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ON THE OUTER-INDEPENDENT TOTAL (ROMAN) DOMINATION NUMBER OF SOME GRAPH OPERATORS

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Abstract. The goal of this article is to obtain closed formulas for the outer-independent total domination number and the outer-independent total Roman domination number of the following well-known graph operators defined from a connected graph G: the central graph $\mathtt{C}(G)$, the middle graph $\mathtt{M}(G)$, the graph $\mathtt{R}(G)$ and the total graph $\mathtt{T}(G)$. We show that these formulas can be obtained for the last three graph operators and for the outer-independent total domination number of $\mathtt{C}(G)$. The picture is quite different when it concerns the outer-independent total Roman domination number of $\mathtt{C}(G)$. In this case, we obtain tight bounds and, imposing some restrictions on G, we obtain closed formulas.

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1. Introduction

The concept of outer-independent total domination in graphs was first introduced in 2012 by Soner, Dhananjaya Murthy and Deepak [29], and subsequently, has been studied in several research works, including [5,8,10,21–23]. An *outer-independent total dominating set* (OITDS) of a nontrivial connected graph G is a set $D \subseteq V(G)$ which satisfies that every vertex of G is adjacent to at least one vertex in D and that $V(G) \setminus D$ is an independent set of G. The set of OITDSs of G will be denoted as $\mathcal{D}_t^{oi}(G)$. The *outer-independent total domination number* of G is defined as $\gamma_t^{oi}(G) = \min\{|D| : D \in \mathcal{D}_t^{oi}(G)\}$. A $\gamma_t^{oi}(G)$*-set* is an OITDS of G of cardinality $\gamma_t^{oi}(G)$.

In the last decade, new parameters related to domination and independence in graphs have been introduced. Several of them come from a Roman domination structure whose "complement" has independent properties. In this article, the outer-independent total Roman domination will be of interest. An *outer-independent total Roman dominating function* (OITRDF) on a nontrivial connected graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the following three conditions.

– Every vertex $v \in V(G)$ for which $f(v) = 0$ is adjacent to at least one vertex $u \in V(G)$ such that $f(u) = 2$.
– Every vertex $v \in V(G)$ for which $f(v) \geq 1$ is adjacent to at least one vertex $u \in V(G)$ such that $f(u) \geq 1$.
– The set $\{v \in V(G) : f(v) = 0\}$ is an independent set of G.

Keywords. Outer-independent total domination, outer-independent total Roman domination, graph operators.

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Let $V_i = \{v \in V(G) : f(v) = i\}$, where $i \in \{0,1,2\}$. We will identify f with the subsets V_0, V_1, V_2, and so we will use the unified notation $f(V_0, V_1, V_2)$ for the function f and these associated subsets. The *weight* of f equals $\omega(f) = \sum_{v\in V(G)} f(v) = 2|V_2| + |V_1|$. The set of OITRDFs on G will be denoted as $\mathcal{F}^{oi}_{tR}(G)$. The *outer-independent total Roman domination number* of G is defined as $\gamma^{oi}_{tR}(G) = \min\{\omega(f) : f \in \mathcal{F}^{oi}_{tR}(G)\}$. A $\gamma^{oi}_{tR}(G)$-function is an OITRDF on G such that $\omega(f) = \gamma^{oi}_{tR}(G)$. This parameter was introduced in 2019 by Cabrera-Martínez, Kuziak and Yero [6] and recent results can be found in [9, 22, 26, 27].

It is well known that the general problems of computing the values $\gamma^{oi}_t(G)$ and $\gamma^{oi}_{tR}(G)$ are NP-hard. This suggests finding closed formulas or giving tight bounds for these domination invariants in specific families of graphs. Such studies are particularly appealing for graph products and graph operators, since their structured nature often makes it possible to obtain closed-form results. The goal of this work is to determine closed formulas for these domination invariants for several well-known graph operators constructed from a connected graph G: the central graph $\mathtt{C}(G)$, the middle graph $\mathtt{M}(G)$, the graph $\mathtt{R}(G)$ and the total graph $\mathtt{T}(G)$.

The article is organized as follows. In Section 1.1 we give some basic definitions and terminology that will be used throughout the article. Section 2 is devoted to the study of these two parameters in central graphs. In particular, we obtain closed formulas for the outer-independent total domination number. The picture is quite different when it concerns the outer-independent total Roman domination number. In this case, we obtain tight bounds and, imposing some restrictions on G, we obtain closed formulas. Finally, in the last three sections we obtain closed formulas for the outer-independent total (Roman) domination number of $\mathtt{M}(G)$, $\mathtt{R}(G)$ and $\mathtt{T}(G)$, respectively.

1.1. Some definitions and terminology

For notation and graph theory terminology, we in general follow [15]. Let G be a nontrivial connected graph of order $n = |V(G)|$ and size $m = |E(G)|$. Let $V(G) = \{v_1, \ldots, v_n\}$ be the vertex set of G and let $V_E(G) = \{v^{i,j} : v_iv_j \in E(G)\}$ (observe that $v^{i,j} = v^{j,i}$). For any subscript $i \in [n] = \{1, \ldots, n\}$, $N_G(v_i)$ and $N_G[v_i]$ represent the *open neighborhood* and the *closed neighborhood* of vertex v_i in G, respectively. The minimum and maximum degrees among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A *leaf* is a vertex of degree one. We denote the set of leaves of a graph G by $\mathcal{L}(G)$, and the set of *support vertices* (vertices adjacent to leaves) by $\mathcal{S}(G)$. We will use the notation K_n and $K_{1,n-1}$ for complete graphs and star graphs of order n, respectively.

Given a connected graph G, a set $I \subseteq V(G)$ is an *independent dominating set* of G if the subgraph induced by I is an edgeless graph and if every vertex in $V(G)\setminus I$ is adjacent to at least one vertex in I. Let $\mathcal{ID}(G)$ be the set of independent dominating sets of G. The *independence number* of G is defined as $\alpha(G) = \max\{|I| : I \in \mathcal{ID}(G)\}$. An $\alpha(G)$-set is a set $I \in \mathcal{ID}(G)$ such that $|I| = \alpha(G)$.

In 2012, Soner *et al.* [29] established the following well-known relationship between the outer-independent total domination number and the independence number of a graph G with no isolated vertex.

Theorem 1.1 ([29]). *If G is a graph with no isolated vertex of order n, then $\gamma^{oi}_t(G) \geq n - \alpha(G)$.*

The previous relationship will be an important tool for deducing the results obtained in the following sections. Now, let us define the following four well-known graph parameters, which will be necessary for the study we are conducting.

- A *vertex cover* of a graph G without isolated vertices is a set $D \subseteq V(G)$ such that every edge of G is incident to at least one vertex in D. The *vertex covering number* of G, denoted by $\beta(G)$, is the minimum cardinality among all vertex covers of G.
- An *edge cover* of a graph G is a set $F \subseteq E(G)$ such that every vertex of G is incident to at least one edge in F. The *edge covering number* of G, denoted by $\beta'(G)$, is the minimum cardinality among all edge covers of G.
- A *dominating set* of a graph G is a set $D \subseteq V(G)$ such that every vertex not in D is adjacent to at least one vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of G.

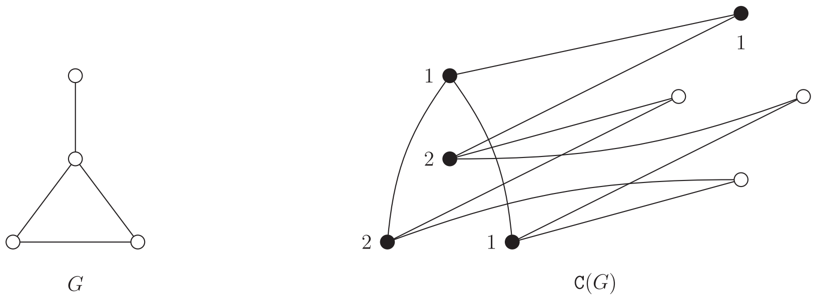

FIGURE 1. A graph G, and the corresponding central graph $\mathtt{C}(G)$.

