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A GAME THEORETIC APPROACH WITH PRICING AND COOPERATIVE ADVERTISEMENT AND WARRANTY PERIOD IN A TWO-LEVEL SUPPLY CHAIN

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Abstract. Cooperative advertisement is one of the market strategies in which the manufacturer pays for a part of the retailer's local advertisements. Offering warranty period to customers is another market strategy for improving company market share. This paper develops a two level supply chain with one manufacturer and one retailer in which warranty period costs such as local advertisements costs are shared among supply chain members. It means that, both a cooperative advertisement and a cooperative warranty period are considered at the same time. So, there are two participation rates in the two-level supply chain model. But unlike the previous research works, this paper investigates the effect of these two participation rates on other important variables and display the relationship among these rates and the variables trend. This is important for decision makers, because investigating about these trends improve their insights about cooperative relation in supply chain. The cooperative two-level supply chain model considers a linear demand function and a nonlinear products return rate. A game theoretic approach is used to study the cooperative two-level supply chain model. Basically, three games including Nash, manufacturer Stackelberg and retailer Stackelberg are applied. The main purpose of this paper is to improve the insights of decision makers and help them to make appropriate decisions when they incorporate cooperative advertisement and a cooperative warranty period.

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1. Introduction

In today's competitive marketplace, supply chain members increasingly rely on collaborative strategies to enhance their market position and operational efficiency. Two prominent cooperative strategies have emerged as particularly significant: cooperative advertising and cooperative warranty programs. Cooperative advertising creates synergies among supply chain members, with many companies implementing this strategy in their

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distribution channels due to the economic benefits it provides for all participants [5]. Simultaneously, warranty provision has become a critical tool for attracting customers and building brand trust, with the warranty period serving as a strategic decision variable that directly impacts product return rates and customer satisfaction.

While these cooperative strategies have been studied extensively in isolation, there exists a notable gap in understanding how they interact when implemented simultaneously within a supply chain context. Most existing research focuses on either cooperative advertising or warranty decisions independently, without considering their joint optimization and mutual effects on supply chain performance.

This research addresses this gap by developing a comprehensive model that integrates both cooperative advertising and cooperative warranty decisions in a two-level supply chain consisting of one manufacturer and one retailer. Unlike previous studies that focus primarily on profit maximization for individual supply chain members, this work examines how cooperative actions impact various model variables and explores the trade-offs between different levels of cooperative participation in advertising *vs.* warranty programs.

The key innovations of this study compared to prior research include: (1) the joint consideration of cooperative advertising and cooperative warranty decisions within a single framework, (2) the development of a shared warranty cost structure between manufacturer and retailer, (3) the analysis of how different levels of cooperative participation rates affect pricing strategies and other decision variables, and (4) the incorporation of product quality as a determinant of return rates in the cooperative warranty model.

This study addresses the following core research questions:

(1) How do cooperative advertising and cooperative warranty decisions interact in a two-level supply chain?
(2) What are the optimal strategies when supply chain members can choose different levels of participation in cooperative advertising *vs.* cooperative warranty programs?
(3) How do varying cooperative participation rates affect pricing strategies, product quality decisions, and overall supply chain performance?
(4) What is the impact of product quality on warranty costs and return rates in a cooperative warranty framework?

The rest of this research work is structured as follows. Section 2 provides a literature review. Section 3 provides the notation, assumptions and the model description. Section 4 solves the model with three games: Manufacturer Stackelberg (MS), Retailer Stackelberg (RS) and Nash (N) games. Section 5 presents some numerical examples and a sensitivity analysis. Finally, Section 6 gives some conclusions.

2. Literature review

To provide a structured analysis, the literature review is organized into two main categories: (1) cooperative advertising and pricing strategies, and (2) warranty and quality considerations. This categorization allows for a focused discussion of key themes relevant to the research.

2.1. Cooperative advertising and pricing

The cooperative advertising literature has established five different meanings for cooperative advertising [2], with vertical cooperative advertising being the most prevalent in both practice and research. Bergen and John [4] and Zhang and Xie [47] provide foundational work on vertical cooperative advertising models. Recent comprehensive reviews by categorize the various model types and their underlying assumptions. The primary focus of cooperative advertising research has been profit maximization for supply chain members through coordinated advertising programs. Huang and Li [14] established early frameworks for profit optimization in cooperative advertising, while subsequent studies by Yue *et al.* [45], Yan [41], Yang *et al.* [42], Zhou *et al.* [48], Farshbaf-Geranmayeh *et al.* [12], Karray *et al.* [21] and Wu *et al.* [39] have extended these models to various supply chain configurations and market conditions. Demand function specification plays a crucial role in cooperative advertising models. Linear demand functions have been predominantly used in the literature, with notable implementations in Marchi and Cohen [25], Wang [37], Tsao and Sheen [36], Karray and Zaccour [19] and

Kennedy *et al.* [22]. The integration of pricing decisions with advertising has been addressed by Yue *et al.* [45], Xie and Neyret [40], Karray *et al.* [20], and Farshbaf-Geranmayeh *et al.* [12], Taleizadeh *et al.* [29, 30, 35] and Asghari *et al.* [1] demonstrating the interconnected nature of these decision variables.

Recently Zaheri and Rafiee [46] model a two-level supply chain (manufacturer & retailer). They simultaneously consider cooperative advertising (shared costs of local and national ads), price, and also incorporate *inventory costs* (holding, shortage, ordering). They compare cooperative *vs.* non-cooperative structures (Stackelberg, Nash, etc.). Their findings include how inventory considerations modify optimal advertising and pricing decisions, and under what conditions the cooperative relationship yields greater combined profit. Du *et al.* [10] build a game-theoretic model of cooperative advertising in a manufacturer–retailer supply chain that incorporates *fairness concerns* of the retailer (*via* Nash bargaining fairness). They find that under certain parameters (fairness concern coefficients), retailer's advertising investment may exceed what would occur under centralized/coordinated control; but there is a trade-off: retailer's profit shows an inverted U-shape as fairness concern increases; manufacturer's profit tends to decline after a point. They also show that a two-way subsidy mechanism can still lead to coordination even when retailer has fairness concerns. Wu and Liu [38] consider a shared manufacturing platform (one manufacturer + a platform to lease idle capacity) and model the joint decisions of pricing and advertising investment. Using differential game methods, they derive optimal strategies under cooperative advertising, show how advertising efficiency and platform characteristics affect the optimal price, participation/subsidy (advertising support) etc. Liang *et al.* [24] study a manufacturer–retailer supply chain where there is a network good (so demand has network externalities). Their model has the manufacturer set wholesale price and advertising cost share, retailer sets retail price and advertising level. They examine how the order of decision-making (price first *vs.* advertising first *vs.* simultaneous) affects outcomes. They find that stronger network externality leads to higher prices and more advertising; also decision order matters: *e.g.* manufacturer prefers price before advertising, retailer prefers simultaneous decisions. Esmaeili and Rasi-Barzoki [11] analyze a model where in addition to pricing and advertising, there is a *service* element which affects customer loyalty. They compare Nash *vs.* Stackelberg (retailer leader) *vs.* cooperative game outcomes. They show that cooperative formulation gives highest total system profit; retailer-led Stackelberg gives higher profits than Nash for individual members; pricing, advertising, and service levels depend on game structure.

