

## STRONG DEFENSIVE ALLIANCES ON GRAPH OPERATORS

RAÚL JUÁREZ MORALES<sup>1,\*</sup>, GERARDO REYNA HERNÁNDEZ<sup>2</sup>,  
OMAR ROSARIO CAYETANO<sup>2</sup> AND JAIR CASTRO SIMÓN<sup>2</sup>

**Abstract.** If  $G = (V(G), E(G))$  is a simple connected graph with vertex set  $V(G)$  and edge set  $E(G)$ , we say that a subset  $D \subseteq V(G)$  is a *strong defensive alliance* if for every vertex  $v \in D$  the condition  $\delta_D(v) \geq \delta_{\overline{D}}(v)$  holds. The *strong defensive alliance number*  $\alpha(G)$  is defined as the minimum cardinality among all the strong defensive alliances. A *unitary operator* of graphs  $\mathcal{O}$  assigns to each graph  $G$  a graph  $\mathcal{O}(G)$ . A few examples of unitary operators of graphs are: Subdivision  $S(G)$ ,  $R(G)$ , Middle  $Q(G)$ , Total  $T(G)$ , and Central  $\mathcal{C}(G)$ . In this paper we determine the exact values of  $\alpha(S(G))$  and  $\alpha(R(G))$ . We also characterize the graphs  $G$  for which the number of strong defensive alliances is 1, 2, or 3 in  $Q(G)$  and  $T(G)$ . We also we give tight bounds for  $\alpha(\overline{S(G)})$ ,  $\alpha(\overline{Q(G)})$ ,  $\alpha(\overline{Q(G)})$ ,  $\alpha(\overline{T(G)})$ , and  $\alpha(\overline{\mathcal{C}(G)})$ .

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### 1. INTRODUCTION

Connections and collaborations naturally emerge in everyday life. At their core, alliances are groups of entities that share goals or common characteristics. Alliances can be found in various contexts: a team united by a shared project, a community of species coexisting in the same ecosystem, a network of entrepreneurs working to boost a specific industry, a group of Instagram followers who frequently interact, or even an online community with shared interests or hobbies. This reflects how individuals or elements naturally unite around shared similarities or common purposes.

A strong defensive alliance in a graph is a subset of vertices where each member is supported by a majority of its neighbors within the set, making it resistant to potential external attacks. The first results on defensive alliances were presented in [16, 21], and since then, several related works have appeared in the literature, such as those in [9, 23–28]. The complexity of computing the minimum cardinality of defensive alliances in graphs was studied in [7, 18, 19], it was shown to be an NP-complete problem.

Throughout this paper,  $G = (V(G), E(G))$  denotes a simple and connected graph of order  $|V(G)| = n > 1$  and size  $|E(G)| = m$ . Two adjacent vertices  $u$  and  $v$  are denoted by  $u \sim v$ . For a nonempty set  $X \subseteq V(G)$  and a vertex  $v \in V(G)$ ,  $N_X(v)$  denotes the set of neighbors of  $v$  in  $X$ , that is,  $N_X(v) := \{u \in X : u \sim v\}$ . The *degree*

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<sup>1</sup> Facultad de Ciencias y Tecnologías de la Información, Universidad Autónoma de Guerrero, Las Colinas 37A, Col. Fracc. Las Playas, Acapulco, Guerrero, México.

<sup>2</sup> Facultad de Matemáticas, Universidad Autónoma de Guerrero, Carlos E. Adame 5, Col. La Garita, Acapulco, Guerrero, México.

\*Corresponding author: [raul.juarezm@uagro.mx](mailto:raul.juarezm@uagro.mx)

of  $v$  in  $X$  is denoted by  $\delta_X(v) = |N_X(v)|$ . The minimum and maximum degree of  $G$  are denoted by  $\delta(G)$  and  $\Delta(G)$  respectively. The subgraph induced by  $D \subset V(G)$  is denoted by  $\langle D \rangle$ , and the complement set  $V(G) - D$  is denoted by  $\bar{D}$ . The complement graph of  $G$  is denoted by  $\bar{G} = (V(G), E'(G))$ , where  $E'(G) = E_k - E(G)$  and  $E_k$  is the edge set of the complete graph  $K_n$ .

In this work, we will denote by  $K_n$ ,  $K_{m,n}$ ,  $S_n$ ,  $S_{m,n}$ ,  $W_n$ ,  $C_n$ ,  $P_n$ , and  $T_n$  the complete, complete bipartite, star, double star, wheel, cycle, path, and tree graphs, respectively.

A nonempty set  $D \subseteq V(G)$  is a *defensive  $k$ -alliance* in  $G$ , if for every  $v \in D$

$$\delta_D(v) \geq \delta_{\bar{D}}(v) + k \quad \text{for } k \in [-\Delta(G), \Delta(G)] \cap \mathbb{Z}. \quad (1)$$

Informally, a defensive  $k$ -alliance in a graph  $G$  is a set  $D$  of vertices in  $G$  such that every vertex in  $D$  has at least  $k$  more neighbors within  $D$  than outside it.

A *defensive alliance* is a defensive  $(-1)$ -alliance in  $G$ . The *defensive alliance number* of  $G$ , denoted by  $a(G)$ , is defined as the minimum cardinality of defensive alliances in  $G$ . A *strong defensive alliance* is a defensive  $0$ -alliance. For a strong defensive alliance  $D$ , each vertex in  $D$  is said to be strongly protected by numerical superiority against a possible attack from its neighbors outside  $D$ . The *strong defensive alliance number* of  $G$ , denoted by  $\alpha(G)$ , is defined as the minimum cardinality of strong defensive alliances in  $G$ . For convenience, we refer to a minimum strong defensive alliance as an  $\alpha$ -set.

