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ON THE DUAL FORMULATION OF RUSSELL MEASURE MODEL

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Abstract. The Russell Measure Model (RM), a nonlinear data envelopment analysis (DEA) model for evaluating decision-making units, allows for independent and disproportionate inputs and outputs, which makes it superior and more accurate than the radial models. The model is formulated as a second-order cone programming (SOCP) problem, and its dual is derived using SOCP duality. Previous studies have noted the complexity and limited interpretability of this dual formulation and have proposed an alternative using semidefinite programming (SDP) problem. This paper demonstrates the equivalence of these dual formulations through variable transformations. In addition, a new SOCP formulation of the dual RM model is introduced, which is in the usual form of multiplier models without any variable transformations. It is shown that this new formulation is equivalent to the SDP model. Moreover, using the conic model, a new approach is proposed to identify the unique maximal reference set and projection by solving one model, thereby improving upon the existing two-stage approach. Two examples demonstrate the advantages of the proposed models.

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1. Introduction

Data Envelopment Analysis (DEA) is a tool for evaluating the relative efficiency of decision-making units (DMUs) with multiple inputs and outputs. It was first presented by Charnes *et al.* [6] and their model is called the CCR model, which assumes constant returns to scale (CRS). Later, Banker *et al.* [2], extended this model to assume variable returns to scale (VRS), which is known as the BCC model. The DEA models are generally classified into radial and non-radial types. Radial models, such as CCR and BCC models, proportionally reduce inputs or increase outputs to reach the efficiency frontier. Meanwhile, non-radial models, by ignoring proportional changes in all inputs and outputs, allow for some inputs to be reduced or some outputs to be increased independently and disproportionately simultaneously, which leads to a more accurate and real evaluation of the performance of the DMU. One of the non-radial models is the Russell Measure (RM) model proposed by Färe and Lovell [9], which is capable of contraction input and expanding the outputs in a disproportionate and independent manner. Moreover, this model, by aggregating input and output efficiencies, possesses the properties of unit invariance and provides a meaningful economic interpretation [13]. Sueyoshi and Sekitani [21] reformulated the RM model as a second-order cone programming (SOCP) problem and extracted its dual.

Keywords. Russell measure, second-order cone programming, semidefinite programming, strong complementary slackness conditions, maximal reference set.

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One of the important aspects of DEA models is the interpretation and utilization of multiplier models. The multiplier formulation provides economic interpretations through the obtained coefficients and plays a pivotal role in deriving various theoretical results. Among these results, we can mention the identification of efficient frontiers, the description of returns to scale, and the guarantee of the strong complementary slackness condition (SCSC) property. However, the dual formulation presented in Sueyoshi and Sekitani [21] appears difficult to compare with standard multiplier models due to its structural complexity. To overcome this problem, an alternative formulation based on semidefinite programming (SDP) problem has been proposed in Haličká and Trnovská [13], which offers a more interpretable structure. However, the precise correspondence and equivalence between these two multiplier formulations have not been comprehensively investigated so far.

In this paper, first we prove the equivalence of the dual formulations proposed in Sueyoshi and Sekitani [21] and Haličká and Trnovská [13]. This equivalence allows us to provide more clarity in understanding the relationships and interpretation of the dual formulas [21]. Furthermore, by introducing a new dual formulation based on SOCP that is equivalent to the aforementioned models and preserves the structure without the need for variable transformation, this model helps to understand the interpretability and applicability of the dual formulations based on SOCP more easily.

It is worth noting that, in addition to discussing the dual conic formulations in this study, the RM model adopts a classical Lagrangian dual representation with nonlinear objective and linear constraints [13]. This dual can be obtained directly using conjugate functions and standard Lagrangian duality techniques, as shown in Trnovská *et al.* [24]. Furthermore, both the original and dual convex models can be solved efficiently using existing software tools such as CVX [10, 11], without the need to reformulate the problem in conic form.

Another challenge with DEA models is the possibility of multiple optimal solutions, which makes the reference sets and projections non-unique in some DMUs under evaluation. To solve this problem, Sueyoshi and Sekitani [22] proposed a primal-dual linear programming method with SCSC to handle complex issues like multiple reference sets and projections in non-radial DEA models for measuring returns to scale. Later, they developed a method using SCSC in the BCC model to identify maximal reference sets (MRSs) and projections for all DMUs under evaluation [20]. Also, Sekitani [19] proposed a two-stage approach to overcome the nonlinear structure and multiple optimal solutions in the RM model. In the first stage, the nonlinear RM model is solved to obtain a unique projected output. In the second stage, a linear programming model based on SCSC is used to establish the uniqueness of the projection and to determine the MRS. Krivonozhko *et al.* [14] presented a new direct and efficient two-stage approach to evaluate returns to scale in non-radial DEA models. In the first stage, an interior point on the boundary belonging to the optimal face is identified. Then, in the second stage, the returns to scale at this interior point are evaluated using standard methods. Mehdilou *et al.* [17] extended Krivonozhko *et al.*'s approach. In the first stage, they start by finding a relative interior point of the minimum facet of the linear problem to identify the global reference set. Then, for the obtained point, they use a method to solve the problem arising from the incomplete dimensions of the minimum facet.

To deal with the above mentioned challenge, first, we prove that all conic constraints are active in both primal and dual models at the optimality, which allows us to identify the MRS and the unique projection simultaneously by solving a model using the special form of SCSC. Also, the approach presented in this study significantly improves the computational efficiency compared to the two-stage method proposed in Sekitani [19]. We apply the new approach to a set of random data and real data from 20 branches of a commercial bank in Iran. The contribution of the paper can be summarized as follows:

- We establish equivalence between the SOCP dual [21] and the SDP dual [13] through a transformation of variables.
- We derive a new dual formulation for the RM model based on SOCP, which is presented in the standard format of multiplier models and is easily interpreted and compared with conventional multiplier models
- Using SCSC, we introduce a new approach to identify the unique MRS, eliminating the need for the two-step method [19].

The remainder of the paper is organized as follows. In Section 2, RM model is presented followed by dual formulations derived in Sueyoshi and Sekitani [21] and Haličká and Trnovská [13]. In Section 3, we show that these two dual formulations can be rewritten in the form of each other and then introduce a new dual formulation based on SOCP. In Section 4, we present a new approach to identify unique MRS and projection that reduces the computational complexity. Section 5 applies the proposed approach to two random and real datasets. Finally, Section 6 concludes the paper.