2.2. Warranty and quality

Warranty research has evolved from basic warranty provision models to more sophisticated approaches that consider quality interactions. Early work by Chen *et al.* [7], Dimitrov *et al.* [9], and Huang *et al.* [15] established fundamental warranty decision frameworks. Yeh *et al.* [44] and Huang *et al.* [16] extended these models to consider supply chain contexts. The relationship between product quality and warranty decisions has been recognized as critical. In the other hand Taleizadeh *et al.* [31–34] and Cárdenas-Barron *et al.* [6] extensively focused on quality issues of items. Zhu *et al.* [50] developed deterministic models where both buyers and suppliers incur quality-related costs, creating incentives for quality improvement investments. Dai *et al.* [8] specifically highlighted how quality levels significantly influence warranty period decisions and associated costs. From a strategic marketing perspective, extended warranties serve as signals of product quality and reliability. Jack and Murthy [17], Maronick [26], Hartman and Laksana [13], Huang *et al.* [16], Keshavarz-Ghorbani *et al.* [23] and Zhou *et al.* [49] have demonstrated that warranty provision functions as an effective marketing tactic to improve product competitiveness, market share, and brand positioning. Qi *et al.* [27] developed a bi-objective optimization model for low-carbon product family design where the decision variables include component reliability, supplier selection, and component warranty period. They show that treating warranty period of critical components as a design variable (rather than fixed) allows manufacturers to balance profit against greenhouse gas emissions. The model is solved with a genetic algorithm, and sensitivity analyses reveal that allowing differentiated warranty lengths across components and product variants increases profit and/or lowers emissions compared to forcing uniform warranty periods. Yang *et al.* [43] examines a Stackelberg game between a key-component manufacturer and an external end-product manufacturer. They allow both the warranty period for the key component and for the rest of the product to be decision variables. They show that a manufacturer with higher brand reputation tends to offer longer warranty periods. However, sometimes even a high-reputation manufacturer may choose a shorter warranty for certain components (or the end product) if warranty servicing cost is too high. Their model highlights how component type, brand reputation, and cost trade-offs determine optimal warranty periods for different parts of the product. Jalapathy and Mubashir-Unnissa [18] integrates extended warranty into a closed-loop supply chain with remanufacturing. The authors consider a non-renewing extended warranty offered by the manufacturer, and model how inventory and remanufacturing decisions interact with warranty promises. Their findings include that offering a longer extended warranty can increase demand and improve customer satisfaction, but also increases costs (both from warranty claims and from needing higher quality/remanufacturable inventory). Optimal warranty length is derived from balancing these effects in their two-phase news-vendor framework. Shang *et al.* [28] introduced models of renewable warranties and post-warranty replacement for self-announcing failure products (*i.e.* products that signal their own failure). They propose several warranty strategies, including limiting the number of failures, mission cycles, or using two-dimensional constraints ("whichever occurs first/last" between usage and time) to define warranty coverage. They find that appropriate thresholds (*e.g.* failure count, mission cycles) can reduce warranty servicing cost and extend effective warranty coverage without necessarily increasing the nominal warranty period. Thus this work ties the warranty period (and its shape) with usage/failure constraints. Becker and Schölzel [3] compared machine learning-based estimators with human expert estimates for warranty cost provisions (*i.e.* accrued costs under a fixed statutory warranty period). Though they do not directly manipulate the warranty period length, their work shows that more accurate assessments of cost over a known warranty period reduce the uncertainty and risk of setting warranty conditions (*e.g.* service costs, liability). They find ML models outperform human experts, especially when aggregating many units, which can influence better decisions about how long warranties should or can be offered profitably.

2.3. Gap analysis

Despite the extensive literature on both cooperative advertising and warranty decisions, several critical gaps remain:

Integration gap: No existing studies have simultaneously considered cooperative advertising and cooperative warranty decisions within a unified framework. The literature treats these strategies as independent decisions, missing potential synergies and trade-offs.

Cooperative warranty gap: While cooperative advertising has been extensively studied, the concept of cooperative warranty programs where costs and decisions are shared between supply chain members remains unexplored.

Participation rate analysis gap: Current research focuses on binary cooperation decisions (cooperate or not) rather than examining the effects of varying levels of cooperative participation rates across different strategies.

Quality-return rate integration gap: Although individual studies have examined quality effects on warranty and return rate effects on demand, no comprehensive model integrates quality, warranty, and return rates in a cooperative supply chain context.

Strategic trade-off gap: The literature lacks analysis of how supply chain members should balance their cooperative participation between advertising and warranty programs when resources are limited.

These gaps represent significant opportunities for advancing both theoretical understanding and practical applications of cooperative strategies in supply chain management.

3. The cooperative two-level supply chain model

This study examines a two-echelon supply chain configuration consisting of a single manufacturer and a single retailer operating in a monopolistic market environment. The manufacturer produces goods and delivers them to the retailer, who then sells these products to end customers. The supply chain operates through a single distribution channel without any competitive manufacturers or retailers, creating a controlled environment for analyzing cooperative decision-making dynamics. The customer demand function in this system is influenced by three critical factors: the retail price of the product, the level of advertising investment, and the warranty period offered. This multi-dimensional demand structure reflects the reality that consumers make purchasing decisions based not only on price considerations but also on the perceived value created through marketing communications and product assurance mechanisms.