A *unitary graph operator*  $\mathcal{O}$  associates each graph  $G$  with a graph  $\mathcal{O}(G)$ . The unitary graph operators we will work with are: Subdivision  $S(G)$ ,  $R(G)$ ,  $Q(G)$ , Total  $T(G)$  (see [12]), and Central  $\mathcal{C}(G)$  (see [1, 20, 30]). Intuitively these are described as follows (Fig. 1):

The *Subdivision graph*  $S(G)$  of a graph  $G$  is obtained from  $G$  by inserting an additional vertex into each edge of  $G$ . The graph  $R(G)$  is obtained from  $G$  by adding a new vertex corresponding to each edge of  $G$  and by joining each new vertex to the endpoints of its corresponding edge. The graph  $Q(G)$  is obtained from  $G$  by inserting a new vertex into each edge of  $G$  and connecting pairs of these new vertices if and only if the corresponding edges in  $G$  are incident. The *Total graph*  $T(G)$  of a graph  $G$  is the graph whose vertices are the vertices and edges of  $G$ , with two vertices of  $T(G)$  adjacent if and only if the corresponding elements of  $G$  are adjacent or incident. The *Central graph* of a graph  $G$  is obtained by inserting a new vertex for each edge of  $G$  and joining all the nonadjacent vertices of  $G$ , it is denoted by  $\mathcal{C}(G)$ .

In [6], Bindusree *et al.* investigated the relationships between the Zagreb polynomials of a graph  $G$  and those of the graphs obtained by applying the operators  $S(G)$ ,  $R(G)$ , and  $Q(G)$ . Further information about  $S(G)$  can be found in [8, 10, 11, 17]. The graph  $R(G)$  has been studied in Basilio *et al.* [4], where the relationship between the differential of  $G$  and the differential of  $R(G)$  is established, along with tight bounds for the differential of  $R(G)$ . Additional studies on  $R(G)$  can be found in [6, 15]. The graph  $Q(G)$  has been analyzed in [2, 3, 13, 14, 29]. In the 2020 work by Basilio *et al.*, they establish the relationship between the differential of a graph and its differential under the  $Q(G)$  operator. Furthermore, they present results connecting the differential of  $G$  with its domination and independence numbers. The Total operator has been studied by Nagarathinam and Parvathi in 2018 [22], where they examine the relationship between the chromatic number and the  $b$ -chromatic number of a graph  $G$  and those of the graphs  $S(G)$ ,  $R(G)$ ,  $Q(G)$ , and  $T(G)$ . Bermudo [5] provides bounds for the total domination number of these graphs and establishes the exact value of the total domination number for some of them.

Throughout this paper, we focus on strong defensive alliances. Note that, since the study of a strong defensive alliance  $D \subseteq V(G)$  (when  $\langle D \rangle$  is not connected) can be carried out separately for each connected component, we will assume henceforth that every strong defensive alliance induces a connected subgraph. Therefore, we will assume henceforth that every strong defensive alliance induces a connected subgraph.

The aim of this work is to analyze the strong defensive alliance number in graphs obtained by applying the previously defined unitary graph operators, which we formally define as follows.

Given a graph  $G = (V(G), E(G))$ , for  $e = v_1 \sim v_2 \in E(G)$ , let  $V(e) := \{v_1, v_2\}$ . Now, let us define:

$$E_E := \{\{e_1, e_2\} : e_1, e_2 \in E(G), e_1 \neq e_2, |V(e_1) \cap V(e_2)| = 1\},$$

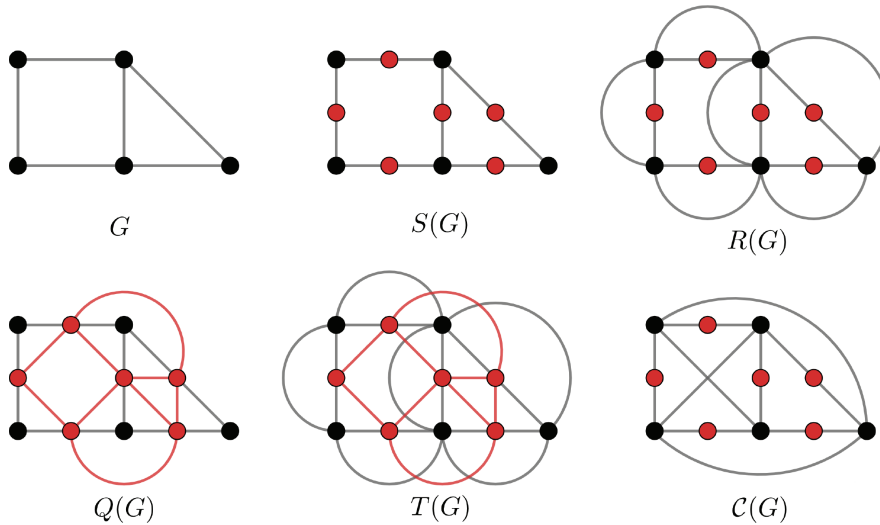


FIGURE 1. A graph  $G$  and the corresponding graphs  $S(G)$ ,  $R(G)$ ,  $Q(G)$ ,  $T(G)$  and  $\mathcal{C}(G)$ .

and:

$$E_V := \{\{e, v\} : e \in E(G), v \in V(e)\}.$$

Then:

- $S(G) = (V(G) \cup E(G), E_V)$ ,
- $R(G) = (V(G) \cup E(G), E(G) \cup E_V)$ ,
- $Q(G) = (V(G) \cup E(G), E_E \cup E_V)$ ,
- $T(G) = (V(G) \cup E(G), E_E \cup E_V \cup E(G))$ ,
- $\mathcal{C}(G) = (V(G) \cup E(G), E_V \cup E(\overline{G}))$ .

Due to the nature of the operators considered in this work, the edges of  $G$  also serve as vertices of  $\mathcal{O}(G)$ . In each case, the context clarifies which role is assumed.

The following lemma is used throughout this paper to simplify several proofs. Although it will not always be cited explicitly, its contribution is evident from the context.

**Lemma 1.** *Let  $I$  be an independent set of a graph  $G$ . If  $D$  is a strong defensive alliance of  $G$ , then  $D \cap \overline{I} \neq \emptyset$ .*

*Proof.* Let  $D$  be a subset of  $V(G)$  such that  $D \subseteq I$ . Note that each element  $v \in D$  satisfies that  $\delta_D(v) = 0$ , since  $D$  is a independent set in  $G$ . Therefore  $\delta_D(v) = 0 < 1 \leq \delta_{\overline{D}}(v)$ .  $\square$

## 2. STRONG DEFENSIVE ALLIANCES IN $S(G)$ AND $\overline{S(G)}$

We begin this section by making some observations on the operator  $S(G)$ .