– An *edge dominating set* of a graph G is a set $F \subseteq E(G)$ such that every edge not in F is adjacent to at least one edge in F. The *edge domination number* of G, denoted by $\gamma'(G)$, is the minimum cardinality among all edge dominating sets of G.

For the rest of the article, definitions will be introduced whenever a concept is needed.

2. Central graphs

The *central graph* of a connected graph G, denoted by $\mathtt{C}(G)$, is the graph with vertex set $V(\mathtt{C}(G)) = V(G) \cup V_E(G)$ and edge set $E(\mathtt{C}(G)) = \{v_iv^{i,j}, v_jv^{i,j} : v^{i,j} \in V_E(G)\} \cup \{v_iv_j : v_iv_j \notin E(G)\}$. This graph operator was first introduced by Vernold in 2007 [32]. Some domination parameters have been studied on central graphs, see, for example, [13,16,25]. In Figure 1 we show a graph G and its corresponding central graph $\mathtt{C}(G)$. The set of black vertices describes a $\gamma_t^{oi}(\mathtt{C}(G))$-set, while the labels correspond to the positive weights of a $\gamma_{tR}^{oi}(\mathtt{C}(G))$-function.

The following theorem provides a closed formula for the outer-independent total domination number of the central graph $\mathtt{C}(G)$, where G is a connected graph of order $n \geq 4$. Before, we need to define two families of graph. For any $i \in [n-1]$, let $V_i(G) = \{v \in V(G) : |N_G(v)| = i\}$. For any integer $k \geq 4$, let $\mathcal{F}_k$ and $\mathcal{G}_k$ be the families of connected graphs G of order k defined as follows.

– $G \in \mathcal{F}_k$ whenever $V_{k-1}(G) = \emptyset$.
– $G \in \mathcal{G}_k$ whenever $|V_{k-1}(G)| = 1$, $V_1(G) \neq \emptyset$ and $V_{k-2}(G) = \emptyset$.

Theorem 2.1. *For any connected graph G of order $n \geq 4$,*

$$\gamma_t^{oi}(\mathtt{C}(G)) = \begin{cases} n & \text{if } G \in \mathcal{F}_n \cup \mathcal{G}_n, \\ n + \left\lceil \frac{|V_{n-1}(G)|}{2} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. We first assume that $G \in \mathcal{F}_n \cup \mathcal{G}_n$. We proceed to prove that $\gamma_t^{oi}(\mathtt{C}(G)) = n$. By Theorem 1.1 and the fact that $\alpha(\mathtt{C}(G)) = |E(G)|$ (see [16], Lem. 1.11) we have that $\gamma_t^{oi}(\mathtt{C}(G)) \geq |V(\mathtt{C}(G))| - \alpha(\mathtt{C}(G)) = n$. We only need to prove that $\gamma_t^{oi}(\mathtt{C}(G)) \leq n$. If $G \in \mathcal{F}_n$, then it is easy to check that $V(G) \in \mathcal{D}_t^{oi}(\mathtt{C}(G))$. Hence, $\gamma_t^{oi}(\mathtt{C}(G)) \leq |V(G)| = n$, as desired. Moreover, let us suppose that $G \in \mathcal{G}_n$. Let $V_{n-1}(G) = \{v_j\}$ and $v_i \in V_1(G)$. Since $V_{n-2}(G) = \emptyset$, it is easy to check that $(V(G) \setminus \{v_i\}) \cup \{v^{i,j}\} \in \mathcal{D}_t^{oi}(\mathtt{C}(G))$. Hence, $\gamma_t^{oi}(\mathtt{C}(G)) \leq n$, as desired. From now on, assume that $G \notin \mathcal{F}_n \cup \mathcal{G}_n$. This implies that $|V_{n-1}(G)| = r \geq 1$. Without loss of generality, let us consider that $V_{n-1}(G) = \{v_1, \ldots, v_r\}$. Let $W \subseteq V_E(G)$ be a set of minimum cardinality such that $N_{\mathtt{C}(G)}(v_i) \cap W \neq \emptyset$ for every $i \in [r]$. Due to the fact that the subgraph induced by $V_{n-1}(G)$ is isomorphic to the complete graph K_r and the minimality of $|W|$, it follows that $|W| \leq \lceil |V_{n-1}(G)|/2 \rceil$. Now, observe that $V(G) \cup W \in \mathcal{D}_t^{oi}(\mathtt{C}(G))$. Hence,

$$\gamma_t^{oi}(\mathtt{C}(G)) \leq |V(G) \cup W| \leq n + \left\lceil \frac{|V_{n-1}(G)|}{2} \right\rceil.$$

To complete the proof, we only need to prove that $\gamma_t^{oi}(\mathtt{C}(G)) \geq n + \lceil |V_{n-1}(G)|/2 \rceil$. Let D be a $\gamma_t^{oi}(\mathtt{C}(G))$-set and we consider the following three complementary cases.

Case 1: $V(G) \subset D$**.** Observe that $\emptyset \neq D \cap N_{\mathtt{C}(G)}(v_i) \subseteq D \cap V_E(G)$ for every subscript $i \in [r]$. Since every vertex in $D \cap V_E(G)$ has exactly two neighbors in $V(G)$, we deduce that

$$2|D \cap V_E(G)| \geq \sum_{i \in \{1,\ldots,r\}} |D \cap N_{\mathtt{C}(G)}(v_i)| \geq r = |V_{n-1}(G)|.$$

Therefore, $|D \cap V_E(G)| \geq \lceil |V_{n-1}(G)|/2 \rceil$, and as a consequence, we obtain that $\gamma_t^{oi}(\mathtt{C}(G)) = |D| = |D \cap V(G)| + |D \cap V_E(G)| \geq n + \left\lceil \frac{|V_{n-1}(G)|}{2} \right\rceil$, as desired.

Case 2: There exists a subscript $i \in [r]$ such that $v_i \notin D$**.** In this case we have that $v_j, v^{i,j} \in D$ for every $j \in [n] \setminus \{i\}$. This implies that $|D| \geq \sum_{j \in [n] \setminus \{i\}} |\{v_j, v^{i,j}\}| = 2(n-1)$. Since $n \geq 4$ and $|V_{n-1}(G)| \leq n$, we deduce that $n - 2 \geq \lceil |V_{n-1}(G)|/2 \rceil$. Therefore, we obtain that $\gamma_t^{oi}(\mathtt{C}(G)) = |D| \geq 2n - 2 \geq n + \left\lceil \frac{|V_{n-1}(G)|}{2} \right\rceil$, as desired.

Case 3: $V_{n-1}(G) \subset D$ and there exists a subscript $i \in \{r+1, \ldots, n\}$ such that $v_i \notin D$**.** In this case, we have that $V(G) \setminus N_G[v_i] \subset D$ and that $v_j, v^{i,j} \in D$ for every $j \in [n]$ such that $v_j \in N_G(v_i)$. Since $V_{n-1}(G) \subseteq N_G(v_i)$, we obtain that

$$|D| \geq |V(G) \setminus N_G[v_i]| + 2|N_G(v_i)| = n - 1 + |N_G(v_i)| \geq n - 1 + |V_{n-1}(G)|. \tag{1}$$

If $|V_{n-1}(G)| \geq 2$, then it is easy to check that $|V_{n-1}(G)| \geq \lceil |V_{n-1}(G)|/2 \rceil + 1$, and combining it with the inequality given in equation (1) we obtain that $\gamma_t^{oi}(\mathtt{C}(G)) = |D| \geq n - 1 + |V_{n-1}(G)| \geq n + \left\lceil \frac{|V_{n-1}(G)|}{2} \right\rceil$, as desired.

