RAIRO-Oper. Res. 60 (2026) 615–624
https://doi.org/10.1051/ro/2026014

RAIRO Operations Research
www.rairo-ro.org

THE ALON–TARSI NUMBER OF PLANAR GRAPHS WITHOUT SOME FORBIDDEN CONFIGURATIONS

Lin Niu*

Abstract. The Alon–Tarsi number of a graph G, denoted $AT(G)$, defined as the smallest integer k admitting an Alon–Tarsi orientation with maximum out-degree at most $k-1$, satisfies the fundamental inequalities $\chi_\ell(G) \leq AT(G) \leq d(G)+1$, where $\chi_\ell(G)$ and $d(G)$ denote the list chromatic number and degeneracy, respectively. Notably, for planar graphs without k-cycles where $k \in \{5,6\}$, the 3-degeneracy implies $AT(G) \leq 4$, and our main result extends this by proving $AT(G) \leq 4$ for planar graphs without 4-cycles adjacent to 3-cycles, thereby improving previous results from Lam *et al.* [*J. Comb. Theory, Ser. B* **76** (1999) 117–126] and Cheng *et al.* [*Discrete Math.* **339** (2016) 3052–3057]. We also show that this result is sharp.

Mathematics Subject Classification. 05C15.

Received April 23, 2025. Accepted January 25, 2026.

1. Introduction

In this paper, all graphs are finite and simple. A *proper k-coloring* of a graph G is a function $c : V(G) \to \{1,\dots,k\} = [k]$ where adjacent vertices are assigned distinct colors. A graph is *k-colorable* if it admits a proper k-coloring. The *chromatic number of G*, denoted $\chi(G)$, is the smallest integer k for which G has a proper k-coloring. Two faces are *adjacent* if they share at least one common edge.

Vizing [1], as well as Erdős *et al.* [2] independently, introduced list coloring, a generalization of classical graph coloring. A *list assignment* L for a graph G assigns each vertex $v \in V(G)$ a set $L(v)$ of admissible colors. If G admits a proper coloring c such that $c(v) \in L(v)$ for every vertex v, then G is said to be *L-colorable*. The graph G is *k-choosable* if it is L-colorable for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$. The *list chromatic number of* G, denoted $\chi_\ell(G)$, is the smallest integer k for which G is k-choosable. Trivially, $\chi(G) \leq \chi_\ell(G)$ holds for all graphs G.

Jensen and Toft [3] introduced the Alon–Tarsi number of a graph G, a further generalization of list coloring. Let D be an orientation of G. A subgraph H of D is called *Eulerian* if it is spanning and satisfies $d^+_H(v) = d^-_H(v)$ for every $v \in V(H)$. Denote by $EE(D)$ (resp. $OE(D)$) the set of all Eulerian subgraphs of D with an even (resp. odd) number of edges. An orientation D is an *Alon–Tarsi orientation* (for short, AT-orientation) if $|EE(D)| \neq |OE(D)|$. The *Alon–Tarsi number of* G, denoted $AT(G)$, is the smallest integer k for which G

Keywords. Alon–Tarsi number, orientation, list coloring, discharging method.

School of Mathematics and Statistics, Weifang University, Weifang 261061, P.R. China.
*Corresponding author: 906024652@qq.com

admits an AT-orientation D with maximum out-degree $\Delta_D^+ \leq k-1$. By the Alon–Tarsi Theorem [4], it follows that $\chi_\ell(G) \leq AT(G)$ for any graph G.

A graph G is *d-degenerate* if every subgraph contains a vertex of degree at most d, with the *degeneracy* $d(G)$ being the smallest such d. For any orientation D of G, the empty subgraph $(V(G), \emptyset)$ forms a trivial even Eulerian subgraph, and when D contains no directed cycles, it is called an *acyclic orientation.* Since every Eulerian subdigraph decomposes into edge-disjoint directed cycles, any acyclic orientation D is necessarily an Alon–Tarsi orientation. Crucially, a graph is d-degenerate if and only if it admits an acyclic orientation D with maximum out-degree $\Delta_D^+ \leq d$, which directly implies the inequality chain $AT(G) \leq d(G)+1$ and consequently establishes the fundamental relationship

$$\chi(G) \leq \chi_\ell(G) \leq AT(G) \leq d(G)+1 \text{ for all graphs } G.$$

The study of list coloring in planar graphs has produced several important results. Thomassen's groundbreaking work [5] established that all planar graphs are 5-choosable, while Voigt [6] demonstrated the existence of planar graphs that are not 4-choosable. The complexity of this problem was further highlighted by Gutner [7], who proved that determining 4-choosability for planar graphs is NP-hard. These results have motivated the search for sufficient conditions guaranteeing 4-choosability. Significant progress includes the result of Cushing and Kierstead [8] showed that every planar graph contains a matching M such that $G-M$ is 4-choosable. Furthermore, It is well known that a planar graph is 4-choosable if it has no i-cycles where $i \in \{3,4,5,6\}$, see [9–12]. Building on these findings, Cheng *et al.* [13] extended the results by proving that planar graphs without 4-cycles adjacent to 3-cycles are also 4-choosable.

Recent advances in the study of Alon–Tarsi numbers have yielded significant results for planar graphs. Zhu [14] first established the upper bound $AT(G) \leq 5$ for all planar graphs. This result was subsequently improved by Grytczuk and Zhu [15], who demonstrated that every planar graph G admits a matching M such that $AT(G-M) \leq 4$. Most recently, Lu and Zhu [16] achieved a further refinement, proving that if G is a planar graph without 4-cycles and l-cycles for some $l \in \{5,6,7\}$, then there exists a matching M such that $AT(G-M) \leq 3$, which marks an important progress in understanding how cycle restrictions affect Alon–Tarsi numbers.

Recent work on planar graph degeneracy has revealed important structural properties. Wang and Lih [12] demonstrated that planar graphs containing no 5-cycles are necessarily 3-degenerate. Independently, Fijavž *et al.* [9] established an analogous result for planar graphs excluding 6-cycles. Combining these degeneracy bounds with the fundamental inequality $AT(G) \leq d(G)+1$, we immediately obtain the following corollary.

