RAIRO-Oper. Res. 60 (2026) 1621–1643
https://doi.org/10.1051/ro/2026036

RAIRO Operations Research
www.rairo-ro.org

CHARACTERIZATIONS OF LOCALLY LIPSCHITZ GEODESIC PSEUDOLINEAR FUNCTIONS ON HADAMARD MANIFOLDS

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Abstract. In this paper, using the properties of Clarke subdifferential, we derive some characterizations of locally Lipschitz geodesic pseudolinear functions on Hadamard manifolds. Moreover, employing these characterizations, we derive some properties of the solution sets of pseudolinear programming problems on the Hadamard manifold. The results established in this paper generalize and extend several known results in the literature. We furnish several examples to illustrate these results. Furthermore, to demonstrate the practical applications of our outcomes, we solve a specific form of the Karcher mean problem by employing the results established in this paper.

Mathematics Subject Classification. 90C30, 58C05, 58D17.

Received July 1, 2022. Accepted April 6, 2026.

1. Introduction

It is well known that convexity plays a central role in optimization theory, as in a convex minimization problem, a local minimum becomes a global minimum. Despite being a central tool in optimization, the notion of convexity is very restrictive for real-life problems (see, [49]). By weakening the notion of convexity, Mangasarian [24] introduced a more general class of functions, namely, pseudoconvex functions. Moreover, Mangasarian [24] introduced pseudoconcave functions by relaxing concavity. Functions, which are both pseudoconvex and pseudoconcave, are named as pseudolinear functions by Chew and Choo [8]. For more exposition about pseudolinear functions, we refer to a monograph by Mishra and Upadhyay [29].

In recent years, researchers have established many characterizations of pseudolinear functions as well as characterizations of solution sets of pseudolinear programming problems, for instance, see [14, 15, 34] and the references cited therein. Kortanek and Evans [17] characterized the functions which are both pseudoconvex and pseudoconcave, and established the Lagrange regularity conditions for constrained pseudoconcave problem. Further, some properties of pseudolinear function and pseudolinear optimization problem have been derived by Chew and Choo [8]. Komlosi [16] presented first and second order characterizations of pseudolinear functions.

Keywords. Hadamard manifold, geodesic pseudolinear functions, Clarke subdifferential, solution sets, Karcher mean problem.

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Characterizations of solution sets of nonlinear programming problems with multiple solutions are very useful for better understanding of solution methods. Mangasarian [25] characterized solution sets of convex programs with differentiable objective function and studied monotone linear complementarity problems in $\mathbb{R}^n$. Burke and Ferris [4] provided several characterizations for nondifferentiable convex programming problem. Jeyakumar and Yang [13] considered differentiable pseudolinear programming problem and obtained some new characterizations of its solution sets. Lalitha and Mehta [18] extended these characterizations for non-differentiable case. Using Clarke subdifferential, Hui and Li [23] characterized locally Lipschitz pseudolinear functions and solution sets of a pseudolinear programming problem on Banach space. Dinh *et al.* [9] studied pseudolinear minimization problem and characterized solution sets of linear fractional programs. Zhao and Tang [51] characterized solution sets of nonsmooth generalized pseudolinear extremum problem. Moreover, Mishra and Upadhyay [26–28] studied fractional and multiobjective programming problems involving generalized pseudolinearity. Mishra *et al.* [30] considered a minimization problem involving nonsmooth pseudolinear function and established Lagrange multiplier characterizations of its solution sets.

In the last few decades, several important concepts of optimization theory have been extended from Euclidean space to Riemannian manifold, (see, for instance, [19, 32, 37, 38]). There are some advantages of these extensions, for example, a nonconvex optimization problem can be reduced into a convex optimization problem by a suitable Riemannian metric, and a constrained optimization problem can be transformed into an unconstrained one, in the view of Riemannian geometry. A generalized concept of convexity known as geodesic convexity was introduced by Rapscák [35] and Udrişte [40]. Barani [1] and Hosseini and Pouryayevali [11] considered the notion of Clarke subdifferential and gave several characterizations of generalized convexity in the framework of the Riemannian manifold. Németh [31] presented the notion of generalized pseudoconvexity for differentiable functions on Riemannian manifolds. Jayswal *et al.* [12] derived some relations between vector variational inequality and vector optimization problem with approximate geodesic pseudoconvex function on a Riemannian manifold. Chen and Fang [5] derived some properties for nondifferentiable generalized pseudoconvex functions on Hadamard manifolds. Li *et al.* [22] characterized the solution sets of pseudoaffine programming problems on the Hadamard manifold. For more exposition and the updated survey about optimization techniques in the framework of Hadamard manifolds, we refer to [5, 6, 10, 41–45, 47, 48] and the references cited therein.

Motivated by [22, 23, 30, 51], we introduce locally Lipschitz geodesic pseudolinear functions on the Hadamard manifold and derive some properties of this class of functions. Moreover, we show the geodesic convexity of the level set of this class of functions and establish positive linear dependence of any two vectors of the Clarke subdifferential of every geodesic pseudolinear function. Further, we consider a minimization problem involving a geodesic pseudolinear function and deduce some important characterizations of its solution sets. In particular, the results presented in this paper extend the corresponding results of [23] from Banach space to the Hadamard manifold. In addition to this, the results deduced in this paper generalize corresponding results of [30, 51] from Euclidean space to the Hadamard manifold. Moreover, our results generalize the result of [22] from smooth to nonsmooth case in the Hadamard manifold setting. We also provide suitable examples to illustrate these results.

This paper is organized in the following manner: In Section 2, we recall some definitions and preliminaries. In Section 3, we derive some characterizations of locally Lipschitz geodesic pseudolinear functions. In Section 4, we formulate a nonlinear programming problem involving a geodesic pseudolinear objective function and derive some characterizations of solution sets of this problem. In Section 5, to illustrate the practical significance of the derived results, we consider a specific form of the Karcher mean problem and solve it by employing the results established in this paper. Conclusions along with future research direction are drawn in Section 6.

2. Preliminaries

In this section, we recall some basic definitions and results related to nonsmooth analysis that will be useful to derive our main results.

The conventional notations $\mathbb{R}^n$ and $\mathbb{N}$ are utilized to represent the Euclidean space of dimension n and the set of all natural numbers, respectively. For any $n \in \mathbb{N}$, let $\mathbb{H}$ denote the n-dimensional smooth manifold. The set of all tangent vectors at the point $x \in \mathbb{H}$, is denoted by $T_x\mathbb{H}$. The tangent bundle of $\mathbb{H}$ is denoted by $T\mathbb{H} = \bigcup_{x\in\mathbb{H}} T_x\mathbb{H}$. On a smooth manifold $\mathbb{H}$, the Riemannian metric is a 2-tensor field, symmetric, and positive definite. We employ the notation $\mathscr{G}$ to denote the Riemannian metric. For any two arbitrary elements $\nu_1,\ \nu_2 \in T_x\mathbb{H}$, the inner product of ν_1 and ν_2 is given by $\langle \nu_1, \nu_2\rangle_x = \mathscr{G}_x(\nu_1, \nu_2)$, where $\mathscr{G}_x$ is the Riemannian metric at the point and $\|\cdot\|_x$ be norm associated with this inner product. For simplicity, from now onwards, unless specified otherwise, we use $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ to denote the inner product on $\mathbb{H}$ and its corresponding norm, respectively.

A smooth manifold $\mathbb{H}$, endowed with a Riemannian metric, is referred to as a Riemannian manifold. Now, we assume that $\mathbb{H}$ is a connected Riemannian manifold of dimension n.

Let $\gamma_{xy} : [a, b] \to \mathbb{H}$ be a piecewise smooth curve joining the two points x and y, such that $\gamma_{xy}(a) = x$ and $\gamma_{xy}(b) = y$. The length of curve γ_{xy} is defined as follows:

$$L(\gamma_{xy}) := \int_a^b ||\gamma'_{xy}(t)||\ \mathrm{d}t.$$

Then, for any $x, y \in \mathbb{H}$, the Riemannian distance $d(x, y)$ is defined by minimizing this length over the set of all such curves γ_{xy} joining x and y.

Let ∇ denote the Levi–Civita connection associated with the Riemannian metric on $\mathbb{H}$. Given a smooth curve γ in $\mathbb{H}$, a vector field X is said to be parallel along γ if $\nabla_{\gamma'}X = 0$. Additionally, it is noteworthy that if γ' itself is parallel along γ, then γ is termed as a geodesic. Furthermore, if the length of a geodesic joining x to y in $\mathbb{H}$ equals $d(x, y)$, the geodesic is called a minimal geodesic. The parallel translation on the tangent bundle $T\mathbb{H}$ along γ, is an isometry induced by ∇ and given by the map:

$$P_{\gamma(t_1)}^{\gamma(t_2)} : T_{\gamma(t_1)}\mathbb{H} \to T_{\gamma(t_2)}\mathbb{H}.$$

A Riemannian manifold is said to be complete if, for any $x \in \mathbb{H}$, all geodesics $\gamma(t)$ emanating from x are defined for all $t \in \mathbb{R}$. In view of the Hopf–Rinow theorem, it follows that if $\mathbb{H}$ is complete, then any two points in $\mathbb{H}$ can be joined by a minimal geodesic. Moreover, $(\mathbb{H}, d)$ forms a complete metric space.

