies } \left\{ \; \bar{\mathcal{F}}_{li}^L(x) + \mathfrak{D}_{li}^{L^-} \geq \hat{\mathcal{F}}_{li}^{L^{(1)}}, \; \underline{\mathcal{F}}_{li}^L(x) + \mathfrak{D}_{li}^{L^-} \leq \hat{\mathcal{F}}_{li}^{L^{(2)}} \right. \tag{4.14}$$

Similarly,

$$\mathcal{F}_{li}^{U}(x)+\mathfrak{d}_{li}^{U^-}\geq\hat{\mathcal{F}}_{li}^{U} \text{ implies } \left\{ \quad \bar{\mathcal{F}}_{li}^{U}(x)+\mathfrak{D}_{li}^{U^-}\geq\hat{\mathcal{F}}_{li}^{U^{(1)}},\ \underline{\mathcal{F}}_{li}^{U}(x)+\mathfrak{D}_{li}^{U^-}\leq\hat{\mathcal{F}}_{li}^{U^{(2)}} \right. \tag{4.15}$$

Based on the formulation of the above linear inequalities (4.14) and (4.15), the sum of the deviations $\mathfrak{D}_{li}^{L^-}$ and $\mathfrak{D}_{li}^{U^-}$ is to be minimized so as to attain the respective aspiration values. Therefore, the model $\mathbf{M_l^{(II)}}$ (4.12) is reformulated as the following linear model $\mathbf{M_l^{(III)}}$.

Model $M_l^{(\mathrm{III})}$, $(l=1,2)$:

$$\min\left(\sum_{i=1}^{m_l}\mathfrak{D}_{li}^{L^-}+\sum_{i=1}^{m_l}\mathfrak{D}_{li}^{U^-}\right) \tag{4.16}$$

subject to

$$\begin{aligned}&\bar{\mathcal{F}}_{li}^{L}(x)+\mathfrak{D}_{li}^{L^-}\geq\hat{\mathcal{F}}_{li}^{L^{(1)}},\underline{\mathcal{F}}_{li}^{L}(x)+\mathfrak{D}_{li}^{L^-}\leq\hat{\mathcal{F}}_{li}^{L^{(2)}}\\ &\bar{\mathcal{F}}_{li}^{U}(x)+\mathfrak{D}_{li}^{U^-}\geq\hat{\mathcal{F}}_{li}^{U^{(1)}},\underline{\mathcal{F}}_{li}^{U}(x)+\mathfrak{D}_{li}^{U^-}\leq\hat{\mathcal{F}}_{li}^{U^{(2)}}, && i=1,2,\ldots m_l\\ &\sum_{j=1}^{n}\mathfrak{t}_{jk}^{L}x_j\leq\mathfrak{p}_k^{L}\quad\sum_{j=1}^{n}\mathfrak{t}_{jk}^{U}x_j\leq\mathfrak{p}_k^{U},\ x_j\geq0,\ \mathfrak{D}_{li}^{L^-},\mathfrak{D}_{li}^{U^-}\geq0, && k=1,2,\ldots,m.\end{aligned}$$

ULDM ($l=1$) and LLDM ($l=2$) formulate and solve the above model $M_l^{(\mathrm{III})}$ *i.e.*, problems (4.16) to obtain the compromise solutions at the respective level. Let, $X^*=(x_1^*,x_2^*,\ldots,x_n^*)=(X_1^*,X_2^*)$ and $X^{**}=(x_1^{**},x_2^{**},\ldots x_n^{**})=(X_1^{**},X_2^{**})$ be the optimal solutions of the model (4.16) obtained by the ULDM and LLDM respectively. Consider, X_1^* as the aspiration values for the decision variables X_1 which are controlled by ULDM. To provide an extended feasible region for deriving the compromise solution, ULDM sets the goals for its decision variables as $X_1=X_1^*$. The aspiration values of the interval valued objective functions formulated at both the levels of BL-MOIFLFPP in model ($\mathbf{M_2}$), are computed at the compromise solution of the respective levels as follows,

$[\mathcal{F}_{1i}^{L^*},\mathcal{F}_{1i}^{U^*}]=[\mathcal{F}_{1i}^{L}(X^*),\mathcal{F}_{1i}^{U}(X^*)]$ for $[\mathcal{F}_{1i}^{L}(X),\mathcal{F}_{1i}^{U}(X)]$ at the upper level

$[\mathcal{F}_{2i}^{L^{**}},\mathcal{F}_{2i}^{U^{**}}]=[\mathcal{F}_{2i}^{L}(X^{**}),\mathcal{F}_{2i}^{U}(X^{**})]$ for $[\mathcal{F}_{2i}^{L}(X),\mathcal{F}_{2i}^{U}(X)]$ at the lower level.

Goals for the objective functions of model ($\mathbf{M_2}$) are determined as,

$[\mathcal{F}_{1i}^{L}(X),\mathcal{F}_{1i}^{U}(X)]\geq[\mathcal{F}_{1i}^{L^*},\mathcal{F}_{1i}^{U^*}]$, $(l=1,i=1,2,\ldots,m_1)$ at the upper level.

$[\mathcal{F}_{2i}^{L}(X),\mathcal{F}_{2i}^{U}(X)]\geq[\mathcal{F}_{2i}^{L^{**}},\mathcal{F}_{2i}^{U^{**}}]$, $(l=2,i=1,2,\ldots,m_2)$ at the lower level.

To solve the BL-MOIFLFPP (3.1) in model ($\mathbf{M_1}$), the process of goal programming is utilized to formulate the problem as follows.

Find $x=(X_1,X_2)$ so as to minimize $\sum[\mathfrak{d}_{li}^{L^-},\mathfrak{d}_{li}^{U^-}]+(\mathfrak{d}_1^-+\mathfrak{d}_1^+)$ subject to,
$[\mathcal{F}_{li}^{L}(X),\mathcal{F}_{li}^{U}(X)]+[\mathfrak{d}_{li}^{L^-},\mathfrak{d}_{li}^{U^-}]\geq[\mathcal{F}_{li}^{L^*},\mathcal{F}_{li}^{U^*}]$, $(l=1,i=1,2,\ldots,m_1),(l=2,i=1,2,\ldots,m_2)$
$X_1+\mathfrak{d}_1^- - \mathfrak{d}_1^+ = X_1^*$
$\sum_{j=1}^{n}\mathfrak{t}_{jk}^{L}x_j\leq\mathfrak{p}_k^{L},\ \sum_{j=1}^{n}\mathfrak{t}_{jk}^{U}x_j\leq\mathfrak{p}_k^{U},\ x_j\geq0,\quad k=1,2,\ldots,m$

It is expressed as the following final model $(\mathbf{M}_{\mathbf{F}}^{(\mathrm{I})})$ to find a compromise or satisfactory solution of BL-MOIFLFPP (3.1).

Final model $M_F^{(\mathbf{I})}$:

$$\begin{aligned}
&\min \left(\sum_{i=1}^{m_1} \mathfrak{d}_{1i}^{L^-} + \sum_{i=1}^{m_1} \mathfrak{d}_{1i}^{U^-} + \sum_{i=1}^{m_2} \mathfrak{d}_{2i}^{L^-} + \sum_{i=1}^{m_2} \mathfrak{d}_{2i}^{U^-} \right) + (\mathfrak{d}_1^- + \mathfrak{d}_1^+) \\
&\text{subject to} \\
&\quad \mathcal{F}_{1i}^L(x) + \mathfrak{d}_{1i}^{L^-} \geq \mathcal{F}_{1i}^{L^*},\ \mathcal{F}_{1i}^U(x) + \mathfrak{d}_{1i}^{U^-} \geq \mathcal{F}_{1i}^{U^*}, \qquad i = 1, 2, \ldots m_1 \\
&\quad \mathcal{F}_{2i}^L(x) + \mathfrak{d}_{2i}^{L^-} \geq \mathcal{F}_{2i}^{L^{**}},\ \mathcal{F}_{2i}^U(x) + \mathfrak{d}_{2i}^{U^-} \geq \mathcal{F}_{2i}^{U^{**}}, \qquad i = 1, 2, \ldots m_2 \\
&\quad X_1 + \mathfrak{d}_1^- - \mathfrak{d}_1^+ = X_1^* \\
&\quad \sum_{j=1}^{n} \mathfrak{t}_{jk}^L x_j \leq \mathfrak{p}_k^L,\ \sum_{j=1}^{n} \mathfrak{t}_{jk}^U x_j \leq \mathfrak{p}_k^U,\ x_j \geq 0, \qquad k = 1, 2, \ldots, m \\
&\quad \mathfrak{d}_{1i}^{L^-}, \mathfrak{d}_{1i}^{U^-}, \mathfrak{d}_{2i}^{L^-}, \mathfrak{d}_{2i}^{U^-} \geq 0,\ \mathfrak{d}_1^- . \mathfrak{d}_1^+ = 0.
\end{aligned} \tag{4.17}$$

Since the functions $\mathcal{F}_{1i}^L(x)$, $\mathcal{F}_{1i}^U(x)$, $\mathcal{F}_{2i}^L(x)$ and $\mathcal{F}_{2i}^U(x)$ of (4.17) exist in linear fractional forms, using the modified linearization method as discussed in Section 2.2 and derived as (4.14)–(4.15), the following linearizations are performed.

$$\mathcal{F}_{1i}^L(x) + \mathfrak{d}_{1i}^{L^-} \geq \mathcal{F}_{1i}^{L^*} \text{ implies } \left\{ \quad \bar{\bar{\mathcal{F}}}_{1i}^L(x) + \mathfrak{D}_{1i}^{L^-} \geq \mathcal{F}_{1i}^{L^{*(1)}},\ \underline{\underline{\mathcal{F}}}_{1i}^L(x) + \mathfrak{D}_{1i}^{L^-} \leq \mathcal{F}_{1i}^{L^{*(2)}} \right. \tag{4.18}$$

$$\mathcal{F}_{1i}^U(x) + \mathfrak{d}_{1i}^{U^-} \geq \mathcal{F}_{1i}^{U^*} \text{ implies } \left\{ \quad \bar{\bar{\mathcal{F}}}_{1i}^U(x) + \mathfrak{D}_{1i}^{U^-} \geq \mathcal{F}_{1i}^{U^{*(1)}},\ \underline{\underline{\mathcal{F}}}_{1i}^U(x) + \mathfrak{D}_{1i}^{U^-} \leq \mathcal{F}_{1i}^{U^{*(2)}} \right. \tag{4.19}$$

$$\mathcal{F}_{2i}^L(x) + \mathfrak{d}_{2i}^{L^-} \geq \mathcal{F}_{2i}^{L^{**}} \text{ implies } \left\{ \quad \bar{\bar{\mathcal{F}}}_{2i}^L(x) + \mathfrak{D}_{2i}^{L^-} \geq \mathcal{F}_{2i}^{L^{**(1)}},\ \underline{\underline{\mathcal{F}}}_{2i}^L(x) + \mathfrak{D}_{2i}^{L^-} \leq \mathcal{F}_{2i}^{L^{**(2)}} \right. \tag{4.20}$$

$$\mathcal{F}_{2i}^U(x) + \mathfrak{d}_{2i}^{U^-} \geq \mathcal{F}_{2i}^{U^{**}} \text{ implies } \left\{ \quad \bar{\bar{\mathcal{F}}}_{2i}^U(x) + \mathfrak{D}_{2i}^{U^-} \geq \mathcal{F}_{2i}^{U^{**(1)}},\ \underline{\underline{\mathcal{F}}}_{2i}^U(x) + \mathfrak{D}_{2i}^{U^-} \leq \mathcal{F}_{2i}^{U^{**(2)}}. \right. \tag{4.21}$$

The above model $(\mathbf{M}_{\mathbf{F}}^{(\mathrm{I})})$ (4.17) can be simplified and transformed into the following linear model $(\mathbf{M}_{\mathbf{F}}^{(\mathrm{II})})$.