**Remark 1.** Let  $G$  be a graph of order  $n$  and size  $m$ . Then:

- (i)  $|V(S(G))| = n + m$ ,
- (ii)  $|E(S(G))| = 2m$ ,
- (iii) If  $v \in V(G)$ , then  $\delta_{S(G)}(v) = \delta_G(v)$ ,
- (iv) If  $e \in E(G)$ , then  $\delta_{S(G)}(e) = 2$ .

**Theorem 2.** For each graph  $G$  it holds that

$$\alpha(S(G)) = 1 + \left\lceil \frac{\delta(G)}{2} \right\rceil.$$

*Proof.* Let  $v \in V(G)$  such that  $\delta(v) = \delta(G)$  and let  $e_1, e_2, \dots, e_{\lceil \frac{\delta(G)}{2} \rceil}$  be edges incident to  $v$  in  $G$ . Then  $D = \{v, e_1, e_2, \dots, e_{\lceil \frac{\delta(G)}{2} \rceil}\}$  is a strong defensive alliance in  $S(G)$ . Hence  $\alpha(S(G)) \leq 1 + \lceil \frac{\delta(G)}{2} \rceil$ . On the other hand, let  $D'$  be an  $\alpha$ -set. By Lemma 1,  $D'$  cannot consist solely of vertices from  $E$ . Therefore,  $D'$  must contain at least one vertex  $v$  from  $G$ , which implies that the upper half of the neighbors of  $v$  must also be included in  $D'$ . Consequently,  $D'$  contains at least  $1 + \lceil \frac{\delta_{S(G)}(v)}{2} \rceil = 1 + \lceil \frac{\delta_G(v)}{2} \rceil \geq 1 + \lceil \frac{\delta(G)}{2} \rceil$  vertices.  $\square$

According to Remark 1, the following properties of  $\overline{S(G)}$  can be deduced.

**Remark 2.** Let  $G$  be a graph of order  $n$  and size  $m$ . Then:

- (i)  $|V(\overline{S(G)})| = n + m,$
- (ii)  $|E(\overline{S(G)})| = \frac{(n+m)(n+m-1)}{2} - 2m,$
- (iii) If  $v \in V(G)$ , then  $\delta_{\overline{S(G)}}(v) = n + m - \delta_G(v) - 1,$
- (iv) If  $e \in E$ , then  $\delta_{\overline{S(G)}}(e) = n + m - 3.$

**Theorem 3.** For any graph  $G$  it holds that

$$\alpha(\overline{S(G)}) \leq 1 + \left\lceil \frac{n + m - 3}{2} \right\rceil.$$

Moreover, this bound is tight.

*Proof.* Let  $D$  be a subset of vertices in  $\overline{S(G)}$  contained in  $E(G)$  with  $|D| = 1 + \lceil \frac{n+m-3}{2} \rceil$ . Such a set  $D$  exists, since we know that in any connected graph,  $m \geq n - 1$ .

Let us prove that  $D$  is a strong defensive alliance.

If  $e \in D$ , then

$$\delta_D(e) = \lceil \frac{n+m-3}{2} \rceil \geq \lfloor \frac{n+m-3}{2} \rfloor = \delta_{\overline{D}}(e).$$

Therefore,  $\alpha(\overline{S(G)}) \leq 1 + \lceil \frac{n+m-3}{2} \rceil$ .

Finally, it can be verified that  $G \simeq P_3$  is a graph for which the bound is attained.  $\square$

The bound in the previous theorem is also attained for a large number of graphs, as the following result shows.

**Proposition 4.** If  $G$  is a  $k$ -regular graph that is not isomorphic to  $K_3$  or  $K_4$ , then

$$\alpha(\overline{S(G)}) = 1 + \left\lceil \frac{n + m - 3}{2} \right\rceil.$$

*Proof.* The result holds clearly for  $G \simeq P_2$ . For  $G \simeq C_n$  with  $n \geq 4$ , if  $D$  is a strong defensive alliance of  $\overline{S(G)}$  and  $v \in D$ , then  $2(|D| - 1) \geq 2\delta_D(v) \geq \delta_{\overline{D}} + \delta_D(v) = \delta(v) = n + m - 2 = 2n - 2$ . From which we obtain that  $|D| \geq n$  and therefore,

$$\alpha(\overline{S(G)}) \geq n = 1 + \left\lceil \frac{n + m - 3}{2} \right\rceil.$$

The other inequality is given by Theorem 3.

Thus, it only remains to prove that the result holds for  $k \geq 3$ . On the one hand, notice first that

$$n - 2 < \frac{nk}{2} - k. \tag{2}$$

Otherwise we obtain  $k \leq 2$ , contrary to our assumption. On the other hand, we assert that a strong defensive alliance of  $\overline{S(G)}$  cannot be entirely contained in  $V(G)$ . Indeed, if  $D \subseteq V(G)$  is one such alliance and  $v \in D$ , then this vertex has at most  $n - 1$  neighbours inside  $D$ , whereas it has at least  $m - \delta_G(v) = \frac{nk}{2} - k$  neighbours in  $\overline{D}$ , which yields the inequality

$$\frac{nk}{2} - k \leq n - 1. \tag{3}$$

Thus, (2) forces equality in (3). From this equality we obtain  $n = 1 + \frac{k}{k-2}$ , which in turn only makes sense for  $k = 3$  or  $k = 4$ . In the first case, we obtain  $G \simeq K_4$ , contrary to the hypothesis. In the second case,  $G$  is a 4-regular graph with 3 vertices, which is absurd.