2. Russell measure model

The RM model for a set of n DMUs (DMU_j, $j = 1, \cdots, n$), each consuming m inputs x_{ij} $(i = 1, \cdots, m)$ to produce s outputs y_{rj} $(r = 1, \cdots, s)$, is as follows:

$$
\begin{aligned}
\min\ & \frac{1}{m+s}\left(\sum_{i=1}^{m}\theta_i + \sum_{r=1}^{s}\frac{1}{\varphi_r}\right) & \\
\text{s.t.}\ & \sum_{j=1}^{n}\lambda_j x_{ij} \le \theta_i x_{io}, & \forall i \\
& \sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi_r y_{ro}, & \forall r \\
& \sum_{j=1}^{n}\lambda_j = 1, & \\
& \theta_i \le 1, & \forall i \\
& \varphi_r \ge 1, & \forall r \\
& \lambda \ge 0, &
\end{aligned}
\tag{1}
$$

where all inputs and outputs are assumed to be positive, Haličká and Trnovská [13].

It is observed that the objective function of model (1) is nonlinear due to the existence of the terms $\frac{1}{\phi_r}$ related to the outputs. The authors in Sueyoshi and Sekitani [21] formulated model (1) as an SOCP by variable transforming $\overline{\varphi}_r = \frac{1}{\phi_r}$ $\forall r$. Then, using the conic duality theory, they presented its dual as follows:

$$
\begin{aligned}
\max\ & \sum_{r=1}^{s}\rho_r\sqrt{y_{ro}} - \sum_{i=1}^{m}v_i x_{io} - \sum_{r=1}^{s}\delta_r + \sigma + \frac{m}{m+s} & \\
\text{s.t.}\ & \sum_{r=1}^{s}\frac{\left(\overline{\mu}_r + \underline{\mu}_r\right)}{2}y_{rj} - \sum_{i=1}^{m}v_i x_{ij} + \sigma \le 0, & \forall j \\
& v_i x_{io} \ge \frac{1}{m+s}, & \forall i \\
& \frac{(\overline{\mu}_r - \underline{\mu}_r)}{2} \le \frac{1}{m+s} + \delta_r, & \forall r \\
& {\underline{\mu}_r}^2 + {\rho_r}^2 \le {\overline{\mu}_r}^2, & \forall r \\
& v, \delta \ge 0. &
\end{aligned}
\tag{2}
$$

In Haličká and Trnovská [13], the authors have claimed that the dual formulation (2) is complicated and allows neither interpretation nor comparison with other multiplier models. Thus they presented a SDP of the dual RM model as follows:

$$\begin{aligned}
\max\ & -\sigma+\sum_{r=1}^{s}u_r y_{ro}-\sum_{i=1}^{m}v_i x_{io}+1-\sum_{r=1}^{s}(z_{1r}+z_{2r}+2z_{3r}) & \\
\text{s.t. } & \sum_{r=1}^{s}u_r y_{rj}-\sum_{i=1}^{m}v_i x_{ij}\le\sigma, & \forall j\\
& v_i x_{io}\ge\frac{1}{m+s}, & \forall i\\
& u_r\ge\frac{z_{2r}}{y_{ro}}, & \forall r\\
& z_{1r}=\frac{1}{m+s}, & \forall r\\
& Z_r=\begin{pmatrix} z_{1r} & z_{3r}\\ z_{3r} & z_{2r}\end{pmatrix}\succeq 0, & \forall r.
\end{aligned} \tag{3}$$

As can be seen, this model is fully interpretable and compatible with other forms of coefficients of linear models (*e.g.*, CCR), where the variables u and v are interpreted as output and input shadow prices, respectively, and virtual profit (x_o, y_o) as $\sum_{r=1}^{s} y_{ro}u_r-\sum_{i=1}^{m}x_{io}v_i$ is defined, which is maximized under the following conditions [13]:

$$\sum_{r=1}^{s} y_{rj}u_r-\sum_{i=1}^{m}x_{ij}v_i\le 0, \qquad \forall j=1,\dots,n.$$

3. Equivalent dual RM formulations

Obviously, since dual models (2) and (3) are obtained from a primal model, they are equivalent to each other. In the following, it is shown that these two models can be rewritten in the form of each other, thus model (2) is interpretable and consistent with other all multiplier models, just like model (3).

First, the following technical lemma is crucial for our derivation, which can be easily proved.

Lemma 3.1. *The following models have equal optimal objective values:*

$$\begin{aligned}
\max_{x,y}\ & -\left(\sqrt{x}-\sqrt{y}\right)^2\\
\text{s.t. } & x\ge\beta,\\
& y=\alpha,
\end{aligned} \tag{4}$$

$$\begin{aligned}
\max_{x,y}\ & -\left(\sqrt{x}-\sqrt{y}\right)^2\\
\text{s.t. } & x=\beta,\\
& y\le\alpha.
\end{aligned} \tag{5}$$

Theorem 3.2. *Model* (2) *can be rewritten in the form of model* (3).

Let

$$u_r:=\frac{\overline{\mu}_r+\underline{\mu}_r}{2},\quad z_{3r}:=\frac{\rho_r\sqrt{y_{ro}}}{2},\quad z_{1r}:=\frac{\overline{\mu}_r-\underline{\mu}_r}{2},\quad z_{2r}:=u_r y_{ro}.$$

Then the fourth constraint of model (2) can be written as

$$z_{1r}z_{2r}-(z_{3r})^2\ge 0.$$

At optimality,

$$\delta_r=z_{1r}-\frac{1}{m+s},\quad \sum_{r=1}^{s}z_{2r}=\sum_{r=1}^{s}u_r y_{ro},$$

so model (2) simplifies to

$$\begin{aligned}
\max\ & \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} - \sum_{r=1}^{s} (z_{1r} + z_{2r} + 2z_{3r}) + \sigma + 1 \\
\text{s.t.}\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sigma \leq 0, && \forall j \\
& v_i x_{io} \geq \frac{1}{m+s}, && \forall i \\
& z_{2r} = u_r y_{ro}, && \forall r \\
& z_{1r} \geq \frac{1}{m+s}, && \forall r \\
& Z_r = \begin{pmatrix} z_{1r} & z_{3r} \\ z_{3r} & z_{2r} \end{pmatrix} \succeq 0, && \forall r.
\end{aligned} \tag{6}$$