Final model $M_F^{(\mathbf{II})}$:

$$\min \left(\sum_{i=1}^{m_1} \mathfrak{D}_{1i}^{L^-} + \sum_{i=1}^{m_1} \mathfrak{D}_{1i}^{U^-} + \sum_{i=1}^{m_2} \mathfrak{D}_{2i}^{L^-} + \sum_{i=1}^{m_2} \mathfrak{D}_{2i}^{U^-} \right) + (\mathfrak{d}_1^- + \mathfrak{d}_1^+)$$

subject to

$$\bar{\bar{\mathcal{F}}}_{1i}^L + \mathfrak{D}_{1i}^{L^-} \geq \mathcal{F}_{1i}^{L^{*(1)}}, \underline{\underline{\mathcal{F}}}_{1i}^L + \mathfrak{D}_{1i}^{L^-} \leq \mathcal{F}_{1i}^{L^{*(2)}}, \qquad i = 1, 2, \ldots m_1$$

$$
\begin{aligned}
&\bar{\bar{\mathcal{F}}}_{1i}^{U} + \mathfrak{D}_{1i}^{U^-} \geq \mathcal{F}_{1i}^{U^{*(1)}}, \underline{\underline{\mathcal{F}}}_{1i}^{U} + \mathfrak{D}_{1i}^{U^-} \leq \mathcal{F}_{1i}^{U^{*(2)}}, && i = 1,2,\ldots m_1\\
&\bar{\bar{\mathcal{F}}}_{2i}^{L} + \mathfrak{D}_{2i}^{L^-} \geq \mathcal{F}_{2i}^{L^{**(1)}}, \underline{\underline{\mathcal{F}}}_{2i}^{L} + \mathfrak{D}_{2i}^{L^-} \leq \mathcal{F}_{2i}^{L^{**(2)}}, && i = 1,2,\ldots m_2\\
&\bar{\bar{\mathcal{F}}}_{2i}^{U} + \mathfrak{D}_{2i}^{U^-} \geq \mathcal{F}_{2i}^{U^{**(1)}}, \underline{\underline{\mathcal{F}}}_{2i}^{U} + \mathfrak{D}_{2i}^{U^-} \leq \mathcal{F}_{2i}^{U^{**(2)}}, && i = 1,2,\ldots m_2\\
&X_1 + \mathfrak{d}_1^- - \mathfrak{d}_1^+ = X_1^*\\
&\sum_{j=1}^{n} \mathfrak{t}_{jk}^{L} x_j \leq \mathfrak{p}_k^L \quad \sum_{j=1}^{n} \mathfrak{t}_{jk}^{U} x_j \leq \mathfrak{p}_k^U, \ x_j \geq 0\\
&\mathfrak{D}_{1i}^{L^-}, \mathfrak{D}_{1i}^{U^-}, \mathfrak{D}_{2i}^{L^-}, \mathfrak{D}_{2i}^{U^-} \geq 0, \ \mathfrak{d}_1^-.\mathfrak{d}_1^+ = 0.
\end{aligned}
\tag{4.22}
$$

On solving the above model (4.22), a compromise solution can be determined for the BL-MOIFLFPP (3.1). If the DM still remains unsatisfied with the obtained solution, then different values of α and β satisfying $\alpha \in [0, \mathfrak{w}]$, $\beta \in [\mathfrak{u}, 1]$ and $0 \leq \alpha + \beta \leq 1$ can be used to generate another compromise solution.

4.2. Algorithm

As the developed methodology consists of multiple models formulated at the upper and lower level of the bi-level problem, the steps need to be carefully followed in a proper sequence. For the sake of convenience and well understanding, an algorithm is presented below to sequentially narrate the steps formulated in the proposed methodology for determining the compromise solution of the BL-MOIFLFPP.

Step 1. Use the concept of intuitionistic fuzzy (α, β)-cuts defined for the TIFNs which are present in the objective functions as well as constraints involved in the BL-MOIFLFPP (3.1) *i.e.*, model $(\mathbf{M_1})$ satisfying $\alpha \in [0, \mathfrak{w}]$, $\beta \in [\mathfrak{u}, 1]$ and $0 \leq \alpha + \beta \leq 1$.

Step 2. Convert the intersection of intervals obtained in the objective functions, constraints of (4.1) and (4.5) in form of single intervals as proposed in (4.2) and (4.6) respectively.

Step 3. Express the objective functions of BL-MOIFLFPP (3.1) in interval-valued forms as (4.3) and constraints in linear forms as (4.8).

Step 4. Convert the model $(\mathbf{M_1})$ (3.1) in form of the bi-level multi-objective interval-valued fractional programming problem *i.e.*, model $(\mathbf{M_2})$ (4.9).

Step 5. Solve the model (4.10) to determine the individual optimal solutions $\hat{X}_{li}$ of each interval valued fractional objective functions $\mathcal{F}_{li}(x)$ at each level $(l = 1, 2)$ using the proposed concept.

Step 6. Compute the aspiration values $[\hat{\mathcal{F}}_{li}^{L}, \hat{\mathcal{F}}_{li}^{U}]$ of each interval valued objective functions $[\mathcal{F}_{li}^{L}(x), \mathcal{F}_{li}^{U}(x)]$ at $x = \hat{X}_{li}$ for each $i = 1, 2, \ldots, m_1$, and $i = 1, 2, \ldots, m_2$ at level $l = 1$, and $l = 2$ respectively.

Step 7. Formulate the models $\mathbf{M_l^{(I)}}$ and $\mathbf{M_l^{(II)}}$ *i.e.*, (4.11)–(4.12) separately at the upper level $(l = 1)$ and lower level $(l = 2)$ and linearize the fractional constraints of (4.12) as described in (4.13)–(4.15).

Step 8. Solve the model $\mathbf{M_l^{(III)}}$ (4.16) at the upper level $(l = 1)$, and lower level $(l = 2)$ to determine their compromise solutions $X^* = (X_1^*, X_2^*)$ and $X^{**} = (X_1^{**}, X_2^{**})$ respectively.

Step 9. Determine the aspiration values of $[\mathcal{F}_{1i}^{L}(X), \mathcal{F}_{1i}^{U}(X)]$ and $[\mathcal{F}_{2i}^{L}(X), \mathcal{F}_{2i}^{U}(X)]$ as $[\mathcal{F}_{1i}^{L^*}, \mathcal{F}_{1i}^{U^*}]$ and $[\mathcal{F}_{2i}^{L^{**}}, \mathcal{F}_{2i}^{U^{**}}]$ at the upper level $(i = 1, 2, \ldots, m_1)$ and lower level $(i = 1, 2, \ldots, m_2)$ of the BL-MOIFLFPP (3.1) respectively.

Step 10. Calculate the aspiration values of the upper level decision variables X_1 as X_1^* obtained from the solution $X^* = (X_1^*, X_2^*)$ at upper level of the model (4.16).

Step 11. Finally formulate the BL-MOIFLFPP (3.1) *i.e.*, model $(\mathbf{M_1})$ as the model $(\mathbf{M_F^{(I)}})$ (4.17) using fuzzy goal programming and linearize its fractional constraints to construct the model $(\mathbf{M_F^{(II)}})$ (4.22).

Step 12. Solve the model $(\mathbf{M_F^{(II)}})$ (4.22) to obtain the comromise solution of the BL-MOIFLFPP (3.1).

Step 13. If DMs do not get satisfied with the calculated compromise solution, the model $(\mathbf{M_1})$ (3.1) is reformulated and solved by changing $\alpha \in [0, \mathfrak{w}]$, $\beta \in [\mathfrak{u}, 1]$ with $0 \leq \alpha + \beta \leq 1$.

A flowchart Figure 2 is incorporated to narrate the proposed algorithm.

5. Numerical examples

In order to illustrate the solution methodology proposed for BL-MOIFLFPP, we solve the following problems and further discuss the results on comparative basis to justify its effectiveness and feasibility.

5.1. Example 1

Consider the following bi-level multi-objective LFPP in an environment of triangular intuitionistic fuzzy numbers which is initially solved in deterministic form by O.E. Emam [22].

$$\text{(ULDM)} : \max_{x_1} \left(\mathcal{F}_{11} = \frac{\tilde{6}^I x_1 + \tilde{5}^I x_2}{\tilde{1}^I x_1 + \tilde{10}^I x_2 + \tilde{8}^I},\ \mathcal{F}_{12} = \frac{\tilde{9}^I x_1 + \tilde{8}^I x_2}{\tilde{1}^I x_1 + \tilde{10}^I x_2 + \tilde{8}^I} \right) \tag{5.1}$$

$$\text{(LLDM)} : \max_{x_2} \left(\mathcal{F}_{21} = \frac{\tilde{1}^I x_1 + \tilde{2}^I x_2}{\tilde{2}^I x_1 + \tilde{3}^I x_2 + \tilde{5}^I},\ \ \mathcal{F}_{22} = \frac{\tilde{2}^I x_1 + \tilde{3}^I x_2}{\tilde{2}^I x_1 + \tilde{3}^I x_2 + \tilde{5}^I} \right)$$

subject to

$$\tilde{1}^I x_1 + \tilde{1}^I x_2 \leq \tilde{5}^I,\ \tilde{3}^I x_1 + \tilde{1}^I x_2 \leq \tilde{10}^I,\ x_1, x_2 \geq 0$$

where, the fuzzy coefficients and constants associated with the decision variables and both the objective functions, constraints, are all considered as TIFNs. Each one is associated with its maximum degree of membership value as well as minimum degree of non-membership value. The left and right spreads of all the TIFNs defined above are not necessarily considered as same.