It follows that every strong defensive alliance must contain at least one vertex of  $E(G)$ . By Remark 2-(iv), any such alliance must have at least  $1 + \lceil \frac{n+m-3}{2} \rceil$  vertices, and hence  $\alpha(\overline{S(G)}) \geq 1 + \lceil \frac{n+m-3}{2} \rceil$ . The result can now be deduced from Theorem 3. □

### 3. STRONG DEFENSIVE ALLIANCE IN $R(G)$

**Remark 3.** Let  $G$  be a graph of order  $n$  and size  $m$ . Then:

- (i)  $|V(R(G))| = n + m$ ,
- (ii)  $|E(R(G))| = 3m$ ,
- (iii) If  $v \in V(G)$  then  $\delta_{R(G)}(v) = 2\delta_G(v)$ ,
- (iv) If  $e \in E(G)$  then  $\delta_{R(G)}(e) = 2$ .

**Theorem 5.** For each graph  $G$ , the following holds

$$\alpha(R(G)) = 1 + \delta(G).$$

*Proof.* Let  $v$  be a vertex of  $R(G)$  which also lies in  $V(G)$  and  $\delta(v) = \delta(G)$ . Let  $e_1, e_2, \dots, e_{\delta(G)}$  be adjacent vertices to  $v$  in  $R(G)$ , then the set  $\{v, e_1, e_2, \dots, e_{\delta(G)}\}$  is a strong defensive alliance in  $R(G)$ . Therefore,  $\alpha(R(G)) \leq 1 + \delta(G)$ . On the other hand, let  $D$  be an  $\alpha$ -set of  $R(G)$ . Since  $E(G)$  is an independent set in  $R(G)$  with  $\overline{E(G)} = V(G)$ , by Lemma 1, there exists  $v \in D \cap V(G)$ . This implies that at least the upper half of neighbors of this vertex must lie in  $D$ . Thus, in  $D$  there are at least  $1 + \lceil \frac{\delta_{R(G)}(v)}{2} \rceil = 1 + \delta_G(v) \geq 1 + \delta(G)$  vertices. □

Using Theorems 2 and 5, we obtain the following.

**Corollary 6.** For any graph  $G$  it holds that

$$\alpha(R(G)) = \alpha(S(G)) + \left\lceil \frac{\delta(G)}{2} \right\rceil.$$

*Proof.* We know that

$$\alpha(R(G)) = 1 + \delta(G) = 1 + \left\lceil \frac{\delta(G)}{2} \right\rceil + \left\lfloor \frac{\delta(G)}{2} \right\rfloor = \alpha(S(G)) + \left\lceil \frac{\delta(G)}{2} \right\rceil.$$

□

Moreover, noting that  $k = \lceil \frac{k}{2} \rceil$  if and only if  $k = 1$ , the following corollary follows immediately.

TABLE 1.  $\alpha(S(G))$  and  $\alpha(R(G))$  for classical graphs  $G$ .

Graphs $G$	$\alpha(S(G))$	$\alpha(R(G))$
$P_n$	2	2
$C_n$	2	3
$T_n$	2	2
$S_n$	2	2
$S_{m,n}$	2	2
$K_n$	$\lceil \frac{n-1}{2} \rceil + 1$	$n$
$K_{m,n}$	$\min\{\lceil \frac{m}{2} \rceil + 1, \lceil \frac{n}{2} \rceil + 1\}$	$\min\{m + 1, n + 1\}$
$W_n$	3	4
$k$ -regular	$\lceil \frac{k}{2} \rceil + 1$	$k + 1$

**Corollary 7.** *If  $G$  is a graph, then the following statements are equivalent:*

- (i)  $\alpha(R(G)) = 2$ ,
- (ii)  $\alpha(R(G)) = \alpha(S(G))$ ,
- (iii)  $\delta(G) = 1$ .

By applying Theorems 2 and 5, we can obtain the strong defensive alliance number for classical graphs. The results are presented in Table 1.

#### 4. STRONG DEFENSIVE ALLIANCES IN $Q(G)$

**Remark 4.** Let  $G$  be a graph of order  $n$  and size  $m$ . Then:

- (i)  $|V(Q(G))| = n + m$ ,
- (ii)  $|E(Q(G))| = 2m + \sum_{i=1}^n \binom{\delta(v_i)}{2}$ ,
- (iii) If  $v \in V(G)$ , then  $\delta_{Q(G)}(v) = \delta_G(v)$ ,
- (iv) If  $e \in E(G)$ , then there exist vertices  $u, v \in V$  such that  $\{u, v\} = N_{Q(G)}(e) \cap V$  and  $\delta_{Q(G)}(e) = \delta_G(u) + \delta_G(v)$ .

**Theorem 8.** *For any graph  $G$*

$$\alpha(Q(G)) \leq \alpha(G) + |E(\langle D \rangle_G)|,$$

where  $D$  is an  $\alpha$ -set of  $G$ . Moreover, this bound is tight.

*Proof.* Let  $D$  be an  $\alpha$ -set of  $G$  and consider  $D' = D \cup E(\langle D \rangle_G)$ . By Remark 4-(iii) it follows that each vertex in  $D$  satisfies the strong defensive alliance condition. Now, if  $e = u \sim v \in E(\langle D \rangle_G)$ , by Remark 4-(iv) it follows that

$$\begin{aligned} \delta_{D'}(e) &= \delta_D(u) + \delta_D(v) \\ &\geq \delta_{\overline{D}}(u) + \delta_{\overline{D}}(v) \\ &= \delta_{\overline{D'}}(e). \end{aligned}$$

The bound is attained if  $G \simeq C_3$ . □

**Theorem 9.** *If  $G$  is the graph  $W_n$ , then  $\alpha(Q(W_n)) = \lceil \frac{n}{2} \rceil + 2$ .*

*Proof.* If  $n = 4, 5$ , the result can be obtained by direct computation. So that, let us suppose that  $n \geq 6$ . Let  $a$  be the apex vertex of  $W_n$ , and consider the sets:

$$\begin{aligned} V_1 &= V - \{a\}, \\ E_1 &= E(W_n - a), \text{ and} \\ E_2 &= \{e \in E(G) \mid e \text{ is incident to } a\}. \end{aligned}$$

Take  $D \subseteq E_2$ , with  $|D| = \lceil \frac{n}{2} \rceil + 2$ , and for  $e \in D$  notice that  $\delta_D(e) = \lceil \frac{n}{2} \rceil + 1$ , while  $\delta_{\overline{D}}(e) = \lfloor \frac{n}{2} \rfloor + 1$ . This show that  $\alpha(Q(W_n)) \leq \lceil \frac{n}{2} \rceil + 2$ .