Corollary 1.1. *For any planar graph G without k-cycles where $k \in \{5,6\}$, $AT(G) \leq 4$.*

Let $\mathcal{G}$ be the family of planar graphs without any 3-cycle adjacent to a 4-cycle. In this paper, we prove the following main result.

Theorem 1.2. *If $G \in \mathcal{G}$, then $AT(G) \leq 4$.*

By Theorem 1.2, we obtain the following corollary.

Corollary 1.3. *If a planar graph G contains no k-cycles for any $k \in \{3,4\}$, then $AT(G) \leq 4$.*

Voigt [17] proved the existence of a planar graph G_0 without 3-cycles that is not 3-choosable, whence $AT(G_0) \geq \chi_\ell(G_0) \geq 4$. As G_0 has no 3-cycles, we have $G_0 \in \mathcal{G}$. By Theorem 1.2, $AT(G) \leq 4$ for every $G \in \mathcal{G}$. Thus, this result is sharp.

This paper is organized as follows. In Section 2, we establish several lemmas in order to prove Theorem 1.2. In Section 3, we prove Theorem 1.2.

In the end of this section, we introduce some terminology and notation. Let u and v be two vertices in G. If uv is an edge of G, then we say that u is a *neighbor* of v. The set of all neighbors of v in G is denoted by $N(v)$. The degree of v, denoted by $d(v)$, is the number of vertices in $N(v)$. A *k-vertex* (resp. *k^+-vertex* or *k^--vertex*) is a vertex of degree k (resp. at least k or at most k). Similarly, a *k-face* (resp. *k^+-face* or *k^--face*) is a face with k (resp. at least k or at most k) vertices. For $f \in F(G)$, let $b(f)$ be the *boundary walk of f* and write $f = [u_1 \dots u_n]$ when $u_1, \dots, u_n$ are the boundary vertices of f in clockwise order. An n-face $[u_1 \dots u_n]$ is called an $(a_1, \dots, a_n)$-face if $d(u_i) = a_i$ for $i \in [n]$. For a cycle C, an edge $xy \in E(G) \setminus E(C)$ is called a *chord* of C if $x, y \in V(C)$. For a k-cycle C, C is called *chord k-cycle* if C has a chord.

Throughout all figures in this paper, a solid vertex represents a vertex that all of its incident edges are depicted in the figure, whereas a hollow circle represents a 4^+-vertex.

2. Reducible configurations

Assume, for contradiction, that Theorem 1.2 is false. Let G be a minimal counterexample with respect to $|V(G)|$, which implies G must be connected.

We call an orientation D of G good if it is an AT-orientation with maximum out-degree $\Delta_D^+ \leq 3$. For any vertex subset $X \subset V(G)$, we denote by $\langle X \rangle$ the induced digraph consisting of X and all edges between X and $V(G) \backslash X$, oriented from X to $V(G) \backslash X$.

The following technique lemma is due to Zhu and Balakrishnan [18].

Lemma 2.1. *Assume that G is a graph and X is a subset of $V(G)$. Suppose that $G - X$ has an AT-orientation D_0 and $G[X]$ has an orientation D_1. Let D be obtained from $D_0 \cup D_1 \cup \langle X \rangle$. If D_1 is an AT-orientation of $G[X]$, then D is an AT-orientation of G.*

The following lemmas will be essential for proving Theorem 1.2.

Lemma 2.2. *Every vertex in G has degree at least 4.*

Proof. Assume to the contrary that there exists a vertex $v \in V(G)$ with $d(v) \leq 3$. Consider the proper subgraph $G - v$. By the minimality of G, there exists a good AT-orientation D_0 of $G - v$ satisfying $|EE(D_0)| \neq |OE(D_0)|$ and $\max_{u \in V(G-v)} d^+_{D_0}(u) \leq 3$.

We construct an orientation D of G by extending D_0 with all edges incident with v oriented outward. This construction has the following properties.

- The Eulerian subgraph counts are preserved. Thus, $|EE(D)| = |EE(D_0)|$ and $|OE(D)| = |OE(D_0)|$;
- The out-degree of v satisfies $d^+_D(v) = d(v) \leq 3$;
- For all other vertices $u \neq v$, $d^+_D(u) = d^+_{D_0}(u) \leq 3$.

Therefore, D is a good AT-orientation of G with maximum out-degree at most 3, contradicting our assumption that G was a counterexample. □

Lemma 2.3. *The minimal counterexample G satisfies:*

(1) *G contains no chord 5-cycles;*
(2) *No 3-face in G is adjacent to any k-face with $k \leq 4$.*

Proof. (1) Suppose G contains a chord 5-cycle $v_1 \cdots v_5 v_1$ with chord $v_1 v_3$. This simultaneously induces a 4-cycle $v_1 v_3 v_4 v_5 v_1$ and a 3-cycle $v_1 v_2 v_3 v_1$ that share the edge $v_1 v_3$, violating the forbidden configuration of graph G.

(2) For any 3-face $[v_1 v_2 v_3]$, if it adjacency to a 3-face $[v_1 v_2 v_4]$ creates a 4-cycle $v_1 v_3 v_2 v_4 v_1$; if it is adjacent to a 4-face $[v_1 v_2 v_4 v_5]$ yields the 4-cycle $v_1 v_2 v_4 v_5 v_1$. In each of case, each of those two 4-cycles is adjacent to the 3-cycle $v_1 v_2 v_3 v_1$ along the edge $v_1 v_2$, which contradicts the properties of G.

□

By Lemma 2.3(2), the following corollary holds.

Corollary 2.4. *A k-vertex in G is incident with at most $\lfloor \frac{k}{2} \rfloor$ 3-faces where $k \geq 4$.*

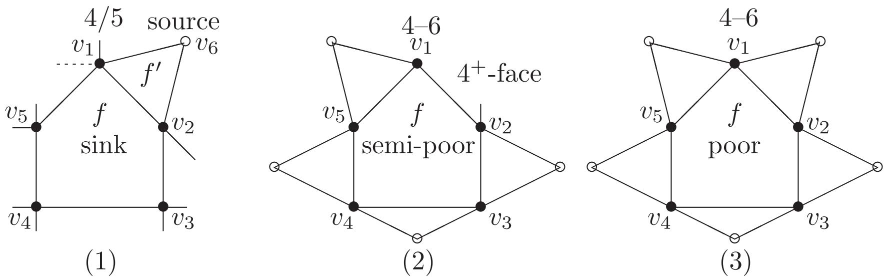

FIGURE 1. Definition of different 5-faces.