The decision-making structure within this supply chain is characterized by a division of responsibilities between the two members. The retailer holds decision-making authority over two key variables: the retail price charged to customers and the warranty period offered with each product. Meanwhile, the manufacturer maintains control over the advertising expenditure level and the wholesale price charged to the retailer. This division creates interdependencies between the supply chain members, as each party's decisions directly impact the other's profitability and market performance. The distinguishing feature of this supply chain model lies in its cooperative approach to cost allocation for both advertising and warranty programs. Unlike traditional models where each party independently bears the costs of their respective activities, this framework allows the manufacturer and retailer to negotiate and share the costs associated with both advertising investments and warranty provision. The cooperative arrangement is governed by participation rates that serve as controllable parameters, enabling both parties to adjust their level of involvement in each cooperative program.

The quality dimension adds another layer of complexity to the model. The warranty period and associated warranty costs are directly dependent on the quality level of the manufactured products. When products exhibit low quality characteristics, the likelihood of failures increases, necessitating product replacements that must be fulfilled by the retailer. This quality-warranty relationship creates a feedback mechanism where quality decisions impact warranty costs, which in turn influence the cooperative cost-sharing arrangements between supply chain members. The flexibility inherent in this cooperative framework allows both the manufacturer and retailer to dynamically adjust their participation levels in advertising and warranty programs. This capability enables them to explore different cooperative strategies, such as high participation in advertising with low participation in warranty programs, or *vice versa*. The model thus captures the strategic trade-offs that supply chain members face when allocating their cooperative efforts and resources across multiple collaborative initiatives.

This problem formulation addresses the gap in existing literature by simultaneously considering cooperative advertising and cooperative warranty decisions within a unified framework, while incorporating the effects of product quality on warranty costs and allowing for variable levels of cooperative participation across different programs.

3.1. Notation

The notation used in the development of mathematical expressions is shown below.

Parameters

α	Baseline demand
β	Effect of the price on demand
γ	Effect of the advertisement on demand
λ	Effect of the warranty period on demand
k	Cooperative advertising cost participation rate of manufacturer; $0 < k < 1$
s	Cooperative warranty period participation rate of manufacturer $0 < s < 1$
q	Product quality level $0 < q < 1$
c_1	Production cost
c_2	Repair cost in warranty period

Functions

D	Demand function for product
R	Product's rate of return function

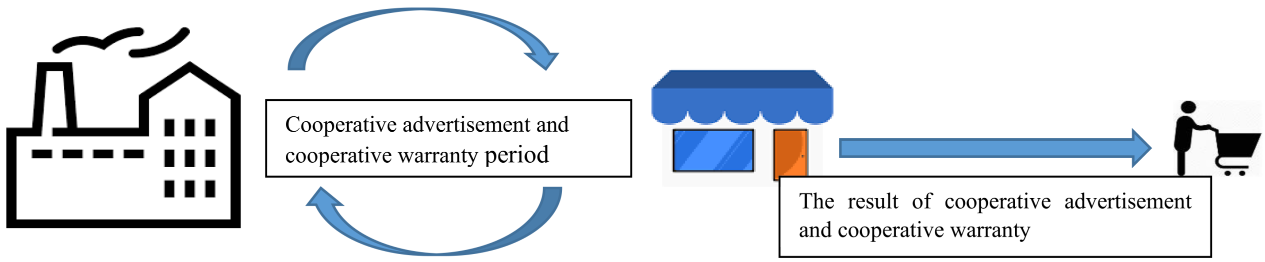

FIGURE 1. The manufacturer–retailer supply chain model.

π_m	Manufacturer profit function
π_r	Retailer profit function

Decision variables

w	Manufacturer price
p	Retailer price
a	Advertising cost
t	Product's warranty period

The cooperative warranty period participation rate of the manufacturer (s, where $0 < s < 1$) is treated as an exogenous parameter rather than an endogenous decision variable. This modeling choice is justified by the following considerations:

First, in practice, cooperative participation rates are established through long-term contractual agreements that specify fixed cost-sharing ratios and remain stable across multiple operational decision periods. Second, participation rates represent strategic-level partnership decisions that operate on different time horizons than operational variables such as price and warranty period. Our model focuses on operational optimization within an established cooperative framework. Third, the parametric approach enables systematic policy analysis of how different cooperation levels affect supply chain performance before managers commit to specific contractual arrangements. Finally, this approach aligns with established practice in cooperative game theory literature, where cost allocation mechanisms are treated as exogenous parameters to focus analysis on their effects while maintaining analytical tractability.

3.2. The development of the cooperative two-level supply chain model

This research considers a manufacturer–retailer supply chain model. This model has one manufacturer and one retailer which are located in a decentralized channel. Figure 1 shows a schematic diagram of the manufacturer–retailer supply chain model.

The retailer vends products to its clients at price p. The manufacturer sells its products to the retailer at price w. Parameters k and s represent the manufacturer cooperative participation rate in retailer local advertising cost and warranty period, respectively.

The demand function is given by

$$D = \alpha - \beta p + \gamma a + \lambda t. \tag{1}$$

This demand functions states that the costumers' demand is dependent on price, advertising, and warranty period. A decrease in the advertising (a) and/or warranty period (t) lead to a reduction in the customer's demand. However, a decrease in the price (p) leads to an increase in the demand. Parameter β shows the customer's sensitivity to price, parameter γ denotes the advertising sensitivity and parameter λ displays the warranty period sensitivity. If the product has abnormal quality then the customers must want to change the product into another one. The product's rate of return function R is defined below.

$$R = t^2(1-q). \tag{2}$$

The product's rate of return function R considers the warranty period (t) and product's quality level (q) as relevant factors in calculating the product's rate of return. According to the equation (2), if the quality level of product is very low and the warranty period is very long, all products will be returned. The warranty costs are shared between the manufacturer and retailer with s representing the cooperative participation rate. To have a convex advertising cost function, the quadratic function form $g(a) = a^2$ is used. In cooperative advertising programs, when the manufacturer offers a cooperative participation rate (k) to the retailer thus the ka^2 part of promotional expenditures is the manufacturer's responsibility and the rest of it ($(1-k)a^2$) is the retailer's duty $k \in (0,1)$. In a similar way, for the case of cooperative warranty period programs, when the manufacturer offers a cooperative participation rate (s) to the retailer, thus the sDc_2R part of warranty cost is the manufacturer's responsibility and the rest of it ($(1-s)Dc_2R$) is the retailer's duty $s \in (0,1)$.