$\tilde{6}^I = \langle 6, 4, 8; 0.7, 0.2\rangle$, $\tilde{5}^I = \langle 5, 3, 7; 0.6, 0.3\rangle$, $\tilde{1}^I = \langle 1, 0.5, 1.5; 0.7, 0.1\rangle$,
$\tilde{10}^I = \langle 10, 7, 13; 0.8, 0.1\rangle$, $\tilde{8}^I = \langle 8, 5, 11; 0.6, 0.2\rangle$, $\tilde{9}^I = \langle 9, 7, 11; 0.6, 0.2\rangle$,
$\tilde{8}^I = \langle 8, 6, 10; 0.5, 0.3\rangle$ $\tilde{1}^I = \langle 1, 0.4, 1.6; 0.7, 0.2\rangle$, $\tilde{2}^I = \langle 2, 1.5, 2.5; 0.8, 0.1\rangle$,
$\tilde{2}^I = \langle 2, 1, 3; 0.6, 0.3\rangle$, $\tilde{3}^I = \langle 3, 1, 5; 0.7, 0.2\rangle$, $\tilde{5}^I = \langle 5, 4, 6; 0.7, 0.2\rangle$, $\tilde{2}^I = \langle 2, 0.5, 3.5; 0.6, 0.3\rangle$,
$\tilde{3}^I = \langle 3, 1.5, 4.5; 0.8, 0.1\rangle$, $\tilde{1}^I = \langle 1, 0.5, 1.5; 0.7, 0.1\rangle$, $\tilde{1}^I = \langle 1, 0.3, 1.8; 0.8, 0.1\rangle$, $\tilde{5}^I = \langle 5, 4, 7; 0.5, 0.3\rangle$,
$\tilde{3}^I = \langle 3, 1, 5; 0.6, 0.2\rangle$, $\tilde{1}^I = \langle 1, 0.2, 2; 0.5, 0.4\rangle$, $\tilde{10}^I = \langle 10, 8, 12; 0.6, 0.1\rangle$.

Substituting the triangular intuitionistic fuzzy numbers in the model (5.1), it is expressed in form of the following optimization model (5.2).

$$\begin{aligned}
\text{(ULDM)} : \max_{x_1} \mathcal{F}_{11} &= \frac{\langle 6, 4, 8; 0.7, 0.2\rangle x_1 + \langle 5, 3, 7; 0.6, 0.3\rangle x_2}{\langle 1, 0.5, 1.5; 0.7, 0.1\rangle x_1 + \langle 10, 7, 13; 0.8, 0.1\rangle x_2 + \langle 8, 5, 11; 0.6, 0.2\rangle}\\
\max_{x_1} \mathcal{F}_{12} &= \frac{\langle 9, 7, 11; 0.6, 0.2\rangle x_1 + \langle 8, 6, 10; 0.5, 0.3\rangle x_2}{\langle 1, 0.5, 1.5; 0.7, 0.1\rangle x_1 + \langle 10, 7, 13; 0.8, 0.1\rangle x_2 + \langle 8, 5, 11; 0.6, 0.2\rangle}\\
\text{(LLDM)} : \max_{x_2} \mathcal{F}_{21} &= \frac{\langle 1, 0.4, 1.6; 0.7, 0.2\rangle x_1 + \langle 2, 1.5, 2.5; 0.8, 0.1\rangle x_2}{\langle 2, 1, 3; 0.6, 0.3\rangle x_1 + \langle 3, 1, 5; 0.7, 0.2\rangle x_2 + \langle 5, 4, 6; 0.7, 0.2\rangle}\\
\max_{x_2} \mathcal{F}_{22} &= \frac{\langle 2, 0.5, 3.5; 0.6, 0.3\rangle x_1 + \langle 3, 1.5, 4.5; 0.8, 0.1\rangle x_2}{\langle 2, 1, 3; 0.6, 0.3\rangle x_1 + \langle 3, 1, 5; 0.7, 0.2\rangle x_2 + \langle 5, 4, 6; 0.7, 0.2\rangle}\\
\text{subject to}&\\
&\langle 1, 0.5, 1.5; 0.7, 0.1\rangle x_1 + \langle 1, 0.3, 1.8; 0.8, 0.1\rangle x_2 \leq \langle 5, 4, 7; 0.5, 0.3\rangle\\
&\langle 3, 1, 5; 0.6, 0.2\rangle x_1 + \langle 1, 0.2, 2; 0.5, 0.4\rangle x_2 \leq \langle 10, 8, 12; 0.6, 0.1\rangle\\
&x_1, x_2 \geq 0.
\end{aligned} \tag{5.2}$$

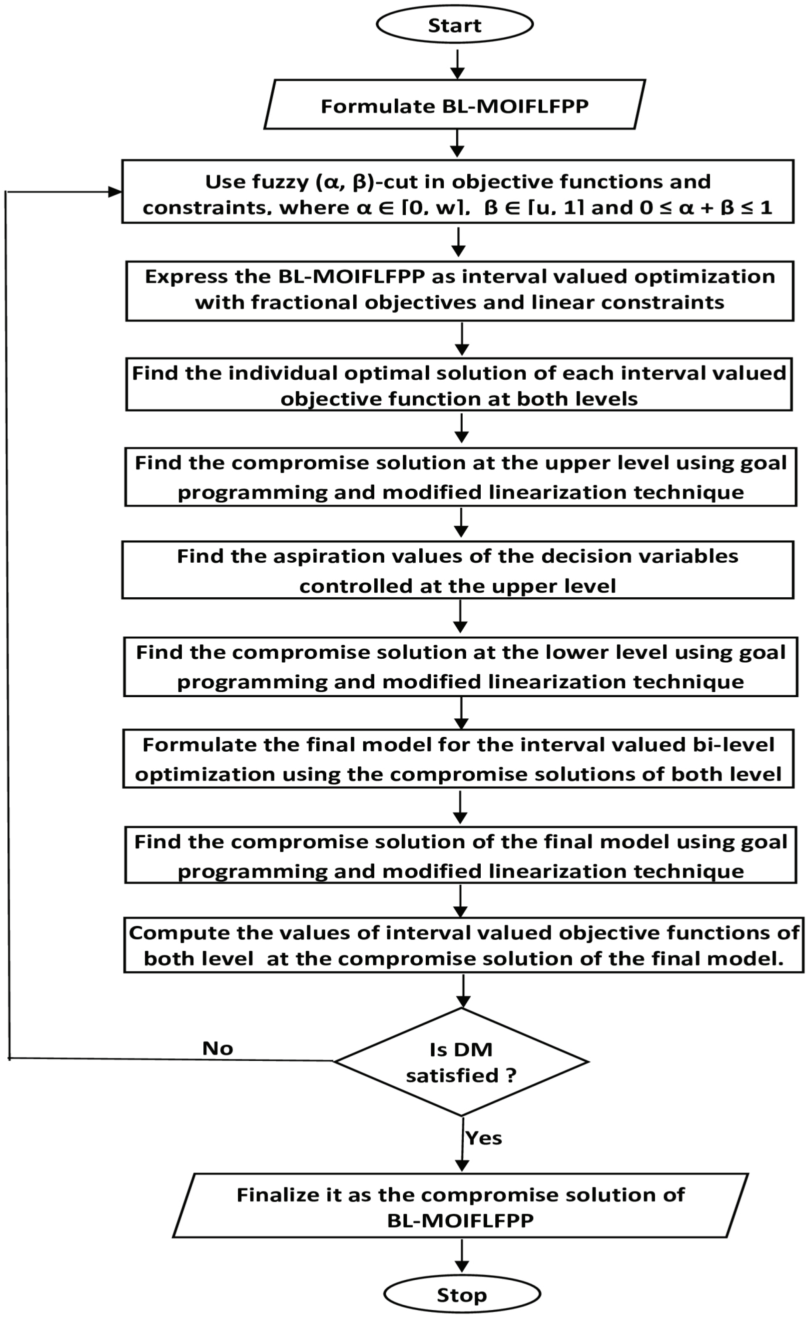

FIGURE 2. Flow chart of the proposed algorithm.

TABLE 1. Optimal solutions of each interval valued objective functions at upper and lower levels, Example 1.

$\hat{X}$	$\hat{\mathcal{F}}^L$	$\hat{\mathcal{F}}^U$
$\hat{X}_{11} = (2.4, 0)$	$\hat{\mathcal{F}}^L_{11} = 0.5363$	$\hat{\mathcal{F}}^U_{11} = 3.3180$
$\hat{X}_{12} = (2.4, 0)$	$\hat{\mathcal{F}}^L_{12} = 0.7374$	$\hat{\mathcal{F}}^U_{12} = 4.8111$
$\hat{X}_{21} = (0, 3.9)$	$\hat{\mathcal{F}}^L_{21} = 0.2207$	$\hat{\mathcal{F}}^U_{21} = 1.0366$
$\hat{X}_{22} = (1.0795, 2.9105)$	$\hat{\mathcal{F}}^L_{22} = 0.2988$	$\hat{\mathcal{F}}^U_{22} = 1.6526$

Solution: Using the proposed concepts of intuitionistic fuzzy (α, β)-cuts with a pair of values $(\alpha = 0.3, \beta = 0.5)$, Theorem 1 and interval analysis, the intuitionistic fuzzy model (5.2) is equivalently expressed in form of the following interval valued bi-level optimization containing linear constraints.

$$\begin{aligned}
(\text{ULDM}) : \max_{x_1} \mathcal{F}_{11} &= \left[\frac{4x_1 + 3.7143x_2}{1.8333x_1 + 17.2222x_2 + 13.5}, \frac{10x_1 + 8x_2}{0.7222x_1 + 6.1111x_2 + 5.5}\right] \\
\max_{x_1} \mathcal{F}_{12} &= \left[\frac{5.5x_1 + 5.6x_2}{1.8333x_1 + 17.2222x_2 + 13.5}, \frac{14.5x_1 + 12x_2}{0.7222x_1 + 6.1111x_2 + 5.5}\right] \\
(\text{LLDM}) : \max_{x_2} \mathcal{F}_{21} &= \left[\frac{0.8x_1 + 1.6667x_2}{3.2857x_1 + 5.5x_2 + 8}, \frac{1.8x_1 + 3.3889x_2}{1.5714x_1 + 2.5x_2 + 3}\right] \\
\max_{x_2} \mathcal{F}_{22} &= \left[\frac{1.7857x_1 + 2.1667x_2}{3.2857x_1 + 5.5x_2 + 8}, \frac{3.5x_1 + 5.5x_2}{1.5714x_1 + 2.5x_2 + 3}\right] \\
&\text{subject to} \\
&0.7222x_1 + 0.8333x_2 \leq 3.4,\ 1.8333x_1 + 2x_2 \leq 7.8 \\
&2.5x_1 + 0.9333x_2 \leq 6,\ 5.5x_1 + 1.6667x_2 \leq 16, x_1, x_2 \geq 0.
\end{aligned} \tag{5.3}$$

At each level of (5.3), interval valued objective functions are individually considered as bi-objective functions as per the proposed concept defined in Section 2.1. Applying the method developed by Chakraborty and Gupta [14], the optimal solution $(\hat{X})$ for each objective function and its optimal lower bound $(\hat{\mathcal{F}}^L)$, upper bound $(\hat{\mathcal{F}}^U)$ are obtained in Table 1.