A 3-face is called *bad* if it is of type $(5^-, 5^-, 4)$-face and *good* otherwise. It can be easily verified that a good 3-face f is either a $(4^+, 4^+, 6^+)$-face or a $(5, 5, 5)$-face.

A 5-vertex v is *bad* if it is incident with two bad 3-faces and *good* otherwise. This definition implies that a good 5-vertex v satisfies one of the following conditions: it is incident with at most one 3-face, or it is incident with two 3-faces, at least one of which is good.

Let $f = [v_1 \dots v_5]$ be a $(k, 4, 4, 4, 4)$-face in G, where $d(v_1) = k \in \{4, 5, 6\}$ and $d(v_i) = 4$ for each $i \in \{2, 3, 4, 5\}$. If $k \in \{4, 5\}$ and f is adjacent to a 3-face $[v_1v_2v_6]$, where $v_6 \notin b(f)$, then we say that v_6 is a *source* of f, and f is a *sink* of v_6 (as illustrated in Fig. 1(1)). The face f is called *semi-poor* if it is adjacent to four 3-faces and one 4^+-face, with the additional condition that v_1 is incident with exactly one 3-face adjacent to f (as shown in Fig. 1(2)). The face f is called *poor* if it is adjacent to five 3-faces (as depicted in Fig. 1(3)). For simplicity, a 5-face is called *rich* if it is neither poor nor semi-poor. Conversely, a 5-face is *non-rich* if it is either poor or semi-poor.

Lemma 2.5. *Every source is a 5^+-vertex in G.*

Proof. Let $f = [v_1 \dots v_5]$ be a $(k, 4, 4, 4, 4)$-face where $d(v_1) = k \in \{4, 5\}$ and $d(v_i) = 4$ for each $i \in \{2, 3, 4, 5\}$. Without loss of generality, suppose the edge v_1v_2 is incident with a 3-face $g = [v_1v_2v_6]$, where v_6 is a source of f and thus $v_6 \notin b(f)$.

For contradiction, assume v_6 is a 4-vertex by Lemma 2.2. Let $X = \{v_1, \dots, v_6\}$. By the minimality of G, there exists a good orientation D_0 of $G - X$.

By Lemma 2.3(1), $v_iv_{i+2} \notin E(G)$ for $i \in [5]$ (indies module 5). If $v_5v_6 \in E(G)$, then G contains a 4-cycle $v_1v_5v_6v_2v_1$ adjacent to the 3-cycle $v_1v_2v_6v_1$, a contradiction. Hence, $v_5v_6 \notin E(G)$. Symmetrically, $v_3v_6 \notin E(G)$. If $v_4v_6 \in E(G)$, then G contains a 4-cycle $v_2v_3v_4v_6v_2$ adjacent to the 3-cycle $v_1v_2v_6v_1$, another contradiction. Thus, $v_4v_6 \notin E(G)$. Therefore, $G[X]$ contains no edges beyond those in the faces f and g.

Let $D_1 = \{\overrightarrow{v_2v_1}, \overrightarrow{v_2v_3}, \overrightarrow{v_3v_4}, \overrightarrow{v_4v_5}, \overrightarrow{v_5v_1}, \overrightarrow{v_1v_6}, \overrightarrow{v_6v_2}\}$. By the forbidden configuration constraint, $G[X]$ contains no edges other than those specified in D_1, which therefore forms an orientation of the subgraph $G[X]$, as shown in Figure 2(1). The odd and even Eulerian subgraphs are $OE(D_1) = (X, \{\overrightarrow{v_1v_6}, \overrightarrow{v_6v_2}, \overrightarrow{v_2v_1}\})$ and $EE(D_1) = \{(X, \emptyset), (X, \{\overrightarrow{v_1v_6}, \overrightarrow{v_6v_2}, \overrightarrow{v_2v_3}, \overrightarrow{v_3v_4}, \overrightarrow{v_4v_5}, \overrightarrow{v_5v_1}\})\}$. Since $|OE(D_1)| \neq |EE(D_1)|$, D_1 is an AT-orientation of $G[X]$.

Let $D = D_0 \cup D_1 \cup \langle X \rangle$. By Lemma 2.1, D is an AT-orientation of G. We verify the out-degrees.

- For $v \in V(G) \setminus X$, $d_D^+(v) = d_{D_0}^+(v) \leq 3$;
- For $i \in \{3, 4, 5, 6\}$, $d_D^+(v_i) = d_{D_1}^+(v_i) + 2 = 3$, $d_D^+(v_2) = d_{D_1}^+(v_2) + 1 = 3$;
- If $d(v_1) = 4$, then $d_D^+(v_1) = d_{D_1}^+(v_1) + 1 = 2$; if $d(v_1) = 5$, then $d_D^+(v_1) = d_{D_1}^+(v_1) + 2 = 3$.

In all cases, $\Delta_D^+ \leq 3$, so D is a good orientation of G. This contradicts the initial assumption, proving that v_6 must be a 5^+-vertex. □

Lemma 2.6. *Let v be a k-vertex in G where $k \in \{5, 6\}$ incident with two adjacent 5-faces, f_1 and f_2. Then at most one of f_1 and f_2 is a semi-poor 5-face.*

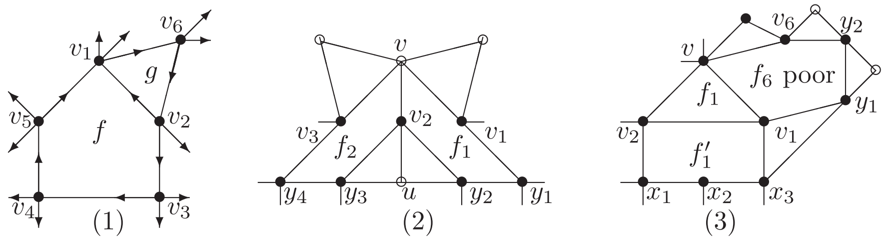

FIGURE 2. Illustrations of Lemmas 2.5–2.8.