Now, if $|V_{n-1}(G)| = 1$ and $|N_G(v_i)| \geq 2$, then by equation (1) we have that $\gamma_t^{oi}(\mathtt{C}(G)) = |D| \geq n - 1 + |N_G(v_i)| \geq n + 1 = n + \left\lceil \frac{|V_{n-1}(G)|}{2} \right\rceil$, as desired.

Finally, assume that $|V_{n-1}(G)| = |N_G(v_i)| = 1$. Since $G \notin \mathcal{G}_n$, there exists a subscript $j \in \{2, \ldots, n\} \setminus \{i\}$ such that $v_j \in V_{n-2}(G)$. Observe that $N_{\mathtt{C}(G)}(v_j) \setminus \{v_i\} \subseteq V_E(G)$. As $v_i \notin D$, it follows that $(V(G) \setminus \{v_i\}) \cup \{v^{1,i}\} \subset D$ and that $N_{\mathtt{C}(G)}(v_j) \cap D \cap V_E(G) \neq \emptyset$. From this previous fact, we obtain that $\gamma_t^{oi}(\mathtt{C}(G)) = |D| \geq |(V(G) \setminus \{v_i\}) \cup \{v^{1,i}\}| + |(D \cap V_E(G)) \setminus \{v^{1,i}\}| \geq n + 1 = n + \left\lceil \frac{|V_{n-1}(G)|}{2} \right\rceil$, which completes the proof. $\square$

Now, we proceed to study the outer-independent total Roman domination number of $\mathtt{C}(G)$. For this purpose, we need to give the following useful lemma.

Lemma 2.2. *Let F be the set of the $\gamma_{tR}^{oi}(\mathtt{C}(G))$-functions g such that $|\{x \in V(G) : g(x) > 0\}|$ is maximum. If $f(V_0, V_1, V_2)$ is a $\gamma_{tR}^{oi}(\mathtt{C}(G))$-function belonging to F, then the following statements hold.*

(i) $|V_0 \cap V(G)| \leq 1$ *and* $V_2 \subseteq V(G)$.
(ii) *If* $\delta(G) \geq 2$ *and* $\Delta(G) \leq n - 2$*, then* $V(G) \subseteq V_1 \cup V_2$.

Proof. If there exist $i, j \in [n]$ such that $v_i, v_j \in V_0$, then $v_i v_j \notin E(\mathtt{C}(G))$, which implies that $v^{i,j} \in V_1 \cup V_2$ and $N_{\mathtt{C}(G)}(v^{i,j}) \subseteq V_0$, a contradiction. Therefore, $|V_0 \cap V(G)| \leq 1$. On the other hand, if there exist $i, j \in [n]$ such that $v_i v_j \in E(G)$ and $v^{i,j} \in V_2$, then $f(v_i) f(v_j) = 0$. Without loss of generality, assume that $f(v_i) = 0$. Observe that the function $f'(V_0', V_1', V_2')$; defined by $f'(v_i) = f'(v^{i,j}) = 1$ and $f'(x) = f(x)$ whenever $x \in V(\mathtt{C}(G)) \setminus \{v_i, v^{i,j}\}$; is a $\gamma_{tR}^{oi}(\mathtt{C}(G))$-function such that $|(V_1' \cup V_2') \cap V(G)| > |(V_1 \cup V_2) \cap V(G)|$, a contradiction. Therefore, $V_2 \subseteq V(G)$, which completes the proof of (i). From now on, assume that $\delta(G) \geq 2$ and $\Delta(G) \leq n - 2$. Suppose that there exists $i \in [n]$ such that $v_i \in V_0$. So, Statement (i) leads to $V_2 \subseteq V(G) \setminus \{v_i\} \subseteq V_1 \cup V_2$. So, there exists a $k \in [n] \setminus \{i\}$ such that $v_i v_k \notin E(G)$ (which implies that $v_i v_k \in E(\mathtt{C}(G))$) and $v_k \in V_2$. Since $\delta(G) \geq 2$, there exist $j, l \in [n] \setminus \{i, k\}$ such that $v_i v_j, v_i v_l \in E(G)$ and $v_j, v_l \in V_1 \cup V_2$. Now note that $v^{i,j}, v^{i,l} \in V_1$. Due to the fact that $\Delta(G) \leq n-2$ and $V(G) \setminus \{v_i\} \subseteq V_1 \cup V_2$ we have that $N_{\mathtt{C}(G)}(v_j) \cap V(G) \cap (V_1 \cup V_2) \neq \emptyset$ and $N_{\mathtt{C}(G)}(v_l) \cap V(G) \cap (V_1 \cup V_2) \neq \emptyset$. This implies that the function $f''(V_0'', V_1'', V_2'')$; defined by $f''(v_i) = 2$, $f''(v^{i,j}) = f''(v^{i,l}) = 0$ and $f''(x) = f(x)$ whenever $x \in V(\mathtt{C}(G)) \setminus \{v_i, v^{i,j}, v^{i,l}\}$; is a $\gamma_{tR}^{oi}(\mathtt{C}(G))$-function such that $|(V_1'' \cup V_2'') \cap V(G)| > |(V_1 \cup V_2) \cap V(G)|$, a contradiction. Therefore, $V(G) \subseteq V_1 \cup V_2$, which completes the proof of (ii). □

The following result provides lower and upper bounds on $\gamma_{tR}^{oi}(\mathtt{C}(G))$ in terms of the order and the vertex covering number of G.

Proposition 2.3. *The following statements hold for any connected graph G of order $n \geq 4$.*

(i) $\gamma_{tR}^{oi}(\mathtt{C}(K_{1,n-1})) = n + 2$.
(ii) *If $G \neq K_{1,n-1}$, then $n + \beta(G) - 1 \leq \gamma_{tR}^{oi}(\mathtt{C}(G)) \leq \gamma_t^{oi}(\mathtt{C}(G)) + \beta(G)$.*

Proof. It is easy to check that $\gamma_{tR}^{oi}(\mathtt{C}(K_{1,n-1})) = n + 2$. We now assume that $G \neq K_{1,n-1}$. Let S be a $\beta(G)$-set. Hence, $\beta(G) = |S| \geq 2$. From the proof of Theorem 2.1, we immediately deduce the following useful statements.

- If $G \in \mathcal{F}_n$, then $V(G)$ is a $\gamma_t^{oi}(\mathtt{C}(G))$-set.
- If $G \in \mathcal{G}_n$, then $(V(G) \setminus \{v_i\}) \cup \{v^{i,j}\}$ is a $\gamma_t^{oi}(\mathtt{C}(G))$-set, where $V_{n-1}(G) = \{v_j\}$ and $v_i \in V_1(G)$.
- If $G \notin \mathcal{F}_n \cup \mathcal{G}_n$, then $V(G) \cup W$ is a $\gamma_t^{oi}(\mathtt{C}(G))$-set, where $W \subseteq V_E(G)$ is a set of cardinality $\lceil |V_{n-1}(G)|/2 \rceil$ such that $N_{\mathtt{C}(G)}(v_i) \cap W \neq \emptyset$ for every $v_i \in V_{n-1}(G)$.

The inequality $\gamma_{tR}^{oi}(\mathtt{C}(G)) \leq \gamma_t^{oi}(\mathtt{C}(G)) + \beta(G)$ follows as a consequence of the following two complementary cases.

Case 1: $G \in \mathcal{G}_n$**.** In this case, we have that $D = (V(G) \setminus \{v_i\}) \cup \{v^{i,j}\}$ is a $\gamma_t^{oi}(\mathtt{C}(G))$-set, where $V_{n-1}(G) = \{v_j\}$ and $v_i \in V_1(G)$. In addition, it is straightforward that $v_j \in S$ and $v_i \notin S$, which implies that $S \subseteq D \setminus \{v^{i,j}\}$. Since $V_{n-2} = \emptyset$ and $|S| \geq 2$, it follows that the function $f(V_0, V_1, V_2)$; defined by $V_2 = S$, $V_1 = D \setminus S$ and $V_0 = V(\mathtt{C}(G)) \setminus (V_1 \cup V_2)$; is an OITRDF on $\mathtt{C}(G)$. Hence, $\gamma_{tR}^{oi}(\mathtt{C}(G)) \leq \omega(f) = |D \setminus S| + 2|S| = |D| + |S| = \gamma_t^{oi}(\mathtt{C}(G)) + \beta(G)$, as desired.