To find out the compromise solution at the upper level, goal programming and the proposed modified linearization technique are implemented to design the following model.

$$\begin{aligned}
&\min\ (\mathfrak{D}^{L^-}_{11} + \mathfrak{D}^{L^-}_{12} + \mathfrak{D}^{U^-}_{11} + \mathfrak{D}^{U^-}_{12}) \\
&\text{subject to} \\
&1.6179x_1 - 2.9614x_2 + \mathfrak{D}^{L^-}_{11} \geq 3.8829,\ 25.2291x_1 - 40.7338x_2 + \mathfrak{D}^{U^-}_{11} \geq 60.5502 \\
&3.0588x_1 - 5.2354x_2 + \mathfrak{D}^{L^-}_{12} \geq 7.3407,\ 53.0443x_1 - 83.7160x_2 + \mathfrak{D}^{U^-}_{12} \geq 127.3070 \\
&-0.5273x_1 - 4.9534x_2 + \mathfrak{D}^{L^-}_{11} \leq 3.8829,\ -7.9509x_1 - 67.2778x_2 + \mathfrak{D}^{U^-}_{11} \leq 60.5502 \\
&-0.9969x_1 - 9.3648x_2 + \mathfrak{D}^{L^-}_{12} \leq 7.3407,\ -16.7166x_1 - 141.4492x_2 + \mathfrak{D}^{U^-}_{12} \leq 127.3070 \\
&0.7222x_1 + 0.8333x_2 \leq 3.4,\ 1.8333x_1 + 2x_2 \leq 7.8\ 2.5x_1 + 0.9333x_2 \leq 6,\ 5.5x_1 + 1.6667x_2 \leq 16 \\
&x_1, x_2 \geq 0,\ \mathfrak{D}^{L^-}_{11} \geq 0, \mathfrak{D}^{L^-}_{12} \geq 0, \mathfrak{D}^{U^-}_{11} \geq 0, \mathfrak{D}^{U^-}_{12} \geq 0.
\end{aligned} \tag{5.4}$$

TABLE 2. Objective values at the compromise solution of upper level, Example 1.

Upper level	$\mathcal{F}^{L*}$	$\mathcal{F}^{U*}$
Objective function–1:$\mathcal{F}_{11}$	$\mathcal{F}_{11}^{L^*} = 0.5363$	$\mathcal{F}_{11}^{U^*} = 3.3180$
Objective function–2:$\mathcal{F}_{12}$	$\mathcal{F}_{12}^{L^*} = 0.7374$	$\mathcal{F}_{12}^{U^*} = 4.8111$

TABLE 3. Objective values at the compromise solution of lower level, Example 1.

Lower level	$\mathcal{F}^{L**}$	$\mathcal{F}^{U**}$
Objective function–1:$\mathcal{F}_{21}$	$\mathcal{F}_{21}^{L^{**}} = 0.2207$	$\mathcal{F}_{21}^{U^{**}} = 1.0366$
Objective function–2:$\mathcal{F}_{22}$	$\mathcal{F}_{22}^{L^{**}} = 0.2869$	$\mathcal{F}_{22}^{U^{**}} = 1.6823$

On solving the above problem (5.4), the upper level compromise solution is found as $X^* = (2.4, 0)$ where the optimal values of corresponding objective functions are computed in Table 2. The aspiration value of the decision variable x_1 controlled by the upper level DM is considered as $x_1^* = 2.4$.

Similarly, the following model is formulated at the lower level to determine its compromise solution.

$$
\begin{aligned}
&\min\ (\mathfrak{D}_{21}^{L^-} + \mathfrak{D}_{22}^{L^-} + \mathfrak{D}_{21}^{U^-} + \mathfrak{D}_{22}^{U^-})\\
&\text{subject to}\\
&\quad 0.0165x_1 + 0.0999x_2 + \mathfrak{D}_{21}^{L^-} \geq 0.3897, 0.1774x_1 + 0.8266x_2 + \mathfrak{D}_{21}^{U^-} \geq 3.2236\\
&\quad 0.2402x_1 + 0.1564x_2 + \mathfrak{D}_{22}^{L^-} \geq 0.7143,\ 1.4925x_1 + 2.2616x_2 + \mathfrak{D}_{22}^{U^-} \geq 8.1933\\
&\quad -0.1601x_1 - 0.2679x_2 + \mathfrak{D}_{21}^{L^-} \leq 0.3897,\ -1.6885x_1 - 2.6863x_2 + \mathfrak{D}_{21}^{U^-} \leq 3.2236\\
&\quad -0.2934x_1 - 0.4910x_2 + \mathfrak{D}_{22}^{L^-} \leq 0.7143,\ -4.2916x_1 - 6.8277x_2 + \mathfrak{D}_{22}^{U^-} \leq 8.1933\\
&\quad 0.7222x_1 + 0.8333x_2 \leq 3.4,\ 1.8333x_1 + 2x_2 \leq 7.8,\\
&\quad 2.5x_1 + 0.9333x_2 \leq 6,\ 5.5x_1 + 1.6667x_2 \leq 16\\
&\quad x_1, x_2 \geq 0,\ \mathfrak{D}_{11}^{L^-} \geq 0, \mathfrak{D}_{12}^{L^-} \geq 0, \mathfrak{D}_{11}^{U^-} \geq 0, \mathfrak{D}_{12}^{U^-} \geq 0.
\end{aligned} \tag{5.5}
$$

The optimal solution of (5.5) is obtained as $X^{**} = (0.0002412533, 3.899779)$ which is considered as the compromise solution of LLDM and optimal values of the corresponding objective functions are computed in Table 3.

According to the proposed methodology, the final model is formulated as follows to solve the interval valued bi-objective optimization (5.3) and get the compromise solution of the BL-MOIFLFPP (5.1).

$$
\begin{aligned}
&\min\ (\mathfrak{D}_{11}^{L^-} + \mathfrak{D}_{12}^{L^-} + \mathfrak{D}_{11}^{U^-} + \mathfrak{D}_{12}^{U^-} + \mathfrak{D}_{21}^{L^-} + \mathfrak{D}_{22}^{L^-} + \mathfrak{D}_{21}^{U^-} + \mathfrak{D}_{22}^{U^-}) + (\mathfrak{d}^- + \mathfrak{d}^+)\\
&\text{subject to}\\
&\quad 1.6179x_1 - 2.9614x_2 + \mathfrak{D}_{11}^{L^-} \geq 3.8829,\ 25.2291x_1 - 40.73386x_2 + \mathfrak{D}_{11}^{U^-} \geq 60.5502\\
&\quad 3.0588x_1 - 5.2354x_2 + \mathfrak{D}_{12}^{L^-} \geq 7.3407,\ 53.0443x_1 - 83.7160x_2 + \mathfrak{D}_{12}^{U^-} \geq 127.3070\\
&\quad 0.0165x_1 + 0.0999x_2 + \mathfrak{D}_{21}^{L^-} \geq 0.3897,\ 0.1774x_1 + 0.8266x_2 + \mathfrak{D}_{21}^{U^-} \geq 3.2236\\
&\quad 0.2418x_1 + 0.1689x_2 + \mathfrak{D}_{22}^{L^-} \geq 0.6585,\ 1.4407x_1 + 2.1772x_2 + \mathfrak{D}_{22}^{U^-} \geq 8.4904\\
&\quad -0.5273x_1 - 4.9534x_2 + \mathfrak{D}_{11}^{L^-} \leq 3.8829,\ -7.9509x_1 - 67.2778x_2 + \mathfrak{D}_{11}^{U^-} \leq 60.5502\\
&\quad -0.9969x_1 - 9.3648x_2 + \mathfrak{D}_{12}^{L^-} \leq 7.3407,\ -16.7166x_1 - 141.4492x_2 + \mathfrak{D}_{12}^{U^-} \leq 127.3070
\end{aligned} \tag{5.6}
$$

TABLE 4. Objective values at the compromise solution of the BL-MOIFLFPP, Example 1.

Level	$\mathcal{F}^L$	$\mathcal{F}^U$
Upper	$\mathcal{F}_{11}^L = 0.5363$	$\mathcal{F}_{11}^U = 3.3180$
	$\mathcal{F}_{12}^L = 0.7374$	$\mathcal{F}_{12}^U = 4.8111$
Lower	$\mathcal{F}_{21}^L = 0.1209$	$\mathcal{F}_{21}^U = 0.6380$
	$\mathcal{F}_{22}^L = 0.2698$	$\mathcal{F}_{22}^U = 1.2405$

TABLE 5. Comparative objective values of the deterministic and intuitionistic fuzzy BL-MOLFPP.

Level	O.E. Emam [22] (Deterministic BL-MOLFPP)	Proposed approach (Intuitionistic fuzzy BL-MOLFPP)
Upper	$\mathcal{F}_{11} = 1.0952$	$\mathcal{F}_{11} = [\mathcal{F}_{11}^L, \mathcal{F}_{11}^U] = [0.5363, 3.3180]$
	$\mathcal{F}_{12} = 1.6667$	$\mathcal{F}_{12} = [\mathcal{F}_{12}^L, \mathcal{F}_{12}^U] = [0.7374, 4.8111]$
Lower	$\mathcal{F}_{21} = 0.3571$	$\mathcal{F}_{21} = [\mathcal{F}_{21}^L, \mathcal{F}_{21}^U] = [0.1209, 0.6380]$
	$\mathcal{F}_{22} = 0.6429$	$\mathcal{F}_{22} = [\mathcal{F}_{22}^L, \mathcal{F}_{22}^U] = [0.2698, 1.2405]$

$$
\begin{aligned}
&-0.1601x_1 - 0.2679x_2 + \mathfrak{D}_{21}^{L^-} \le 0.3897,\ -1.6885x_1 - 2.6863x_2 + \mathfrak{D}_{21}^{U^-} \le 3.2236\\
&-0.2705x_1 - 0.4527x_2 + \mathfrak{D}_{22}^{L^-} \le 0.6585,\ -4.4473x_1 - 7.0754x_2 + \mathfrak{D}_{22}^{U^-} \le 8.4904\\
&x_1 + \mathfrak{d}^- - \mathfrak{d}^+ = 2.4\\
&0.7222x_1 + 0.8333x_2 \le 3.4,\ 1.8333x_1 + 2x_2 \le 7.8\\
&2.5x_1 + 0.9333x_2 \le 6,\ 5.5x_1 + 1.6667x_2 \le 16\\
&x_1, x_2 \ge 0, \mathfrak{D}_{11}^{L^-} \ge 0, \mathfrak{D}_{12}^{L^-} \ge 0, \mathfrak{D}_{11}^{U^-} \ge 0, \mathfrak{D}_{12}^{U^-} \ge 0,\ \mathfrak{D}_{21}^{L^-} \ge 0,\\
&\mathfrak{D}_{22}^{L^-} \ge 0, \mathfrak{D}_{21}^{U^-} \ge 0, \mathfrak{D}_{22}^{U^-} \ge 0, \mathfrak{d}^-, \mathfrak{d}^+ \ge 0, \mathfrak{d}^-.\mathfrak{d}^+ = 0.
\end{aligned}
$$

On solving the above problem (5.6), the compromise solution for the BL-MOIFLFPP (5.1) is found as $x_1 = 2.4, x_2 = 0$ where the optimal values of the corresponding objective functions are obtained in Table 4.