Proof. Assume for contradiction that both f_1 and f_2 are semi-poor 5-faces. Let $f_1 = [vv_1y_1y_2v_2]$ and $f_2 = [vv_2y_3y_4v_3]$, as illustrated in Figure 2(2). By definition of semi-poor 5-faces, v_2 must be a 4-vertex.

If $y_2 = y_3$, then v_2 would have only two distinct neighbors (y_2 and v), making it a 2-vertex, which contradicts $d(v_2) = 4$. Thus, $y_2 \neq y_3$. By the semi-poor property, each edge v_2y_2 and v_2y_3 is incident with a 3-face. Let u be the fourth neighbor of v_2 (distinct from v, y_2, y_3). The 3-faces must be $[v_2y_2u]$ and $[v_2y_3u]$. The 4-cycle $v_2y_2uy_3v_2$ shares a common edge v_2y_2 with the 3-cycle $v_2y_2uv_2$. This violates the property that G contains no 4-cycle adjacent to a 3-cycle, yielding the desired contradiction. □

Lemma 2.7. *Let v be a good 5-vertex with two 3-faces and three 5-faces. Then v has at most one non-rich 5-face.*

Proof. Let v be a 5-vertex with neighbors $v_1, \ldots, v_5$ in clockwise order, where f_i denotes the face incident with v and bounded by edges vv_i and vv_{i+1} where $i \in [5]$ (indices modulo 5). By Lemma 2.3(2), we may assume f_1 and f_3 are 3-faces, which implies f_2, f_4, and f_5 are all 5-faces. By the definition of poor 5-face, neither f_4 nor f_5 is a poor 5-face.

First assume that f_2 is a rich 5-face. By Lemma 2.6, at most one of $\{f_4, f_5\}$ can be semi-poor 5-face. Thus, v is incident with at most one non-rich 5-face.

Next assume that f_2 is a non-rich 5-face. Then f_2 must be a (5,4,4,4,4)-face. Lemma 2.5 shows both v_1 and v_4 are 5^+-vertices. Consequently, both the faces f_4 and f_5 contain at least two 5^+-vertices, making them rich 5-faces. The result holds. □

Lemma 2.8. *If a 6-vertex v in G is incident with three 3-faces and three poor 5-faces, then v cannot have any sink.*

Proof. Let v be a 6-vertex with neighbors $v_1, \ldots, v_6$ in clockwise order. For each $i \in [6]$ (indices modulo 6), let f_i denote the face incident with edges vv_i and vv_{i+1}. By Lemma 2.3(2) and symmetry, we may assume that f_i is a 3-face for $i \in \{1, 3, 5\}$, and f_j is a poor 5-face for $j \in \{2, 4, 6\}$. By the definition of poor 5-faces, each v_i is a 4-vertex for $i \in [6]$.

Without loss of generality, suppose v has a sink f'_1 adjacent to f_1. Let $f_6 = [vv_1y_1y_2v_6]$ and $f'_1 = [v_1v_2x_1x_2x_3]$, as shown in Figure 2(3). Since f'_1 is a sink of v and, both of v_1 and v_2 are 4-vertices, making f'_1 a $(4, 4, 4, 4, 4)$-face.

Because f_6 is poor, v_1 is incident with two 3-faces. One is $f_1 = [vv_1v_2]$ and the other must be $[v_1x_3y_1]$. Clearly, $y_1 \notin \{v_1, v_2, x_3\}$. If $y_1 = x_2$, then x_3 would have only two neighbors (x_2 and v_1), making it a 2-vertex, a contradiction. Thus, $y_1 \neq x_2$. If $y_1 = x_1$, then $x_1x_3 \in E(G)$, and then G contains a 4-cycle $v_2v_1x_3x_1v_2$ adjacent to the 3-cycle $x_1x_2x_3x_1$, which violates the properties of graph G. Thus, $y_1 \neq x_1$. So far, $y_1 \notin b(f'_1)$.

Since $y_1 \notin b(f'_1)$ and f'_1 is a $(4, 4, 4, 4, 4)$-face, y_1 must be a source of f'_1. By Lemma 2.5, any source must be a 5^+-vertex. However, y_1 is a 4-vertex (as f_6 is poor), yielding the final contradiction. □

3. Proof of Theorem 1.2

For each $x \in V(G) \cup E(G)$, define an initial weight function $\mu(x)$ on $V(G) \cup F(G)$ by $\mu(v) = 2d(v) - 6$ and $\mu(f) = d(f) - 6$ where $v \in V(G)$ and $f \in F(G)$. Using the Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we derive

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

Let $\mu'(x)$ be the charge after applying discharging rules for each $x \in V(G) \cup F(G)$. To prove the Theorem 1.2, we design discharging rules to preserve the total weight sum and ensure $\mu'(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This leads to

$$0 \leq \sum_{x \in V(G) \cup F(G)} \mu'(x) = \sum_{x \in V(G) \cup F(G)} \mu(x) = -12$$

which is impossible, completing the proof by contradiction.

We employ the discharging method to establish the non-existence of the graph G. For any k-vertex v in G where $k \geq 4$, let m, n, and s denote the number of incident 3-faces, 4-faces, and 5-faces, respectively, with α representing the number of sinks incident with v. It can verify that $m + n + s \leq k$. By the definition of sink and Corollary 2.4, it gets $\alpha \leq m \leq \lfloor \frac{k}{2} \rfloor$. A 6-vertex v is denoted as 6^*-*vertex* if it is incident with exactly one 5^+-face other than poor 5-face, and as 6^{**}-*vertex* if it is incident with two or more such faces.

Now, we define the discharge rules as following.