Finally, at optimality $z_{3r} = -\sqrt{z_{1r} z_{2r}}$ (in the following, it is proved), yielding

$$\sum_r (z_{1r} + z_{2r} + 2z_{3r}) = \sum_r (\sqrt{z_{1r}} - \sqrt{z_{2r}})^2.$$

Using Lemma 3.1, the second and third constraints can be treated as equality and inequality, respectively, giving model (3). □

In the following, we present another form of dual RM model which is in the standard multiplier form without extra change of variables or any need to SDP formulation. To do so, in (1) let

$$\frac{1}{\varphi_r} \leq t_r, \qquad \forall r$$

then model (1) can be written as follows:

$$\begin{aligned}
\min\ & \frac{1}{m+s} \left(\sum_{i=1}^{m} \theta_i + \sum_{r=1}^{s} t_r \right) \\
\text{s.t.}\ & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_i x_{io}, && \forall i \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi_r y_{ro}, && \forall r \\
& \left\| \begin{bmatrix} 1 \\ \dfrac{\varphi_r - t_r}{2} \end{bmatrix} \right\| \leq \frac{\varphi_r + t_r}{2}, && \forall r \\
& \sum_{i=1}^{n} \lambda_j = 1, \\
& \theta_j \leq 1, && \forall i \\
& \varphi_r \geq 1, && \forall r \\
& \lambda \geq 0.
\end{aligned} \tag{7}$$

Following the conic duality [4], its dual becomes

$$\begin{aligned}
\max\ & \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} - \sum_{r=1}^{s} \rho_r - \sum_{r=1}^{s}\left(\frac{(\overline{\mu}_r + \underline{\mu}_r)}{2} + \frac{(\overline{\mu}_r - \underline{\mu}_r)}{2}\right) + \sigma + 1 \\
\text{s.t. } & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sigma \leq 0, && \forall j \\
& v_i x_{io} \geq \frac{1}{m+s}, && \forall i \\
& \frac{(\overline{\mu}_r - \underline{\mu}_r)}{2} = \frac{1}{m+s}, && \forall r \\
& \frac{\left(\overline{\mu}_r + \underline{\mu}_r\right)}{2} - u_r y_{ro} \leq 0, && \forall r \\
& {\underline{\mu}_r}^2 + {\rho_r}^2 \leq {\overline{\mu}_r}^2, && \forall r.
\end{aligned} \tag{8}$$

The proposed model has a similar structure to the multiplier model in Haličká and Trnovská [13] and therefore allows for an economic interpretation equivalent to that discussed in Haličká and Trnovská [13].

Theorem 3.3. *Model (8) can be rewritten in the form of model (3).*

Proof. Let $z_{3r} := \frac{\rho_r}{2}$, $z_{1r} := \dfrac{(\overline{\mu}_r - \underline{\mu}_r)}{2}$ and $z_{2r} := \dfrac{(\overline{\mu}_r + \underline{\mu}_r)}{2}$. Then the result follow. □

Corollary 3.4. *Based on Theorems 3.2 and 3.3, model (2) can be rewritten in the form of model (8).*

4. Finding an MRS

One of the issues with DEA models is the existence of multiple optimal solutions which causes the existence of multiple reference sets for some DMUs. The importance of this issue becomes more obvious when inefficient DMUs change the state from inefficiency to the efficiency frontier [20]. For the linear DEA model, in Sueyoshi and Sekitani [20] using SCSC between the primal and dual models, an approach is proposed to deal with the occurrence of multiple projections. However, for RM model, in Sekitani [19] the author first solved model (7) and using its optimal solution and removing conic constraint proposed a linear programming approach to find the unique MRS.

Here, we propose an approach that finds an MRS by solving just one model instead of two models in Sekitani [19]. The following results, that are easy to prove, are crucial for our goal.

Lemma 4.1. *Let $(\theta^*, \varphi^*, \lambda^*, t^*)$ be an optimal solution of model (7) for DMU_o. Then we have*

$$\sum_{j=1}^{n} \lambda_j^* x_{ij} = {\theta_i}^* x_{io}, \qquad \forall i \tag{9}$$

$$\frac{1}{\varphi_r^*} = t_r^*, \qquad \forall r \tag{10}$$

$$\sum_{j=1}^{n} \lambda_j^* y_{rj} = \varphi_r^* y_{ro}, \qquad \forall r. \tag{11}$$

Lemma 4.2. *Let $(u^*, v^*, \sigma^*, \underline{\mu}^* \rho^*, \overline{\mu}^*)$ be an optimal solution of model (8) for DMU_o. Then we have*

$$\rho_r^* = -\sqrt{{\overline{\mu}_r^*}^2 - {\underline{\mu}_r^*}^2} \qquad \forall r. \tag{12}$$

Proof. Model (8) can be rewritten as follows:

$$\max_{u,v,\overline{\mu},\underline{\mu},\sigma} \max_{\rho} \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} - \sum_{r=1}^{s} \left(\frac{(\overline{\mu}_r + \underline{\mu}_r)}{2} + \frac{(\overline{\mu}_r - \underline{\mu}_r)}{2} \right) + \sigma + 1 - \sum_{r=1}^{s} \rho_r \tag{13}$$

$$\text{s.t. constraints of model (8).} \tag{14}$$

The inner maximization happens when $\rho_r^* = -\sqrt{\overline{\mu}_r^{*2} - \underline{\mu}_r^{*2}} \quad \forall r.$ □

Corollary 4.3. *Let $(\theta^*, \varphi^*, \lambda^*, t^*)$ and $(u^*, v^*, \sigma^*, \underline{\mu}^*, \rho^*, \overline{\mu}^*)$ be optimal solutions of models (7) and (8) for DMU$_o$, respectively. Since in both models all the second-order cone constraints are active the strict complementarity holds condition, i.e.,*

$$\begin{pmatrix} \dfrac{\varphi_r^* + t_r^*}{2} \\ 1 \\ \dfrac{\varphi_r^* - t_r^*}{2} \end{pmatrix} + \begin{pmatrix} \overline{\mu}_r^* \\ \rho_r^* \\ \underline{\mu}_r^* \end{pmatrix} >_{L^3} 0, \tag{15}$$

where $L^n = \left\{ x \in \mathcal{R}^n \,\middle|\, \sqrt{\sum_{i=2}^{n} x_i^2} \leq x_1 \right\}$.