The manufacturer profit function is expressed as:

$$\pi_m = Dw - Dc_1 - ka^2 - sDc_2R. \tag{3}$$

Substituting D and R given by equations (1) and (2), respectively into equation (3), thus,

$$\pi_m = (\alpha - \beta p + \gamma a + \lambda t)(w - c_1) - ka^2 - s(\alpha - \beta p + \gamma a + \lambda t)c_2(t^2(1-q)). \tag{4}$$

The retailer profit function is written as:

$$\pi_r = D(p-w) - (1-k)a^2 - (1-s)Dc_2R. \tag{5}$$

By replacing D and R given by equations (1) and (2), respectively into equation (5), hence,

$$\pi_r = (\alpha - \beta p + \gamma a + \lambda t)(p-w) - (1-k)a^2 - (1-s)(\alpha - \beta p + \gamma a + \lambda t)c_2(t^2(1-q)). \tag{6}$$

As it can be seen, the equations (4) and (6) have allocated to supply chain members two cooperative programs which are related to advertising costs and warranty costs. This is due to the fact that the manufacturer offers to retailer to have in common two cooperative programs. By doing this, manufacturer has better control to its distribution channel.

4. Game models

This section presents three game models including Manufacturer Stackelberg (MS) game, Retailer Stackelberg (RS) game and Nash (N) game.

4.1. Manufacturer Stackelberg (MS) game

In the Manufacturer Stackelberg (MS) game model, the manufacturer is the leader and the retailer is the follower. The solution to this model is called the Stackelberg equilibrium. The relationship of the manufacturer and retailer in this type of game model is sequential. It means that, firstly the best response of the retailer is obtained. After, the manufacturer obtains his/her best response according to the retailer's best response. The retailer makes decision about retailer price (p) and warranty period (t). Consequently, the manufacturer determines the manufacturer price (w) and local advertisements (a). So, the solution to the retailer's problem is given by solving simultaneously the following equations.

$$\frac{\partial \pi_r}{\partial p} = (c_2(s-1)(q-1)t^2 - 2p + w)\beta + a\gamma + \lambda t + \alpha = 0 \tag{7}$$

$$\frac{\partial \pi_r}{\partial t} = -(2(s-1))(a\gamma + (3/2)\lambda t - \beta p + \alpha)(q-1)tc_2 + \lambda(p-w) = 0. \tag{8}$$

Therefore, the retailer's best responses related to retailer price (p) and warranty period (t) are:

$$p^*_{ms} = \frac{4wc_2(s-1)(q-1)\beta^2 + 4c_2(s-1)(q-1)(a\gamma+\alpha)\beta + 3\lambda^2}{8(s-1)(q-1)c_2\beta^2} \tag{9}$$

$$t^*_{ms} = \frac{\lambda}{2(c_2(s-1)(q-1)\beta)}. \tag{10}$$

Now, by inserting retailer's best response into the manufacturer profit function the optimization problem Max π_m is obtained. In order to maximize the manufacturer profit function, the manufacturer makes decision about manufacturer price (w) and local advertisement (a).

Thus, the manufacturer profit function becomes

$$\begin{aligned}\pi_m^{ms} = \frac{1}{32(\beta^3(s-1)^3(q-1)^2c_2^2)}\Bigg(& -16wc_2^2(s-1)^3(q-1)^2(w-c_1)\beta^4 \\ & -32c_2^2(s-1)^3(q-1)^2\left(\left(\frac{-1}{2}a\gamma - \frac{1}{2}\alpha\right)w + \left(\frac{1}{2}a\gamma + \frac{1}{2}\alpha\right)c_1 + ka^2\right)\beta^3 \\ & -4\lambda^2c_2(s-1)(c_1s - c_1 + w)(q-1)\beta^2 + 4\lambda^2sc_2(s-1)(q-1)(a\gamma+\alpha)\beta + \lambda^4 s\Bigg).\end{aligned} \tag{11}$$

Hence, the solution to the manufacturer's problem is given by solving simultaneously the following equations.

$$\begin{aligned}\frac{\partial \pi_m^{\mathrm{ms}}}{\partial w} &= \frac{1}{32c_2^2\beta^3(s-1)^3(q-1)^2}\Big\{-16c_2^2\beta^4(s-1)^3(q-1)^2(2w-c_1) \\ &\quad + 16c_2^2\beta^4(s-1)^3(q-1)^2(a\gamma+\alpha) - 4c_2\lambda^2\beta^2(s-1)(q-1)\Big\} = 0. \end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial \pi_m^{\mathrm{ms}}}{\partial a} &= \frac{1}{32c_2^2\beta^3(s-1)^3(q-1)^2}\Big\{-16c_2^2\beta^3(s-1)^3(q-1)^2(4ka + c_1\gamma - w\gamma) \\ &\quad + 4c_2\lambda^2 s\beta\gamma(s-1)(q-1)\Big\} = 0. \end{aligned} \tag{13}$$

Thus, the solution to manufacturer price (w) and local advertisement (a) are:

$$w^*_{ms} = \frac{16kc_1\sigma\beta^3 + 16\sigma\beta^2\left(\frac{-1}{4}c_1\gamma^2 + k\alpha\right) - 4\beta k\lambda^2 + \lambda^2\gamma^2 s}{32\sigma\beta^2(\beta k - \frac{1}{8}\gamma^2)} \tag{14}$$

$$a^*_{ms} = \frac{-\gamma(c_1\sigma\beta^2 - \alpha\sigma\beta - \frac{1}{2}\lambda^2(s-\frac{1}{2}))}{8\beta\sigma(\beta k - \frac{1}{8}\gamma^2)} \tag{15}$$

where ($\sigma = c_2(q-1)(s-1)^2$).

The Appendix A presents the detailed derivation to proof the optimality. Figure 2 shows the relationship among retailer price (p), manufacturer price (w) and local advertisements (a) in the Manufacturer Stackelberg (MS) game model. Please note that the decision variables, time and price, are constrained to non-negative values in all practical applications. The negative values shown in Figures 2–4 are included solely to illustrate the shape of the functions (*e.g.* concavity), and do not represent feasible solutions.