The following table incorporates the optimal objective values generated by O.E. Emam [22] and the proposed solution approach for the deterministic and intuitionistic fuzzy BL-MOLFPP respectively. A result analysis section is further included after the solution of the practical problem to comparatively study the results in detail (see Tab. 5).

5.2. Example 2 (Practical problem)

A company manufactures three products P_1, P_2 and P_3. The company's goal is to increase the profit while producing more in less time, less energy and less waste so that decision-makers of both levels are satisfied with the production process. The upper level of the company is the governing body who aims to maximize the ratios of profit per cost and production per time. The lower level of the company is the production authority who aims to maximize the ratios of production per power consumption and production per waste product. Each product requires some raw materials and transportation costs for the manufacturing process. At the upper level, P_1, P_2 and P_3 each makes profits of \$4, \$5 and \$6, while the costs of producing each unit are \$10, \$12 and \$13 respectively. A profit of \$2 and a fixed manufacturing cost of \$8 are added during the production process. It produces 15, 17 and 10 units of goods with production times of 10, 9 and 8 mins respectively. A production of 4 units with a fixed production time of 7 mins are added during the manufacturing process. Lower level production authority produces 7, 8 and 9 units of goods with energy consumption of 5 KW, 4 KW and 6 KW, waste materials of 3 KG, 4 KG and 5 KG required respectively. A production of 4 units, power consumption of 5 KW with waste product 2 KG are added during the production process. Each unit of products require 14 KG, 15 KG and 11 KG of raw materials respectively with the supply of raw materials are restricted to 16 KG and fixed transportation cost of \$1, \$2 and \$1 respectively which is restricted to \$2.

This problem can be mathematically designed in the following form with intuitionistic fuzzy parameters.

$$\begin{aligned}
&(\text{ULDM}): \max_{x_1,\ x_2} \mathcal{F}_1 = \left(\frac{\tilde{4}^I x_1 + \tilde{5}^I x_2 + \tilde{6}^I x_3 + \tilde{2}^I}{\widetilde{10}^I x_1 + \tilde{12}^I x_2 + \tilde{13}^I x_3 + \tilde{8}^I},\ \frac{\tilde{15}^I x_1 + \tilde{17}^I x_2 + \tilde{10}^I x_3 + \tilde{4}^I}{\tilde{10}^I x_1 + \tilde{9}^I x_2 + \tilde{8}^I x_3 + \tilde{7}^I} \right)\\
&(\text{LLDM}): \max_{x_3} \mathcal{F}_2 = \left(\frac{\tilde{7}^I x_1 + \tilde{8}^I x_2 + \tilde{9}^I x_3 + 4^I}{\tilde{5}^I x_1 + \tilde{4}^I x_2 + \tilde{6}^I x_3 + 5^I},\ \frac{\tilde{7}^I x_1 + \tilde{8}^I x_2 + \tilde{9}^I x_3 + 4^I}{\tilde{3}^I x_1 + \tilde{4}^I x_2 + \tilde{5}^I x_3 + 2^I} \right) \qquad (5.7)\\
&\quad \text{subject to}\\
&\qquad \tilde{14}^I x_1 + \tilde{15}^I x_2 + \tilde{11}^I x_3 \leq \tilde{16}^I,\ \tilde{1}^I x_1 + \tilde{2}^I x_2 + \tilde{1}^I x_3 \leq \tilde{2}^I,\\
&\qquad x_1, x_2, x_3 \geq 0
\end{aligned}$$

where, the constants and coefficients involved are all considered as intuitionistic fuzzy numbers as follows.

$\tilde{4}^I = \langle 4,2,6;0.7,0.2\rangle$, $\tilde{5}^I = \langle 5,3,7;0.8,0.1\rangle$, $\tilde{6}^I = \langle 6,4,7.5;0.6,0.3\rangle$,
$\tilde{2}^I = \langle 2,1,3;0.6,0.4\rangle$, $\tilde{10}^I = \langle 10,9,12;0.7,0.2\rangle$, $\tilde{12}^I = \langle 12,10,14;0.8,0.1\rangle$,
$\tilde{13}^I = \langle 13,10,16;0.7,0.3\rangle$ $\tilde{8}^I = \langle 8,6,9;0.6,0.3\rangle$, $\tilde{15}^I = \langle 15,11,16;0.8,0.1\rangle$,
$\tilde{17}^I = \langle 17,15,20;0.7,0.3\rangle$, $\tilde{10}^I = \langle 10,7,12;0.8,0.1\rangle$, $\tilde{4}^I = \langle 4,2,6;0.6,0.3\rangle$,
$\tilde{10}^I = \langle 10,7,11;0.6,0.3\rangle$, $\tilde{9}^I = \langle 9,7,10;0.7,0.2\rangle$, $\tilde{8}^I = \langle 8,7,10;0.7,0.3\rangle$,
$\tilde{7}^I = \langle 7,5,10;0.7,0.2\rangle$, $\tilde{7}^I = \langle 7,4,9;0.6,0.3\rangle$, $\tilde{8}^I = \langle 8,5,11;0.7,0.2\rangle$,
$\tilde{9}^I = \langle 9,7,12;0.8,0.1\rangle$, $\tilde{4}^I = \langle 4,2,5;0.7,0.1\rangle$ $\tilde{5}^I = \langle 5,3,6;0.6,0.2\rangle$,
$\tilde{4}^I = \langle 4,3,6;0.8,0.1\rangle$, $\tilde{6}^I = \langle 6,4,7;0.7,0.2\rangle$, $\tilde{5}^I = \langle 5,2,7;0.7,0.2\rangle$,
$\tilde{3}^I = \langle 3,2,5;0.7,0.1\rangle$, $\tilde{4}^I = \langle 4,2,7;0.6,0.3\rangle$, $\tilde{5}^I = \langle 5,2,6;0.6,0.3\rangle$,
$\tilde{2}^I = \langle 2,1,4;0.6,0.3\rangle$, $\tilde{14}^I = \langle 14,10,15;0.7,0.2\rangle$, $\tilde{15}^I = \langle 15,13,17;0.6,0.3\rangle$,
$\tilde{11}^I = \langle 11,10,12;0.6,0.1\rangle$, $\tilde{16}^I = \langle 16,15,6;0.6,0.2\rangle$, $\tilde{1}^I = \langle 1,1,1;0.6,0.3\rangle$,
$\tilde{2}^I = \langle 2,1,3;0.7,0.3\rangle$, $\tilde{1}^I = \langle 1,1,1;0.7,0.3\rangle$, $\tilde{2}^I = \langle 2,2,2;0.7,0.2\rangle$

Solution:

The BL-MOIFLFPP (5.7) is mathematically expressed in form of the following optimization model (5.8) by substituting the values of the intuitionistic triangular fuzzy coefficients and constants.

$$\begin{aligned}
(\text{ULDM}): \max_{x_1,\ x_2} \mathcal{F}_{11} &= \frac{\langle 4,2,6;0.7,0.2\rangle x_1 + \langle 5,3,7;0.8,0.1\rangle x_2 + \langle 6,4,7.5;0.6,0.3\rangle x_3 + \langle 2,1,3;0.6,0.4\rangle}{\langle 10,9,12;0.7,0.2\rangle x_1 + \langle 12,10,14;0.8,0.1\rangle x_2 + \langle 13,10,16;0.7,0.3\rangle x_3 + \langle 8,6,9;0.6,0.3\rangle}\\
\max_{x_1,\ x_2} \mathcal{F}_{12} &= \frac{\langle 15,11,16;0.8,0.1\rangle x_1 + \langle 17,15,20;0.7,0.3\rangle x_2 + \langle 10,7,12;0.8,0.1\rangle x_3 + \langle 4,2,6;0.6,0.3\rangle}{\langle 10,7,11;0.6,0.3\rangle x_1 + \langle 9,7,10;0.7,0.2\rangle x_2 + \langle 8,7,10;0.7,0.3\rangle x_3 + \langle 7,5,10;0.7,0.2\rangle}\\
(\text{LLDM}): \max_{x_3} \mathcal{F}_{21} &= \frac{\langle 7,4,9;0.6,0.3\rangle x_1 + \langle 8,5,11;0.7,0.2\rangle x_2 + \langle 9,7,12;0.8,0.1\rangle x_3 + \langle 4,2,5;0.7,0.1\rangle}{\langle 5,3,6;0.6,0.2\rangle x_1 + \langle 4,3,6;0.8,0.1\rangle x_2 + \langle 6,4,7;0.7,0.2\rangle x_3 + \langle 5,2,7;0.7,0.2\rangle}\\
\max_{x_3} \mathcal{F}_{22} &= \frac{\langle 7,4,9;0.6,0.3\rangle x_1 + \langle 8,5,11;0.7,0.2\rangle x_2 + \langle 9,7,12;0.8,0.1\rangle x_3 + \langle 4,2,5;0.7,0.1\rangle}{\langle 3,2,5;0.7,0.1\rangle x_1 + \langle 4,2,7;0.6,0.3\rangle x_2 + \langle 5,2,6;0.6,0.3\rangle x_3 + \langle 2,1,4;0.6,0.3\rangle}
\end{aligned}$$

subject to

TABLE 6. Optimal solutions of each interval valued objective functions at upper and lower levels, Example 2.