The discharging rules:

(R1) Each 4^+-vertex sends 1 or $\frac{1}{2}$ to each incident 3- or 4-face, respectively.
(R2) Each 4-vertex sends $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{4}$ to each incident 5-face when $m = 0$, $m = 1$ and $n = 0$, or $m = n = 1$, respectively.
(R3) Let v be a 5-vertex. If v is bad, it sends $\frac{1}{2}$ to each incident 5-face; if v is good, it sends $\frac{1}{2}$ or $\frac{3}{4}$ to each incident rich 5-face or non-rich 5-face, respectively.
(R4) Each 6-vertex sends 1, $\frac{3}{4}$ or $\frac{1}{2}$ to each incident poor 5-face, semi-poor 5-face or rich 5-face, respectively.
(R5) Each 7^+-vertex sends 1 to each incident 5-face.
(R6) Let f be a 5-face with source vertex v. Then v sends $\frac{1}{4}$, $\frac{1}{6}$, or $\frac{1}{3}$ to f if v is a 5- or 7^+-vertex, 6^*-vertex, or 6^{**}-vertex, respectively.

We first check $\mu'(v) \geq 0$ for all $v \in V(G)$. Let a k-vertex v be adjacent to $v_1, \ldots, v_k$ in clockwise order and f_i be the face incident with v with vv_i and vv_{i+1} as boundary edge where $i \in [k]$ (indices modulo k). By Lemma 2.2, $k \geq 4$. By Corollary 2.4, $m \leq \lfloor \frac{k}{2} \rfloor$.

(1) Let v be a 4-vertex. Then $m \leq 2$.
- If $m = 0$, then $n + s \leq 4$. By (R1) and (R2), v sends $\frac{1}{2}$ to each incident 4- or 5-face. Thus, $\mu'(v) = 2 - \frac{1}{2}(n + s) \geq 0$.
- If $m = 1$, then $n \leq 1$ by Lemma 2.3(2). If $n = 0$, then $s \leq 3$. By (R1) and (R2), v sends 1 or $\frac{1}{3}$ to each incident 3- or 5-face, respectively. Thus, $\mu'(v) = 2 - 1 - \frac{1}{3}s = \frac{1}{3}(3 - s) \geq 0$. If $n = 1$, then $s \leq 2$ by Lemma 2.3(2). By (R1) and (R2), v sends 1, $\frac{1}{2}$ or $\frac{1}{4}$ to each incident 3-, 4- or 5-face, respectively. Thus, $\mu'(v) = 2 - 1 - \frac{1}{2} - \frac{1}{4}s = \frac{1}{4}(2 - s) \geq 0$.
- If $m = 2$, then $n = 0$ by Lemma 2.3(2). By (R1), v sends 1 to each incident 3-face. Note that v sends nothing to each incident 5-face by (R2). Thus, $\mu'(v) = 2 - 2 = 0$.

(2) Let v be a 5-vertex. Then $m \leq 2$.
First assume that v is a bad 5-vertex. By the definition of bad 5-vertex and sink, v is incident with exactly two 3-faces, and v has at most two sinks. Thus, $\alpha \leq m = 2$. By Lemma 2.3(2), v is not incident with any 4-face. Thus, $n = 0$. It implies that v is incident with at most three 5-faces. Thus, $s \leq 3$. So far, $\alpha \leq m = 2, n = 0$ and $s \leq 3$. By (R1) and (R3), v sends 1 or $\frac{1}{2}$ to each incident 3- or 5-face, respectively. By (R6), v sends $\frac{1}{4}$ to each of its sink. Thus, $\mu'(v) = 4 - 2 - \frac{1}{2}s - \frac{1}{4}\alpha \geq 0$.

Next assume that v is a good 5-vertex. By the definition of good 5-vertex, v is incident with at most one bad 3-face. By the definition of sink, α is less than or equal to the number of bad 3-face incident with v. Thus, $\alpha \leq 1$.

– Suppose first that $m = 0$. Then $n + s \leq 5$, and $\alpha = 0$ since $0 \leq \alpha \leq m = 0$. By (R1) and (R3), v sends at most $\frac{3}{4}$ to each incident 4- or 5-face. Thus, $\mu'(v) \geq 4 - \frac{3}{4}(n + s) \geq \frac{1}{4} > 0$.
– Suppose next that $m = 1$. Then $n + s \leq 4$ and $\alpha \leq 1$. Let $f_1 = [vv_1v_2]$ be a 3-face. Then v is incident with at most two $(5, 4, 4, 4, 4)$-faces share an edge with vv_1 or vv_2. By the definition of non-rich 5-face, v is incident with at most two non-rich 5-faces. By (R1) and (R3), v sends 1 to each incident 3-face, $\frac{1}{2}$ to each incident 4-face or rich 5-face, and $\frac{3}{4}$ to each incident non-rich 5-face. By (R6), v sends $\frac{1}{4}$ to its sink. Thus, $\mu'(v) \geq 4 - 1 - 2 \times \frac{1}{2} - 2 \times \frac{3}{4} - \frac{1}{4}\alpha \geq \frac{1}{4} > 0$.
– Finally, suppose that $m = 2$. Then $s + n \leq 3$. It follows from Lemma 2.3(2) that $n = 0$.
  If $s \leq 2$, then by (R1) and (R3), v sends 1 or at most $\frac{3}{4}$ to each incident 3- or 5-face, respectively, and by (R6), v sends $\frac{1}{4}$ to each of its sink. Thus, $\mu'(v) \geq 4 - 2 - \frac{3}{4}s - \frac{1}{4}\alpha \geq \frac{1}{4} > 0$.
  If $s = 3$, then v is incident with at most one non-rich 5-face by Lemma 2.7. By (R1) and (R3), v sends $1, \frac{3}{4}$ or $\frac{1}{2}$ to each incident 3-face, non-rich 5-face or rich 5-face, respectively, and by (R6), v sends $\frac{1}{4}$ to each of its sink. Thus, $\mu'(v) \geq 4 - 2 - 2 \times \frac{1}{2} - \frac{3}{4} - \frac{1}{4}\alpha \geq 0$.