4.2. Retailer Stackelberg (RS) game

In the Retailer Stackelberg (RS) game model, the retailer dominates manufacturer. Thus, the retailer operates as the leader and the manufacturer as the follower. To acquire the retailer Stackelberg equilibrium, the retailer

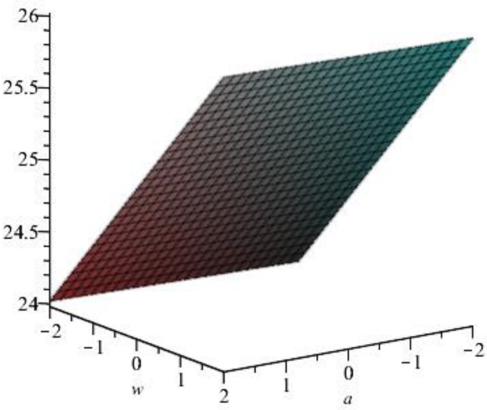

FIGURE 2. Relation among p, w and a in Manufacturer Stackelberg (MS) game model.

optimizes his/her profits as to manufacturer's act in manufacturer price (w) and advertising (a). Here, it is assumed that $= w + m$. This means that the retailer price is the manufacturer price (w) plus the retailer marginal price (m). In retailer Stackelberg (RS) game, $w + m$ is inserted in equation (4) instead of p. As a result, the solution to the manufacturer's problem is determined by solving simultaneously the following equations:

$$\frac{\partial \pi_m}{\partial w} = (-sc_2(q-1)t^2 - m - 2w + c_1)\beta + a\gamma + \lambda t + \alpha = 0 \tag{16}$$

$$\frac{\partial \pi_m}{\partial a} = (sc_2(q-1)t^2 + w - c_1)\gamma - 2ka = 0. \tag{17}$$

So, the manufacturer's best response about manufacturer price (w) and advertising (a) is:

$$w^*_{rs} = \frac{2\beta\big(c_2kqst^2 - c_2kst^2 - c_1k + km\big) + c_2\gamma^2st^2 - c_2\gamma^2qst^2 + c_1\gamma^2 - 2k\lambda t - 2\alpha k}{-4\beta k + \gamma^2} \tag{18}$$

$$a^*_{rs} = -\frac{\gamma(\beta c_2qst^2 - \beta c_2st^2 - \beta c_1 - \beta m + \lambda t + \alpha)}{-4\beta k + \gamma^2}. \tag{19}$$

Now, the manufacturer's best response is inserted into the retailer's optimization problem to get Max π_r. To maximize the retailer profit function, the retailer should makes decision about the retailer margin price (m) and warranty period (t). Hence, the retailer profit function is determined by the following equation

$$\begin{aligned}\pi_r^{rs} = \frac{1}{(4\beta k - \gamma^2)^2}\Big\{&\big(2c_2ks\beta^2t^2(1-q) + 2c_1k\beta^2 + 2mk\beta^2 - 2k\lambda t\beta - 2k\alpha\beta\big)\big(m\gamma^2 \\ &- c_2\gamma^2t^2(1-s)(1-q) - 4km\beta + 4c_2k\beta t^2(1-s)(1-q)\big) \\ &- \big(\gamma^2 - k\gamma^2\big)\big(c_2sq\beta t^2 - c_2s\beta t^2 - \beta c_1 - \beta m + \lambda t + \alpha\big)^2\Big\}.\end{aligned} \tag{20}$$

The solution to the retailer's problem is computing by solving simultaneously the following equations:

$$\begin{aligned}\frac{\partial \pi_r^{rs}}{\partial m} = {}& 2k\beta^2\big(m\gamma^2 - c_2\gamma^2t^2(1-s)(1-q) - 4km\beta + 4c_2k\beta t^2(1-s)(1-q)\big) \\ &+ \big(2c_2ks\beta^2t^2(1-q) + 2c_1k\beta^2 + 2mk\beta^2 - 2k\lambda t\beta - 2k\alpha\beta\big)\big(\gamma^2 - 4k\beta\big) + 2\beta\big(\gamma^2 \\ &- k\gamma^2\big)\big(c_2sq\beta t^2 - c_2s\beta t^2 - \beta c_1 - \beta m + \lambda t + \alpha\big) = 0\end{aligned} \tag{21}$$

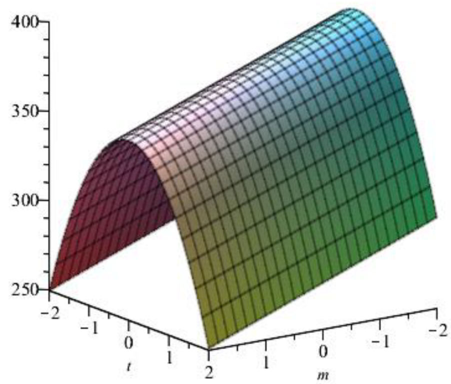

FIGURE 3. Relation between a, m and t in Retailer Stackelberg (RS) game model.

$$\begin{aligned}\frac{\partial \pi_r^{rs}}{\partial t} &= \big(4c_2ks\beta^2t(1-q)-2k\lambda\beta\big)\big(m\gamma^2-c_2\gamma^2t^2(1-s)(1-q)-4km\beta+4c_2k\beta t^2(1-s)(1-q)\big)\\ &+\big(2c_2ks\beta^2t^2(1-q)+2c_1k\beta^2+2mk\beta^2-2k\lambda t\beta-2k\alpha\beta\big)\big(8c_2k\beta t(1-s)(1-q)-2c_2t\gamma^2(1-s)\\ &\times(1-q)\big)-2\big(\gamma^2-k\gamma^2\big)\big(c_2sq\beta t^2-c_2s\beta t^2-\beta c_1-\beta m+\lambda t+\alpha\big)(2c_2sq\beta t-2c_2s\beta t+\lambda)=0. \qquad (22)\end{aligned}$$

Therefore, the retailer's solution related to retailer marginal price (m) and warranty period (t) are as follows:

$$\begin{aligned}m_{rs}^* &= \frac{1}{2c_2(1-q)(4k\beta^2(\gamma^2-4k\beta)-2\beta^2(\gamma^2-k\gamma^2))}\Big\{\big(k\lambda^2-k\lambda^2s\big)\big(\gamma^2-4k\beta\big)-\big(\gamma^2-4k\beta\big)\big(k\lambda^2(s-2)\\ &+4c_1c_2k\beta^2(1-q)-4c_2k\alpha\beta(1-q)\big)-\big(\gamma^2-k\gamma^2\big)\big(2\lambda^2-s\lambda^2+4c_2(1-q)\big(\alpha\beta-c_1\beta^2\big)\big)\Big\} \qquad (23)\\ t_{rs}^* &= \frac{-\lambda}{2\beta c_2(q-1)}. \qquad (24)\end{aligned}$$

And the retailer price (p) is obtained with

$$p_{rs}^* = m_{rs}^* + w_{rs}^*. \qquad (25)$$

The Appendix B provides the calculations to proof the optimality. Figures 3 and 4 display the relationship between the retailer margin price (m) and warranty period (t) in the Retailer Stackelberg (RS).