$\hat{X}$	$\hat{\mathcal{F}}^L$	$\hat{\mathcal{F}}^U$
$\hat{X}_{11} = (0, 0.0931, 1.18)$	$\hat{\mathcal{F}}^L_{11} = 0.2666$	$\hat{\mathcal{F}}^U_{11} = 0.5911$
$\hat{X}_{12} = (0.2224, 0.7252, 0)$	$\hat{\mathcal{F}}^L_{12} = 0.7258$	$\hat{\mathcal{F}}^U_{12} = 2.0728$
$\hat{X}_{21} = (0, 0.6363, 0.4348)$	$\hat{\mathcal{F}}^L_{21} = 0.7182$	$\hat{\mathcal{F}}^U_{21} = 2.1970$
$\hat{X}_{22} = (0.2432, 0.7039, 0)$	$\hat{\mathcal{F}}^L_{22} = 1.3000$	$\hat{\mathcal{F}}^U_{22} = 3.0648$

$$\begin{aligned}
&\langle 14, 10, 15; 0.7, 0.2\rangle x_1 + \langle 15, 13, 17; 0.6, 0.3\rangle x_2 + \langle 11, 10, 12; 0.6, 0.1\rangle x_3 \leq \langle 16, 15, 6; 0.6, 0.2\rangle \\
&\langle 1, 1, 1; 0.6, 0.3\rangle x_1 + \langle 2, 1, 3; 0.7, 0.3\rangle x_2 + \langle 1, 1, 1; 0.7, 0.3\rangle x_3 \leq \langle 2, 2, 2; 0.7, 0.2\rangle \\
&x_1, x_2, x_3 \geq 0.
\end{aligned} \tag{5.8}$$

On implementing the proposed concepts of intuitionistic fuzzy (α, β)-cuts with $\alpha = \beta = 0.5$, Theorem 1 and interval analysis, the intuitionistic fuzzy model (5.8) is equivalently transformed into an interval valued bi-level fractional optimization model having linear constraints.

$$\begin{aligned}
(ULDM) : \max_{x_1,\ x_2} \mathcal{F}_{11} &= \left[\frac{3.4286x_1 + 3.8750x_2 + 5.3333x_3 + 1.8333}{13.4286x_1 + 17.25x_2 + 17.5714x_3 + 9.5}, \frac{5.7143x_1 + 7.6250x_2 + 7.1667x_3 + 2.5}{7.4286x_1 + 8.25x_2 + 10.1429x_3 + 7}\right] \\
\max_{x_1,\ x_2} \mathcal{F}_{12} &= \left[\frac{10.8750x_1 + 12.7143x_2 + 7.3750x_3 + 3.6667}{11.8333x_1 + 11.8571x_2 + 10.8571x_3 + 9.8571}, \frac{21x_1 + 22.7143x_2 + 14.5x_3 + 5}{8.8333x_1 + 7x_2 + 6x_3 + 5.5714}\right] \\
(LLDM) : \max_{x_3} \mathcal{F}_{21} &= \left[\frac{6.3333x_1 + 6.5714x_2 + 6.3750x_3 + 3.4286}{6x_1 + 6.25x_2 + 8x_3 + 7}, \frac{8.5x_1 + 11.1429x_2 + 13.5x_3 + 5.4286}{4.5x_1 + 2.8750x_2 + 4.8571x_3 + 4.4286}\right] \\
\max_{x_3} \mathcal{F}_{22} &= \left[\frac{6.3333x_1 + 6.5714x_2 + 6.3750x_3 + 3.4286}{4.4286x_1 + 5.1667x_2 + 6x_3 + 2.6667}, \frac{8.5x_1 + 11.1429x_2 + 13.5x_3 + 5.4286}{2.4286x_1 + 3.6667x_2 + 4.6667x_3 + 1.8333}\right]
\end{aligned} \tag{5.9}$$

subject to

$$\begin{aligned}
&11.1429x_1 + 12.8333x_2 + 9.3333x_3 \leq 13.5,\ 18.2857x_1 + 17.8333x_2 + 13x_3 \leq 17 \\
&0.8333x_1 + 1.7143x_2 + 0.7143x_3 \leq 1.4286,\ 1.1667x_1 + 2.8571x_2 + 1.2857x_3 \leq 2.5714, \\
&x_1, x_2, x_3 \geq 0.
\end{aligned}$$

At each level of (5.9), interval valued objective functions are individually considered as bi-objective functions based on the proposed concept defined in Section 2.1. Using the method designed by Chakraborty and Gupta [14], the optimal solution $(\hat{X})$ for each objective function and its optimal lower bound $(\hat{\mathcal{F}}^L)$, upper bound $(\hat{\mathcal{F}}^U)$ are computed in Table 6.

To find the required compromise solution at the upper level, a model is formulated below implementing goal programming approach and the modified method of linearization by the proposed approach.

$$\min\ (\mathfrak{D}^{L^-}_{11} + \mathfrak{D}^{L^-}_{12} + \mathfrak{D}^{U^-}_{11} + \mathfrak{D}^{U^-}_{12})$$

subject to

$$\begin{aligned}
&-0.0403x_1 - 0.1930x_2 + 0.1730x_3 + \mathfrak{D}^{L^-}_{11} \geq 0.1864,\ 0.7822x_1 + 1.6246x_2 + 0.6923x_3 + \mathfrak{D}^{U^-}_{11} \geq 0.9681 \\
&1.6595x_1 + 2.9818x_2 - 0.3666x_3 + \mathfrak{D}^{L^-}_{12} \geq 2.5313,\ 5.5765x_1 + 17.0067x_2 + 4.2766x_3 + \mathfrak{D}^{U^-}_{12} \geq 13.5735
\end{aligned}$$

TABLE 7. Objective values at the compromise solution of upper level, Example 2.

Upper level	$\mathcal{F}^{L*}$	$\mathcal{F}^{U*}$
Objective function–1:$\mathcal{F}_{11}$	$\mathcal{F}_{11}^{L^{**}} = 0.2163$	$\mathcal{F}_{11}^{U^{**}} = 0.6355$
Objective function–2:$\mathcal{F}_{12}$	$\mathcal{F}_{12}^{L^{**}} = 0.7258$	$\mathcal{F}_{12}^{U^{**}} = 2.0728$

TABLE 8. Objective values at the compromise solution of lower level, Example 2.

Lower level	$\mathcal{F}^{L**}$	$\mathcal{F}^{U**}$
Objective function–1:$\mathcal{F}_{21}$	$\mathcal{F}_{21}^{L^{**}} = 0.7294$	$\mathcal{F}_{21}^{U^{**}} = 2.1561$
Objective function–2:$\mathcal{F}_{22}$	$\mathcal{F}_{22}^{L^{**}} = 1.2772$	$\mathcal{F}_{22}^{U^{**}} = 2.1709$

$$
\begin{aligned}
&-0.9544x_1 - 1.2261x_2 - 1.2489x_3 + \mathfrak{D}_{11}^{L^-} \leq 0.6752,\ -2.5955x_1 - 2.8825x_2 - 3.5439x_3 + \mathfrak{D}_{11}^{U^-} \leq 2.4458\\
&-6.2336x_1 - 6.2462x_2 - 5.7194x_3 + \mathfrak{D}_{12}^{L^-} \leq 5.1926,\ -37.9523x_1 - 30.0755x_2 - 25.7790x_3 + \mathfrak{D}_{12}^{U^-} \leq 23.9375\\
&11.1429x_1 + 12.8333x_2 + 9.3333x_3 \leq 13.5,\ 18.2857x_1 + 17.8333x_2 + 13x_3 \leq 17,\\
&0.8333x_1 + 1.7143x_2 + 0.7143x_3 \leq 1.4286,\ 1.1667x_1 + 2.8571x_2 + 1.2857x_3 \leq 2.5714\\
&x_1, x_2, x_3 \geq 0, \mathfrak{D}_{11}^{L^-} \geq 0, \mathfrak{D}_{12}^{L^-} \geq 0, \mathfrak{D}_{11}^{U^-} \geq 0, \mathfrak{D}_{12}^{U^-} \geq 0.
\end{aligned} \tag{5.10}
$$

On solving the above problem (5.10), $X^* = (0.2222864, 0.7252190, 0.0001761036)$ is obtained as the optimal solution which is also the compromise solution at the upper level, where the optimal values of the objective functions are computed in the following Table 7. So, $x_1^* = 0.2222864$ and $x_2^* = 0.7252190$ are taken as the aspiration values for the decision variables x_1 and x_2 respectively which are controlled at the upper level.

Similarly, the following model is constructed at the lower level to find its compromise solution.

$$
\min\ (\mathfrak{D}_{21}^{L^-} + \mathfrak{D}_{22}^{L^-} + \mathfrak{D}_{21}^{U^-} + \mathfrak{D}_{22}^{U^-})
$$

subject to

$$
\begin{aligned}
&1.4537x_1 + 1.4958x_2 + 0.4520x_3 + \mathfrak{D}_{21}^{L^-} \geq 1.1483,\ -3.0461x_1 + 10.6039x_2 + 6.2152x_3 + \mathfrak{D}_{21}^{U^-} \geq 9.4494\\
&0.7490x_1 - 0.1889x_2 - 1.8525x_3 + \mathfrak{D}_{22}^{L^-} \geq 0.0495,\ 3.2390x_1 - 0.2905x_2 - 2.4595x_3 + \mathfrak{D}_{22}^{U^-} \geq 0.5826\\
&-3.0949x_1 - 3.2238x_2 - 4.1265x_3 + \mathfrak{D}_{21}^{L^-} \leq 3.6107,\ -21.7206x_1 - 13.8771x_2 - 23.4443x_3 + \mathfrak{D}_{21}^{U^-} \leq 21.3760\\
&-7.4843x_1 - 8.7317x_2 - 10.1400x_3 + \mathfrak{D}_{22}^{L^-} \leq 4.5067,\ -22.8118x_1 - 34.4413x_2 - 43.8343x_3 + \mathfrak{D}_{22}^{U^-} \leq 17.2202\\
&11.1429x_1 + 12.8333x_2 + 9.3333x_3 \leq 13.5,\ 18.2857x_1 + 17.8333x_2 + 13x_3 \leq 17\\
&0.8333x_1 + 1.7143x_2 + 0.7143x_3 \leq 1.4286,\ 1.1667x_1 + 2.8571x_2 + 1.2857x_3 \leq 2.5714\\
&x_1, x_2, x_3 \geq 0, \mathfrak{D}_{11}^{L^-} \geq 0, \mathfrak{D}_{12}^{L^-} \geq 0, \mathfrak{D}_{11}^{U^-} \geq 0, \mathfrak{D}_{12}^{U^-} \geq 0.
\end{aligned} \tag{5.11}
$$

The optimal solution of (5.11) is obtained as $X^{**} = (0, 0.8333431, 0)$ which is considered as the compromise solution of the LLDM and optimal values of the corresponding objective functions are calculated in Table 8.