On the other hand, by Lemma 1, no strong defensive alliance  $D$  can be entirely contained within  $V_1$ . Neither can  $D$  be contained within  $E_1$ , since  $\delta_D(e) = 2 < 4 = \delta_{\overline{D}}(e)$ . Furthermore,  $V_1 \cup E_1$  is the unique strong defensive alliance that includes vertices of  $V_1$  and  $E_1$ . However, this set has a cardinality  $2(n - 1) > \lceil \frac{n}{2} \rceil + 2$ . Thus, if  $D$  is an  $\alpha$ -set of  $Q(W_n)$ , then  $D \cap E_2 \neq \emptyset$ , and the result can be now deduced.  $\square$

**Theorem 10.** *If  $G$  is a  $k$ -regular graph of order  $n \geq 2$ , then  $\alpha(Q(G)) = k + 1$ .*

*Proof.* If  $v \in V(G)$  and  $e_1, e_2, \dots, e_k$  are vertices in  $E(G)$  adjacent to  $v$ , then  $D = \{v, e_1, e_2, \dots, e_k\}$  is a strong defensive alliance in  $Q(G)$ . Finally, if  $D$  is any strong defensive alliance of  $Q(G)$  notice that  $D \cap E(G) \neq \emptyset$  and the result follows.  $\square$

**Corollary 11.** *For any positive integer  $n$ ,  $\alpha(Q(K_n)) = n$ .*

**Proposition 12.** *For every graph  $G$ ,*

- (i)  $\alpha(Q(G)) = 1$  if and only if  $G$  is the empty graph,
- (ii)  $\alpha(Q(G)) = 2$  if and only if  $G \simeq K_2$ ,
- (iii)  $\alpha(Q(G)) = 3$  if and only if one of the following conditions hold:
  - (a) In  $G$  there exist vertices  $u, v$  such that  $u \sim v \in E(G)$  and  $\delta_G(u) = 2$  and  $\delta_G(v) \leq 2$ ,
  - (b) In  $G$  there exist vertices  $u, v, w$  such that  $u \sim v, u \sim w \in E(G)$ ,  $\delta(u) = 3$ ,  $\delta(v) = 1$ , and  $\delta(w) = 1$ .

*Proof.* (i) If  $G$  is the empty graph, then  $Q(G) \simeq G$ , the result can be obtained by direct computation.

(ii) If  $G \simeq K_2$ , then  $Q(G) \simeq P_3$  and the result is clear. Now, if  $\alpha(Q(G)) = 2$ , any  $\alpha$ -set must contain an edge  $e \in E$  with end vertices, say,  $u, v \in V$ . If there were a third vertex in  $V$ , since  $G$  is a connected graph, there must be a third vertex  $e_1 \in E$  adjacent to  $e$ . Therefore,  $G \simeq K_2$ .

(iii) Suppose that condition (a) is satisfied. Then the vertex  $e = uv \in E$  is such that  $\delta_{Q(G)}(e) \leq 4$ . It can be verified that  $D = \{u, e, v\}$  is a strong defensive alliance, and thus  $\alpha(Q(G)) \leq 3$ . Parts (i) and (ii) imply the inequality  $\alpha(Q(G)) \geq 3$ .

Suppose now that (b) is satisfied. In  $Q(G)$ , the vertices  $e_1 = uv$  and  $e_2 = uw$  are such that  $\delta_{Q(G)}(e_1) = \delta_{Q(G)}(e_2) = 4$ . The set  $D = \{e_1, u, e_2\}$  is a strong defensive alliance, and the desired equality is again justified by parts (i) and (ii).

On the other hand, assume that  $\alpha(Q(G)) = 3$ , but conditions (a) and (b) are not satisfied. If  $e \in E$  is an edge whose end vertices have degree at least two, then  $\delta_{Q(G)}(e) \geq 5$  and therefore this vertex cannot belong to any  $\alpha$ -set. Hence, any edge in an  $\alpha$ -set must contain a leaf as an end vertex.

Let  $D$  be an  $\alpha$ -set of  $Q(G)$ , and let  $e = uv \in D$ , where  $u$  is a leaf. Observe that  $\delta_{Q(G)}(v) = 3$ , otherwise,  $v$  would be a leaf, or condition (a) would be satisfied. Moreover, if  $\delta_{Q(G)}(v) \geq 4$ , then  $e$  would not be in  $D$ . Let  $e_1, e_2 \in E$  be the two remaining neighbors of  $v$ , which are adjacent to  $e$ . These vertices cannot belong to  $D$ , as their remaining neighbors must be leaves, meaning that condition (b) would be satisfied. Therefore, the strong defensive alliance condition is not satisfied in  $D$ .  $\square$

5. STRONG DEFENSIVE ALLIANCES IN TOTAL GRAPH  $T(G)$ 

**Remark 5.** Let  $G$  be a graph of order  $n$  and size  $m$ . Then:

- (i)  $|V(T(G))| = n + m$ ,
- (ii)  $|E(T(G))| = 3m + \sum_{i=1}^n \binom{\delta(v_i)}{2}$ ,
- (iii) If  $v \in V(G)$  then  $\delta_{T(G)}(v) = 2\delta_G(v)$ ,
- (iv) If  $e \in E(G)$ , then there exist vertices  $u, v \in V$  such that  $\{u, v\} = N_{T(G)}(e) \cap V$  and  $\delta_{T(G)}(e) = \delta_G(u) + \delta_G(v)$ .

**Theorem 13.** For every graph  $G$ , the following holds:

$$\alpha(T(G)) \leq \alpha(G) + |E(\langle D \rangle_G)|,$$

where  $D$  is an  $\alpha$ -set of  $G$ . Moreover, the bound is sharp.