4.3. Nash (N) game

In Nash game, the manufacturer and the retailer are equivalent and they make decisions simultaneously. To solve the Nash equilibrium, the decisions of the manufacturer and the retailer are decided individually and simultaneously. Here, it is defined $p = w + m$. In Nash game, the manufacturer makes decision about manufacturer price (w) and advertising (a); and the retailer makes decision about retailer margin price (m) and warranty period (t). The first-order conditions for optimization of manufacturer's profit function are $\frac{\partial \pi_m}{\partial w} = 0$ and $\frac{\partial \pi_m}{\partial a} = 0$. And the conditions for the optimization of retailer's profit function are $\frac{\partial \pi_r}{\partial m} = 0$ and $\frac{\partial \pi_r}{\partial t} = 0$. Thus,

$$\frac{\partial \pi_m}{\partial w} = \big(-sc_2(q-1)t^2-m-2w+c_1\big)\beta+a\gamma+t\lambda+\alpha=0 \qquad (26)$$

$$\frac{\partial \pi_m}{\partial a} = (sc_2(q-1)t^2+w-c_1)\gamma-2ka=0 \qquad (27)$$

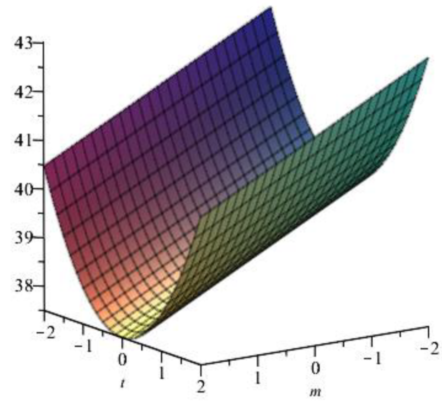

FIGURE 4. Relation between w, m and t in Retailer Stackelberg (RS) game model.

$$\frac{\partial \pi_r}{\partial m} = (c_2(s-1)(q-1)t^2 - 2m - w)\beta + a\gamma + t\lambda + \alpha = 0 \tag{28}$$

$$\frac{\partial \pi_r}{\partial t} = -(2(q-1))(s-1)\left(\left(\frac{3}{2}\right)t\lambda + (-m-w)\beta + a\gamma + \alpha\right)tc_2 + m\lambda = 0. \tag{29}$$

By solving the above equation simultaneously, the best responses for manufacturer and retailer in the Nash game are as follows:

$$w_n^* = \frac{16kc_1\sigma\beta^3 + 8\sigma\beta^2\left(\alpha k - \frac{1}{2}\gamma^2 c_1\right) - 2k\lambda^2(s+1)\beta + \lambda^2\gamma^2 s}{24(\beta k - \frac{1}{6}\gamma^2)\sigma\beta^2} \tag{30}$$

$$a_n^* = \frac{-\gamma\left(c_1\sigma\beta^2 - \alpha\sigma\beta - \frac{1}{2}\lambda^2(s - \frac{1}{2})\right)}{6(\beta k - \frac{1}{6}\gamma^2)\sigma\beta} \tag{31}$$

$$m_n^* = \frac{-8kc_1\sigma\beta^3 + 8k\alpha\sigma\beta^2 + 10\lambda^2\left(s - \frac{4}{5}\right)k\beta - \lambda^2\gamma^2(s-1)}{24(\beta k - \frac{1}{6}\gamma^2)\sigma\beta^2} \tag{32}$$

$$t_n^* = \frac{\lambda}{2\beta c_2(s-1)(q-1)}. \tag{33}$$

The Appendix C provides the computations to prove the optimality.

5. Numerical examples and sensitivity analysis

This section presents the numerical example to illustrate the applicability of the games models. It is assumed that the values of parameters are

$$\alpha = 100\,000,\ \beta = 1000,\ \gamma = 20,\ \lambda = 20,\ c_1 = 10,\ c_2 = 10,\ q = 0.5,\ k = 0.3,\ s = 0.1.$$

The solutions for the decisions variables for the Manufacturer Stackelberg (MS), Retailer Stackelberg (RS) and Nash (N) games are presented in Table 1.

Now, consider the following data in a fixed way: $\alpha = 50\,000$, $\beta = 1000$, $\gamma = 20$, $\lambda = 20$, $c_1 = 20$, $c_2 = 20$. The main aim is to check how the cooperative advertising cost participation rate (k) and the cooperative warranty period the participation rate (s) influence the decisions variables of the three models: Manufacturer Stackelberg (MS), Retailer Stackelberg (RS) and Nash (N) games.

TABLE 1. Optimal results for the decisions variables for the three game models.

Game model	Manufacturer price (\$) (w)	Retailer price (\$) (p)	Advertisement cost (\$) (a)	Warranty period (t)	Manufacturer's profit (\$) (π_m)	Retailer's profit (\$) (π_r)
MS	64.00001	91.00004	900.0002	0.00222	1 215 001	162 000.1
RS	31.3158	92.89478	710.5265	0.002	302 908.7	959 211
N	48.57144	87.1429	1285.715	0.00222	991 837.2	330 612.4