According to the proposed methodology, the final model is formulated as follows to solve the model (5.9) and find the compromise solution of the BL-MOIFLFPP (5.7).

$$
\min\ (\mathfrak{D}_{11}^{L^-} + \mathfrak{D}_{12}^{L^-} + \mathfrak{D}_{11}^{U^-} + \mathfrak{D}_{12}^{U^-} + \mathfrak{D}_{21}^{L^-} + \mathfrak{D}_{22}^{L^-} + \mathfrak{D}_{21}^{U^-} + \mathfrak{D}_{22}^{U^-}) + (\mathfrak{d}_1{}^- + \mathfrak{d}_1{}^+) + (\mathfrak{d}_2{}^- + \mathfrak{d}_2{}^+)
$$

subject to

$$
0.1133x_1 + 0.0311x_2 + 0.3315x_3 + \mathfrak{D}_{11}^{L^-} \geq 0.0480,\ 0.6313x_1 + 1.5138x_2 + 0.4581x_3 + \mathfrak{D}_{11}^{U^-} \geq 1.2383
$$

TABLE 9. Objective values at the compromise solution of the BL-MOIFLFPP, Example 2.

Level	$\mathcal{F}^L$	$\mathcal{F}^U$
Upper	$\mathcal{F}_{11}^L = 0.2120$	$\mathcal{F}_{11}^U = 0.6381$
	$\mathcal{F}_{12}^L = 0.7226$	$\mathcal{F}_{12}^U = 2.0981$
Lower	$\mathcal{F}_{21}^L = 0.7294$	$\mathcal{F}_{21}^U = 2.1561$
	$\mathcal{F}_{22}^L = 1.2772$	$\mathcal{F}_{22}^U = 3.0098$

$$
\begin{aligned}
&1.6595x_1 + 2.9818x_2 - 0.3666x_3 + \mathfrak{D}_{12}^{L^-} \geq 2.5313,\ 5.5765x_1 + 17.0067x_2 + 4.2766x_3 + \mathfrak{D}_{12}^{U^-} \geq 13.5735\\
&1.4274x_1 + 1.4681x_2 + 0.3937x_3 + \mathfrak{D}_{21}^{L^-} \geq 1.2234,\ -2.5924x_1 + 10.66x_2 + 6.5279x_3 + \mathfrak{D}_{21}^{U^-} \geq 8.8829\\
&0.8648x_1 - 0.0351x_2 - 1.6453x_3 + \mathfrak{D}_{22}^{L^-} \geq -0.0290,\ 7.0073x_1 + 6.9097x_2 + 7.3140x_3 + \mathfrak{D}_{22}^{U^-} \geq -3.1449\\
&- 0.6283x_1 - 0.8071x_2 - 0.8221x_3 + \mathfrak{D}_{11}^{L^-} \leq 0.4445,\ -3.0001x_1 - 3.3319x_2 - 4.0963x_3 + \mathfrak{D}_{11}^{U^-} \leq 2.8270\\
&- 6.2336x_1 - 6.2462x_2 - 5.7194x_3 + \mathfrak{D}_{12}^{L^-} \leq 5.1926,\ -37.9523x_1 - 30.0755x_2 - 25.7790x_3 + \mathfrak{D}_{12}^{U^-} \leq 23.9375\\
&- 3.1921x_1 - 3.3251x_2 - 4.2562x_3 + \mathfrak{D}_{21}^{L^-} \leq 3.7242,\ -20.9193x_1 - 13.3652x_2 - 22.5795x_3 + \mathfrak{D}_{21}^{U^-} \leq 20.5875\\
&- 7.2241x_1 - 8.4281x_2 - 9.7874x_3 + \mathfrak{D}_{22}^{L^-} \leq 4.35,\ -11.4454x_1 - 17.2804x_2 - 21.9932x_3 + \mathfrak{D}_{22}^{U^-} \leq 8.64\\
&x_1 + \mathfrak{d}_1^- - \mathfrak{d}_1^+ = 0.2222864\\
&x_2 + \mathfrak{d}_2^- - \mathfrak{d}_2^+ = 0.7252190\\
&11.1429x_1 + 12.8333x_2 + 9.3333x_3 \leq 13.5,\ 18.2857x_1 + 17.8333x_2 + 13x_3 \leq 17\\
&0.8333x_1 + 1.7143x_2 + 0.7143x_3 \leq 1.4286,\ 1.1667x_1 + 2.8571x_2 + 1.2857x_3 \leq 2.5714\\
&x_1, x_2, x_3 \geq 0, \mathfrak{D}_{11}^{L^-} \geq 0, \mathfrak{D}_{12}^{L^-} \geq 0, \mathfrak{D}_{11}^{U^-} \geq 0,\ \mathfrak{D}_{12}^{U^-} \geq 0, \mathfrak{D}_{21}^{L^-} \geq 0, \mathfrak{D}_{22}^{L^-} \geq 0, \mathfrak{D}_{21}^{U^-} \geq 0, \mathfrak{D}_{22}^{U^-} \geq 0,\\
&\mathfrak{d_1}^-, \mathfrak{d_1}^+, \mathfrak{d_2}^-, \mathfrak{d_2}^+ \geq 0, \mathfrak{d}_1^-.\mathfrak{d}_1^+ = 0, \mathfrak{d}_2^-.\mathfrak{d}_2^+ = 0.
\end{aligned} \tag{5.12}
$$

On solving the above problem (5.12), the compromise solution for the BL-MOIFLFPP (5.7) is found as $x_1 = 0.00006907189, x_2 = 0.8333095, x_3 = 0$ where optimal values of the corresponding objective functions are obtained in Table 9.

5.3. Result analysis

In Example 1, O.E. Emam [22] found $x = (3, 1)$ as the compromise solution of the bi-level multi-objective LFPP comprising its parameters in deterministic forms where the corresponding optimal objective values are computed as $\mathcal{F}_{11} = 1.0952$, $\mathcal{F}_{12} = 1.6667$, $\mathcal{F}_{21} = 0.3571$ and $\mathcal{F}_{22} = 0.6429$.

In this paper, we have considered the parameters in form of intuitionistic triangular fuzzy numbers and computed its compromise solution as $x = (2.4, 0)$ using the proposed methodology where the corresponding optimal objective values are obtained in the form of specific range instead of fixed values *i.e.*, $\mathcal{F}_{11} = [0.5363, 3.3180]$, $\mathcal{F}_{12} = [0.7374, 4.8111]$ and $\mathcal{F}_{21} = [0.1209, 0.6380]$, $\mathcal{F}_{22} = [0.2698, 1.2405]$.

It is found that, the optimal objective values found for the deterministic BL-MOLFPP due the existing method proposed by [22], lie within the range of the interval valued optimal objective functions computed using the proposed solution approach *i.e.*, $\mathcal{F}_{11} = 1.0952 \in [0.5363, 3.3180]$, $\mathcal{F}_{12} = 1.6667 \in [0.7374, 4.8111]$, $\mathcal{F}_{21} = 0.3571 \in [0.1209, 0.6380]$ and $\mathcal{F}_{22} = 0.6429 \in [0.2698, 1.2405]$ which validate the feasibility of the proposed solution algorithm.

The following Figure 3 represents that the crisp objective values ($\mathcal{F}_{11} = 1.0952$, $\mathcal{F}_{12} = 1.6667$, $\mathcal{F}_{21} = 0.3571$, $\mathcal{F}_{22} = 0.6429$) of the deterministic problem due to the existing method proposed by O.E. Emam [22], lie within the respective lower bounds ($\mathcal{F}_{11}^L = 0.5363$, $\mathcal{F}_{12}^L = 0.7374$, $\mathcal{F}_{21}^L = 0.1209$, $\mathcal{F}_{22}^L = 0.2698$) and upper

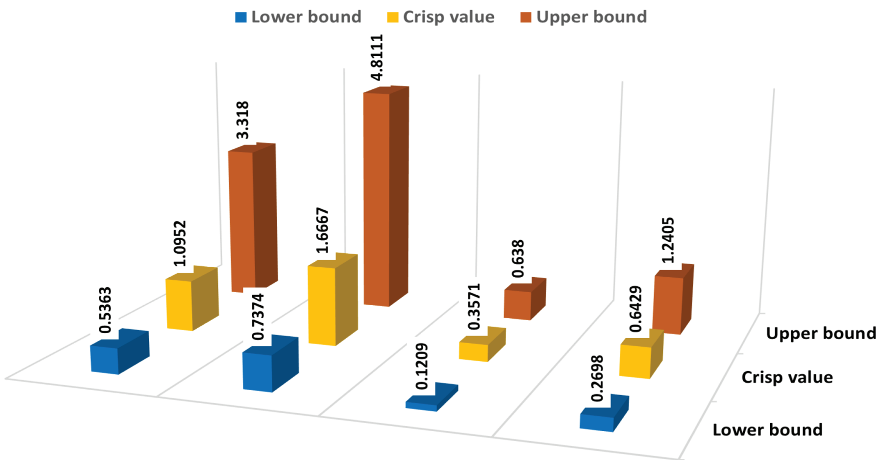

FIGURE 3. Crisp and interval valued objective values.

bounds ($\mathcal{F}_{11}^U = 3.318$, $\mathcal{F}_{12}^U = 4.8111$, $\mathcal{F}_{21}^U = 0.638$, $\mathcal{F}_{22}^U = 1.2405$) for their interval valued optimal objective values of the fuzzy problem due to the proposed solution algorithm.

The majority of the existing approaches in literature deal with BL-MOLFPP in either deterministic form (*i.e.*, with real parameters) or uncertain form with classical fuzzy parameters having degrees of membership only. Besides, some other existing methodologies solve bi-level FPP or only MOLFPP in deterministic or fuzzy environment. But, the proposed methodology deals with BL-MOLFPP having intuitionistic fuzzy parameters which involve both the degrees of membership as well as non-membership to practically fit with many real life cases *i.e.*, the main advantage of the proposed solution methodology.

The solutions of the practical, numerical problems generated by the proposed methodology and the above comparative analysis of results both validate the computational efficiency of the proposed approach. It also comprises multiple models throughout this process which are not much complex in nature but need to be sequentially designed with appropriate formulations. The methodology developed may be comparatively complex with respect to other methodologies as it attempts to derive the compromise solution of BL-MOLFPP in intuitionistic fuzzy environment which is a complex form of practical problem in hierarchical organizations and mostly unsolved in literature.

6. CONCLUSIONS

The uncertainty of many practical problems in decision making context, can be suitably expressed using intuitionistic fuzziness in formulation of the appropriate optimization models. The decision makers at each level of a hierarchical organization often require to optimize multiple fractional functions comprising ambiguous data. Majority of the existing methodologies in literature solve LPP, bi-level LPP or MOLPP with intuitionistic fuzzy parameters but the methodology developed in this paper attempts to solve and derive a compromise solution of a bi-level MOLFPP in an intuitionistic fuzzy environment. The intuitionistic triangular fuzzy model of the bi-level multi-objective optimization is transformed into an equivalent interval valued optimization using fuzzy α, β-cuts for the degrees of membership and also non-membership respectively. It generates the compromise solution using the proposed concepts, modified linearization technique and fuzzy goal programming. The computational results in the numerical section and its analysis incorporated based on the comparisons with an existing deterministic problem, validate the acceptability and feasibility of the proposed method. To derive the computational results of the numerical section, LINGO optimization tool is utilized. As the bi-level MOLFPP intuitionistic fuzzy model is almost unsolved in literature, the proposed methodology can be considered very useful in decision making hierarchical systems. Bi-level MOLFPP comprising trapezoidal, pentagonal, hexagonal etc. types of IFNs can also be solved based on the solution methodology proposed in this paper. In future scope of research, multilevel MOLFPP can be studied in various fully fuzzy environments for developing some efficient and novel solution methodologies.