(3) Let v be a 6-vertex. Then $m \leq 3$. Let v be incident with s_p poor 5-faces, s_{sp} semi-poor 5-faces and s_r rich 5-faces. Then $s = s_p + s_{sp} + s_r$. By (R1) and (R4), v sends 1 to each incident 3-face or poor 5-face, $\frac{3}{4}$ to each incident semi-poor 5-face, and $\frac{1}{2}$ to each incident 4-face or rich 5-face. By (R6), v sends at most $\frac{1}{3}$ to each of its sink. Thus,

$$\mu'(v) \geq 6 - (m + s_p) - \frac{3}{4}s_{sp} - \frac{1}{2}(n + s_r) - \frac{1}{3}\alpha. \tag{1}$$

– If $m \leq 2$, then $\alpha \leq m \leq 2$ and $s_p \leq 1$. By equation (1), $\mu'(v) \geq 6 - (m + s_p) - \frac{3}{4}(6 - m - s_p) - \frac{1}{3}\alpha = \frac{3}{2} - \frac{7}{12}m - \frac{1}{4}s_p \geq \frac{1}{12}(15 - 7m) \geq \frac{1}{12} > 0$.
– If $m = 3$, then $s, \alpha \leq 3$. By symmetry and Lemma 2.3(2), let f_i be a 3-face for each $i \in \{1, 3, 5\}$. By the definition of semi-poor 5-face, v is not incident with any semi-poor 5-face. Thus, $s_{sp} = 0$. It implies that $s = s_p + s_r \leq 3$. By Lemma 2.3(2), v is not incident with any 4-face. Thus, $n = 0$. So far, $\alpha \leq 3, s = s_p + s_r \leq 3$ and $s_{sp} = n = 0$. By equation (1), it follows that

$$\mu'(v) \geq 3 - s_p - \frac{1}{2}s_r - \frac{1}{3}\alpha = 3 - s + \frac{1}{2}s_r - \frac{1}{3}\alpha. \tag{2}$$

  If either $s \leq 2$ or $s = 3$ and $\alpha = 0$, then $\mu'(v) \geq \frac{1}{2}s_r \geq 0$ by equation (2). Note that $s, \alpha \leq 3$. It is sufficient to consider the case of $s = 3$ and $1 \leq \alpha \leq 3$. Then $\mu'(v) \geq \frac{1}{2}s_r - \frac{1}{3}\alpha$ by equation (2). By Lemma 2.8, it gets $s_p \leq 2$ and $s_r \geq 1$ since $1 \leq \alpha \leq 3$ and $s = s_p + s_r = 3$. To prove $\mu'(v) \geq 0$, it is sufficient to consider the case of $s_r = 1$. In this case, it follows that $s_p = 2$ since $s_p = s - s_r = 3 - 1 = 2$, which implies that v is incident with one 5-face other than poor 5-face. By (R1) and (R4), v sends 1 to each incident 3-face or poor 5-face, and $\frac{1}{2}$ to each incident rich 5-face. By (R6), v sends $\frac{1}{6}$ to each of its sink. Thus, $\mu'(v) \geq 6 - 3 - 2 - \frac{1}{2} - \frac{1}{6}\alpha = \frac{1}{6}(3 - \alpha) \geq 0$.

(4) Let v be a k-vertex where $k \geq 7$. By (R1) and (R5), v sends at most 1 to each incident face, and by (R6), v sends $\frac{1}{4}$ to each of its sink. Recall that v has at most $\lfloor \frac{k}{2} \rfloor$ sinks since $\alpha \leq m \leq \lfloor \frac{k}{2} \rfloor$. Thus, $\mu'(v) \geq 2k - 6 - k - \frac{1}{4}\alpha \geq \frac{1}{8}(7k - 48) \geq \frac{1}{8} > 0$.

Further check that $\mu'(f) \geq 0$ for each $f \in F(G)$. Let $f = [v_1 \dots v_k]$ be a k-face in G where $k \geq 3$. By Lemma 2.2, f is a $(4^+, \dots, 4^+)$-face.

(1) Let f be a 3-face. By (R1), each incident 4^+-vertex of f sends 1 to f. Thus, $\mu'(f) = -3 + 3 \times 1 = 0$.
(2) Let f be a 4-face. By (R1), each incident 4^+-vertex of f sends $\frac{1}{2}$ to f. Thus, $\mu'(f) = -2 + 4 \times \frac{1}{2} = 0$.
(3) Let f be a 5-face. If f is incident with at least one 7^+-vertex, then each incident 7^+-vertex of f sends 1 to f by (R5). Thus, $\mu'(f) \geq -1 + 1 = 0$. Thus, assume that f is a $(6^-, 6^-, 6^-, 6^-, 6^-)$-face. If f is incident

with at least two 5^+-vertices, then each incident 5^+-vertex of f sends at least $\frac{1}{2}$ to f by (R3) and (R4). Thus, $\mu'(f) \geq -1 + 2 \times \frac{1}{2} = 0$. Without loss of generality, we may assume that f is a $(k_1, 4, 4, 4, 4)$-face where $d(v_1) = k_1 \in \{4, 5, 6\}$ and $d(v_i) = 4$ for each $i \in \{2, 3, 4, 5\}$.

Suppose that v_i is incident with at most one 3-face for some $i \in \{2, 3, 4, 5\}$. If v_i is incident with exactly one 3-face where $i \in \{2, 3, 4, 5\}$, then by Lemma 2.3(2), it is incident with at most one 4-face. Thus, v_i sends at least $\frac{1}{4}$ to f by (R2). Similarly, if v_1 is also a 4-vertex incident with at most one 3-face, then v_1 sends at least $\frac{1}{4}$ to f.

Let f be adjacent to $n_3(f)$ 3-faces, and be incident with β 4-vertices where each such 4-vertex is incident with at most one 3-face. Depending on the degree of the vertex v_1, it is divided into the following three cases.

Case 1. The face f is a $(6, 4, 4, 4, 4)$-face where $d(v_1) = 6$ and $d(v_i) = 4$ for $i \in \{2, 3, 4, 5\}$.