Let $\mathbb{H}$ be complete. Then the exponential map $\exp_z : T_z\mathbb{H} \to \mathbb{H}$ at $z \in \mathbb{H}$ is defined as $\exp_z v = \gamma_v(1, z)$ for each $v \in T_z\mathbb{H}$, where $\gamma(\cdot) = \gamma_v(\cdot, z)$ represents the geodesic starting at z with velocity v, satisfying $\gamma(0) = z$ and $\gamma'(0) = v$. It is evident that $\exp_z(tv) = \gamma_v(t, z)$ for each real number t. Note that the map $\exp_z$ is differentiable on $T_z\mathbb{H}$ for any $z \in \mathbb{H}$.

An n-dimensional simply connected and complete Riemannian manifold $\mathbb{H}$ with nonpositive sectional curvature is referred to as Hadamard manifold of dimension n. Henceforth, we are assuming that $\mathbb{H}$ denotes the Hadamard manifold of dimension n.

The following proposition is from [36].

Proposition 2.1. *The exponential mapping* $\exp_x : T_x\mathbb{H} \to \mathbb{H}$ *is a diffeomorphism for every* $x \in \mathbb{H}$. *Moreover, for any* $x, y \in \mathbb{H}$, *there exists a unique minimal geodesic joining* x *to* y, *given by:*

$$\gamma(t) = \exp_x(t \exp_x^{-1} y), \quad \forall t \in [0, 1].$$

We recall that a function $h : \mathbb{H} \to \mathbb{R}$ is said to be Lipschitz of rank K on a given subset Γ of $\mathbb{H}$, if there exists a constant $K \geq 0$ with $K = K(\Gamma)$, such that

$$|h(x) - h(y)| \leq Kd(x, y), \quad \forall x, y \in \Gamma,$$

where $d(x, y)$ represents the Riemannian distance on $\mathbb{H}$. A function h is said to be Lipschitz near $x' \in \mathbb{H}$, if there exists $K(x') \geq 0$ such that the above inequality holds with $K = K(x')$ for every x, y in the neighbourhood of x'. A function $h : \mathbb{H} \to \mathbb{R}$ is said to be locally Lipschitz on $\mathbb{H}$, if h is locally Lipschitz near x, for every $x \in \mathbb{H}$.

The following notions of generalized directional derivative and Clarke subdifferential in the framework of Hadamard manifolds are from [2].

Definition 2.2. Given $x, y \in \mathbb{H}$ and a locally Lipschitz function $h : \mathbb{H} \to \mathbb{R}$, the generalized directional derivative $h^o(x; v)$ of h at x in the direction $v \in T_x\mathbb{H}$, is defined as follows:

$$h^o(x; v) = \lim_{y \to x} \sup_{t \downarrow 0} \frac{h\Big(\exp_y t(d\exp_x)_{\exp_x^{-1} y} v\Big) - h(y)}{t},$$

where $(d\exp_x)_{\exp_x^{-1} y} : T_{\exp_x^{-1} y}(T_x\mathbb{H}) \simeq T_x\mathbb{H} \to T_y\mathbb{H}$ is the differential of exponential mapping at $\exp_x^{-1} y$.

Definition 2.3. Given a locally Lipschitz function $h : \mathbb{H} \to \mathbb{R}$, the generalized gradient (or Clarke subdifferential) of h at $z \in \mathbb{H}$ is the subset $\partial_c h(z)$ of $T_z\mathbb{H}$, defined by:

$$\partial_c h(z) = \{\zeta \in T_z\mathbb{H} \mid h^\circ(z; v) \geq \langle \zeta, v \rangle, \ \forall v \in T_z\mathbb{H}\}.$$

Remark 2.4. Notably, for a locally Lipschitz function $h : \mathbb{H} \to \mathbb{R}$, the following holds:

$$\partial_c(-h)(x) = -\partial_c(h)(x).$$

Moreover, $\xi \in \partial_c h(x)$ if and only if $-\xi \in \partial_c(-h)(x)$.

The following result from Barani [1] is an extension of the classical Lebourg's mean value theorem in the framework of Hadamard manifolds.

Lemma 2.5 (Mean Value Theorem)**.** *Let $x, y \in \mathbb{H}$ and $h : \mathbb{H} \to \mathbb{R}$ be a locally Lipschitz function. Then there exist $t_0 \in (0, 1)$ and $\xi \in \partial_c h(\gamma(t_0))$, such that*

$$h(y) - h(x) = \langle \xi, \gamma'(t_0) \rangle,$$

where $\gamma(t) = \exp_x\big(t \exp_x^{-1} y\big), \ \forall t \in [0, 1]$.

The following lemma from [44] provides some basic properties of the Clarke subdifferential in the framework of Hadamard manifolds.

Lemma 2.6. *Let $x \in \mathbb{H}$ and $h : \mathbb{H} \to \mathbb{R}$ be a locally Lipschitz function of rank K near x. Then the following assertions hold:*

(i) *$\partial_c h(x)$ is a nonempty, convex, compact subset of $T_x\mathbb{H}$, and $\|\xi\|_x \leq K$ for every $\xi \in \partial_c h(x)$.*
(ii) *Let $\{x_i\}$ and $\{\xi_i\}$ be sequences in $\mathbb{H}$ and $T_x\mathbb{H}$, respectively, such that $\xi_i \in \partial_c h(x_i)$ for each i. Suppose $\{x_i\}$ converges to x, and let ξ be a cluster point of the sequence $\{\xi_i\}$. Then, it follows that $\xi \in \partial_c h(x)$.*

The following definition is from [6].

Definition 2.7. A set Γ $(\subseteq \mathbb{H})$ is said to be geodesic convex, if the following inclusion holds:

$$\exp_x\big(t \exp_x^{-1} y\big) \in \Gamma,$$

for any two points $x, y \in \Gamma$ and for all $t \in [0, 1]$.

Throughout this paper, unless stated otherwise, we assume that $\mathcal{D}$ is a nonempty open subset of $\mathbb{H}$ and Γ is a geodesic convex subset of $\mathcal{D}$.

In the Hadamard manifold setting, the notion of generalized convexity for a locally Lipschitz function is recalled in the following definition (see, [5]).

Definition 2.8. Let $h : \mathcal{D} \to \mathbb{R}$ be a locally Lipschitz function on $\mathcal{D}$. Then h is said to be
(i) geodesic quasiconvex on Γ if

$$h(\gamma(t)) \leq \max\{h(x), h(y)\}, \qquad \forall x, y \in \Gamma \text{ and } t \in [0,1];$$

(ii) geodesic semistrictly quasiconvex on Γ, if $\forall x, y \in \Gamma$ with $h(x) \neq h(y)$,

$$h(\gamma(t)) < \max\{h(x), h(y)\}, \qquad \forall t \in [0,1],$$

where $\gamma(t) = \exp_x\left(t \exp_x^{-1} y\right)$;
(iii) geodesic pseudoconvex on Γ, if for any $x, y \in \Gamma$

$$\exists\ \xi \in \partial_c h(x) \text{ such that } \left\langle \xi, \exp_x^{-1} y \right\rangle \geq 0 \Rightarrow h(y) \geq h(x),$$

or equivalently,

$$h(y) < h(x) \implies \langle \xi, \exp_x^{-1} y \rangle < 0, \qquad \forall \xi \in \partial_c h(x).$$

Definition 2.9. A function $h : \mathcal{D} \to \mathbb{R}$ is said to be geodesic quasiconcave if $-h$ is geodesic quasiconvex.

Definition 2.10. A function $h : \mathcal{D} \to \mathbb{R}$ is said to be geodesic pseudoconcave on Γ if $-h$ is geodesic pseudoconvex on Γ.

Definition 2.11. A function $h : \mathcal{D} \to \mathbb{R}$ is said to be geodesic pseudolinear on Γ if h is geodesic pseudoconvex and geodesic pseudoconcave on Γ.

The following result from [5] demonstrates the relationship between geodesic quasiconvex, geodesic semistrictly quasiconvex, and geodesic pseudoconvex functions.

Lemma 2.12. *Let $h : \mathcal{D} \to \mathbb{R}$ be a locally Lipschitz and geodesic pseudoconvex function on Γ. Then h is both geodesic quasiconvex and geodesic semistrictly quasiconvex on Γ.*

Let $X(\mathbb{H})$ be the set of all multivalued vector fields $\sigma : \mathbb{H} \to 2^{T\mathbb{H}}$, such that $\sigma(x) \subseteq T_x\mathbb{H}$ for all $x \in \mathbb{H}$, and

$$D(\sigma) = \{x \in \mathbb{H} : \sigma(x) \neq \emptyset\},$$

defines the domain of σ.

The following definition from Li *et al.* [21] represents the concept of pseudomonotonicity for multivalued vector fields in the framework of Hadamard manifolds.

Definition 2.13. A multivalued vector field σ is said to be pseudomonotone, if for any $x, y \in D(\sigma)$ and $u \in \sigma(x)$, the following implication holds:

$$\left\langle u, \exp_x^{-1} y \right\rangle \geq 0 \Rightarrow \left\langle v, \exp_y^{-1} x \right\rangle \leq 0, \qquad \forall v \in \sigma(y).$$

The following Lemma from [5] establishes the relationship between the geodesic pseudoconvexity of a locally Lipschitz function and the pseudomonotonicity of its Clarke subdifferential.

Lemma 2.14. *Let $h : \mathcal{D} \to \mathbb{R}$ be a locally Lipschitz function on Γ. Then, h is geodesic pseudoconvex if and only if $\partial_c h$ is pseudomonotone on Γ.*

3. Locally Lipschitz geodesic pseudolinear functions on Hadamard manifolds

In this section, we establish some characterizations of geodesic pseudolinear functions.