*Proof.* Let  $D' = D \cup E(\langle D \rangle_G)$ . For any  $v \in D$ , by Remark 5-(iii), we have:

$$\delta_{D'}(v) = 2\delta_D(v) \geq 2\delta_{\overline{D}}(v) = \delta_{\overline{D'}}(v).$$

According to Remark 5(iv), for each  $e = uv \in E(\langle D \rangle_G)$ , it holds that:

$$\delta_{D'}(e) = \delta_D(u) + \delta_D(v) \geq \delta_{\overline{D}}(u) + \delta_{\overline{D}}(v) = \delta_{\overline{D'}}(e).$$

The bound is attained for the graph  $G$  isomorphic to  $C_3$ . □

**Theorem 14.** If  $G$  is a  $k$ -regular graph, then:

$$\alpha(T(G)) = k + 1.$$

*Proof.* Let  $D = \{v\} \cup N_{E(G)}(v)$ , where  $v \in V(G)$ .  $D$  is a strong defensive alliance, since  $\delta_D(v) = k = \delta_{\overline{D}}(v)$ . Therefore,  $\alpha(T(G)) \leq k + 1$ . However, any vertex in  $T(G)$  has degree  $2k$ , and the result can now be deduced. □

**Corollary 15.** If  $G \simeq K_n$ , then  $\alpha(T(K_n)) = n$ .

**Theorem 16.** If the graph  $G$  is isomorphic to  $W_n$ , then

$$\alpha(T(W_n)) = \left\lceil \frac{n}{2} \right\rceil + 2.$$

*Proof.* If  $n = 4$ , then  $G$  is a  $(n - 1)$ -regular graph, and by Theorem 14, we have obtained result. For  $n = 5$ , the result can be obtained by direct computation.

Let  $a$  be the apex vertex of the wheel graph  $W_n$ , and consider the sets:

$$\begin{aligned} V_1 &= V - \{a\}, \\ E_1 &= E(W_n - a), \text{ and} \\ E_2 &= \{e \in E(G) \mid e \text{ is incident to } a\}. \end{aligned}$$

Assume  $n \geq 6$  and let  $D \subseteq E_2$ , with  $|D| = \left\lceil \frac{n}{2} \right\rceil + 2$ . We claim that  $D$  is a strong defensive alliance. Indeed, for any  $e \in D$ , we have:  $\delta_D(e) = \left\lceil \frac{n}{2} \right\rceil + 1 \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 = \delta_{\overline{D}}(e)$ , which implies that  $\alpha(T(W_n)) \leq \left\lceil \frac{n}{2} \right\rceil + 2$ .

On the other hand, no strong defensive alliance  $D$  can be entirely contained in  $E_1$ , since for any  $e \in D$ ,  $\delta_D(e) = 2 < \delta_{\overline{D}}(e)$ . Similarly,  $D$  cannot be contained in  $V_1$  because for any  $e \in D$ ,  $\delta_D(v) = 2 < \delta_{\overline{D}}(v)$ . If  $D \subseteq V$ , then the only possible strong defensive alliance is  $D = V$ , which has cardinality  $|D| = n > \left\lceil \frac{n}{2} \right\rceil + 2$ . Moreover, the set  $V_1 \cup E_1$  is the only strong defensive alliance that includes elements from both  $V_1$  and  $E_1$ . However, its cardinality is  $|V_1 \cup E_1| = 2(n - 1) > \left\lceil \frac{n}{2} \right\rceil + 2$ . Thus, if  $D$  is an  $\alpha$ -set of  $T(W_n)$ , then  $D \cap E_2 \neq \emptyset$ , and the result follows. □

**Proposition 17.** *For every graph  $G$ ,*

- (i)  $\alpha(T(G)) = 1$  if and only if  $G$  is the empty graph,
- (ii)  $\alpha(T(G)) = 2$  if and only if  $G \simeq K_2$ ,
- (iii)  $\alpha(T(G)) = 3$  if and only if one of the following conditions holds:
  - (a) There exist  $u, v \in V(G)$  such that  $uv \in E(G)$ ,  $\delta_G(u) = 2$ , and  $\delta_G(v) \leq 2$ ,
  - (b)  $G \simeq S_4$ .

*Proof.* (i) The result can be obtained by direct computation.

(ii) If  $G \simeq K_2$ , then  $T(G) \simeq C_3$  and the result is clear. Now, if  $\alpha(T(G)) = 2$ , this means that there exists a set  $D \subseteq V(G) \cup E(G)$  with  $|D| = 2$ , which is an  $\alpha$ -set. Supposed  $G \not\simeq K_2$ , we analyze two cases:

Case 1:  $G$  has vertices of degree one.

Let  $X = \{v \in V \mid \delta_G(v) = 1\}$ , then  $D \not\subseteq X$ , since  $X$  is an independent set. Thus,  $D$  must contain a vertex  $e$  such that  $\delta_{T(G)}(e) \geq 3$ , which does not satisfy the strong defensive alliance condition.

Case 2:  $G$  has no vertices of degree one.

In this case, for any vertex  $v \in T(G)$ , it holds that  $\delta_{T(G)}(v) \geq 4$ .

(iii) Suppose that condition (a) is satisfied. Then, there exists a vertex  $e = uv \in E(G)$  such that  $\delta_{T(G)}(e) \leq 4$ . It can be verified that  $D = \{u, e, v\}$  is a strong defensive alliance, and thus  $\alpha(T(G)) \leq 3$ . Parts (i) and (ii) imply the inequality  $\alpha(T(G)) \geq 3$ . If  $G \simeq S_4$ , the result can be obtained by direct computation.