Proof. This proof follows from Lemmas 4.1 and 4.2, which show that both primal and dual conic constraints are active. Consequently, by Corollary 24 of [1], the strict complementarity holds condition. □

Definition 4.4. Let $(\theta^*, \varphi^*, \lambda^*, t^*)$ be an optimal solution of model (7) for DMU$_o$. The reference set for the DMU$_o$ is defined as follows

$$R^o = \left\{ \text{DMU}_j \mid \lambda_j^* > 0 \right\}, \tag{16}$$

where each DMU member of the set R^o is considered as the reference DMU$_o$.

Since the occurrence of multiple optimal values of the λ variable in model (7) is possible, DMU$_o$ can have multiple reference sets [19].

Definition 4.5. Let $(\theta^*, \varphi^*, \lambda^*, t^*)$ be an optimal solution of model (7) for DMU$_o$. The projection of (x_o, y_o) is

$$(\widehat{x}_o, \widehat{y}_o) = (\theta^* x_o, \varphi^* y_o) = \sum_{j \in J} \lambda_j^* (x_j, y_j) \tag{17}$$

where J is the set of indices j such that DMU$_j \in R^o$, i.e, the reference of DMU$_o$.

Note that in the model (7) for DMU$_o$, the optimal value of θ is always unique. However, due to the linearity of the input variables, multiple optimal values of θ may occur, which can lead to multiple projections for DMU$_o$ [19].

Definition 4.6. The union of all sets R^o for a DMU$_o$ is defined as the MRS for DMU$_o$.

In the following lemma, we prove strong duality [5] for models (7) and (8).

Lemma 4.7. *Both models (7) and (8) satisfy strict feasibility. Thus strong duality holds. i.e., both models are solvable with equal optimal objective function values.*

Proof. One can easily check that

$$\theta_i = 1, \quad \forall i, \ \varphi_r = 1, \quad \forall r, \ t_r = 2, \quad \forall r, \ \lambda_o = 1, \ \lambda_j = 0, \quad \forall j \neq o$$

is feasible for the linear constraint and strictly feasible for the conic constraints in model (7). Also,

$$v_i = \frac{1}{\overline{x}_i}\left(\frac{s}{m+s} + \frac{s}{m+s}\sum_{r=1}^{s}\sum_{j=1}^{n}\frac{y_{rj}}{y_{ro}}\right), \quad \forall i, \ \sigma = 0, u_r = \frac{1}{m+s}\frac{1}{y_{ro}}, \ \rho_r = \underline{\mu}_r = 0,$$
$$\overline{\mu}_r = \frac{2}{m+s}, \quad \forall r, \ \overline{x}_i = \min\{x_{i1}, \ldots, x_{in}\}, \quad \forall i,$$

is feasible for the linear constraint and strictly feasible for the conic constraints in model (8). Thus both models are solvable with equal optimal objective values [5]. □

According to Lemmas 4.1, 4.2 and Corollary 4.3, conic constraints are active in both model (7) and (8). Thus, SCSC between models (7) and (8) for a pair of optimal solutions is defined as follows:

$$\lambda_j + \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} - \sigma > 0, \qquad \forall j \tag{18}$$

$$1 - \theta_i + v_i x_{io} - \frac{1}{m+s} > 0, \qquad \forall i \tag{19}$$

$$\varphi_r + u_r y_{ro} - \frac{(\overline{\mu} + \underline{\mu}_r)}{2} - 1 > 0, \qquad \forall r \tag{20}$$

$$v_i - \sum_{j=1}^{n} \lambda_j x_{ij} + \theta_i x_{io} > 0, \qquad \forall i \tag{21}$$

$$u_r + \sum_{j=1}^{n} \lambda_j y_{rj} - \varphi_r y_{ro} > 0, \qquad \forall r. \tag{22}$$

Based on Lemma 4.1, constraints (21) and (22) are rewritten as follows:

$$v_i - \sum_{j=1}^{n} \lambda_j x_{ij} + \theta_i x_{io} = v_i > 0, \qquad \forall i \tag{23}$$

$$u_r + \sum_{j=1}^{n} \lambda_j y_{rj} - \varphi_r y_{ro} = u_r > 0, \qquad \forall r. \tag{24}$$

In the following, we prove that constraints (18)–(20) play a key role in dealing with multiple optimal solutions in model (7), and constraints (21) and (22) are redundant.

Lemma 4.8. *Let $(\lambda', \theta', \varphi', t')$ be an optimal solution of model (7) for DMU_o and $(v', u', \sigma', \underline{\mu}' \ \rho', \overline{\mu}')$ be an optimal solution of model (8) for DMU_o satisfying SCSC (18)–(20). Also, let $(\theta^*, \varphi^*, \lambda^*, t^*)$ and $(u^*, v^*, \sigma^*, \underline{\mu}^*, \rho^*, \overline{\mu}^*)$ be another optimal solutions of models (7) and (8) for DMU_o, respectively. Then we have*

$$\{j\,|\lambda_j^* > 0\} \subseteq \{j\,|\lambda_j' > 0\} = \left\{j \,\middle|\, \sum_{i=1}^{m} v_i' x_{ij} = \sum_{r=1}^{s} u_r' y_{rj} + \sigma'\right\} \subseteq \left\{j \,\middle|\, \sum_{i=1}^{m} v_i^* x_{ij} = \sum_{r=1}^{s} u_r^* y_{rj} + \sigma^*\right\} \tag{25}$$

and

$$\{i\,|\theta_i' = 1\} \subseteq \{i\,|\theta_i^* = 1\}. \tag{26}$$

Proof. First, we prove

$$\left\{j\left|\lambda_j' > 0\right.\right\} = \left\{j\left|\sum_{i=1}^{m} v_i' x_{ij} = \sum_{r=1}^{s} u_r' y_{rj} + \sigma'\right.\right\}. \tag{27}$$

By the complementary slackness theorem [3], we have

$$\lambda_j' \left(\sum_{i=1}^{m} v_i' x_{ij} - \sum_{r=1}^{s} u_r' y_{rj} - \sigma'\right) = 0, \qquad \forall j. \tag{28}$$

According to (18), the following cases can occur:

$$\lambda_j' > 0 \Leftrightarrow \sum_{r=1}^{s} u_r' y_{rj} - \sum_{i=1}^{m} v_i' x_{ij} + \sigma' = 0, \qquad \forall j, \tag{29}$$

$$\sum_{i=1}^{m} v_i' x_{ij} - \sum_{r=1}^{s} u_r' y_{rj} - \sigma' > 0 \Leftrightarrow \lambda_j' = 0, \qquad \forall j. \tag{30}$$