Theorem 3.1. *Let $h : \mathcal{D} \to \mathbb{R}$ be a locally Lipschitz geodesic pseudolinear function on Γ. Then for all $x, y \in \Gamma$, the following statements are equivalent:*

(i) *There exists some $\xi \in \partial_c h(x)$, such that $\langle \xi, \exp_x^{-1} y\rangle = 0$,*
(ii) *$h(x) = h(y)$.*

Proof. To begin with, we assume that (i) holds and we show that

$$h(x) = h(y), \quad \forall x, y \in \Gamma.$$

From the hypothesis, it follows that h is a geodesic pseudoconvex function and hence for all $x, y \in \Gamma$, there exists some $\xi \in \partial_c h(x)$, such that

$$\langle \xi, \exp_x^{-1} y\rangle \geq 0 \Rightarrow h(y) \geq h(x). \tag{1}$$

Moreover, by employing the geodesic pseudoconcavity of h, there exists some $\xi \in \partial_c h(x)$, such that

$$\langle \xi, \exp_x^{-1} y\rangle \leq 0 \Rightarrow h(y) \leq h(x), \tag{2}$$

for all $x, y \in \Gamma$. Combining (1) with (2), it follows that

$$\exists\, \xi \in \partial_c h(x), \text{ such that } \langle \xi, \exp_x^{-1} y\rangle = 0 \Rightarrow h(y) = h(x), \quad \forall x, y \in \Gamma.$$

(ii) $\Rightarrow$ (i) On the contrary, let $h(x) = h(y)$ for some $x, y \in \Gamma$, but $\langle \xi, \exp_x^{-1} y\rangle \neq 0$. Without loss of generality, one can assume that

$$\forall \xi \in \partial_c h(x),\ \langle \xi, \exp_x^{-1} y\rangle > 0.$$

For any $t \in [0, 1]$, we set

$$x_t := \exp_x(t \exp_x^{-1} y).$$

Now, our claim is to show that $h(x_t) = h(x),\ \forall t \in (0, 1)$. To prove, $h(x_t) = h(x),\ \forall t \in (0, 1)$, We consider the following possible cases:

Case 1: Let $h(x_t) > h(x),\ \forall t \in (0, 1)$.

By invoking the geodesic pseudoconvexity of h, for all $\xi' \in \partial_c h(x_t)$, we have

$$\langle \xi', \exp_{x_t}^{-1} x\rangle < 0. \tag{3}$$

Moreover,

$$\exp_{x_t}^{-1} x = \frac{-t}{1-t} \exp_{x_t}^{-1} y, \quad \forall t \in (0, 1). \tag{4}$$

Employing (3) and (4), one can conclude that

$$\left\langle \xi', \frac{-t}{1-t} \exp_{x_t}^{-1} y \right\rangle < 0,$$

or

$$\langle \xi', \exp_{x_t}^{-1} y\rangle > 0.$$

Utilizing the geodesic pseudoconvexity of h, it follows that

$$h(y) \geq h(x_t),$$

but $h(x_t) > h(x) = h(y)$, which is a contradiction.

Case 2: Let $h(x_t) < h(x),\ \forall t \in (0,1)$.

By employing similar arguments to those in Case 1, we conclude that $h(x_t) = h(x) = h(y)$. Now,

$$h(x_t) = h(\exp_x(t \exp_x^{-1} y)) = h(x), \qquad \forall t \in (0,1),$$
$$\implies h(\exp_x(t \exp_x^{-1} y)) - h(x) = 0.$$

In the light of Lemma 2.5, there exists some $t_0 \in (0,t)$ and $\zeta \in \partial_c h(x_{t_0})$, such that the following equality holds:

$$\langle \zeta, x'_{t_0} \rangle = \langle \zeta, \exp_x^{-1} y \rangle = 0. \tag{5}$$

Since $t \to 0,\ t_0 \to 0,\ x_{t_0} \to x$, therefore from (5) and Lemma 2.6, it follows that

$$\exists\, \xi \in \partial_c h(x) \text{ such that } \langle \xi, \exp_x^{-1} y \rangle = 0.$$

This completes the proof.

□

Remark 3.2. From Theorem 3.1, we can conclude that if h is locally Lipschitz geodesic pseudolinear function on a geodesic convex set $\Gamma \subseteq \mathbb{H}$ and $0 \in \partial_c h(x)$ for any $x \in \Gamma$, then h is a constant function on Γ.

Theorem 3.3. *Let $h : \mathcal{D} \to \mathbb{R}$ be a locally Lipschitz function, then the following statements are equivalent:*

(i) *h is geodesic pseudolinear on Γ,*
(ii) *there exists a function $P : \Gamma \times \Gamma \to \mathbb{R}$, such that for every $x, y \in \Gamma$, $P(x,y) > 0$ and there exists $\xi \in \partial_c h(x)$, satisfying:*

$$h(y) = h(x) + P(x,y)\langle \xi, \exp_x^{-1} y \rangle. \tag{6}$$

Proof. We assume that h is a geodesic pseudolinear function on Γ and we show that the assertion (ii) holds. If $\langle \xi, \exp_x^{-1} y \rangle = 0$, for some $\xi \in \partial_c h(x)$ and for any $x, y \in \Gamma$, then we define $P(x,y) := 1$. Therefore, invoking Theorem 3.1, one can conclude that $h(x) = h(y)$ and equation (6) holds.

If $\langle \xi, \exp_x^{-1} y \rangle \neq 0$, for some $\xi \in \partial_c h(x)$ and for any $x, y \in \Gamma$, we define $P : \Gamma \times \Gamma \to \mathbb{R}$ as follows:

$$P(x,y) := \frac{h(y) - h(x)}{\langle \xi, \exp_x^{-1} y \rangle}.$$

From the definition of $P(x,y)$, one can observe that the Equation (6) is satisfied. To complete the proof, it remains to show that $P(x,y) > 0$, $\forall x, y \in \Gamma$. The following cases may arise:

Case 1: Let $h(y) > h(x)$, then by employing the geodesic pseudoconcavity of h, we have:

$$\langle \xi, \exp_x^{-1} y \rangle > 0 \implies \frac{h(y) - h(x)}{\langle \xi, \exp_x^{-1} y \rangle} > 0.$$

Case 2: Let $h(y) < h(x)$, then by using the geodesic pseudoconvexity of h, it follows that:

$$\langle \xi, \exp_x^{-1} y \rangle < 0 \implies \frac{h(y) - h(x)}{\langle \xi, \exp_x^{-1} y \rangle} > 0.$$

Case 3: Let $h(y) = h(x)$, then in view of Theorem 3.1, there exists some $\xi \in \partial_c h(x)$, such that

$$\langle \xi, \exp_x^{-1} y \rangle = 0 \implies P(x,y) = 1 > 0.$$

This ensures that $P(x,y) > 0$ for all $x, y \in \Gamma$.

Conversely, assume that, there exists $P : \Gamma \times \Gamma \to \mathbb{R}$ and $\xi \in \partial_c h(x)$, such that

$$P(x,y) > 0 \text{ and } h(y) = h(x) + P(x,y)\langle \xi, \exp_x^{-1} y\rangle.$$

We show that h is a geodesic pseudolinear function on Γ.

Now, if $\langle \xi, \exp_x^{-1} y\rangle \geq 0$, then from (6), one has

$$h(y) - h(x) = P(x,y)\langle \xi, \exp_x^{-1} y\rangle \geq 0 \Rightarrow h(y) \geq h(x),$$

which exhibits the fact that h is a geodesic pseudoconvex. In the similar manner, when $\langle \xi, \exp_x^{-1} y\rangle \leq 0$, then from (6), one has:

$$h(y) - h(x) = P(x,y)\langle \xi, \exp_x^{-1} y\rangle \leq 0 \Rightarrow h(y) \leq h(x),$$

which implies that h is a geodesic pseudoconcave. Hence, h is geodesic pseudolinear on Γ. This completes the proof. □

Theorem 3.4. *Let $h : \mathcal{D} \to \mathbb{R}$ be a locally Lipschitz function. If h is geodesic pseudolinear on Γ and there exists $\xi \in \partial_c h(x)$, such that $\langle \xi, \exp_x^{-1} y\rangle = 0$. Then, for all $t \in [0,1]$ and $x, y \in \Gamma$*

$$h(\gamma(t)) = h(y) = h(x),$$

where $\gamma(t) = \exp_x(t \exp_x^{-1} y)$.

Proof. Let $x, y \in \Gamma$ and $\xi \in \partial_c h(x)$, such that $\langle \xi, \exp_x^{-1} y\rangle = 0$. Then, by geodesic pseudoconvexity of h, it follows that

$$h(y) \geq h(x). \tag{7}$$

From Lemma 2.12, it follows that a geodesic pseudoconvex function is also geodesic quasiconvex, and hence, by geodesic quasiconvexity of h, we have:

$$h(y) \geq h(\gamma(t)), \quad \forall t \in [0,1].$$

Similarly, by geodesic pseudoconcavity and geodesic quasiconcavity of h, we obtain:

$$h(y) \leq h(x),$$

and

$$h(y) \leq h(\gamma(t)), \quad \forall t \in [0,1]. \tag{8}$$

From (7) and (8), we get

$$h(\gamma(t)) = h(y) = h(x), \quad \forall t \in [0,1].$$

This completes the proof. □

Remark 3.5. (i) Theorems 3.1, 3.3 and 3.4 extend the results of Propositions 2.1 and 2.2 of [8], respectively, from smooth optimization problems to nonsmooth optimization problems and also generalize them from the context of Euclidean space to the framework of Hadamard manifolds.