On the other hand, assume that  $\alpha(T(G)) = 3$ , but conditions (a) and (b) are not satisfied. If  $v \in V(G)$  with  $\delta_G(v) \geq 3$  then  $v$  can not belongs to any  $\alpha$ -set. Let  $X = \{v \in V \mid \delta_G(v) = \{1, 2\}\}$ . Then, no  $\alpha$ -set can be entirely contained in  $X$ , since  $X$  is an independent set. Moreover, no two elements of  $X$  can simultaneously be in  $\alpha$ -set. If  $e = uv \in E(G)$  is an edge where  $\delta_G(u) \geq 2$  and  $\delta_G(v) \geq 3$ , then  $\delta_{T(G)}(e) \geq 5$  and hence  $e$  cannot belong to any  $\alpha$ -set. Therefore, any edge in an  $\alpha$ -set must have a leaf as one of its endpoints. Let  $D$  be an  $\alpha$ -set of  $T(G)$ , and let  $e_1 = uv$ , where  $u$  is a leaf. Observe that  $\delta_{T(G)}(v) = 3$ , since otherwise  $v$  would either be a leaf or condition (a) would be satisfied. Moreover, if  $\delta_{T(G)}(v) \geq 4$  then  $e$  would not be in  $D$ . The only remaining option is to choose a vertex  $e_2$  with  $\delta_{T(G)}(e_2) = 4$  adjacent to  $e_1$ . However,  $e_2$  does not satisfy the strong defensive alliance condition. □

## 6. STRONG DEFENSIVE ALLIANCES IN THE CENTRAL GRAPH $\mathcal{C}(G)$

We start by listing some basic properties of  $\mathcal{C}(G)$ , which can be deduced from the definition.

**Remark 6.** Let  $G$  be a graph of order  $n \geq 2$  and size  $m$ . Then:

- (i)  $|V(\mathcal{C}(G))| = n + m$ ,
- (ii)  $|E(\mathcal{C}(G))| = m + \frac{n(n-1)}{2}$ ,
- (iii) For each  $v \in V$  and  $e \in E$  it holds that  $\delta_{\mathcal{C}(G)}(v) = n - 1$  and  $\delta_{\mathcal{C}(G)}(e) = 2$ .

Also, note that if  $n \geq 4$ , then  $\alpha(\mathcal{C}(G)) \geq 3$ , and this bound is attained, for example if  $G \simeq K_4$ .

**Theorem 18.** *For any graph  $G$ , either  $\alpha(\mathcal{C}(G)) = 1 + \left\lceil \frac{n-1}{2} \right\rceil$  or  $\alpha(\mathcal{C}(\overline{G})) = 1 + \left\lceil \frac{n-1}{2} \right\rceil$ .*

*Proof.* Let  $v \in V(G)$  and assume that  $\delta_G(v) \geq \delta_{\overline{G}}(v)$ . Consider the vertices  $e_1, e_2, \dots, e_{\lceil \frac{n-1}{2} \rceil} \in N_{\mathcal{C}(G)}(v) \cap E$ . Observe that  $D = \{v, e_1, \dots, e_{\lceil \frac{n-1}{2} \rceil}\}$  is a strong defensive alliance in  $\mathcal{C}(G)$ , hence  $\alpha(\mathcal{C}(G)) \leq 1 + \lceil \frac{n-1}{2} \rceil$ . Now, let  $D'$  be an arbitrary strong defensive alliance in  $\mathcal{C}(G)$ . There must exist at least one vertex  $v \in V \cap D'$ , and from this assumption, the other inequality can be derived. □

From the proof of the above theorem, the following criteria naturally arise to determine when the number  $\alpha(\mathcal{C}(G))$  is precisely  $1 + \left\lceil \frac{n-1}{2} \right\rceil$  and when  $\alpha(\mathcal{C}(G)) = \alpha(\mathcal{C}(\overline{G}))$ .

**Corollary 19.** For any graph  $G$ , the followings statements hold:

- (i) If for some vertex  $v \in V$ ,  $\delta_G(v) \geq \lceil \frac{n-1}{2} \rceil$ , then  $\alpha(\mathcal{C}(G)) = 1 + \lceil \frac{n-1}{2} \rceil$ .
- (ii) If there exists any vertex  $v \in V$  such that  $\delta_G(v) \geq \delta_{\overline{G}}(v)$ , then  $\alpha(\mathcal{C}(G)) = 1 + \lceil \frac{n-1}{2} \rceil$ .
- (iii) If for a vertex  $v \in V$ ,  $\delta_G(v) = \delta_{\overline{G}}(v)$ , then  $\alpha(\mathcal{C}(G)) = 1 + \lceil \frac{n-1}{2} \rceil = \alpha(\mathcal{C}(\overline{G}))$ .
- (iv) If for some vertices  $u, v \in V$ ,  $\delta_G(u) \geq \delta_{\overline{G}}(u)$  and  $\delta_{\overline{G}}(v) \geq \delta_G(v)$ , then  $\alpha(\mathcal{C}(G)) = 1 + \lceil \frac{n-1}{2} \rceil = \alpha(\mathcal{C}(\overline{G}))$ .

For the following result,  $ind(G)$  denotes the independence number of the graph  $G$ .

**Theorem 20.** Let  $G$  be a graph with  $ind(G) \geq 1 + \lceil \frac{n-1}{2} \rceil$ . Then:

$$\alpha(\mathcal{C}(G)) = 1 + \lceil \frac{n-1}{2} \rceil = \alpha(\mathcal{C}(\overline{G})).$$

*Proof.* Let  $I$  be a maximum independent set in  $G$  with  $|I| \geq 1 + \lceil \frac{n-1}{2} \rceil$ . In  $\mathcal{C}(G)$ , the set  $I$  induces a subgraph isomorphic to  $K_{|I|}$  which contains a copy of  $K_{1+\lceil \frac{n-1}{2} \rceil}$ . Note that the vertex set of this copy forms a strong defensive alliance in  $\mathcal{C}(G)$ , hence  $\alpha(\mathcal{C}(G)) \leq 1 + \lceil \frac{n-1}{2} \rceil$ .