Case 2: $G \notin \mathcal{G}_n$**.** Depending on whether or not G belongs to $\mathcal{F}_n$, let us consider D' as the $\gamma_t^{oi}(\mathtt{C}(G))$-set defined in the first or third previous statement. In any case, we have that $V(G) \subseteq D'$. As a consequence, the function $f'(V_0', V_1', V_2')$; defined by $V_2' = S$, $V_1' = D' \setminus S$ and $V_0' = V(\mathtt{C}(G)) \setminus (V_1' \cup V_2')$; is an OITRDF on $\mathtt{C}(G)$. Hence, $\gamma_{tR}^{oi}(\mathtt{C}(G)) \leq \omega(f') = |D' \setminus S| + 2|S| = |D'| + |S| = \gamma_t^{oi}(\mathtt{C}(G)) + \beta(G)$, as desired.

Now, we proceed to prove that $\gamma_{tR}^{oi}(\mathtt{C}(G)) \geq n + \beta(G) - 1$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}^{oi}(\mathtt{C}(G))$-function that satisfies the conditions given in Lemma 2.2. Let $E' = \{x \in V_E(G) : N_{\mathtt{C}(G)}[x] \subseteq V_1\}$ and let $R \subseteq V(G)$ be a set of minimum cardinality such that $N_{\mathtt{C}(G)}(x) \cap R \neq \emptyset$ for every vertex $x \in E'$. Observe that $R \subseteq V_1 \cap V(G)$. We claim that $V_2 \cup R \cup (V_0 \cap V(G))$ is a vertex cover of G. Let $i, j \in [n]$ such that $v_i v_j \in E(G)$. If $v_i, v_j \in V_1$ then $v^{i,j} \in V_1$. Hence, $v^{i,j} \in E'$, which implies that $\{v_i, v_j\} \cap R \neq \emptyset$. Moreover, if $\{v_i, v_j\} \cap (V_0 \cup V_2) \neq \emptyset$, then we are done. Therefore, $V_2 \cup R \cup (V_0 \cap V(G))$ is a vertex cover of G, as desired. So, $\beta(G) \leq |V_2| + |V_0 \cap V(G)| + |R|$. In addition, it can also be deduced that $|V_2| + |V_1 \cap V(G)| \geq n - 1$ and $|(V_1 \cap V_E(G)) \setminus E'| \geq |V_0 \cap V(G)|$, and by the minimality of $|R|$ it follows that $|R| \leq |E'|$. As a consequence, we obtain that

$$\begin{aligned}
\gamma_{tR}^{oi}(\mathtt{C}(G)) &= 2|V_2| + |V_1 \cap V(G)| + |(V_1 \cap V_E(G)) \setminus E'| + |E'| \\
&\geq n - 1 + |V_2| + |V_0 \cap V(G)| + |R| \\
&\geq n + \beta(G) - 1,
\end{aligned}$$

which completes the proof. □

The following result shows that if G is a connected graph such that $\Delta(G) \leq n-2$, then the outer-independent total Roman domination number of $\mathtt{C}(G)$ can only take two possible values.

Theorem 2.4. *The following statements hold for any connected graph G of order $n \geq 4$ and $\Delta(G) \leq n - 2$.*

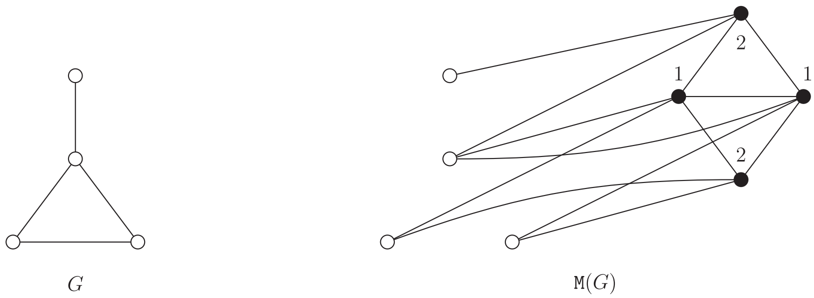

FIGURE 2. A graph G, and the corresponding middle graph $\mathtt{M}(G)$.

(i) $\gamma_{tR}^{oi}(\mathtt{C}(G)) \in \{n+\beta(G)-1, n+\beta(G)\}$.
(ii) *If* $\delta(G) \geq 2$, *then* $\gamma_{tR}^{oi}(\mathtt{C}(G)) = n+\beta(G)$.

Proof. Let G be a connected graph of order $n \geq 4$ and $\Delta(G) \leq n-2$. By Theorem 2.1 we have that $\gamma_t^{oi}(\mathtt{C}(G)) = n$. Therefore, Proposition 2.3 leads to $\gamma_{tR}^{oi}(\mathtt{C}(G)) \in \{n+\beta(G)-1, n+\beta(G)\}$, which completes the proof of (i). Let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}^{oi}(\mathtt{C}(G))$-function that satisfies the conditions given in Lemma 2.2. Also assume that $|V_2 \cap V(G)|$ is maximum. Now, we assume that $\delta(G) \geq 2$. By Lemma 2.2 (ii) we have that $V_2 \subseteq V(G) \subseteq V_1 \cup V_2$. Suppose that V_2 is not a vertex cover of G. So, there exists $i, j \in [n]$ such that $v_iv_j \in E(G)$ and $v_i, v_j, v^{i,j} \in V_1$. Since $\Delta(G) \leq n-2$, there exist $k, l \in [n] \setminus \{i, j\}$ such that $v_iv_k \in E(\mathtt{C}(G))$ and $v_jv_l \in E(\mathtt{C}(G))$. Due to the fact that $v_k, v_l \in V_1 \cup V_2$, it follows that the function $f'(V_0', V_1', V_2')$; defined by $f'(v_i) = 2$, $f'(v^{i,j}) = 0$ and $f'(x) = f(x)$ whenever $x \in V(\mathtt{C}(G)) \setminus \{v_i, v^{i,j}\}$, is a $\gamma_{tR}^{oi}(\mathtt{C}(G))$-function such that $|V_2'| > |V_2|$, a contradiction. Hence, V_2 is a vertex cover of G, which implies that $\beta(G) \leq |V_2|$. Therefore, $\gamma_{tR}^{oi}(\mathtt{C}(G)) = |V_1 \cup V_2| + |V_2| \geq |V(G)| + \beta(G) = n + \beta(G)$, and by Statement (i) it follows that $\gamma_{tR}^{oi}(\mathtt{C}(G)) = n+\beta(G)$, which completes the proof of (ii). □

3. MIDDLE GRAPHS

The *middle graph* of a nontrivial connected graph G, denoted by $\mathtt{M}(G)$, is the graph with vertex set $V(\mathtt{M}(G)) = V(G) \cup V_E(G)$ and edge set $E(\mathtt{M}(G)) = \{v_iv^{i,j}, v_jv^{i,j} : v^{i,j} \in V_E(G)\} \cup \{v^{i,j}v^{k,l} : v^{i,j}, v^{k,l} \in V_E(G) \wedge |\{i,j\} \cap \{k,l\}| = 1\}$. This graph operator was initially proposed by Hamada and Yoshimura in 1976 [14]. Various domination parameters have also been investigated on middle graphs; for instance, see [1,17–19]. In Figure 2 we show a graph G and its corresponding middle graph $\mathtt{M}(G)$. The set of black vertices describes a $\gamma_t^{oi}(\mathtt{M}(G))$-set, while the labels correspond to the positive weights of a $\gamma_{tR}^{oi}(\mathtt{M}(G))$-function.

The following theorem provides a closed formula for the outer-independent total domination number of the middle graph $\mathtt{M}(G)$.