Now, for any $j \in \left\{j\left|\lambda_j' > 0\right.\right\}$ by (29), we have

$$\sum_{r=1}^{s} u_r' y_{rj} - \sum_{i=1}^{m} v_i' x_{ij} + \sigma' = 0. \tag{31}$$

This implies that

$$\left\{j\left|\lambda_j' > 0\right.\right\} \subseteq \left\{j\left|\sum_{i=1}^{m} v_i' x_{ij} = \sum_{r=1}^{s} u_r' y_{rj} + \sigma'\right.\right\}. \tag{32}$$

Similarly, for any $j \in \left\{j\left|\sum_{i=1}^{m} v_i' x_{ij} = \sum_{r=1}^{s} u_r' y_{rj} + \sigma'\right.\right\}$, by (29) we have

$$\left\{j\left|\sum_{i=1}^{m} v_i' x_{ij} = \sum_{r=1}^{s} u_r' y_{rj} + \sigma'\right.\right\} \subseteq \left\{j\left|\lambda_j' > 0\right.\right\}. \tag{33}$$

Thus (27) is proved. Since $(\lambda^*, \theta^*, \varphi^*, t^*)$ and $(v', u', \sigma', \underline{\mu}'\ \rho', \overline{\mu}')$ are optimal solutions for models (7) and (8), respectively. By the complement slackness theorem, we have

$$\lambda_j^* \left(\sum_{i=1}^{m} v_i' x_{ij} - \sum_{r=1}^{s} u_r' y_{rj} - \sigma'\right) = 0, \qquad \forall j. \tag{34}$$

According to the fact that any optimal solution from model (7) and $(v', u', \sigma', \underline{\mu}'\ \rho', \overline{\mu}')$ do not necessarily satisfy SCSC constraints (18)–(20), the following cases can occur:

$$\lambda_j^* > 0 \Rightarrow \sum_{r=1}^{s} u_r' y_{rj} - \sum_{i=1}^{m} v_i' x_{ij} + \sigma' = 0, \qquad \forall j \tag{35}$$

$$\sum_{i=1}^{m} v_i' x_{ij} - \sum_{r=1}^{s} u_r' y_{rj} - \sigma' > 0 \Rightarrow \lambda_j^* = 0, \qquad \forall j \tag{36}$$

$$\sum_{i=1}^{m} v_i' x_{ij} - \sum_{r=1}^{s} u_r' y_{rj} - \sigma' = 0 \text{ and } \lambda_j^* = 0, \qquad \forall j. \tag{37}$$

It follows from (29) and (37) that

$$\forall j \in \left\{ j \left| \sum_{i=1}^{m} v'_i x_{ij} = \sum_{r=1}^{s} u'_r y_{rj} + \sigma' \right. \right\} \Rightarrow \lambda'_j > 0, \text{ also } \left(\lambda^*_j > 0 \text{ or } \lambda^*_j = 0\right). \tag{38}$$

This show that, $\left\{ j \left| \lambda^*_j > 0 \right. \right\} \subseteq \left\{ j \left| \lambda'_j > 0 \right. \right\}$. Thus $\left\{ j \left| \sum_{i=1}^{m} v'_i x_{ij} = \sum_{r=1}^{s} u'_r y_{rj} + \sigma' \right. \right\} \subseteq \left\{ j \left| \sum_{i=1}^{m} v^*_i x_{ij} = \sum_{r=1}^{s} u^*_r y_{rj} + \sigma^* \right. \right\}$. For the next part, by the complementary slackness theorem, we have

$$(\theta'_i - 1)\left(v'_i x_{io} - \frac{1}{m+s}\right) = 0, \qquad \forall i. \tag{39}$$

According to (18), the following cases can be considered:

$$\theta'_i < 1 \Leftrightarrow v'_i x_{io} - \frac{1}{m+s} = 0 \qquad \forall i \tag{40}$$

$$v'_i x_{io} > \frac{1}{m+s} \Leftrightarrow \theta'_i - 1 = 0 \qquad \forall i. \tag{41}$$

It follows From (41) that

$$\{ i \left| \theta'_i = 1 \right. \} = \left\{ i \left| v'_i x_{io} > \frac{1}{m+s} \right. \right\}. \tag{42}$$

As previously mentioned, every optimal solution of model (7) and $(v', u', \sigma', \underline{\mu}'\ \rho', \overline{\mu}')$ does not necessarily satisfy the SCSC conditions (18)–(20). Therefore, the following can occur:

$$\theta^*_i < 1 \Rightarrow v'_i x_{io} - \frac{1}{m+s} = 0 \qquad \forall i \tag{43}$$

$$v'_i x_{io} > \frac{1}{m+s} \Rightarrow \theta^*_i - 1 = 0 \qquad \forall i \tag{44}$$

$$\theta^*_i = 1 \text{ and } v'_i x_{io} - \frac{1}{m+s} = 0 \qquad \forall i. \tag{45}$$

From (41) and (45), we have:

$$\forall i \in \left\{ i \left| v'_i x_{io} = \frac{1}{m+s} \right. \right\} \Rightarrow \theta'_i < 1 \text{ also } (\theta^*_i = 1 \text{ or } \theta^*_i < 1). \tag{46}$$

This leads to the conclusion that $\left\{ i \left| \theta'_i = 1 \right. \right\} \subseteq \left\{ i \left| \theta^*_i = 1 \right. \right\}$. □

Corollary 4.9. *Let $(\lambda', \theta', \varphi', t')$ and $(v', u', \sigma', \underline{\mu}'\ \rho', \overline{\mu}')$ be optimal solutions to models (7) and (8) for DMU$_o$, respectively, satisfying SCSC constraints (18)–(20). Also, let $\{j|\lambda'_j > 0\}$ be the reference set obtained by the optimal solution $(\lambda', \theta', \varphi', t')$. Then, the set $J' = \{j|\lambda'_j > 0\}$ is the unique MRS. Furthermore, the set $I' = \left\{ i \left| \theta'_i = 1 \right. \right\}$ is the unique minimal set of input indices.*

Proof. By on Lemma 4.8, for any optimal solution $(\lambda^*, \theta^*, \varphi^*, t^*)$ of model (7), the following holds:

$$\left\{ j \left| \lambda^*_j > 0 \right. \right\} \subseteq J'. \tag{47}$$

This implies that J' contains every possible reference set of model (7), which indicates that the union of all reference sets is a subset of J'. Thus, J' is the MRS for DMU$_o$.