(ii) Theorems 3.1, 3.3, and 3.4 are extensions of Theorems 3.1, 3.2, and 3.3 of [23], respectively, from the context of Banach space to the framework of Hadamard manifolds.

(iii) Theorems 3.1, 3.3, and 3.4 generalize the results of [30] from the context of Euclidean space to the framework of Hadamard manifolds.

The following result extends Proposition 2.1 of [30] from the Euclidean space to the framework of Hadamard manifolds.

Proposition 3.6. *Let $h : \mathcal{D} \to \mathbb{R}$ be a locally Lipschitz and geodesic pseudolinear on Γ. Then, for any $z \in \Gamma$, the set $L := \{x \in \Gamma \mid h(x) = h(z)\}$ is geodesic convex subset of $\mathbb{H}$.*

Proof. In order to prove that L is a geodesic convex subset of $\mathbb{H}$, we show that

$$\gamma(t) = \exp_x(t \exp_x^{-1} y) \in L, \qquad \forall t \in [0,1],$$

for any $x, y \in L$. Since $x, y \in L$, therefore $h(x) = h(y) = h(z)$. From Theorem 3.1, there exists some $\xi \in \partial_c h(x)$, such that:

$$\langle \xi, \exp_x^{-1} y \rangle = 0.$$

In view of the fact, $\exp_x^{-1} \gamma(t) = t \exp_x^{-1} y$, it follows that

$$\begin{aligned} \langle \xi, \exp_x^{-1} \gamma(t) \rangle &= \langle \xi, t \exp_x^{-1} y \rangle \\ &= t \langle \xi, \exp_x^{-1} y \rangle = 0. \end{aligned}$$

From Theorem 3.1, we have

$$h(\gamma(t)) = h(x) = h(z).$$

This shows that $\gamma(t) \in L$, $\forall t \in [0,1]$. Therefore, L is a geodesic convex subset of Γ. This completes the proof. $\square$

Remark 3.7. The Cauchy–Schwarz inequality for any inner product space states that

$$\langle u, v \rangle \leq ||u|| ||v||,$$

where u, v are vectors of inner product space. When u, v are nonzero, then equality holds if and only if $u = \lambda v$, for some $\lambda > 0$.

The following proposition establishes positive linear dependence between any two vectors associated with the Clarke subdifferential of every locally Lipschitz geodesic pseudolinear function and generalizes Proposition 2.2 of [30] from the context of Euclidean space to the framework of Hadamard manifolds.

Proposition 3.8. *Let Γ $(\subseteq \mathbb{H})$ be an open geodesic convex set and $h : \Gamma \to \mathbb{R}$ be a real-valued locally Lipschitz function. Moreover, assume the geodesic pseudolinearity of h on Γ. Then for all $x \in \Gamma$ and any $\xi_1, \xi_2 \in \partial_c h(x)$, there exists some $\lambda > 0$, such that $\xi_2 = \lambda \xi_1$.*

Proof. Let ξ_1, ξ_2 be two arbitrary vectors in $\partial_c h(x)$. If $\xi_1 = 0$, then by Remark 3.2, h is a constant function on Γ. Therefore

$$\partial_c h(x) = \{0\} \text{ and } \xi_1 = \xi_2 = 0,$$

and the conclusion of the proposition holds for any $\lambda > 0$. Similarly, when $\xi_2 = 0$, it follows that $\xi_1 = 0$ and hence $\xi_2 = \lambda \xi_1$ holds for any $\lambda > 0$.

On the contrary, let us assume that $\xi_1 \neq 0$, $\xi_2 \neq 0$ and there does not exist any $\lambda > 0$, such that $\xi_2 = \lambda \xi_1$. Now, we show that there exists some $u \in \mathbb{H}$, such that $\langle \xi_1, u \rangle > 0$ and $\langle \xi_2, u \rangle < 0$. Let us define:

$$u = \frac{\xi_1}{||\xi_1||} - \frac{\xi_2}{||\xi_2||}.$$

In the Riemannian manifold, the tangent space is equipped with a positive definite inner product. Therefore, we can apply Cauchy–Schwarz inequality on the Hadamard manifold. Hence, by using Remark 3.7, it follows that when $\xi_2 \neq \lambda \xi_1$, for all $\lambda > 0$, one has

$$\begin{aligned} \langle \xi_1, u \rangle &= ||\xi_1|| - \frac{\langle \xi_1, \xi_2 \rangle}{||\xi_2||} \\ &> ||\xi_1|| - \frac{||\xi_1|| ||\xi_2||}{||\xi_2||} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned}\langle \xi_2, u\rangle &= \frac{\langle \xi_2, \xi_1\rangle}{||\xi_1||} - ||\xi_2|| \\ &< \frac{||\xi_2||||\xi_1||}{||\xi_1||} - ||\xi_2|| \\ &= 0.\end{aligned}$$

Here $\langle\cdot,\cdot\rangle$ is inner product on $T_x\mathbb{H}$ and $\|\cdot\|$ is norm associated with this inner product.

From the hypothesis Γ is open, and hence one can choose $t > 0$, such that

$$z_0 = \exp_x tu \in \Gamma \text{ and } \exp_x^{-1} z_0 = tu.$$

Now, we have

$$\langle \xi_1, \exp_x^{-1} z_0\rangle = \langle \xi_1, tu\rangle > 0, \tag{9}$$

and

$$\langle \xi_2, \exp_x^{-1} z_0\rangle = \langle \xi_2, tu\rangle < 0. \tag{10}$$

Let us define an open set B in $\mathbb{H}$ as follows:

$$B := \Gamma \cap \{z \in \mathbb{H} \mid \langle \xi_1, \exp_x^{-1} z\rangle > 0\ \} \cap \{z \in \mathbb{H} \mid \langle \xi_2, \exp_x^{-1} z\rangle < 0\}.$$

From (9) and (10), it follows that $z_0 \in B$ and hence the set B is nonempty. Moreover, for all $y \in B$, we have

$$\begin{aligned}\langle \xi_1, \exp_x^{-1} y\rangle > 0 &\implies h(y) \geq h(x), \\ \langle \xi_2, \exp_x^{-1} y\rangle < 0 &\implies h(y) \leq h(x).\end{aligned}$$

Therefore, it implies that h is constant on B and $\partial_c h(z) = \{0\}$ for every $z \in B$. Since, $B \subseteq \Gamma$ and h is geodesic pseudolinear on Γ, therefore from Remark 3.2, h is a constant function on Γ. Hence, $\partial_c h(x) = \{0\}$ and this contradicts our assumption. This completes the proof. □

The following example illustrates the significance of Theorems 3.1, 3.3, and Proposition 3.8.

Example 3.9. Let us consider the Poincaré half-plane:

$$\mathbb{H} := \{(z_1, z_2) \in \mathbb{R}^2 : z_2 > 0\}.$$

From [40], it follows that $\mathbb{H}$ is a Riemannian manifold endowed with the metric induced by the family of local scalar products $\langle u, v\rangle_z$, given by:

$$\langle u, v\rangle_z = \langle \mathscr{G}(z)u, v\rangle,\ \forall u, v \in T_z\mathbb{H} = \mathbb{R}^2,$$

where $\mathscr{G}(z) = \begin{bmatrix} \frac{1}{z_2^2} & 0 \\ 0 & \frac{1}{z_2^2}\end{bmatrix}$ and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^2$. The sectional curvature of $\mathbb{H}$ is -1 (see, [40]), hence $\mathbb{H}$ is also a Hadamard manifold. The exponential map $\exp : T_z\mathbb{H} \to \mathbb{H}$ is defined as follows (see, [40]):

If $v_1 = 0$

$$\exp_z(v) = \left(z_1, z_2 e^{\frac{v_2}{z_2}}\right).$$

If $v_1 \neq 0$, then $\exp_z(v)$ is given as:

$$\left(z_1 + \frac{v_2}{v_1} + \sqrt{1 + \left(\frac{v_2}{v_1}\right)^2}\tanh(p_{v_1,v_2}(1)), z_2\sqrt{1 + \left(\frac{v_2}{v_1}\right)^2}\frac{1}{\cosh(q_{v_1,v_2}(1))}\right),$$

where

$$p_{v_1,v_2}(t) = \begin{cases} t\sqrt{v_1^2+v_2^2} - \operatorname{arcsinh}\frac{v_2}{v_1} & \text{if } v_1 > 0, \\ -t\sqrt{v_1^2+v_2^2} - \operatorname{arcsinh}\frac{v_2}{v_1} & \text{if } v_1 < 0, \end{cases}$$

$$q_{v_1,v_2}(t) = \begin{cases} t\frac{\sqrt{v_1^2+v_2^2}}{z_2} - \operatorname{arcsinh}\frac{v_2}{v_1} & \text{if } v_1 > 0, \\ -t\frac{\sqrt{v_1^2+v_2^2}}{z_2} - \operatorname{arcsinh}\frac{v_2}{v_1} & \text{if } v_1 < 0. \end{cases}$$

The inverse exponential map is

$$\exp_z^{-1}(y) = \begin{cases} \left(0, z_2 \ln\frac{y_2}{z_2}\right) & \text{if } z_1 = y_1, \\ \frac{z_2}{r}\left(\operatorname{arctanh}\frac{b-z_1}{r} - \operatorname{arctanh}\frac{b-y_1}{r}\right)(z_2, b-z_1) & \text{if } z_1 \neq y_1, \end{cases}$$

where

$$b = \frac{z_1^2 + z_2^2 - \left(y_1^2+y_2^2\right)}{2(z_1-y_1)},\ r = \sqrt{\left(z_1-b\right)^2 + z_2^2}.$$