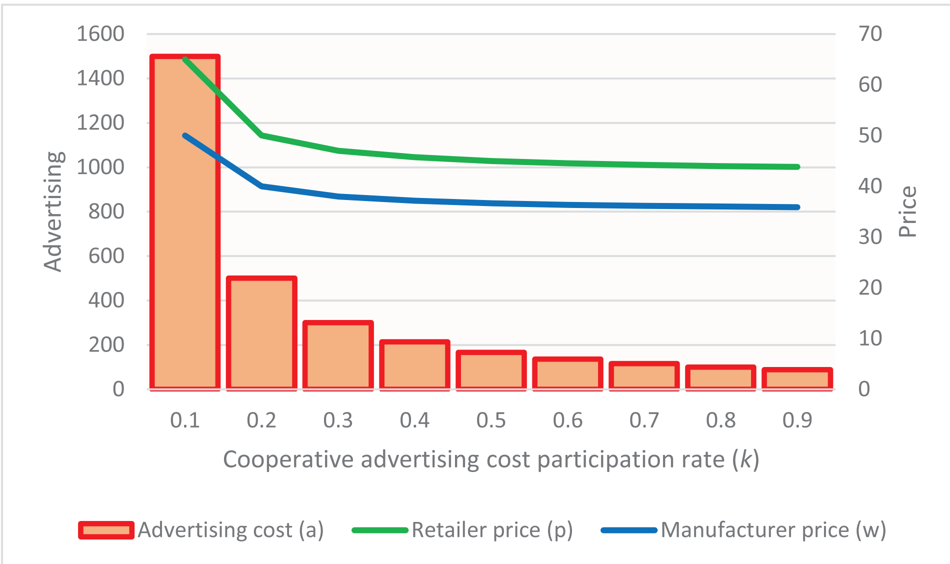

FIGURE 5. The effect of cooperative advertising participation rate (k) on advertising costs and prices in Manufacturer Stackelberg (MS) model.

5.1. Manufacturer Stackelberg (MS)

Here, the quality level (q) is 0.5 and the cooperative warranty period participation rate (s) is assumed as 0.1. Figures 5–7 show how the decision variables change when participation rate in cooperative advertising and cooperative warranty period change.

Figure 5 displays that, in Manufacturer Stackelberg (MS) model (with fix warranty period participation rate (s) and quality level (q)), an increase in amount of cooperative advertisement participation rate (k) will result in a decrease in advertising cost, retailer price and manufacturer price. Specifically, the manufacturer has a preference of having less advertising costs. This is due to the fact that the manufacturer has a leadership position, therefore he controls the level of advertising costs in a high advertising participation rate. This idea from manufacturer is rational because he or she does not want to pay large portion of advertising costs.

Figure 6 exhibits the behavior of advertising costs and prices when the cooperative warranty participation rate (s) changes. In this part, it is assumed that cooperative advertising participation rate (k) is 0.5. As you see, cooperative warranty participation rate (s) does not change the manufacturer and retailer pricing strategy. In Figure 7, the relation between warranty time and cooperative warranty participation rate (s) in Manufacturer Stackelberg (MS) model is shown. Here, it is assumed that cooperative advertising participation rate (k) is 0.5. This figure explains that if cooperative warranty participation rate (s) increases then the warranty time

FIGURE 6. The effect of cooperative warranty participation rate (s) on advertising costs and prices in Manufacturer Stackelberg (MS) model.

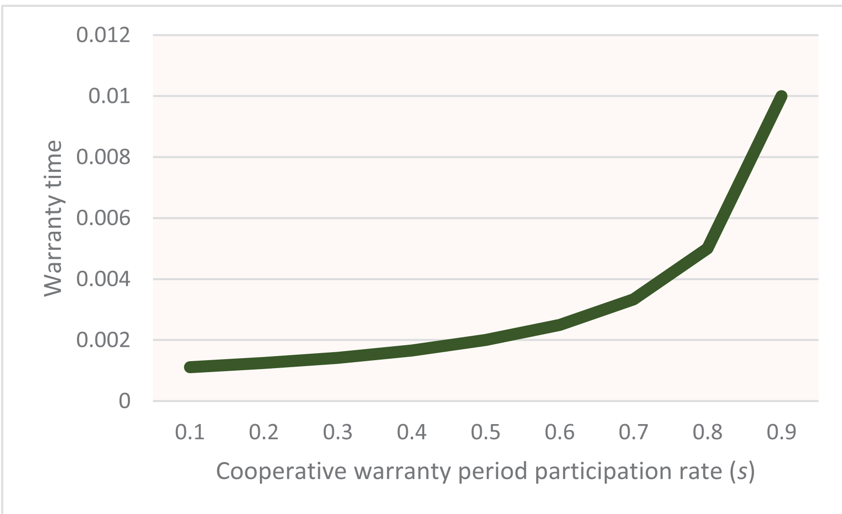

FIGURE 7. The effect of cooperative warranty participation rate (s) on warranty time in Manufacturer Stackelberg (MS) model.

increases. Figure 8 shows that both warranty time (t) and manufacturer price (w) increase when the quality level (q) increases. It means that, if the product's quality is at high level, the supply chain members can offer a larger warranty time to customers. Here, it is used that $k = 0.5$ and $s = 0.1$.

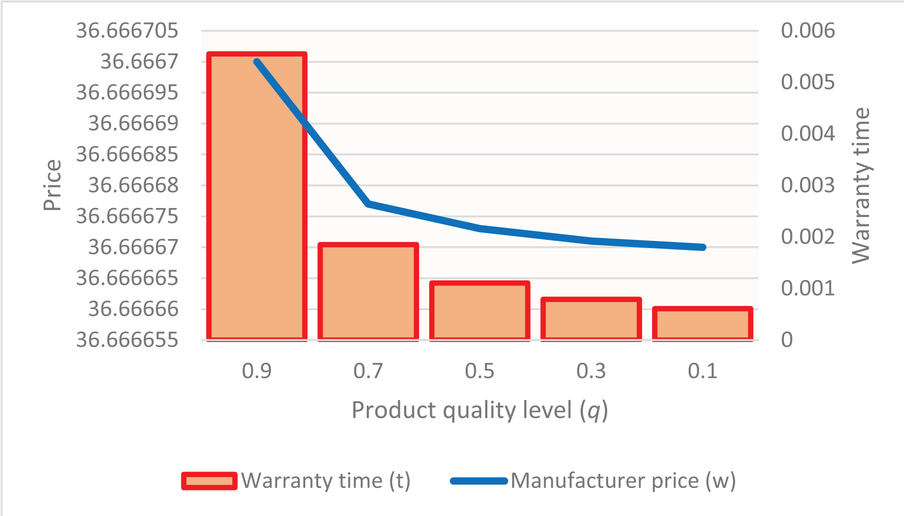

FIGURE 8. The effect of quality level (q) on manufacturer price (w) and warranty time (t) in Manufacturer Stackelberg (MS) model.