Now we prove that the set J' is unique. Suppose this is not true. Then, there exists another MRS $\widehat{J} = \{j|\widehat{\lambda}_j > 0\}$ for DMU$_o$ such that $J \neq J'$ and $\left|\widehat{J}\right| = |J'|$. Consequently, there must exist an element $\widehat{j} \in \widehat{J}$ with $\lambda_{\widehat{j}} > 0$ but $\widehat{j} \notin J'$. This means there exists a reference set identified from model (7) which is not a subset of J', which contradicts the maximality of J'. Similarly, it can be shown that the set I' is the unique minimal set of input indices. □

TABLE 1. Data for 5 DMUs.

DMU	A	B	C	D	E
x_1	2	4	5	1	3
x_2	2	4	1	5	3
x_3	1	1	1	1	1
y_1	4	8	6	6	6
y_2	1	1	1	1	1

TABLE 2. Model (7) results.

DMU	Efficiency	θ_1^*	θ_2^*	θ_3^*	φ_1^*	φ_2^*	λ_A^*	λ_B^*	λ_C^*	λ_D^*	λ_E^*
A	1	1	1	1	1	1	1	0	0	0	0
B	1	1	1	1	1	1	0	1	0	0	0
C	1	1	1	1	1	1	0	0	1	0	0
D	1	1	1	1	1	1	0	0	0	1	0
E	1	1	1	1	1	1	0.2059	0.2059	0.19961	0.19961	0.1961

So, to deal with the multiple reference sets of models (7) and (8), it is sufficient to solve the following model:

$$
\begin{aligned}
&\max \nu \\
&\text{ s.t. constraints of (7) and (8)} \\
&\quad \frac{1}{m+s}\left(\sum_{i=1}^{m}\theta_i+\sum_{r=1}^{s}t_r\right) \\
&\quad = -\sum_{r=1}^{s}\rho_r-\sum_{i=1}^{m}v_i x_{io}+\sum_{r=1}^{s}\delta_r+\sigma+\frac{m}{m+s} \\
&\quad \lambda_j+\sum_{i=1}^{m}v_i x_{ij}-\sum_{r=1}^{s}u_r y_{rj}-\sigma\geq\nu, \qquad \forall j \\
&\quad 1-\theta_i+v_i x_{io}-\frac{1}{m+s}\geq\nu, \qquad \forall i \\
&\quad \varphi_r+u_r y_{ro}-\frac{(\overline{\mu}+\underline{\mu}_r)}{2}-1\geq\nu, \qquad \forall r
\end{aligned}
\tag{48}
$$

According to Lemma 4.7, model (48) is always feasible.

5. NUMERICAL EXAMPLES

In this section, we evaluate our proposed model using an example taken from Sekitani [19] and a real data taken from Lotfi *et al.* [16] to demonstrate its practical application. To solve all models, we used SEDUMI solver of CVX version 2.1 software [10]. All computations are performed in MATLAB 2022a on a 2.20 GHz laptop with 12 GB RAM.

Example 5.1. Consider the data of 5 DMUs with 3 inputs and 2 outputs taken from Sekitani [19] as shown in Table 1.

TABLE 3. Model (48) results.

DMU	Efficiency	θ_1^*	θ_2^*	θ_3^*	φ_1^*	φ_2^*	ν^*	λ_A	λ_B	λ_C	λ_D	λ_E
A	1	1	1	1	1	1	1	1	0	0	0	0
B	1	1	1	1	1	1	1	0	1	0	0	0
C	1	1	1	1	1	1	1	0	0	1	0	0
D	1	1	1	1	1	1	1	0	0	0	1	0
E	1	1	1	1	1	1	0.2	0.2	0.2	0.2	0.2	0.2

TABLE 4. Data of 20 commercial bank branches.

DMU	x_1	x_2	x_3	y_1	y_2	y_3	y_4	y_5
1	5007.37	36.29	87 243	2 696 995	263 643	1 675 519	108634.8	965.97
2	2926.81	18.8	9945	340 377	95 978	377 309	32396.65	304.67
3	8732.7	25.74	47 575	1 027 546	37 911	1 233 548	96842.33	2285.03
4	945.93	20.81	19 292	1 145 235	229 646	468 520	32362.8	207.98
5	8487.07	14.16	3428	390 902	4929	129 751	12662.71	63.32
6	13759.35	19.46	13 929	988 115	74 133	507 502	53591.3	480.16
7	587.69	27.29	27 827	144 906	180 530	288 513	40507.97	176.58
8	4646.39	24.52	9070	408 163	405 396	1 044 221	56260.09	4654.71
9	1554.29	20.47	412 036	335 070	337 971	1 584 722	176436.8	560.26
10	17528.31	14.84	8638	700 842	14 378	2 290 745	662725.2	58.89
11	2444.34	20.42	500	641 680	114 183	1 579 961	17527.58	1070.81
12	7303.27	22.87	16 148	453 170	27 196	245 726	35757.83	375.07
13	9852.15	18.47	17 163	553 167	21 298	425 886	45652.24	438.43
14	4540.75	22.83	17 919	309 670	20 168	124 188	8143.79	936.62
15	3039.58	39.32	51 582	286 149	149 183	787 959	106798.6	1203.79
16	6585.81	25.57	20 975	321 435	66 169	360 880	89971.47	200.36
17	4209.18	27.59	41 960	618 105	244 250	9 136 507	33036.79	2781.24
18	1015.52	13.63	18 641	248 125	3063	26 687	9525.6	240.04
19	5800.38	27.12	19 500	640 890	490 508	2 946 797	66097.16	961.56
20	1445.65	28.96	31 700	119 948	14 943	297 674	21991.53	282.73

The results of solving model (7) are summarized in Table 2. It can be seen that all DMUs are efficient, and the reference set for all DMUs is themselves, except for DMU$_E$. For DMU$_E$, $\lambda_A^* = 0, \lambda_B^* = 0, \lambda_C^* = 0, \lambda_D^* = 0, \lambda_E^* = 1$ is the optimal value for λ. However, because of the convex nature of model (7), any convex combination of this optimal value and the optimal value λ in Table 2, can be an optimal value for λ. Thus, there exists an infinite number of optimal solutions for DMU$_E$ in model (7), indicating the existence of multiple reference sets for DMU$_E$. For example, $\{E\}$, $\{A, B\}$, $\{A, B, E\}$, $\{A, B, C, D\}$ and $\{A, B, C, D, E\}$ are multiple reference sets for DMU$_E$. According to these results, it can be said that model (7) can have multiple optimal solutions for some evaluated DMUs. This means some DMUs have multiple projections. In the following, using model (48), we obtain the MRS for DMUs, as shown in Table 3.