- If f is a poor 5-face, then v_1 sends 1 to f by (R4). Thus, $\mu'(f) \geq -1 + 1 = 0$.
- If f is a semi-poor 5-face, then v_1 is incident with exactly one 3-face adjacent to f. By symmetry, let edge v_1v_2 be not incident with 3-face and each other edge v_iv_{i+1} be incident with one 3-face where $i \in \{2, 3, 4, 5\}$ (indices modulo 5). In this case, the 4-vertex v_2 is incident with exactly one 3-face by Lemma 2.3(2). So, v_2 sends at least $\frac{1}{4}$ to f. By (R4), v_1 sends $\frac{3}{4}$ to f. Thus, $\mu'(f) \geq -1 + \frac{3}{4} + \frac{1}{4} = 0$.
- If f is a rich 5-face, then f contains at least two 4-vertices in $\{v_2, v_3, v_3, v_5\}$, say v_i and v_j, each incident with at most one 3-face. So, each of v_i and v_j sends at least $\frac{1}{4}$ to f. By (R4), v_1 sends $\frac{1}{2}$ to f. Thus, $\mu'(f) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$.

Case 2. The face f is a $(5, 4, 4, 4, 4)$-face where $d(v_1) = 5$ and $d(v_i) = 4$ for $i \in \{2, 3, 4, 5\}$.

By (R3), v_1 sends at least $\frac{1}{2}$ to f. If v_i is incident with at most one 3-face for some $i \in \{2, 3, 4, 5\}$, then v_i sends at least $\frac{1}{4}$ to f. Thus, $\mu'(f) \geq -1 + \frac{1}{2} + \frac{1}{4}\beta = \frac{1}{4}(\beta - 2)$. To prove $\mu'(f) \geq 0$, it is sufficient to consider the case of $\beta \leq 1$. This condition immediately implies that $n_3(f) \geq 4$.

- Suppose first that $n_3(f) = 4$. It follows from $\beta \leq 1$ that f is a semi-poor 5-face. By symmetry, let edge v_1v_2 be not incident with 3-face and each other edge v_iv_{i+1} be incident with one 3-face where $i \in \{2, 3, 4, 5\}$ (indices modulo 5). Then v_2 is incident with exactly one 3-face by Lemma 2.3(2). So, v_2 sends at least $\frac{1}{4}$ to f. If v_1 is a good 5-vertex, then v_1 sends $\frac{3}{4}$ to f by (R3). Thus, $\mu'(f) \geq -1 + \frac{1}{4} + \frac{3}{4} = 0$. Thus, assume that v_1 is a bad 5-vertex. By (R3), v_1 sends $\frac{1}{2}$ to f. By the definition of bad 5-vertex, the edge v_1v_5 is incident with one $(5, 5^-, 4)$-face $[v_1v_5x]$. Then x is a 5^--vertex. By Lemma 2.3(1), x is not a vertex on the face f. Since f is a $(5, 4, 4, 4, 4)$-face, it follows that the vertex x is a source of the face f. By Lemma 2.5, x is a 5-vertex. By (R6), x sends $\frac{1}{4}$ to f. Thus, $\mu'(f) \geq -1 + \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 0$.
- Next suppose that $n_3(f) = 5$. It can verify that f is a poor 5-face where each edge v_iv_{i+1} is incident with a 3-face $g_i = [v_iv_{i+1}x_i]$ for each $i \in [5]$ (indices modulo 5). By Lemma 2.3(1), the vertices x_1 and x_5 lie outside f, making them sources of f. Lemma 2.5 further establishes that both x_1 and x_5 must be 5^+-vertices.

  Let y denote the remaining neighbor of v_1 not on f, and consider the faces h_1 and h_5 containing paths yv_1x_1 and yv_1x_5, respectively. The definition of rich 5-faces combined with Lemma 2.3(2) shows that each h_i is either a 6^+-face or a rich 5-face. Consequently, each x_i is incident with at least one 5^+-face that is not poor 5-face for each $i \in \{1, 5\}$, and by (R6), must send at least $\frac{1}{6}$ to f.

  When v_1 is a good 5-vertex, by (R3), it sends $\frac{3}{4}$ to f. Thus, $\mu'(f) \geq -1 + \frac{3}{4} + 2 \times \frac{1}{6} = \frac{1}{12} > 0$. Now consider that v_1 is a bad 5-vertex. By definition, its adjacent faces g_1 and g_5 are $(5, 5^-, 4)$-faces, which would imply x_1 and x_5 are 5^--vertices. However, this contradicts our earlier conclusion that they must be 5^+-vertices. This implies that x_1 and x_5 are exactly 5-vertices. As sources of f, they each send $\frac{1}{4}$ to f by (R6), while the bad vertex v_1 sends $\frac{1}{2}$ to f by (R3). Thus, $\mu'(f) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$.

Case 3. The face f is a $(4, 4, 4, 4, 4)$-face where $d(v_i) = 4$ for $i \in [5]$.

If any vertex v_i is incident with at most one 3-face where $i \in [5]$, then it sends at least $\frac{1}{4}$ to f. Consequently, we have $\mu'(f) \geq -1 + \frac{1}{4}\beta = \frac{1}{4}(\beta - 4)$. To ensure $\mu'(f) \geq 0$, it suffices to consider the case where $\beta \leq 3$.

By Lemma 2.3(2), we can conclude that the face f must be adjacent to at least three 3-faces. Therefore, $n_3(f) \geq 3$. For some $i \in [5]$, the vertex v_i lies on two 3-faces.
Suppose v_i is incident with two 3-faces $[v_i v_{i-1} x_{i-1}]$ and $[v_i v_{i+1} x_{i+1}]$ for some $i \in [5]$ (indices modulo 5). By Lemma 2.3(1), neither x_{i-1} nor x_i is a vertex on the face f. Thus, both x_{i-1} and x_i are sources of f. By Lemma 2.5, each of x_{i-1} and x_i is a 5^+-vertex. Let h_i be the face containing the path $x_{i-1}v_i x_i$, where $i \in [5]$. By Lemma 2.3(2), h_i is a 5^+-face. Recall that h_i is incident with at least two 5^+-vertices. Therefore, h_i is either a 6^+-face or a rich 5-face. This implies that each of x_{i-1} and x_i is incident with at least one 5^+-face other than poor 5-face. By (R6), each of x_{i-1} and x_i sends at least $\frac{1}{6}$ to f.