Theorem 3.1. *For any connected graph G of size $m \geq 2$,*

$$\gamma_t^{oi}(\mathtt{M}(G)) = m.$$

Proof. By Theorem 1.1 and the fact that $\alpha(\mathtt{M}(G)) = |V(G)|$ (see [12], Thm. 6.1 (ii)) we have that $\gamma_t^{oi}(\mathtt{M}(G)) \geq |V(\mathtt{M}(G))| - \alpha(\mathtt{M}(G)) = m$. Now, observe that $V_E(G) \in \mathcal{D}_t^{oi}(\mathtt{M}(G))$. Hence, $\gamma_t^{oi}(\mathtt{M}(G)) \leq |V_E(G)| = m$. Therefore, $\gamma_t^{oi}(\mathtt{M}(G)) = m$, as desired. □

Now, we proceed to study the outer-independent total Roman domination number of $\mathtt{M}(G)$. For this purpose, we need to give the following useful lemma.

Lemma 3.2. *For any connected graph G of size $m \geq 2$, there exists a $\gamma_{tR}^{oi}(\mathtt{M}(G))$-function $f(V_0, V_1, V_2)$ such that $V(G) \subseteq V_0$.*

Proof. Let G be a connected graph of order $n \geq 3$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}^{oi}(\mathtt{M}(G))$-function such that $|V_0 \cap V(G)|$ is maximum. We now proceed with the following claims.

Claim I: $V(G) \cap V_2 = \emptyset$.

Proof of Claim I. If there exists a subscript $i \in [n]$ such that $v_i \in V_2$, then there exists a subscript $j \in [n] \setminus \{i\}$ such that $v^{i,j} \in V_0$, which implies that $\emptyset \neq N_{\mathtt{M}(G)}(v_i) \setminus \{v^{i,j}\} \subseteq V_1 \cup V_2$. So, it is easy to check that the function $f'(V_0', V_1', V_2')$; defined by $f'(v_i) = 0$, $f'(v^{i,j}) = 2$ and $f'(x) = f(x)$ whenever $x \in V(\mathtt{M}(G)) \setminus \{v_i, v^{i,j}\}$; is a $\gamma_{tR}^{oi}(\mathtt{M}(G))$-function such that $|V_0' \cap V(G)| > |V_0 \cap V(G)|$, a contradiction. Therefore, $V(G) \cap V_2 = \emptyset$, as desired. □

Claim II: $V(G) \cap V_1 = \emptyset$.

Proof of Claim II. From Claim I we have that $V(G) \subseteq V_0 \cup V_1$. Suppose that there exists a subscript $i \in [n]$ such that $v_i \in V_1$. Let $j \in [n] \setminus \{i\}$ such that $f(v^{i,j})$ is maximum among all vertices in $N_{\mathtt{M}(G)}(v_i)$. It is straightforward that $f(v^{i,j}) > 0$. Now, we consider the following two possible cases.

Case 1: $|N_{\mathtt{M}(G)}(v_i)| = 1$. Observe that $(N_{\mathtt{M}(G)}[v_j] \setminus \{v^{i,j}\}) \cap (V_1 \cup V_2) \neq \emptyset$. So, the function $f''(V_0'', V_1'', V_2'')$; defined by $f''(v_i) = 0$, $f''(v^{i,j}) = 2$ and $f''(x) = f(x)$ whenever $x \in V(\mathtt{M}(G)) \setminus \{v_i, v^{i,j}\}$; is a $\gamma_{tR}^{oi}(\mathtt{M}(G))$-function such that $|V_0'' \cap V(G)| > |V_0 \cap V(G)|$, a contradiction. Therefore, $V(G) \cap V_1 = \emptyset$, as desired.

Case 2: $|N_{\mathtt{M}(G)}(v_i)| \geq 2$. Let $l \in [n] \setminus \{i, j\}$ such that $f(v^{i,l})$ is minimum among all vertices in $N_{\mathtt{M}(G)}(v_i)$. If $f(v^{i,j}) = 2$ or $f(v^{i,l}) > 0$, then the function $g(W_0, W_1, W_2)$, defined by $g(v_i) = 0$, $g(v^{i,j}) = 2$, $g(v^{i,l}) = 1$ and $g(x) = f(x)$ whenever $x \in V(\mathtt{M}(G)) \setminus \{v_i, v^{i,j}, v^{i,l}\}$; is a $\gamma_{tR}^{oi}(\mathtt{M}(G))$-function such that $|W_0 \cap V(G)| > |V_0 \cap V(G)|$, a contradiction. Hence, $f(v^{i,j}) = 1$ and $f(v^{i,l}) = 0$. As a consequence, $f(v_l) > 0$ and there exists $k \in [n] \setminus \{i, l\}$ such that $f(v^{l,k}) = 2$. Observe that the function $g'(W_0', W_1', W_2')$, defined by $g'(v_l) = 0$, $g'(v^{i,l}) = 1$ and $g'(x) = f(x)$ whenever $x \in V(\mathtt{M}(G)) \setminus \{v_l, v^{i,l}\}$; is a $\gamma_{tR}^{oi}(\mathtt{M}(G))$-function such that $|W_0' \cap V(G)| > |V_0 \cap V(G)|$, a contradiction too. Therefore, $V(G) \cap V_1 = \emptyset$, as desired.

□

By Claims I and II, it follows that $V(G) \subseteq V_0$, which completes the proof. □

We conclude this section with the following theorem, which provides a closed formula for the outer-independent total Roman domination number of the middle graph $\mathtt{M}(G)$.

Theorem 3.3. *For any connected graph G of size $m \geq 2$,*

$$\gamma_{tR}^{oi}(\mathtt{M}(G)) = m + \beta'(G).$$

Proof. Let G be a connected graph of order n and size $m \geq 2$. In Theorem 2.8 of [9] and Theorem 6.1 (i) of [12], the authors showed that $\gamma_{tR}^{oi}(\mathtt{M}(G)) \leq \gamma_t^{oi}(\mathtt{M}(G)) + \gamma(\mathtt{M}(G))$ and $\gamma(\mathtt{M}(G)) = \beta'(G)$, respectively. As a consequence, it follows that $\gamma_{tR}^{oi}(\mathtt{M}(G)) \leq \gamma_t^{oi}(\mathtt{M}(G)) + \beta'(G)$. We next proceed to prove that $\gamma_{tR}^{oi}(\mathtt{M}(G)) \geq \gamma_t^{oi}(\mathtt{M}(G)) + \beta'(G)$. By Lemma 3.2, there exists a $\gamma_{tR}^{oi}(\mathtt{M}(G))$-function $f(V_0, V_1, V_2)$ such that $V(G) \subseteq V_0$. Let $E_2 = \{v_iv_j \in E(G) : v^{i,j} \in V_2\}$. Notice that $|E_2| = |V_2|$. Now, observe that in $\mathtt{M}(G)$ every vertex $v_i \in V(G) \subseteq V_0$ has a neighbor $v^{i,j} \in V_2$. This implies that v_i is incident to $v_iv_j \in E_2$ in G. So, E_2 is an edge cover of G, and as a consequence, $\beta'(G) \leq |E_2| = |V_2|$. Moreover, and due to the fact that $f \in \mathcal{F}_{tR}^{oi}(\mathtt{M}(G))$, it follows that $V_1 \cup V_2 \in \mathcal{D}_t^{oi}(\mathtt{M}(G))$, which implies that $\gamma_t^{oi}(\mathtt{M}(G)) \leq |V_1| + |V_2|$. Therefore, $\gamma_{tR}^{oi}(\mathtt{M}(G)) = 2|V_2| + |V_1| \geq \gamma_t^{oi}(\mathtt{M}(G)) + \beta'(G)$, as desired. As a consequence, we have that $\gamma_{tR}^{oi}(\mathtt{M}(G)) = \gamma_t^{oi}(\mathtt{M}(G)) + \beta'(G)$. Thus, Theorem 3.1 leads to $\gamma_{tR}^{oi}(\mathtt{M}(G)) = m + \beta'(G)$, which completes the proof. □

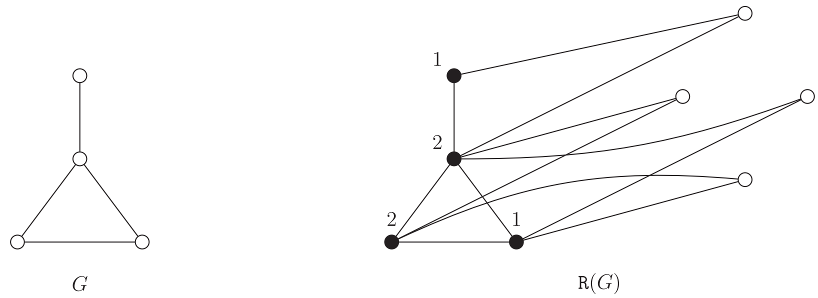

FIGURE 3. A graph G, and the corresponding graph $\mathtt{R}(G)$.