The other inequality follows from the observation that any strong defensive alliance in  $\mathcal{C}(G)$  must contain at least one vertex  $v \in V$ . Finally, to establish the remaining equality, consider a vertex  $v \in I$ . Then, since  $\delta_{\overline{G}}(v) \geq \delta_G(v)$ , Corollary 19-(ii) applies. □

**Corollary 21.** Let  $E_n, P_n, C_n, S_n, S_{p,q}, K_{p,q}$ , and  $K_n$  be the empty, path, cycle, star, double star, complete bipartite and complete graphs respectively, then:

- (i)  $\alpha(\mathcal{C}(E_n)) = 1 + \lceil \frac{n-1}{2} \rceil$ ,
- (ii)  $\alpha(\mathcal{C}(P_n)) = \begin{cases} 3 & \text{if } n = 4, 5, \\ 1 + \lceil \frac{n-1}{2} \rceil & \text{if } n \geq 6, \end{cases}$
- (iii)  $\alpha(\mathcal{C}(C_n)) = \begin{cases} 3 & \text{if } n = 4, 5, \\ 3 + \lceil \frac{n-1}{2} \rceil & \text{if } n \geq 6, \end{cases}$
- (iv)  $\alpha(\mathcal{C}(S_n)) = 1 + \lceil \frac{n-1}{2} \rceil$ ,
- (v)  $\alpha(\mathcal{C}(S_{p,q})) = 1 + \lceil \frac{n-1}{2} \rceil$  where  $p \leq q$ ,
- (vi)  $\alpha(\mathcal{C}(K_{p,q})) = 1 + \lceil \frac{n-1}{2} \rceil$ , where  $p \geq 3$  and  $q \geq 2$ ,
- (vii)  $\alpha(\mathcal{C}(W_n)) = 1 + \lceil \frac{n-1}{2} \rceil$ ,
- (viii)  $\alpha(\mathcal{C}(K_n)) = 1 + \lceil \frac{n-1}{2} \rceil$ .

*Proof.* If  $G \in \{S_n, S_{p,q}, K_{p,q}, W_n\}$ , then there exists a vertex  $v \in V(G)$  such that  $\delta_G(v) \geq \delta_{\overline{G}}(v)$ . From Corollary 19-(ii) we obtain statements (iv), (v), (vi) and (vii).

- (i) Since  $G$  is the empty graph of order  $n$ , then  $\mathcal{C}(G)$  is the complete graph of order  $n$ , thus  $\alpha(\mathcal{C}(G)) = \alpha(K_n) = 1 + \lceil \frac{n-1}{2} \rceil$ .
- (ii) If  $n = 4, 5$ , the result holds. Suppose that  $n \geq 6$ .  
Let  $D = \{v_0, v_1, v_2, \dots, v_{\lceil \frac{n-1}{2} \rceil}\} \subset V$  be such that  $v_0$  and  $v_1$  are the leaves of  $P_n$ , and the remaining vertices are not adjacent to either  $v_0$  or  $v_1$  in  $P_n$ . This set satisfies the strong defensive alliance condition in  $\mathcal{C}(P_n)$ . On the other hand, any strong defensive alliance in  $\mathcal{C}(P_n)$  must contain at least one element of  $V$ , thus yielding the desired equality.
- (iii) If  $n = 4, 5$ , the result holds. Suppose that  $n \geq 6$ .  
The set  $D = \{v_1, v_2, \dots, v_{3+\lceil \frac{n-1}{2} \rceil}\} \subset V$  is a strong defensive alliance in  $\mathcal{C}(C_n)$ . Moreover, any strong defensive alliance in  $\mathcal{C}(C_n)$  must contain at least one element of  $V$ . Therefore,  $\alpha(\mathcal{C}(C_n)) = 3 + \lceil \frac{n-1}{2} \rceil$ .
- (viii) Since  $\mathcal{C}(K_n) \simeq S(K_n)$ , by Theorem 2 it follows that  $\alpha(\mathcal{C}(K_n)) = 1 + \lceil \frac{n-1}{2} \rceil$ . □

## 7. CONCLUSIONS AND FUTURE WORK

In this work, we conducted a comprehensive study of the strong defensive alliance number on several fundamental graph operators, including the subdivision graph  $S(G)$ , the operator  $R(G)$ , the middle graph  $Q(G)$ , the total graph  $T(G)$ , and the central graph  $C(G)$ . Our main contribution was to establish exact values or tight bounds for  $\alpha(O(G))$  under each operator, supported by structural characterizations that highlight when these bounds are attained.

For the operators  $S(G)$  and  $R(G)$ , we obtained closed formulas that hold for every graph, thus providing a complete description of strong defensive alliances in these transformations. For the operators  $Q(G)$  and  $T(G)$ , we identified all graphs for which the strong defensive alliance number takes the values 1, 2, or 3, thereby extending previous partial characterizations in the literature. In the case of the central graph  $C(G)$ , we proved that  $\alpha(C(G))$  takes one of two possible values and established structural conditions that determine precisely when these values coincide. We also computed explicit results for several classical families of graphs, including paths, cycles, wheels, stars, complete graphs, and bipartite complete graphs.

Overall, the results obtained in this paper contribute to the theory of cohesive sets in graphs by showing how the structure of strong defensive alliances evolves under different operator-based graph transformations. Our findings illustrate the central role played by vertex degrees, adjacency patterns among inserted vertices, and the distribution of edges in determining the cohesion properties of operator-generated graphs. By uncovering these relationships, we deepen the understanding of how alliances behave in more complex derived graph structures.

The results presented here open several promising avenues for future research:

- (1) Extending the analysis to other classical graph constructions, such as corona products, Cartesian products, lexicographic products, and generalized line graphs, could provide a more complete picture of how alliances behave under graph transformations.
- (2) Since the computation of defensive alliances is known to be NP-complete for general graphs, it would be valuable to determine whether the same hardness persists for the operators considered in this work or whether certain operator generated graphs admit polynomial time algorithms or approximation schemes.
- (3) Investigating the relationship between strong defensive alliances and other cohesion related parameters, such as domination, total domination, differential, independence, or vulnerability indices, may yield further insights into the structural robustness of transformed graphs.
- (4) Given that strong defensive alliances capture local cohesion and resistance to external influence, exploring applications in social networks, biological systems, and communication networks could connect the theoretical results to practical problems involving cluster resilience and fault tolerance.
- (5) Further research could focus on identifying graphs for which  $\alpha(O(G))$  is maximized or minimized under each operator, and understanding the structural reasons behind these extremal behaviors.

We hope that the results presented here will serve as a foundation for subsequent investigations into strong defensive alliances in graph operators and will stimulate continued research on the structural and algorithmic properties of cohesive sets in derived graphs.

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No new data/codes were created or analyzed in this study.

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