- First assume that $n_3(f) = 3$. Let $v_i v_{i+1}$ be incident with a 3-face $g_i = [v_i v_{i+1} x_i]$ for some $i \in [5]$, with indices taken modulo 5. By symmetry, we may assume without loss of generality that either all three faces g_1, g_3, g_4 are 3-faces, or all three faces g_3, g_4, g_5 are 3-faces.
  In the former case (where g_1, g_3, g_4 are 3-faces), Lemma 2.3(2) implies that each of v_1, v_2, v_3, and v_5 is incident with exactly one 3-face. However, this contradicts our initial assumption that $\beta \leq 3$.
  In the latter case (where g_3, g_4, g_5 are 3-faces), Lemma 2.3(2) yields the following configuration: each of v_1, v_2, v_3 is incident with at most one 3-face, while each of v_4, v_5 is incident with two 3-faces. So, each of v_1, v_2, v_3 sends at least $\frac{1}{4}$ to f, and each of x_3, x_4, x_5 contributes at least $\frac{1}{6}$ to f. Thus, $\mu'(f) \geq -1 + 3 \times \frac{1}{4} + 3 \times \frac{1}{6} = -1 + \frac{3}{4} + \frac{1}{2} = \frac{1}{4} > 0$.
- Next assume that $n_3(f) = 4$. Without loss of generality, we may assume that the edge $v_1 v_2$ is incident with a 4^+-face distinct from f, while each edge $v_i v_{i+1}$ is incident with a 3-face $f_i = [v_i v_{i+1} x_i]$ where $i \in \{2, 3, 4, 5\}$ with indices modulo 5.
  By Lemma 2.3(2), we can verify the following configuration: each of v_1 and v_2 is incident with exactly one 3-face, and while each of v_3, v_4, and v_5 is incident with two 3-faces. So, both v_1 and v_2 send at least $\frac{1}{4}$ to f. Meanwhile, each of the vertices x_2, x_3, x_4, and x_5 contributes at least $\frac{1}{6}$ to f. Combining these contributions, we obtain $\mu'(f) \geq -1 + 2 \times \frac{1}{4} + 4 \times \frac{1}{6} = \frac{1}{6} > 0$.
- Finally, we consider the case when $n_3(f) = 5$. Then each vertex v_i is incident with exactly two 3-faces for $i \in [5]$. Let $v_i v_{i+1}$ be incident with one 3-face $[v_i v_{i+1} x_i]$ where $i \in [5]$ (indices modulo 5). Let h_{i+1} be the face containing the path $x_i v_{i+1} x_{i+1}$, where $i \in [5]$ (indices modulo 5). Similarly, h_{i+1} must be either a 6^+-face or a rich 5-face. Since h_i is also a 6^+-face or rich 5-face, x_i is incident with at least two such 5^+-faces (excluding poor 5-faces). By (R6), x_i sends at least $\frac{1}{4}$ to f. Therefore, $\mu'(f) \geq -1 + 5 \times \frac{1}{4} = \frac{1}{4} > 0$.

(4) Let f be a k-face where $k \geq 6$. Since no 6^+-face participates in the discharge procedure, the charge of f remains unchanged. Thus, $\mu'(f) = \mu(f) = k - 6 \geq 0$.

Acknowledgments

The author would like to thank the referees for valuable suggestions to improve this work.

References

[1] V.G. Vizing, Vertex colorings with given colors. *Metody Diskret. Anal. Novosibirsk* **29** (1976) 3–10 (in Russian).
[2] P. Erdős, A. Rubin and H. Taylor, Choosability in graphs. *Congr. Numer.* **26** (1979) 125–157.
[3] T. Jensen and B. Toft, Graph Coloring Problems. Wiley, New York (1995).
[4] N. Alon and M. Tarsi, Colorings and orientations of graphs. *Combinatorica* **12** (1992) 125–134.
[5] C. Thomassen, Every planar graph is 5-choosable. *J. Comb. Theory, Ser. B* **62** (1994) 180–181.
[6] M. Voigt, List colourings of planar graphs. *Discrete Math.* **120** (1993) 215–219.
[7] S. Gutner, The complexity of planar graph choosability. *Discrete Math.* **159** (1996) 119–130.
[8] W. Cushing and H.A. Kierstead, Planar graphs are 1-relaxed, 4-choosable. *Eur. J. Comb.* **31** (2010) 1385–1397.
[9] G. Fijavž, M. Juvan, B. Mohar and R. Škrekovski, Planar graphs without cycles of specific lengths. *Eur. J. Comb.* **23** (2002) 377–388.

[10] P.C.B. Lam, B. Xu and J. Liu, The 4-choosability of plane graphs without 4-cycles. *J. Comb. Theory, Ser. B* **76** (1999) 117–126.

[11] W. Wang and K.-W. Lih, The 4-choosability of planar graphs without 6-cycles. *Australas. J. Comb.* **24** (2001) 157–164.

[12] W. Wang and K.-W. Lih, Choosability and edge choosability of planar graphs without five cycles. *Appl. Math. Lett.* **15** (2002) 561–565.

[13] P. Cheng, M. Chen and Y. Wang, Planar graphs without 4-cycles adjacent to triangles are 4-choosable. *Discrete Math.* **339** (2016) 3052–3057.

[14] X. Zhu, The Alon–Tarsi number of planar graphs. *J. Comb. Theory, Ser. B* **134** (2019) 354–358.

[15] J. Grytczuk and X. Zhu, The Alon–Tarsi number of a planar graph minus a matching. *J. Comb. Theory, Ser. B* (2020) 511–520.

[16] H. Lu and X. Zhu, The Alon–Tarsi number of planar graphs without cycles of length 4 and l. *Discrete Math.* **343** (2020).

[17] M. Voigt, A not 3-choosable planar graph without 3-cycles. *Discrete Math.* **146** (1995) 325–328.

[18] X. Zhu and R. Balakrishnan, Alon–Tarsi Theorem and Its Applications (2021).