4. The graph $\mathtt{R}(G)$

The graph $\mathtt{R}(G)$ is the graph obtained from a connected graph G with vertex set $V(\mathtt{R}(G)) = V(G) \cup V_E(G)$ and edge set $E(\mathtt{R}(G)) = E(G) \cup \{v_i v^{i,j}, v_j v^{i,j} : v^{i,j} \in V_E(G)\}$. Some domination parameters have also been investigated on this graph; for instance, see [2,4,28]. In Figure 3 we show a graph G and its corresponding graph $\mathtt{R}(G)$. The set of black vertices describes a $\gamma_t^{oi}(\mathtt{R}(G))$-set, while the labels correspond to the positive weights of a $\gamma_{tR}^{oi}(\mathtt{R}(G))$-function.

We begin this section by obtaining the exact values for the outer-independent total domination number and the independence number of the graph $\mathtt{R}(G)$.

Theorem 4.1. *For any nontrivial connected graph G of order n and size m,*

(i) $\alpha(\mathtt{R}(G)) = m$.
(ii) $\gamma_t^{oi}(\mathtt{R}(G)) = n$.

Proof. By definition, it is straightforward that $V_E(G) \in \mathcal{ID}(\mathtt{R}(G))$. This implies that $\alpha(\mathtt{R}(G)) \geq |V_E(G)| = m$. On the other hand, let I be an $\alpha(\mathtt{R}(G))$-set such that $|I \cap V_E(G)|$ is maximum among all $\alpha(\mathtt{R}(G))$-sets. If there exists a subscript $i \in [n]$ such that $v_i \in V(G) \cap I$, then for every $j \in [n] \setminus \{i\}$ such that $v_i v_j \in E(G)$ it follows that $I' = (I \setminus \{v_i\}) \cup \{v^{i,j}\}$ is an $\alpha(\mathtt{R}(G))$-set with $|I' \cap V_E(G)| > |I \cap V_E(G)|$, a contradiction. Thus, $I \subseteq V_E(G)$, which implies that $\alpha(\mathtt{R}(G)) = |I| \leq |V_E(G)| = m$. Therefore, $\alpha(\mathtt{R}(G)) = m$, which completes the proof of (i). Now, it is easy to check that $V(G) \in \mathcal{D}_t^{oi}(\mathtt{R}(G))$. Hence, $\gamma_t^{oi}(\mathtt{R}(G)) \leq n$. Moreover, Theorem 1.1 and the fact that $\alpha(\mathtt{R}(G)) = m$ lead to $\gamma_t^{oi}(\mathtt{R}(G)) \geq |V(\mathtt{R}(G))| - \alpha(\mathtt{R}(G)) = n$. Therefore, $\gamma_t^{oi}(\mathtt{R}(G)) = n$, which completes the proof. □

The next theorem provides a closed formula for the outer-independent total Roman domination number of the graph $\mathtt{R}(G)$. Before, we need to give the following useful lemma.

Lemma 4.2. *For any nontrivial connected graph G, there exists a $\gamma_{tR}^{oi}(\mathtt{R}(G))$-function $f(V_0, V_1, V_2)$ such that $V(G) \subseteq V_1 \cup V_2$.*

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}^{oi}(\mathtt{R}(G))$-function such that $|(V_1 \cup V_2) \cap V(G)|$ is maximum among all $\gamma_{tR}^{oi}(\mathtt{R}(G))$-functions. Now, we suppose that there exists $i \in [n]$ such that $v_i \in V(G) \cap V_0$. So, there exists $j \in [n] \setminus \{i\}$ such that $v_i v_j \in E(G)$ and $f(v_j) + f(v^{i,j}) \geq 3$. Observe that the function $f'(V_0', V_1', V_2')$; defined by $f'(v_i) = \min\{f(v_j), f(v^{i,j})\}$, $f'(v^{i,j}) = 0$, $f'(v_j) = 2$ and $f'(x) = f(x)$ whenever $x \in V(\mathtt{R}(G)) \setminus \{v_i, v_j, v^{i,j}\}$; is a $\gamma_{tR}^{oi}(\mathtt{R}(G))$-function such that $|(V_1' \cup V_2') \cap V(G)| > |(V_1 \cup V_2) \cap V(G)|$, a contradiction. Therefore $V(G) \subseteq V_1 \cup V_2$, which completes the proof. □

Theorem 4.3. *For any nontrivial connected graph G of order n,*

$$\gamma_{tR}^{oi}(\mathtt{R}(G)) = n + \beta(G).$$

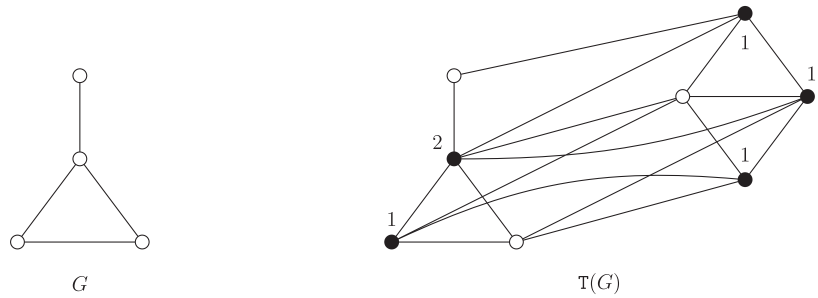

FIGURE 4. A graph G, and the corresponding total graph $\mathtt{T}(G)$.

Proof. Observe that from any $\beta(G)$-set D, the function $g(W_0, W_1, W_2)$; defined by $W_2 = D$, $W_1 = V(G) \setminus D$ and $W_0 = V_E(G)$; is an OITRDF on $\mathtt{R}(G)$. Hence, $\gamma_{tR}^{oi}(\mathtt{R}(G)) \leq \omega(g) = |W_1 \cup W_2| + |W_2| = n + \beta(G)$. Now, we proceed to prove that $\gamma_{tR}^{oi}(\mathtt{R}(G)) \geq n + \beta(G)$. By Lemma 4.2, there exists a $\gamma_{tR}^{oi}(\mathtt{R}(G))$-function $f(V_0, V_1, V_2)$ such that $V(G) \subseteq V_1 \cup V_2$. Without loss of generality, assume that $|V_2 \cap V(G)|$ is maximum among all $\gamma_{tR}^{oi}(\mathtt{R}(G))$-functions that satisfy the condition given in Lemma 4.2. If there exist subscripts $i, j \in [n]$ such that $v_iv_j \in E(G)$ and $f(v^{i,j}) > 0$, then $f(v_i) = f(v_j) = 1$. Notice that the function $f'(V_0', V_1', V_2')$; defined by $f'(v^{i,j}) = 0$, $f'(v_i) = 2$ and $f'(x) = f(x)$ whenever $x \in V(\mathtt{R}(G)) \setminus \{v_i, v^{i,j}\}$; is a $\gamma_{tR}^{oi}(\mathtt{R}(G))$-function such that $|V_2' \cap V(G)| > |V_2 \cap V(G)|$, a contradiction. As a consequence, it follows that $V_E(G) \subseteq V_0$, which implies that V_2 is a vertex cover of G. Therefore, $\gamma_{tR}^{oi}(\mathtt{R}(G)) = \omega(f) = |V_1 \cup V_2| + |V_2| \geq n + \beta(G)$, which completes the proof. □