Let $\mathcal{D}$ ($\subset \mathbb{H}$) be a nonempty open set, defined as follows:

$$\mathcal{D} := \left\{(z_1, z_2) \in \mathbb{H} \mid z_1 = 0,\ z_2 > \frac{1}{4}\right\}.$$

Consider the function $h : \mathcal{D} \to \mathbb{R}$, defined as follows:

$$h(z) := |z_2 - 3| - 2(z_1 + z_2).$$

Now, we define the set Γ ($\subset \mathcal{D}$) as follows:

$$\Gamma := \left\{(z_1, z_2) \in \mathbb{H} \mid z_1 = 0,\ z_2 > \frac{1}{2}\right\}.$$

Then, the Clarke subdifferential of h is given by

$$\partial_c h(z) = \begin{cases} (0, -z_2^2), & z_2 > 3, \\ \{(0,k) : k \in [-27, -9]\}, & z_2 = 3, \\ (0, -3z_2^2), & z_2 < 3. \end{cases}$$

First, we show that h is a geodesic pseudolinear function on Γ. For $z = (z_1, z_2)$ and $y = (y_1, y_2) \in \Gamma$, we have

$$\langle \xi, exp_z^{-1} y\rangle = \begin{cases} -z_2^3 \ln\frac{y_2}{z_2}, & z_2 > 3, \\ \left\{k \ln\frac{y_2}{z_2},\ k \in [-27, -9]\right\} & z_2 = 3, \\ -z_2^3 \ln\frac{y_2}{z_2}, & z_2 < 3. \end{cases}$$

We observe that

$$\langle \xi, exp_z^{-1} y\rangle \geq 0 \implies \ln\frac{y_2}{z_2} \leq 0 \implies y_2 \leq z_2 \implies h(y) \geq h(z).$$

Hence, h is geodesic pseudoconvex on Γ. Similarly, we can verify that h is a geodesic pseudoconcave function on Γ. Moreover, if

$$\langle \xi, exp_z^{-1} y\rangle = 0,$$

it follows that $y = z$ and $h(y) = h(z)$. Conversely, when $h(z) = h(y)$, we have $z = y$. Therefore, we have

$$\langle \xi, \exp_z^{-1} y\rangle = 0 \iff h(z) = h(y).$$

Now, we define $P : \Gamma \times \Gamma \to \mathbb{R}$ as follows:

$$P(z,y) = \begin{cases} \frac{h(y)-h(z)}{\langle \xi, \exp_z^{-1} y\rangle}, & \text{when } z \neq y,\\ 1, & \text{otherwise.}\end{cases}$$

If $z \neq y$, the numerator and denominator in the above expression have the same sign, hence $P(z,y) > 0$. On the other hand, if $y = z$, $P(z,y) = 1 > 0$. Therefore, $P(z,y) > 0,\ \forall z,y \in \Gamma$ and this ensures the satisfaction of the identity in equation (6).

If $z \neq 3$, $\partial_c h(z)$ is a singleton set and for any $\xi_1, \xi_2 \in \partial_c h(z)$, it follows that $\xi_1 = \xi_2$. For $z = 3$, $\partial_c h(z) = \{(0,k) : k \in [-27,-9]\}$ and for any $\xi_1, \xi_2 \in \partial_c h(z)$, there exists some $\lambda > 0$, such that $\xi_1 = \lambda \xi_2$.

4. Solution sets of geodesic pseudolinear programming problem

In this section, we establish some properties of solution sets of geodesic pseudolinear programming problems on the Hadamard manifold.

We consider the following optimization problem in the Hadamard manifold setting:

$$\text{(P)} \qquad \min\ h(x)$$
$$\text{subject to } x \in \Gamma,$$

where, Γ is a geodesic convex subset of an open set $\mathcal{D}(\subseteq \mathbb{H})$, and $h : \mathcal{D} \to \mathbb{R}$ is a locally Lipschitz geodesic pseudolinear function on Γ. The solution set of (P), denoted by $\overline{T}$, is defined as

$$\overline{T} := \arg\min_{x\in\Gamma} h(x).$$

Throughout this section, we assume that $\overline{T} \neq \emptyset$.

Theorem 4.1. *Let $\overline{x} \in \overline{T}$. We define the following sets as follows:*

$$T^a := \{x \in \Gamma \mid \exists\, \xi \in \partial_c h(x)\ :\ \langle \xi, \exp_x^{-1} \overline{x}\rangle = 0\},$$
$$T^b := \{x \in \Gamma \mid \exists\, \overline{\xi} \in \partial_c h(\overline{x})\ :\ \langle \overline{\xi}, \exp_{\overline{x}}^{-1} x\rangle = 0\},$$
$$T^c := \{x \in \Gamma \mid \forall t \in [0,1],\ \exists\, \xi_t \in \partial_c h(\gamma(t))\ :\ \langle P^x_{\gamma(t)}\xi_t, \exp_x^{-1} \overline{x}\rangle = 0\},$$

where $\gamma(t) = \exp_x(t \exp_x^{-1} \overline{x})$. Then, $\overline{T} = T^a = T^b = T^c$.

Proof. Let $x \in \overline{T}$, then $h(x) = h(\overline{x})$. From Theorem 3.1, one has

$$x \in \overline{T} \iff \exists\, \xi \in \partial_c h(x),\ \text{such that } \langle \xi, \exp_x^{-1} \overline{x}\rangle = 0,$$

which concludes that $\overline{T} = T^a$.

To show $T^a = T^b$, let $x \in T^a$. Evidently, from Theorem 3.1, we have:

$$\exists\, \xi \in \partial_c h(x),\ \text{such that } \langle \xi, \exp_x^{-1} \overline{x}\rangle = 0 \implies h(x) = h(\overline{x}),$$

and

$$h(\overline{x}) = h(x) \implies \exists\, \overline{\xi} \in \partial_c h(\overline{x}),\ \text{such that } \langle \overline{\xi}, \exp_{\overline{x}}^{-1} x\rangle = 0.$$

Then, it follows that

$$x \in T^b \text{ and } T^a \subseteq T^b.$$

In a similar manner, we can show that $T^b \subseteq T^a$. Hence, we have

$$T^a = T^b.$$

Next, we will show that $\overline{T} = T^c$. Let $x \in \overline{T}$, then

$$h(x) = h(\overline{x}) \implies \exists\ \xi \in \partial_c h(x),\ \text{such that}\ \langle \xi, \exp_x^{-1} \overline{x} \rangle = 0.$$

By Theorem 3.4, we obtain

$$h(\gamma(t)) = h(x) \text{ for every } t \in [0,1],$$

and from Theorem 3.1, there is $\xi_t \in \partial_c h(\gamma(t))$, such that

$$\langle \xi_t, \exp_{\gamma(t)}^{-1} x \rangle = 0.$$

Then, it follows that

$$-\langle P_{\gamma(t)}^{x} \xi_t, \exp_x^{-1} \gamma(t) \rangle = 0.$$

In view of the following fact,

$$\exp_x^{-1}(\gamma(t)) = t \exp_x^{-1} \overline{x},\ \forall t \in [0,1], \tag{11}$$

it follows that for all $t \in [0,1]$,

$$\langle P_{\gamma(t)}^{x} \xi_t, t \exp_x^{-1} \overline{x} \rangle = 0,$$

or

$$\langle P_{\gamma(t)}^{x} \xi_t, \exp_x^{-1} \overline{x} \rangle = 0.$$

Hence, $x \in T^c$ and $\overline{T} \subseteq T^c$. Now, let $x \in T^c$, then for all $t \in (0,1]$, we get:

$$\exists\ \xi_t \in \partial_c h(\gamma(t)),\ \text{such that}\ \langle P_{\gamma(t)}^{x} \xi_t, \exp_x^{-1} \overline{x} \rangle = 0.$$

From (11), we have

$$\left\langle P_{\gamma(t)}^{x} \xi_t, \exp_x^{-1} \overline{x} \right\rangle = \left\langle P_{\gamma(t)}^{x} \xi_t, \frac{1}{t} \exp_x^{-1}(\gamma(t)) \right\rangle = 0,$$

or

$$\left\langle -\xi_t, \exp_{\gamma(t)}^{-1} x \right\rangle = 0.$$

Now, from Theorem 3.1, we have

$$\langle \xi_t, \exp_{\gamma(t)}^{-1} x \rangle = 0 \Rightarrow h(x) = h(\gamma(t)),\ \forall t \in (0,1].$$

Then,

$$\exists\ \xi \in \partial_c h(x),\ \text{such that}\ \langle \xi, \exp_x^{-1}(\gamma(t)) \rangle = 0,\ \forall t \in [0,1],$$

and using (11), one can write

$$\langle \xi, t \exp_x^{-1} \overline{x} \rangle = 0,$$

or

$$\langle \xi, \exp_x^{-1} \overline{x} \rangle = 0.$$

From Theorem 3.1, it follows that

$$h(x) = h(\overline{x}).$$

That is, $x \in \overline{T}$ and $T^c \subseteq \overline{T}$. Hence, $T^c = \overline{T}$ and

$$T^a = T^b = T^c = \overline{T}.$$

This completes the proof. $\square$

Remark 4.2. (i) Theorem 4.1 extends the results of Theorem 3.1 from [13] from smooth optimization problems to the nonsmooth optimization problems and also generalizes them from the context of Euclidean space to the framework of Hadamard manifolds.