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References

[1] A. Arora, R. Gupta, and R.R. Saxena, A technique to solve transshipment problem with Asymmetric Pentagonal Fuzzy Numbers. *RAIRO Oper. Res.* **58** (2024) 3487–3499.

[2] A.Y. Adhami, A. Melethil and F. Ahmad, Neutrosophic programming approach to multilevel decision-making model for supplier selection problem in a fuzzy situation. *RAIRO Oper. Res.* **57** (2023) 1307–1328.

[3] S. Aggarwal and C. Gupta, Bi-level multi-objective linear programming under intuitionistic fuzzy environment. *Int. J. Pure Appl. Sci. Tech.* **17** (2013) 45.

[4] F. Ahmad, S. Ahmad, A.T. Soliman and M. Abdollahian, Solving multi-level multiobjective fractional programming problem with rough interval parameter in neutrosophic environment. *RAIRO Oper. Res.* **55** (2021) 2567–2581.

[5] N.A. Alessa, Bi-level linear programming of intuitionistic fuzzy. *Soft Comput.* **25** (2021) 8635–8641.

[6] E.E. Ammar and H.A. Khalifa, Solving fully fuzzy multi-objective linear fractional programming problems based on fuzzy programming approach. *J. Fuzzy Math.* **27** (2019) 301–311.

[7] K. Atanassov, Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **20** (1986) 87–96.

[8] A. Barman, V. Shukla, A. Chakraborty and S. Alam, Multi-objective optimization based decision-making process and its application to optimally select suitable greenhouse site for tomato crops. *RAIRO Oper. Res.* **59** (2025) 877–906.

[9] S.K. Bharati, A.K. Nishad and S.R. Singh, Solution of multi-objective linear programming problems in intuitionistic fuzzy environment, in Proceedings of the Second International Conference on Soft Computing for Problem Solving (SocProS 2012), Vol. 28–30. Springer, India (2012) 161–171.

[10] S.K. Bharati and S.R. Singh, Solution of multi-objective linear programming problems in interval-valued intuitionistic fuzzy environment. *Soft Comput.* **23** (2019) 77–84.

[11] D. Bhati and P. Singh, Branch and bound computational method for multi-objective linear fractional optimization problem. *Neural Comput. Appl.* **28** (2017) 3341–3351.

[12] A. Biswas and A.K. De, An efficient ranking technique for intuitionistic fuzzy numbers with its application in chance constrained bilevel programming.*Adv. Fuzzy Syst.* **1** (2016) 6475403.

[13] H.I. Calvete and C. Gale, The bilevel linear/linear fractional programming problem. *Eur. J. Oper. Res.* **114** (1999) 188–197.

[14] M. Chakraborty and S. Gupta, Fuzzy mathematical programming for multi objective linear fractional programming problem. *Fuzzy Sets Syst.* **125** (2002) 335–342.

[15] A. Charnes and W.W. Cooper, Programming with linear fractional functionals. *Nav. Res. Logist. Q.* **9** (1962) 181–186.

[16] Y. Collette and P. Siarry, Multiobjective Optimization: principles and Case Studies. Springer Science & Business Media (2004).

[17] S.Das and D. Guha, A centroid-based ranking method of trapezoidal intuitionistic fuzzy numbers and its application to MCDM problems. *Fuzzy Inform. Eng.* **8** (2016) 41–74.

[18] S.K. Das, T. Mandal and S.A. Edalatpanah, A new approach for solving fully fuzzy linear fractional programming problems using the multi-objective linear programming. *RAIRO Oper. Res.* **51** (2017) 285–297.

[19] S.K. Das, F.Y. Vincent, S.K. Roy and G.W. Weber, Location–allocation problem for green efficient two-stage vehicle-based logistics system: A type-2 neutrosophic multi-objective modeling approach. *Expert Syst. Appl.* **238** (2024) 122–174.

[20] S. Dempe, V. Kalashnikov, G.A. Perez-Valdes and N. Kalashnykova, Bilevel programming problems. Vol. 10 of *Energy Systems*. Springer, Berlin (2015) 978–3.

[21] S. Dempe, Bilevel optimization: theory, algorithms, applications and a bibliography. Bilevel Optimization: Advances and Next Challenges (2020) 581–672.

[22] O.E. Emam, Interactive approach to bi-level integer multi-objective fractional programming problem. *Appl. Math. Comput.* **223** (2013) 17–24.

[23] E. Fathy, A new method for solving the linear programming problem in an interval-valued intuitionistic fuzzy environment. *Alex. Eng. J.* **61** (2022) 10419–10432.

[24] N. Guzel and M. Sivri, Taylor Series Solution of Multi Objective Linear Fractional Programming Problem. *Trak. Univ. J. Sci.* **6** (2005) 91–98.

[25] S. Islam and W.A. Mandal, Fuzzy Geometric Programming Techniques and Applications. Springer, Singapore (2019).

[26] D.F. Li, A ratio ranking method of triangular intuitionistic fuzzy numbers and its application to MADM problems. *Comput. Math. Appl.* **60** (2010) 1557–1570.

[27] D.F. Li, J.X. Nan and M.J. Zhang, A ranking method of triangular intuitionistic fuzzy numbers and application to decision making. *Int. J. Comput. Intell. Syst.* **3** (2010) 522–530.

[28] S.K. Maiti and S.K. Roy, Multi-choice stochastic bi-level programming problem in cooperative nature via fuzzy programming approach. *J. Ind. Eng. Int.* **12** (2016) 287–298.

[29] S.K. Maiti and S.K. Roy, Analysing interval and multi-choice bi-level programming for Stackelberg game using intuitionistic fuzzy programming. *Int. J. Math. Oper. Res.* **16** (2020) 354–375.

[30] S.K. Maiti, S.K. Roy and G.W. Weber, Gaussian type-2 fuzzy cooperative game based on reduction method: An application to multi-drug resistance problem. *J. Dyn. Games.* **12** (2025) 215–242.

[31] S.K. Maiti and S.K. Roy, Bi-level programming for Stackelberg game with intuitionistic fuzzy number: a ranking approach. *J. Oper. Res. Soc. of China.* **9** (2021) 131–149.

[32] M. Malik and S.K. Gupta, An application of fully intuitionistic fuzzy multi-objective linear fractional programming problem in e-education system. *Int. J. Fuzzy Syst.* **24** (2022) 3544–3563.

[33] D. Mardanya and S.K. Roy, New approach to solve fuzzy multi-objective multi-item solid transportation problem. *RAIRO Oper. Res.* **57** (2023) 99–120.

[34] A. Mehra, S. Chandra and C.R. Bector, Acceptable optimality in linear fractional programming with fuzzy coefficients. *Fuzzy Opt. Dec. Mak.* **6** (2007) 5–16.

[35] K. Miettinen, Nonlinear Multiobjective Optimization. Springer Science & Business Media (1999) 12.

[36] S. Mishra, Weighting method for bi-level linear fractional programming problems. *Eur. J. Oper. Res.* **183** (2007) 296–302.

[37] D.M. Moges, A.R. Mushi and B.G. Wordofa, A new method for intuitionistic fuzzy multi-objective linear fractional optimization problem and its application in agricultural land allocation problem. *Inform. Sci.* **625** (2023) 457–475.

[38] D.M. Moges, B.G. Wordofa and A.R. Mushi, Solving multi-objective linear fractional decentralized bi-level decision-making problems through compensatory intuitionistic fuzzy mathematical method. *J. Comput. Sci.* 71 (2023) 102075.

[39] D. Mollalign, A. Mushi and B. Guta, Solving multi-objective multilevel programming problems using two-phase intuitionistic fuzzy goal programming method. *J. Comput. Sci.* **63** (2022) 101786.

[40] H. Omrani, S. Mohammadi and A. Emrouznejad, A bi-level multi-objective data envelopment analysis model for estimating profit and operational efficiency of bank branches. *RAIRO Oper. Res.* **53** (2019) 1633–1648.

[41] B.B. Pal, B.N. Moitra and U. Maulik, A goal programming procedure for fuzzy multiobjective linear fractional programming problem. *Fuzzy Sets Syst.* **139** (2023) 395–405.

[42] S.K. Roy and S.K. Maiti, Reduction methods of type-2 fuzzy variables and their applications to Stackelberg game. *Appl. Intell.* **50** (2020) 1398–1415.

[43] D. Sahoo, A.K. Tripathy and J.K. Pati, Study on multi-objective linear fractional programming problem involving pentagonal intuitionistic fuzzy number. *Results Control Opt.* **6** (2022) 100091.

[44] S. Schaible, Fractional programming. I duality. *Manag. Sci.* **22** (1976) 858–867.

[45] M.R. Seikh, P.K. Nayak and M. Pal, Notes on triangular intuitionistic fuzzy numbers. *Int. J. Math. Oper. Res.* **5** (2013) 446–465.

[46] S.K. Singh and S.P. Yadav, Fuzzy programming approach for solving intuitionistic fuzzy linear fractional programming problem. *Int. J. Fuzzy Syst.* **18** (2016) 263–269.

[47] V.P. Singh, K. Sharma, D. Chakraborty and A. Ebrahimnejad, A novel multi-objective bi-level programming problem under intuitionistic fuzzy environment and its application in production planning problem. *Complex Intell. Syst.* **8** (2022) 3263–3278.

[48] I.M. Stancu-Minasian, Fractional Programming: theory, Methods and Applications. Springer Science & Business Media (2012) 409.

[49] B. Stanojevic, S. Dzitac and I. Dzitac, Fuzzy numbers and fractional programming in making decisions. *Int. J. Inform. Tech. Dec. Mak.* **19** (2020) 1123–1147.

[50] M.D. Toksari, Taylor series approach to fuzzy multiobjective linear fractional programming. *Inform. Sci.* **178** (2008) 1189-1204.

[51] H.C. Wu, On interval-valued nonlinear programming problems. *J. Math. Anal. Appl.* **338** (2008) 299–316.

[52] P. Yuvashri and A. Saraswathi, Multiobjective linear fractional programming model with equality and inequality constraints under pentagonal intuitionistic fuzzy environment. *OPSEARCH.* **62** (2025) 1991–2028.

[53] H.J. Zimmermann, Description and optimization of fuzzy systems. *Int. J. General Syst.* **2** (1975) 209–215.