5. Total graphs

The *total graph* of a nontrivial connected graph G, denoted by $\mathtt{T}(G)$, is the graph with vertex set $V(\mathtt{T}(G)) = V(G) \cup V_E(G)$ and edge set $E(\mathtt{T}(G)) = E(G) \cup E(\mathtt{M}(G))$. This graph operator was introduced by Behzad [3] in 1967. Subsequently, several domination parameters were studied for this graph operator in different works, including [4,11,24,30,31]. In Figure 4 we show a graph G and its corresponding total graph $\mathtt{T}(G)$. The set of black vertices describes a $\gamma_t^{oi}(\mathtt{T}(G))$-set, while the labels correspond to the positive weights of a $\gamma_{tR}^{oi}(\mathtt{T}(G))$-function.

The following theorem provides a closed formula for the independence number of $\mathtt{T}(G)$. This result is necessary in order to deduce the value of $\gamma_t^{oi}(\mathtt{T}(G))$.

Theorem 5.1. *For any nontrivial connected graph G of order n,*

$$\alpha(\mathtt{T}(G)) = n - \gamma'(G).$$

Proof. Let X be a maximal independent edge set of G of minimum cardinality. As pointed out in [15], page 342, $|X| = \gamma'(G)$. Let $V_X \subseteq V(G)$ be the set formed by the end-vertices of edges belonging to X and $X' = \{v^{i,j} \in V_E(G) : \ v_iv_j \in X\}$. Note that $|V_X| = 2|X|$ and $|X'| = |X|$. Now, let us consider the set $I = (V(G) \setminus V_X) \cup X'$. If there exist two adjacent vertices $v_i, v_j \in V(G) \setminus V_X$ in $\mathtt{T}(G)$, then $v_iv_j \in E(G) \setminus X$ is not adjacent to some edge in X, a contradiction. Hence, $V(G) \setminus V_X$ is an independent set of $\mathtt{T}(G)$. In addition, from the previous definitions it can be easily verified that every vertex $x \in X'$ satisfies that $N_{\mathtt{T}(G)}(x) \cap I = \emptyset$. As a consequence, I is also an independent set of $\mathtt{T}(G)$, which implies that

$$\alpha(\mathtt{T}(G)) \geq |I| = |V(G) \setminus V_X| + |X'| = n - 2|X| + |X| = n - \gamma'(G). \tag{2}$$

We only need to prove that $\alpha(\mathtt{T}(G)) \leq n - \gamma'(G)$. Let Y be an $\alpha(\mathtt{T}(G))$-set such that $|Y \cap V_E(G)|$ is maximum among all $\alpha(\mathtt{T}(G))$-sets. Let $E_Y = \{v_iv_j \in E(G) : \ v^{i,j} \in Y \cap V_E(G)\}$. Note that $|E_Y| = |Y \cap V_E(G)|$. We now proceed with the following claims.

Claim I: E_Y is an edge dominating set of G.

Proof of Claim I. Suppose that there exists an edge $v_iv_j \in E(G) \setminus E_Y$ that is not adjacent to some edge in E_Y. This implies that $N_{\mathtt{T}(G)}(v^{i,j}) \cap Y \cap V_E(G) = \emptyset$. So, $Y \cap \{v_i, v_j, v^{i,j}\} \neq \emptyset$ and by the maximality of $|Y \cap V_E(G)|$ it follows that $v^{i,j} \in Y$, which contradicts the fact that $v_iv_j \in E(G) \setminus E_Y$. Therefore, every edge in $E(G) \setminus E_Y$ is adjacent to some edge in E_Y, which implies that E_Y is an edge dominating set of G, as desired. □

Claim II: $V(G) = V_Y \cup (Y \cap V(G))$, where $V_Y = N_{\mathtt{T}(G)}(Y \cap V_E(G)) \cap V(G)$.

Proof of Claim II. It is straightforward that $V_Y \cup (Y \cap V(G)) \subseteq V(G)$. Suppose that there exists a vertex $v_i \in V(G) \setminus (V_Y \cup Y)$. Since Y is an $\alpha(\mathtt{T}(G))$-set, there exists a vertex $v_j \in Y \cap V(G)$ such that $v_iv_j \in E(G)$. Observe that the set $Y' = (Y \setminus \{v_j\}) \cup \{v^{i,j}\}$ is an $\alpha(\mathtt{T}(G))$-set with $|Y' \cap V_E(G)| > |Y \cap V_E(G)|$, a contradiction. Therefore, $V(G) = V_Y \cup (Y \cap V(G))$, as desired. □

By Claim I we deduce that $\gamma'(G) \leq |E_Y| = |Y \cap V_E(G)|$. Moreover, it is easy to check that $V_Y \cap (Y \cap V(G)) = \emptyset$. So, Claim II leads to $|V_Y| + |Y \cap V(G)| = n$. In addition, observe that $|N_{\mathtt{T}(G)}(v_i) \cap (Y \cap V_E(G))| = 1$ for every $v_i \in V_Y$. This implies that $|V_Y| = 2|Y \cap V_E(G)|$. Therefore,

$$\begin{aligned}\gamma'(G) &\leq |Y \cap V_E(G)| \\ &= 2|Y \cap V_E(G)| + |Y \cap V(G)| - (|Y \cap V_E(G)| + |Y \cap V(G)|) \\ &= |V_Y| + |Y \cap V(G)| - |Y| \\ &= n - \alpha(\mathtt{T}(G)).\end{aligned}$$

Thus, $\alpha(\mathtt{T}(G)) \leq n - \gamma'(G)$, which completes the proof. □

Now, we are able to provide the exact value of the outer-independent total domination number of the total graph $\mathtt{T}(G)$.

Theorem 5.2. *For any nontrivial connected graph G of size m,*

$$\gamma_t^{oi}(\mathtt{T}(G)) = m + \gamma'(G).$$

Proof. By Theorems 1.1 and 5.1 we have that

$$\gamma_t^{oi}(\mathtt{T}(G)) \geq |V(\mathtt{T}(G))| - \alpha(\mathtt{T}(G)) = |V(G)| + m - (|V(G)| - \gamma'(G)) = m + \gamma'(G).$$

We only need to prove that $\gamma_t^{oi}(\mathtt{T}(G)) \leq m + \gamma'(G)$. Let I be an $\alpha(\mathtt{T}(G))$-set and let $D = V(\mathtt{T}(G)) \setminus I$. By Theorem 5.1 we have that $|I| = |V(G)| - \gamma'(G)$. We proceed to prove that $D \in \mathcal{D}_t^{oi}(\mathtt{T}(G))$. Since I is an independent set of $\mathtt{T}(G)$, we only need to show that the subgraph induced by D has no isolated vertex. If there exists a vertex $v \in D$ such that $N_{\mathtt{T}(G)}(v) \subseteq I$, then the subgraph induced by $N_{\mathtt{T}(G)}(v)$ is isomorphic to an edgeless graph, which contradicts the fact that every vertex in $\mathtt{T}(G)$ is contained in a triangle. Hence, the subgraph induced by D has no isolated vertex, as desired. As a consequence, $D \in \mathcal{D}_t^{oi}(\mathtt{T}(G))$, which implies that

$$\gamma_t^{oi}(\mathtt{T}(G)) \leq |D| = |V(\mathtt{T}(G))| - |I| = |V(G)| + m - (|V(G)| - \gamma'(G)) = m + \gamma'(G).$$

Therefore, the proof is complete. □

Finally, we proceed to study the outer-independent total Roman domination number of $\mathtt{T}(G)$. For this purpose, we need to give the following useful lemma.