(ii) Theorem 4.1 extends Theorem 3.5 of [22] from smooth to nonsmooth geodesic pseudolinear programming problems in the framework of Hadamard manifolds.

(iii) Theorem 4.1 extends Theorem 4.1 of [23] from the context of Banach space to the framework of Hadamard manifolds.

Corollary 4.3. *Let* $\overline{x} \in \overline{T}$. *Then* $\overline{T} = T^d = T^e = T^f$, *where*

$$\begin{aligned} T^d &:= \{x \in \Gamma \mid \exists\ \xi \in \partial_c h(x)\ :\ \langle \xi, \exp_x^{-1} \overline{x}\rangle \geq 0\},\\ T^e &:= \{x \in \Gamma \mid \exists\ \overline{\xi} \in \partial_c h(\overline{x})\ :\ \langle \overline{\xi}, \exp_{\overline{x}}^{-1} x\rangle \leq 0\},\\ T^f &:= \{x \in \Gamma \mid \forall t \in [0,1],\ \exists\ \xi_t \in \partial_c h(\gamma(t))\ :\ \langle P^x_{\gamma(t)}\xi_t, \exp_x^{-1} \overline{x}\rangle \geq 0\}, \end{aligned}$$

and $\gamma(t) = \exp_x(t \exp_x^{-1} \overline{x})$.

Proof. It is evident from Theorem 4.1 that, $\overline{T} \subseteq T^d$ (these two sets are equal when $\langle \xi, \exp_x^{-1} \overline{x}\rangle = 0$ in T^d). Let $x \in T^d$, then $\exists\ \xi \in \partial_c h(x)$, such that

$$\langle \xi, \exp_x^{-1} \overline{x}\rangle \geq 0.$$

Moreover, it follows from Theorem 3.3 that, there is a function $P(x, \overline{x}) > 0$, such that

$$\begin{aligned} h(\overline{x}) &= h(x) + P(x,\overline{x})\langle \xi, \exp_x^{-1} \overline{x}\rangle \\ &\geq h(x). \end{aligned}$$

Since $\overline{x} \in \overline{T}$, therefore $h(\overline{x}) \leq h(x)$. Hence, we get

$$h(\overline{x}) = h(x).$$

Therefore, it implies that

$$x \in \overline{T} \text{ and } T^d \subseteq \overline{T}.$$

Hence, one can conclude that $T^d = \overline{T}$.

From Lemma 2.14 and taking into account that h exhibits geodesic pseudoconvexity on Γ, it follows that $\partial_c h$ is pseudomonotone. Hence, $T^e = T^d$. Now, we will show that $T^d = T^f$. Let $x \in T^f$, then

$$\exists\ \xi_t \in \partial_c h(\gamma(t)) \text{ such that } \langle P^x_{\gamma(t)}\xi_t, \exp_x^{-1} \overline{x}\rangle \geq 0,\ \forall t \in [0,1].$$

From (11), it follows that, for all $t \in (0,1]$, we have

$$\left\langle P^x_{\gamma(t)}\xi_t, \frac{1}{t} \exp_x^{-1} \gamma(t)\right\rangle \text{ or } \left\langle -\xi_t, \exp^{-1}_{\gamma(t)} x\right\rangle \geq 0.$$

Since, $\xi_t \in \partial_c(h(\gamma(t))$ and h is locally Lipschitz therefore, Remark 2.4 yields, $-\xi_t \in \partial_c(-h)(\gamma(t))$. Moreover, as h is geodesic pseudoconcave, hence $-h$ is geodesic pseudoconvex and this ensures that $\partial_c(-h)(x)$ is pseudomonotone. Therefore, for all $\xi \in \partial_c h(x)$, we have

$$\langle -\xi, \exp_x^{-1} \gamma(t)\rangle \leq 0 \text{ or } \langle \xi, t \exp_x^{-1} \overline{x}\rangle \geq 0,$$

hence, we get

$$\langle \xi, \exp_x^{-1} \overline{x}\rangle \geq 0.$$

Therefore, it follows that $x \in T^d$ and $T^f \subseteq T^d$. From Theorem 4.1, we have $T^d = \overline{T} = T^c \subseteq T^f$. Hence, $T^d \subseteq T^f$ and

$$\overline{T} = T^d = T^e = T^f.$$

This completes the proof. $\square$

Remark 4.4. (i) Corollary 4.3 generalizes Corollary 3.6 of [22] from smooth to nonsmooth case on Hadamard manifold.
(ii) Corollary 4.3 extends Corollary 4.2 of [23] from the context of Banach space to Hadamard manifold setting.

Theorem 4.5. *Let $\overline{x} \in \overline{T}$ and define*

$$T^* := \left\{x \in \Gamma \mid \exists\, \xi \in \partial_c h(x),\ \exists\, \overline{\xi} \in \partial_c h(\overline{x}),\ \langle \overline{\xi}, \exp_x^{-1} \overline{x}\rangle = \langle \xi, \exp_{\overline{x}}^{-1} x\rangle\right\},$$
$$T^{**} := \left\{x \in \Gamma \mid \exists\, \xi \in \partial_c h(x),\ \exists\, \overline{\xi} \in \partial_c h(\overline{x}),\ \langle \overline{\xi}, \exp_{\overline{x}}^{-1} x\rangle \leq \langle \xi, \exp_x^{-1} \overline{x}\rangle\right\}.$$

Then, $\overline{T} = T^* = T^{**}$.

Proof. Let $x \in \overline{T}$, then from Theorem 4.1, it follows that

$$\exists\, \overline{\xi} \in \partial_c h(\overline{x}) \text{ such that } \langle \overline{\xi}, \exp_{\overline{x}}^{-1} x\rangle = 0,$$

and

$$\exists\, \xi \in \partial_c h(x) \text{ such that } \langle \xi, \exp_x^{-1} \overline{x}\rangle = 0.$$

Thus

$$\langle \overline{\xi}, \exp_{\overline{x}}^{-1} x\rangle = \langle \xi, \exp_x^{-1} \overline{x}\rangle,$$

gives, $x \in T^*$ and $\overline{T} \subseteq T^*$.

From the definition, we have $T^* \subseteq T^{**}$. Now, let $x \in T^{**}$, then

$$\langle \xi, \exp_x^{-1} \overline{x}\rangle \geq \langle \overline{\xi}, \exp_{\overline{x}}^{-1} x\rangle. \tag{12}$$

Let $x \notin \overline{T}$, thus $h(x) > h(\overline{x})$. Then the pseudoconcavity of h on Γ, yields

$$\langle \overline{\xi}, \exp_{\overline{x}}^{-1} x\rangle > 0.$$

Hence, from (12) we get

$$\langle \xi, \exp_x^{-1} \overline{x}\rangle > 0.$$

From Theorem 3.3, we have

$$h(\overline{x}) = h(x) + P(x, \overline{x})\langle \xi, \exp_x^{-1} \overline{x}\rangle \implies h(\overline{x}) > h(x),$$

which is a contradiction, and this implies that $x \in \overline{T}$ and $T^{**} \subseteq \overline{T}$. Therefore, we have

$$\overline{T} \subseteq T^* \subseteq T^{**} \subseteq \overline{T},$$

and

$$\overline{T} = T^* = T^{**}.$$

This completes the proof. □

Remark 4.6. (i) Theorem 4.5 extends Theorem 3.7 of [22] from a smooth optimization problem to a nonsmooth optimization problem in the framework of Hadamard manifolds.

(ii) The result of Theorem 4.5 extends Theorem 4.3 of [23] from the context of Banach space to the framework of Hadamard manifolds.

The following example illustrates the significance of the above results.

Example 4.7. Let us denote the set of all symmetric matrices of order 2×2 as $\mathcal{S}^2$. We define the set $\mathcal{P}_2^+ \subset \mathcal{S}^2$ as the collection of all symmetric positive definite matrices of order 2×2. Let $\mathcal{A} \in \mathcal{P}_+^2$. The set $\mathcal{P}_+^2$ is a Hadamard manifold endowed with the following metric known as affine invariant metric (see, for instance, [3,46]):

$$\langle X_1, X_2\rangle_{\mathcal{A}} := \operatorname{trace}\big(X_2\mathcal{A}^{-1}X_1\mathcal{A}^{-1}\big),$$

where $X_1, X_2 \in T_{\mathcal{A}}\mathcal{P}_+^2 = \mathcal{S}^2$. Now, consider any arbitrary pair of elements $X, Y \in \mathcal{P}_+^2$ and $Z \in T_X\mathcal{P}_+^2$.

The corresponding exponential function $\exp_X(Z) : T_X\mathcal{P}_+^2 \to \mathcal{P}_+^2$ is given by:

$$\exp_X(Z) = X^{\frac{1}{2}}\operatorname{Exp}\Big(X^{-\frac{1}{2}}ZX^{-\frac{1}{2}}\Big)X^{\frac{1}{2}},$$

where the symbol Exp represents the matrix exponential (see, [3]). Moreover, the inverse exponential function $\exp_X^{-1} : \mathcal{P}_+^2 \to T_X\mathcal{P}_+^2$ is given by:

$$\exp_X^{-1}(Y) = X^{\frac{1}{2}}\operatorname{Log}\Big(X^{-\frac{1}{2}}YX^{-\frac{1}{2}}\Big)X^{\frac{1}{2}},$$

where the symbol Log represents the usual logarithm on $\mathcal{P}_+^2$ (see, [3]). For any real-valued function $G : \mathcal{P}_+^2 \to \mathbb{R}$, the Riemannian gradient is given by:

$$\operatorname{grad}(G(X)) = XG'(X)X,$$

for any $X \in \mathcal{P}_2^+$, where $G'(X)$ denotes the Euclidean gradient of the function F at X.