The results in Table 3 show that the reference set of all DMUs except DMU$_E$ is the same as the reference set obtained in Table 2. Therefore, we can conclude that all DMUs except DMU$_E$ have MRSs, with DMU$_E$ having a reference set of $\{A, B, C, D, E\}$. In addition, we observe that for DMU$_E$, the reference sets obtained by model (7) are all subsets of the reference set obtained by model (48). Note that the unique MRS obtained in Table 3 and the approach in Sekitani [19] are the same for all DMUs.

TABLE 5. Model (7) results.

DMU	Efficiency score	θ_1^*	θ_2^*	θ_3^*	φ_1^*	φ_2^*	φ_3^*	φ_4^*	φ_5^*	t_1^*	t_2^*	t_3^*	t_4^*	t_5^*	Reference set
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_1=1
2	0.6794	1	1	0.6147	1.6775	1.0493	3.186	1.5846	3.0633	0.5961	0.953	0.3139	0.6311	0.3264	λ_4=0.0446, λ_8=0.0384, λ_{10}=0.0518, λ_{11}=0.6433, λ_{18}=0.2218
3	0.6884	0.8319	1	0.6750	1	7.0726	2.1942	1.9350	1.1274	1	0.1414	0.4557	0.5168	0.8870	λ_1=0.2299 λ_4=0.0016, λ_8=0.4097, λ_{10}=0.2025, λ_{17}=0.1564
4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_4=1
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_5=1
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_6=1
7	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_7=1
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_8=1
9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_9=1
10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_{10}=1
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_{11}=1
12	0.4213	0.6024	0.8845	0.1459	1.375	4.942	6.5225	2.6609	3.584	0.7273	0.2023	0.1533	0.3758	0.279	λ_8=0.1084, λ_{10}=0.1138, λ_{11}=0.7778
13	0.5081	0.7803	1	0.2224	1.1227	4.4524	3.8253	4.7028	2.1159	0.8907	0.2246	0.2614	0.2126	0.4726	λ_5=0.0882, λ_8=0.0701, λ_{10}=0.3020, λ_{11}=0.5396
14	0.4028	0.7045	0.9307	0.1434	1.9055	8.8299	11.8485	4.5808	1.9868	0.5248	0.1133	0.0844	0.2183	0.5033	λ_8=0.2253, λ_{10}=0.0171, λ_{11}=0.7576
15	0.8925	1	0.628	1	1.1513	1.5542	1	1	1	0.8686	0.6434	1	1	1	λ_4=0.0616 λ_7=0.5487, λ_8=0.2212, λ_9=0.078, λ_{10}=0.0848, λ_{17}=0.0057
16	0.4701	0.8778	0.7921	0.1862	1.8732	2.3966	4.4285	1.647	8.2760	0.5338	0.4173	0.2258	0.6072	0.1208	λ_8=0.2174, λ_{10}=0.1895, λ_{11}=0.5931,
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_{17}=1
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_{18}=1
19	1	1	1	1	1	1	1	1	1	1	1	1	1	1	λ_{19}=1
20	0.4824	1	0.7432	0.6256	8.7800	16.3875	3.4951	1.5622	2.5708	0.1139	0.061	0.2861	0.6401	0.389	λ_4=0.8578, λ_8=0.0816, λ_{17}=0.0606

Example 5.2. The data for this example, which is presented in Table 4, includes information of 20 commercial bank branches in Iran [16]. Each branch includes three inputs: Payable interest (x_1), Personnel (x_2), Non-performing loans (x_3) and 5 outputs: The total some of four main deposits (y_1), Other deposits (y_2), Loans granted (y_3), Received interest (y_4) and Fee (y_5).

Table 5 presents the results obtained using model (7). The second column shows the efficiency scores of DMUs. Optimal values of θ, φ, and t are reported in columns 3–15, respectively. The last column contains the reference set for each DMU. As can be seen, 12 DMUs are efficient and 8 DMUs are inefficient. DMU_{14} with an efficiency value of 0.4028 has the lowest efficiency compared to the other DMUs. It can be seen that, based on the obtained reference set, inefficient DMUs refer only to efficient DMUs to improve efficiency. For example, DMU_5 has an overall efficiency of 0.7058, and its reference set is $\{4, 11, 18\}$. Note that efficient DMUs have the least references in their reference sets, with only one element, while DMU_{15} has the highest number of references, with six elements. The results of model (48) and the stage 2 of the approach in Sekitani [19] are given

TABLE 6. MRS.