Lemma 5.3. *Let G be a nontrivial connected graph. The following statements hold.*

(i) *If $D \in \mathcal{ID}(\mathtt{T}(G))$, then $V(\mathtt{T}(G)) \setminus D \in \mathcal{D}_t^{oi}(\mathtt{T}(G))$.*
(ii) *If $f(V_0, V_1, V_2)$ is a $\gamma_{tR}^{oi}(\mathtt{T}(G))$-function such that $|V_0|$ is maximum, then $V_0 \in \mathcal{ID}(\mathtt{T}(G))$ and as a consequence, $V_1 \cup V_2 \in \mathcal{D}_t^{oi}(\mathtt{T}(G))$.*

Proof. Let $D \in \mathcal{ID}(\mathtt{T}(G))$. This implies that $V(\mathtt{T}(G)) \setminus D$ is a vertex cover of $\mathtt{T}(G)$. Since every vertex in $\mathtt{T}(G)$ is contained in a triangle, it follows that $V(\mathtt{T}(G)) \setminus D \in \mathcal{D}_t^{oi}(\mathtt{T}(G))$, which completes the proof of (i). Now, let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}^{oi}(\mathtt{T}(G))$-function such that $|V_0|$ is maximum. By definition we have that V_0 is an independent set of $\mathtt{T}(G)$. Now, it is straightforward that $N_{\mathtt{T}(G)}(v) \cap V_0 \neq \emptyset$ for every vertex $v \in V_2$. If there exists a vertex $v \in V_1$ such that $N_{\mathtt{T}(G)}(v) \subseteq V_1 \cup V_2$, then for any vertex $u \in N_{\mathtt{T}(G)}(v)$ we have that the function $f'(V_0', V_1', V_2')$; defined by $f'(v) = 0$, $f'(u) = 2$ and $f'(x) = f(x)$ whenever $x \in V(\mathtt{T}(G)) \setminus \{v, u\}$; is a $\gamma_{tR}^{oi}(\mathtt{T}(G))$-function such that $|V_0'| > |V_0|$, which is a contradiction. Hence, $N_{\mathtt{T}(G)}(v) \cap V_0 \neq \emptyset$ for every vertex $v \in V_1$. So, $V_0 \in \mathcal{ID}(\mathtt{T}(G))$ and as a consequence, Statement (i) leads to $V_1 \cup V_2 \in \mathcal{D}_t^{oi}(\mathtt{T}(G))$, which completes the proof of (ii). □

We now consider a kind of partial total domination version in a graph H introduced in [7]. Let $A \subseteq V(H)$ be any given set of a nontrivial connected graph H. We say that $D_A \subseteq V(H)$ is a *total dominating set with respect to A in H* (A-total dominating set in H for short) if every vertex in A is adjacent to at least one vertex in D_A. The *total domination number with respect to A in H*, denoted by $\gamma_t(A, H)$, is the minimum cardinality among all A-total dominating sets in H. A $\gamma_t(A, H)$-set is an A-total dominating set in H of cardinality $\gamma_t(A, H)$. Observe that if $A = V(H)$, then $\gamma_t(A, H)$ becomes the standard total domination number of H.

As shown in the next theorem, the above concept is fundamental to obtain a closed formula for the outer-independent total Roman domination number of $\mathtt{T}(G)$.

Theorem 5.4. *For any nontrivial connected graph G of order n and size m,*

$$\gamma_{tR}^{oi}(\mathtt{T}(G)) = n + m - \max\{|I| - \gamma_t(I, \mathtt{T}(G)) : I \in \mathcal{ID}(\mathtt{T}(G))\}.$$

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}^{oi}(\mathtt{T}(G))$-function such that $|V_0|$ is maximum. By Lemma 5.3 (ii) we have that $V_0 \in \mathcal{ID}(\mathtt{T}(G))$ and $V_1 \cup V_2 \in \mathcal{D}_t^{oi}(\mathtt{T}(G))$. Observe that V_2 is a V_0-total dominating set in $\mathtt{T}(G)$. Hence, $\gamma_t(V_0, \mathtt{T}(G)) \leq |V_2|$. If $\gamma_t(V_0, \mathtt{T}(G)) < |V_2|$, then from any $\gamma_t(V_0, \mathtt{T}(G))$-set I, the function f'; defined by $f'(x) = 2$ if $x \in I$, $f'(x) = 1$ if $x \in (V_1 \cup V_2) \setminus I$ and $f'(x) = 0$ whenever $x \in V_0$; is an OITRDF on $\mathtt{T}(G)$ such that $\omega(f') = 2|I| + |(V_1 \cup V_2) \setminus I| = |I| + |V_2| + |V_1| < 2|V_2| + |V_1| = \gamma_{tR}^{oi}(\mathtt{T}(G))$, a contradiction. Hence, $\gamma_t(V_0, \mathtt{T}(G)) = |V_2|$, which implies that V_2 is a $\gamma_t(V_0, \mathtt{T}(G))$-set. Due to the fact that $\gamma_{tR}^{oi}(\mathtt{T}(G)) = 2|V_2| + |V_1|$ and $|V_0| + |V_1| + |V_2| = |V(\mathtt{T}(G))| = n + m$, it follows that

$$\gamma_{tR}^{oi}(\mathtt{T}(G)) = n + m - (|V_0| - |V_2|) \geq n + m - \max\{|I| - \gamma_t(I, \mathtt{T}(G)) : I \in \mathcal{ID}(\mathtt{T}(G))\}.$$

On the other hand, let $I' \in \mathcal{ID}(\mathtt{T}(G))$ such that $|I'| - \gamma_t(I', \mathtt{T}(G))$ is maximum. Let D' be a $\gamma_t(I', \mathtt{T}(G))$-set. Observe that the function g; defined by $g(x) = 2$ if $x \in D'$, $g(x) = 1$ if $x \in V(\mathtt{T}(G)) \setminus (I' \cup D')$ and $g(x) = 0$ whenever $x \in I'$; is an OITRDF on $\mathtt{T}(G)$. Thus,

$$\begin{aligned}\gamma_{tR}^{oi}(\mathtt{T}(G)) \leq \omega(g) &= 2|D'| + |V(\mathtt{T}(G)) \setminus (I' \cup D')| \\ &= n + m - (|I'| - |D'|) \\ &= n + m - \max\{|I| - \gamma_t(I, \mathtt{T}(G)) : I \in \mathcal{ID}(\mathtt{T}(G))\}.\end{aligned}$$

As a consequence, it follows that $\gamma_{tR}^{oi}(\mathtt{T}(G)) = n+m-\max\{|I|-\gamma_t(I, \mathtt{T}(G)) : I \in \mathcal{ID}(\mathtt{T}(G))\}$, which completes the proof. □

6. Concluding remarks

In this article, we address the problem of obtaining closed formulas for the outer-independent total domination number and the outer-independent total Roman domination number of the central graph $\mathtt{C}(G)$, the middle graph $\mathtt{M}(G)$, the graph operator $\mathtt{R}(G)$ and the total graph $\mathtt{T}(G)$. As a consequence of this study, several open questions arise, which we outline below:

– Extend the study of the two parameters considered in this paper to other well-known graph operators.

– Study other domination parameters on these graph operators and analyze how the results relate to the ones obtained here. For instance, it is straightforward to verify that $\gamma(\mathtt{R}(G)) = \beta(G)$, which consequently implies by Theorem 4.3 that $\gamma_{tR}^{oi}(\mathtt{R}(G)) = n + \gamma(\mathtt{R}(G))$.
– From Theorem 5.4, it follows that $\gamma_t^{oi}(\mathtt{T}(G)) + 1 \leq \gamma_{tR}^{oi}(\mathtt{T}(G)) \leq 2\gamma_t^{oi}(\mathtt{T}(G))$. We propose the problem of characterizing all graphs for which these equalities hold.

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