Let $\mathcal{D}$ ($\subset \mathcal{P}_2^+$) be a nonempty open set, defined as follows:

$$\mathcal{D} := \left\{X = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} : x_1, x_2 \in (0,2)\right\}.$$

Consider the function $h : \mathcal{D} \to \mathbb{R}$ defined as follows:

$$h(X) := -\log(x_1x_2),$$

Now, we define the set Γ ($\subset \mathcal{D}$) as follows:

$$\Gamma := \left\{X = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} : x_1, x_2 \in (0,1]\right\}.$$

Then, the Clarke subdifferential of h is given by:

$$\begin{aligned}\partial_c h(X) &= \left\{\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}\begin{bmatrix} -\frac{1}{x_1} & 0 \\ 0 & -\frac{1}{x_2} \end{bmatrix}\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}\right\} \\ &= \left\{\begin{bmatrix} -x_1 & 0 \\ 0 & -x_2 \end{bmatrix}\right\}.\end{aligned}$$

Now, the inverse exponential map is determined as follows:

$$\begin{aligned}\exp_X^{-1}(Y) &= X^{\frac{1}{2}}\log\Big(X^{-\frac{1}{2}}YX^{-\frac{1}{2}}\Big)X^{\frac{1}{2}} \\ &= \begin{bmatrix} \sqrt{x_1} & 0 \\ 0 & \sqrt{x_2} \end{bmatrix}\begin{bmatrix} \log\frac{y_1}{x_1} & 0 \\ 0 & \log\frac{y_2}{x_2} \end{bmatrix}\begin{bmatrix} \sqrt{x_1} & 0 \\ 0 & \sqrt{x_2} \end{bmatrix} \\ &= \begin{bmatrix} x_1\log\frac{y_1}{x_1} & 0 \\ 0 & x_2\log\frac{y_2}{x_2} \end{bmatrix}.\end{aligned}$$

If $Y = \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}$, then the inner product is given by:

$$\begin{aligned}\left\langle \begin{bmatrix} -x_1 & 0 \\ 0 & -x_2 \end{bmatrix},\ \exp_X^{-1}(Y) \right\rangle_X &= \left\langle \begin{bmatrix} -x_1 & 0 \\ 0 & -x_2 \end{bmatrix},\ \begin{bmatrix} x_1 \log \frac{y_1}{x_1} & 0 \\ 0 & x_2 \log \frac{y_2}{x_2} \end{bmatrix} \right\rangle_X \\ &= \operatorname{trace}\left(\begin{bmatrix} -\log \frac{y_1}{x_1} & 0 \\ 0 & -\log \frac{y_2}{x_2} \end{bmatrix} \right) \\ &= -\log \frac{y_1}{x_1} - \log \frac{y_2}{x_2} \cdot \end{aligned}$$

First, we show that h is a geodesic pseudolinear function on Γ. If

$$0 \leq -\log \frac{y_1}{x_1} - \log \frac{y_2}{x_2},$$

we have

$$-\log\left(\frac{y_1 y_2}{x_1 x_2} \right) \geq 0 \implies y_1 y_2 \leq x_1 x_2,$$

and

$$h(Y) \geq h(X).$$

Hence h is a geodesic pseudoconvex function on Γ. In a similar manner, we can show that h is a geodesic pseudoconcave function on the set Γ.

Let us consider the following problem

$$\begin{aligned} (\text{P1}) \qquad & \min\ h(z) \\ & \text{subject to } x \in \Gamma. \end{aligned}$$

The solution set of the problem (P1) is

$$\overline{T} = \{(1,1)\}.$$

Let $\overline{z} \in \overline{T}$, then the set T^a is given by

$$\begin{aligned} T^a &= \{ z \in \Gamma \mid \exists\ \xi \in \partial_c h(z)\ :\ \langle \xi, \exp_z^{-1} \overline{z} \rangle = 0 \} \\ &= \{ z \in \Gamma \mid h(z) = h(\overline{z}) \} \\ &= \{(1,1)\} \\ &= \overline{T}. \end{aligned}$$

In the above the second equality follows from Theorem 3.1. Similarly, we can show that $T^b = T^c = \overline{T}$. The set T^d for the problem (P1) is given by

$$\begin{aligned} T^d &= \{ z \in \Gamma \mid \exists\ \xi \in \partial_c h(z)\ :\ \langle \xi, \exp_z^{-1} \overline{z} \rangle \geq 0 \} \\ &= \left\{ z \in \Gamma \mid \exists\ \xi \in \partial_c h(z)\ :\ -z_1^2 \log \frac{\overline{z}_1}{z_1} - z_2^2 \log \frac{\overline{z}_2}{z_2} \geq 0 \right\} \\ &= \{ z \in \Gamma \mid \overline{z}_1 \overline{z}_2 \leq z_1 z_2 \} \\ &= \overline{T}. \end{aligned}$$

In a similar manner, we can show that $T^e = T^f = T^* = T^{**} = \overline{T}$.

5. Application

In this section, we consider the Karcher mean problem and elucidate its practical implications. Moreover, we establish that under certain assumptions, the objective function associated with this problem becomes a geodesic pseudolinear. Furthermore, to demonstrate the practical significance of the derived results in this paper, we verify their applicability to this specific form of the Karcher mean problem.

Let $\mathcal{P}_+^n$ denote the manifold of $n \times n$ symmetric positive definite matrices. The tangent space of $\mathcal{P}_+^n$ at the point $X \in \mathcal{P}_+^n$ is defined as the set of all $n \times n$ symmetric matrices $\mathcal{S}^n$. The manifold $\mathcal{P}_+^n$ becomes a Hadamard manifold when endowed with the affine-invariant metric, which is defined as follows:

$$\langle A, B\rangle_X = \operatorname{trace}\big(AX^{-1}BX^{-1}\big),\ A, B \in \mathcal{S}^n,\ X \in \mathcal{P}_+^n. \tag{13}$$

If X is an identity matrix of order $n \times n$, then from (13), it follows that,

$$\langle A, B\rangle = \operatorname{trace}(AB). \tag{14}$$

The Karcher mean (in other terms, the Riemannian center of mass) of a set of symmetric positive definite matrices, namely, $\{A_1, \ldots, A_K\}$, is defined as the point that minimizes the sum of squared distances. Equivalently,

$$\mu = \arg\min_{X \in \mathcal{P}_+^n} \Big(\tfrac{1}{2K}\textstyle\sum_{i=1}^K \delta^2(X, A_i)\Big),$$

where $\delta(X, A_i) = \big\|\operatorname{Log}\big(X^{-1/2}A_iX^{-1/2}\big)\big\|_2$ is the geodesic distance associated with the affine-invariant metric and the symbol Log represents the usual logarithm on $\mathcal{P}_+^2$ (see, [3]). Here, $\|A\|_2 = (\operatorname{trace}(A^TA))^{\frac{1}{2}}$, for any $A \in \mathcal{S}^n$ and A^T represents the transpose of the matrix A.

The Karcher mean serves as a powerful tool for averaging and studying positive definite matrices, extending the concept of matrix mean to a larger number of variables. For more details about the Karcher mean problem, we refer to [7, 20, 33] and the references cited therein.

The primary motivations for studying the Karcher mean are threefold: Firstly, in statistical analysis, it relates to models on the manifold of restricted positive semi-definite matrices, which commonly appear in applications such as statistics (for example, covariance matrices) and machine learning (for example, kernel matrices). Understanding the Karcher mean enables a meaningful geometric interpretation of data points within the manifold of positive semi-definite matrices, crucial for various algorithms and statistical methods (see, [7]). Secondly, the Karcher mean plays a pivotal role in longitudinal redundancy check-distributed principal component analysis (abbreviated as, LRC-dPCA). By analyzing the Karcher mean, researchers have shown that LRC-dPCA achieves the same performance as the full-sample principal component analysis algorithm (see, [7, 33]). Essentially, understanding the Karcher mean helps design efficient algorithms for analyzing large-scale data with covariance structures. Thirdly, the Karcher mean has applications in several fields, including statistics, computer vision, medical imaging, and shape analysis, where it is used to compute representative or central points for data sets on curved spaces, see, for instance, [39, 50] and the references cited therein.

Let $\mathcal{D}$ $(\subset \mathcal{P}_2^+)$ be a nonempty open set, defined as follows:

$$\mathcal{D} := \left\{X = \begin{bmatrix} x & 0 \\ 0 & 1\end{bmatrix} : x > \frac{1}{2}\right\}.$$

Now, we consider two subsets of $\mathcal{P}_2^+$, namely, $\mathcal{B}$ and Γ, defined as follows:

$$\mathcal{B} := \left\{\begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix} : 0 < a \leq 1,\ 0 < b \leq 1\right\},\ \Gamma := \left\{\begin{bmatrix} x & 0 \\ 0 & 1\end{bmatrix} : x > 1\right\}.$$

Let us consider matrices $B_1, B_2, \ldots, B_K$, such that $B_1, B_2, \ldots, B_K \in \mathcal{B}$. Then, the Karcher mean of $\{B_1, B_2, \ldots, B_K\}$ on the set Γ is defined as the minimizer of the following optimization problem:

$$\begin{aligned}\text{(KMP):}\ &\min\ H(X)\\ &\text{subject to } X \in \Gamma,\end{aligned}$$

where $H : \mathcal{D} \to \mathbb{R}$ is defined as follows:

$$H(X) := \frac{1}{2K}\sum_{i=1}^{K}\delta^2(X, B_i). \tag{15}$$

In our forthcoming discussions, we verify Theorems 3.1 and 4.1 for the considered optimization problem (KMP).