DMU	Model (48) results											Stage 2 of proposed approach in Sekitani [19]				
	Efficiency score	θ_1^*	θ_2^*	θ_3^*	φ_1^*	φ_2^*	φ_3^*	φ_4^*	φ_5^*	ν^*	Reference set	θ_1^*	θ_2^*	θ_3^*	Reference set	η^*
1	1	1	1	1	1	1	1	1	1	1	λ_1=1	1	1	1	λ_1=1	1
2	0.6794	1	1	0.6145	1.677	1.0494	3.186	1.5842	3.0644	0.0121	λ_4=0.0446, λ_8=0.0384, λ_{10}=0.0518, λ_{11}=0.6433, λ_{18}=0.2218	1	1	0.6147	λ_4=0.0446, λ_8=0.0384, λ_{10}=0.0518, λ_{11}=0.6433, λ_{18}=0.2218	0.0384
3	0.6884	0.8265	1	0.6751	1	7.0788	2.1977	1.9176	1.126	0.0021	λ_1=0.2299 λ_4=0.0015, λ_8=0.4096, λ_{10}=0.2025, λ_{17}=0.1565	0.8317	1.0000	0.6751	λ_1=0.2299 λ_4=0.0015, λ_8=0.4096, λ_{10}=0.2025, λ_{17}=0.1565	0.0015
4	1	1	1	1	1	1	1	1	1	1	λ_4=1	1	1	1	λ_4=1	1
5	1	1	1	1	1	1	1	1	1	1	λ_5=1	1	1	1	λ_5=1	1
6	1	1	1	1	1	1	1	1	1	1	λ_6=1	1	1	1	λ_6=1	1
7	1	1	1	1	1	1	1	1	1	1	λ_7=1	1	1	1	λ_7=1	1
8	1	1	1	1	1	1	1	1	1	1	λ_8=1	1	1	1	λ_8=1	1
9	1	1	1	1	1	1	1	1	1	1	λ_9=1	1	1	1	λ_9=1	1
10	1	1	1	1	1	1	1	1	1	1	λ_{10}=1	1	1	1	λ_{10}=1	1
11	1	1	1	1	1	1	1	1	1	1	λ_{11}=1	1	1	1	λ_{11}=1	1
12	0.4213	0.6024	0.8845	0.1459	1.3749	4.9417	6.5224	2.6608	3.5841	0.0553	λ_8=0.1084, ‘λ_{10}=0.1138, λ_{11}=0.7778	0.6024	0.8845	0.1458	λ_8=0.1084, ‘λ_{10}=0.1138, λ_{11}=0.7778	0.1096
13	0.5081	0.7803	1	0.2225	1.1228	4.451	3.8266	4.7039	2.116	0.0106	λ_5=0.0882, λ_8=0.0701, λ_{10}=0.3021, λ_{11}=0.5396	0.7803	1.0000	0.2224	λ_5=0.0882, λ_8=0.0701, λ_{10}=0.3021, λ_{11}=0.5396	0.071
14	0.4028	0.7046	0.9307	0.1435	1.9055	8.8286	11.8484	4.5849	1.9868	0.0172	λ_8=0.2253, λ_{10}=0.0171, λ_{11}=0.7576	0.7045	0.9307	0.1434	λ_8=0.2253, λ_{10}=0.0171, λ_{11}=0.7576	0.0173
15	0.8925	1	0.628	1	1.1513	1.5541	1	1	1	0.0057	λ_4=0.0616 λ_7=0.5487, λ_8=0.2212, λ_9=0.078, λ_{10}=0.0848, λ_{17}=0.0057	1	0.628	1	λ_4=0.0616 λ_7=0.5487, λ_8=0.2212, λ_9=0.078, λ_{10}=0.0848, λ_{17}=0.0057	0.0057
16	0.4701	0.8778	0.7921	0.1862	1.8732	2.3967	4.4282	1.647	8.2762	0.0119	λ_8=0.2174, λ_{10}=0.1895, λ_{11}=0.5931,	0.8778	0.7921	0.1862	λ_8=0.2174, λ_{10}=0.1895, λ_{11}=0.5931,	0.1222
17	1	1	1	1	1	1	1	1	1	1	λ_{17}=1	1	1	1	λ_{17}=1	1
18	1	1	1	1	1	1	1	1	1	1	λ_{18}=1	1	1	1	λ_{18}=1	1
19	1	1	1	1	1	1	1	1	1	1	λ_{19}=1	1	1	1	λ_{19}=1	1
20	0.4824	1	0.7432	0.6256	8.7799	16.3868	3.4953	1.5622	2.5708	0.0013	λ_4=0.8578, λ_8=0.0816, λ_{17}=0.0606	1	0.7432	0.6256	λ_4=0.8578, λ_8=0.0816, λ_{17}=0.0606	0.0.0280

TABLE 7. Time comparison of model (48) with the approach in Sekitani [19].

DMU	Model (48)	Approach in Sekitani [19]
1	2.6210	3.1949
2	1.4953	2.0595
3	1.4171	2.1702
4	1.1732	1.7645
5	1.6128	1.8495
6	1.1740	2.013
7	1.2431	1.8652
8	1.1223	1.6989
9	1.3117	1.8792
10	1.1466	1.8524
11	1.0961	1.7902
12	1.2740	1.6626
13	1.0843	1.8720
14	1.2591	1.7988
15	1.1928	1.7574
16	1.2843	1.8987
17	1.1369	1.8298
18	1.1567	1.7406
19	1.0266	1.8392
20	1.2687	1.8194
Average	1.3048	1.9178
Total	26.0966	38.3560

in Table 6. Columns 2–10 respectively show the efficiency scores of DMUs along with the optimal values of θ^*, φ^*, and μ^*, calculated by model (48). In addition, the unique MRS obtained from model (48) is also included in the eleventh column. Columns 12–15 present, respectively, the optimal values of θ, the unique MRS, and the optimal value of η calculated by the stage 2 of the approach in Sekitani [19]. The efficiency for all DMUs in Table 5 and model (48) in Table 6 show that model (48) is a reliable tool to evaluate the efficiency of DMUs. The optimal values of θ obtained by both model (48) and stage 2 of the approach in Sekitani [19] are identical to the optimal values of θ presented in Table 5. According to the feature of having multiple optimal solutions in model (7), based on the optimal value of θ obtained in Tables 5 and 6, it can be concluded that all DMUs have unique θ. Hence, model (48) effectively prevents the occurrence of multiple projections.

As can be seen, the unique MRS and projection for each DMU is identical in both models (48) and the approach in Sekitani [19]. This implies that, model (48) is sufficient to determine the unique MRS for each DMU.

Table 7 presents a time performance comparison (in seconds) of model (48) and the approach in Sekitani [19]. As observed, model (48) is faster than the approach in Sekitani [19]. As demonstrated by the results, model (48) has all the features of the approach in Sekitani [19], it effectively evaluates DMU efficiency and identifies the unique MRS.

6. CONCLUSION

All multiplier models in DEA have two common features, first the definition of vectors u and v which are interpreted as shadow prices and second the virtual profit (x_o, y_o) which is defined as $\sum_{r=1}^{s} y_{ro}u_r - \sum_{i=1}^{m} x_{io}v_i$ is maximized under the condition $\sum_{r=1}^{s} y_{rj}u_r - \sum_{i=1}^{m} x_{ij}v_i \leq 0 \quad$ for all $j = 1, \ldots, n$.

In Haličká and Trnovská [13], the authors derived a dual RM model that, unlike the existing model in Sueyoshi and Sekitani [21], for the first time allowed economic interpretations of the derived dual RM model.

In this paper, we presented a new dual RM model based on the SOCP form that is similar in structure and economic interpretations to the dual model of Haličká and Trnovská [13]. Furthermore, we showed that the dual model of Sueyoshi and Sekitani [21] can achieve similar structure and economic interpretations to these two models with some modifications. Furthermore, by introducing an effective method for determining unique MRS and projection, this research took an effective step in the development of non-radial DEA models and paved the way for more practical applications and deeper theoretical analyses in the future. One may consider extending the proposed approach to the network DEA models by considering undesirable inputs and outputs.

Conflicts of interest

The authors of this paper declare that they have no conflicts of interest.

Data availability statement

The data used in this paper are provided in the text.

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