It is apparent that the function H exhibits local Lipschitz continuity on the set Γ. In order to verify Theorem 4.1, it is necessary to show that the set Γ is a geodesic convex subset of $\mathcal{P}_2^+$ and H is a geodesic pseudolinear function on Γ.

To show, Γ is a geodesic convex set, we choose two arbitrary points $X, Y \in \Gamma$. Then, for some $x, y > 1$, one has

$$X = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix},\ Y = \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}. \tag{16}$$

From [3], it follows that the geodesic $\gamma(t)$ joining X and Y is given by:

$$\gamma(t) := X(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^t X,\ \forall t \in [0,1].$$

Employing (16), one has

$$\gamma(t) = \begin{bmatrix} x^{2-t}y^t & 0 \\ 0 & 1 \end{bmatrix}.$$

Evidently, $x^{2-t}y^t > 1$ and hence it follows that $\gamma(t) \in \Gamma$, for all $t \in [0,1]$. This shows that Γ is a geodesic convex set in $\mathcal{P}_2^+$. For any $1 \le i \le K$, let us set $B_i = \begin{bmatrix} b_i^1 & 0 \\ 0 & b_i^4 \end{bmatrix} \in \mathcal{B}$.

To show H is geodesic pseudolinear on Γ, we consider any two elements $X, Y \in \Gamma$. Then, from Definition 2.8, H is geodesic pseudoconvex function on Γ, provided the following implication holds:

$$\langle \text{grad } H(X),\ \exp_X^{-1}(Y)\rangle_X \ge 0 \implies H(Y) \ge H(X),\ \forall X, Y \in \Gamma.$$

Moreover, it follows from [50] that grad $H(X) = -\frac{1}{K}\sum\limits_{i=1}^{K}\exp_X^{-1}(B_i)$. From [3], the inverse of exponential map on $\mathcal{P}_2^+$ is given by:

$$\exp_X^{-1}(Y) = X^{\frac{1}{2}}\text{Log}\Big(X^{-\frac{1}{2}}YX^{-\frac{1}{2}}\Big)X^{\frac{1}{2}}.$$

Furthermore, by (14), it can be inferred that

$$\langle \text{grad } H(X), \exp_X^{-1}(Y)\rangle_X = \frac{1}{K}\log\Big(\frac{y}{x}\Big)\log\left(\frac{x^k}{\prod_{i=1}^{K} b_i^1}\right).$$

Since $\langle \text{grad } H(X),\ \exp_X^{-1}(Y)\rangle_X \ge 0$, therefore $\log\big(\frac{y}{x}\big) \ge 0$. Hence, $y \ge x$. Now, for any $X, Y \in \Gamma$, it can be shown that

$$\begin{aligned} H(Y) - H(X) &= \frac{1}{2K}\sum_{i=1}^{K}\left\{\left(\log\left(\frac{b_i^1}{y}\right)\right)^2 - \left(\log\left(\frac{b_i^1}{x}\right)\right)^2\right\} \\ &= \frac{1}{2K}\log\left(\frac{x}{y}\right)\sum_{i=1}^{K}\log\left(\frac{(b_i^1)^2}{xy}\right). \end{aligned}$$

From the fact that $\log\Big(\frac{x}{y}\Big) \le 0$ and $\log\Big(\frac{(b_i^1)^2}{xy}\Big) < 0$, $\forall i \in \{1, 2, \dots, K\}$, it follows that $H(Y) \ge H(X)$, $\forall X, Y \in \Gamma$. Hence, H is a geodesic pseudoconvex on Γ. In a similar manner, one can verify that H is a geodesic pseudoconcave on Γ. This exhibits the fact that H is a geodesic pseudolinear function on Γ.

To verify Theorem 3.1, let us consider $X, Y \in \Gamma$ such that $H(X) = H(Y)$. Then, we get $\log\left(\frac{x}{y}\right) = 0$. This implies that $\langle \text{grad } H(X),\ \exp_X^{-1}(Y)\rangle_X = 0$. Conversely, for

$$\langle \text{grad } H(X),\ \exp_X^{-1}(Y)\rangle_X = 0,$$

one has $\log\left(\frac{x}{y}\right) = 0$ as $\log\left(\frac{x^k}{\prod_{i=1}^{K} b_i^1}\right) > 0$. Therefore, $H(X) = H(Y)$ for all $X, Y \in \Gamma$.

Let $\overline{Y} = \begin{bmatrix} \overline{y} & 0 \\ 0 & 1 \end{bmatrix} \in \overline{T} := \arg \min_{X \in \Gamma} H(X)$ and $X \in T^a$. Then, from Theorem 4.1, it follows that

$$\langle \text{grad } H(X),\ \exp_X^{-1}(\overline{Y})\rangle_X = 0. \tag{17}$$

Invoking (17), one has $x = \overline{y}$. This implies that $X = \overline{Y}$. Hence, $T^a \subseteq \{\overline{Y}\}$ and since $\{\overline{Y}\} \subseteq T^a$. Therefore $\overline{T} = T^a$. Similarly, one can verify that $T^c = \overline{T} = T^b$. Therefore, $\overline{T} = T^a = T^b = T^c$.

In view of the fact that the Karcher mean problem has several applications in machine learning and engineering (see [7, 33, 39, 50]), the results derived in this paper may be applicable to a wide range of practical problems in science and engineering.

Remark 5.1. It is worth noting that the function H, defined in (15), may fail to be geodesic pseudolinear on Γ if the matrices $B_i \notin \mathcal{B}$ $(i = 1, 2, \ldots, K)$. To demonstrate this fact, we consider the sets S_1 and S_2, such that $S_1 \subset \mathcal{P}_2^+ \setminus \mathcal{B}$ and $S_2 \subset \mathcal{P}_2^+ \setminus \Gamma$. Let there exists some $i \in \{1, 2, \ldots, K\}$, such that $M_i \in S_1$. Then, the function $H : \mathcal{P}_2^+ \to \mathbb{R}$, defined by:

$$H(X) := \frac{1}{2K} \sum_{i=1}^{K} \delta^2(X, M_i),$$

may not be geodesic pseudolinear on S_2.

For instance, let us consider $K = 2$, and matrices $M_1 \in S_1,\ M_2 \in \mathcal{B}$, and $\overline{X} \in S_2$, defined as follows:

$$M_1 := \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad M_2 := \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \overline{X} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To show that H is not geodesic pseudolinear on S_2, it is sufficient to show that H is not geodesic pseudolinear at $\overline{X}$. Let $V \in T_{\overline{X}}\mathcal{P}_2^+$ and $Y \in S_2$, be defined as follows:

$$V := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y := \exp_{\overline{X}} V = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}.$$

Now, we have:

$$\begin{aligned} \text{grad } H(\overline{X}) &= -\frac{1}{2} \sum_{i=1}^{2} \exp_{\overline{X}}^{-1} M_i \\ &= -\frac{1}{2}(\text{Log}(M_1) + \text{Log}(M_2)) \\ &= -\frac{1}{2} \begin{bmatrix} 0 & \log\sqrt{3} \\ \log\sqrt{3} & 0 \end{bmatrix}. \end{aligned}$$

Then, it can be verified that:

$$\left\langle \text{grad } H(\overline{X}),\ \exp_{\overline{X}}^{-1}(Y) \right\rangle_{\overline{X}} = 0.$$

If H is geodesic pseudolinear at $\overline{X}$, then we have:

$$H(\overline{X}) = H(Y).$$

However, we have:

$$H(\overline{X}) \approx 1.3578, \quad H(Y) \approx 1.8085,$$

which yields that $H(Y) > H(\overline{X})$. Hence, H is not geodesic pseudolinear at $\overline{X}$.

6. Conclusions and future directions

In this paper, we have established some characterizations of locally Lipschitz geodesic pseudolinear functions on the Hadamard manifold. We have shown the geodesic convexity of the level set and established that any two vectors of the Clarke subdifferential of a geodesic pseudolinear function, differ by a positive scalar product. Further, we have obtained some properties of solution sets of geodesic pseudolinear programming problems on the Hadamard manifold. The results of this paper have been demonstrated with suitable examples. To validate the applicability of the results established in this paper, we have solved a particular type of Karcher mean problem by utilizing certain characterizations of geodesic pseudolinear functions established in this paper.

The results derived in this paper open up several avenues for future exploration. For instance, in view of the work of Farrokhiniya and Barani [10], the results of this paper could be explored by employing the powerful tools of Mordukhovich limiting subdifferentials in the framework of Riemannian manifolds. Moreover, it would be of significant interest to study the viability of the approach to a non-diagonal case in more details, in particular, developing the necessary and sufficient conditions for the function H to be geodesic pseudolinear on a more general subset of symmetric positive definite matrices, more specifically, not necessarily diagonal than those considered in this paper, This will allow the study of the Karcher mean problem for a broader subset of the set of symmetric positive definite matrices.

Acknowledgments

The authors sincerely thank the anonymous reviewers for their careful reading and valuable suggestions, which have significantly enhanced the quality of the manuscript.

Funding

The fifth author extends gratitude to the Ministry of Education, Government of India, for their financial support through the Prime Minister Research Fellowship (PMRF), granted under PMRF ID-2703573.

Conflicts of interest

The authors affirm that there are no actual or potential conflicts of interest associated with this article.

Data availability statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Author contribution statement

Each author contributed equally to the article.

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