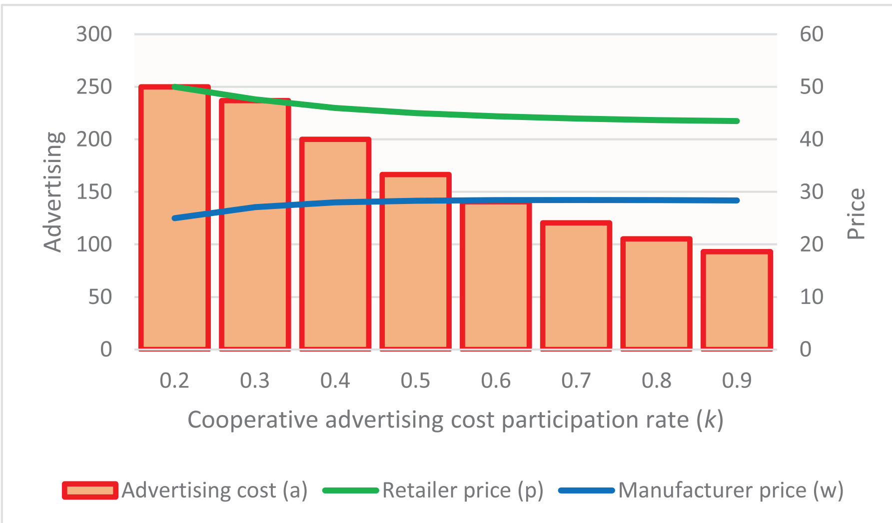

FIGURE 9. The effect of cooperative advertising participation rate (s) on advertising costs and prices in retailer Stackelberg (RS) game.

5.2. Retailer Stackelberg (RS) game

For the retailer Stackelberg game, Figure 9 shows that cooperative advertising participation rate (k) variations make that manufacturer price (w) increases and both advertising cost (a) and retail price (p) decrease when $s = 0.1$. Another important fact is that, with an increase in cooperative participation rate (k), the difference between the manufacturer price (w) and retail price (p) decreases.

Figure 10 explains how cooperative warranty participation rate (s) affects only the advertising cost (a) considering that $k = 0.5$. It means that, with an increment in cooperative warranty participation rate (s), the

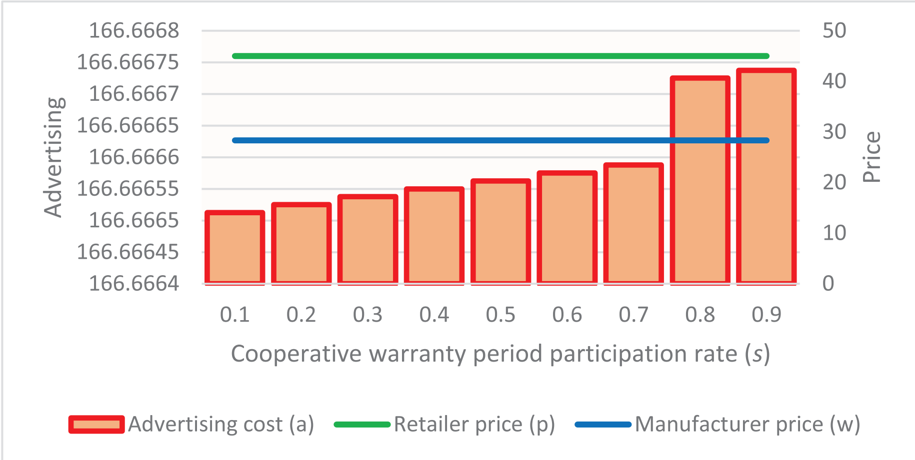

FIGURE 10. The effect of cooperative warranty participation rate (s) on advertising costs and prices in retailer Stackelberg (RS) game.

FIGURE 11. The effect of quality level (q) on retailer price (p) and warranty (t) time in retailer Stackelberg (RS) game.

advertising cost increases. On the other hand, the manufacturer price (w) and the retailer price (p) permanence without change.

Figure 11 displays two important facts. First of all, a high quality level needs more investment and it is obvious that the price become high in the high level of quality. And if the quality level increases, the warranty time increases.

5.3. Nash (N) game

Figure 12 presents the impact of cooperative advertising participation rate (k) on advertising costs and prices in Nash (N) game. It is assumed that cooperative warranty participation rate (s) is $s = 0.1$. In Nash (N) game,

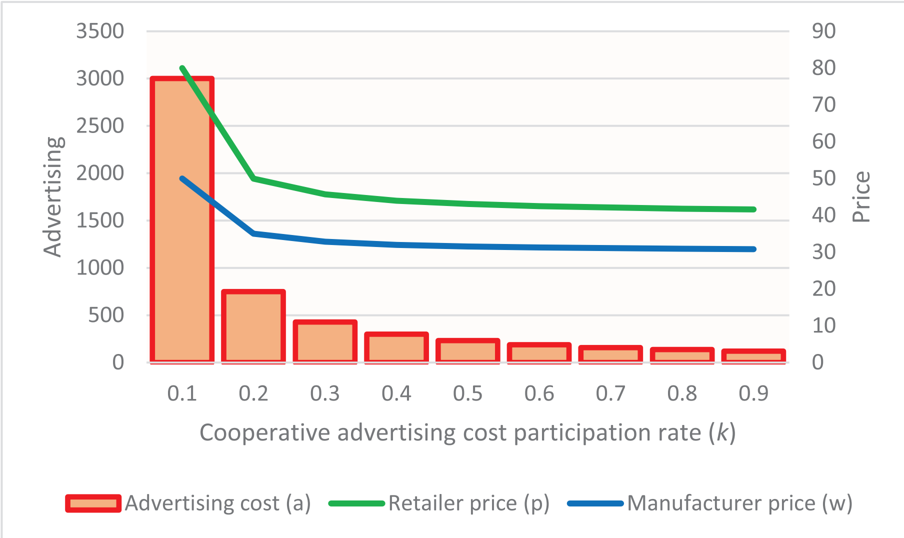

FIGURE 12. The effect of cooperative advertising participation rate (k) on advertising cost, retailer price (p) and warranty (t) in Nash (N) game.

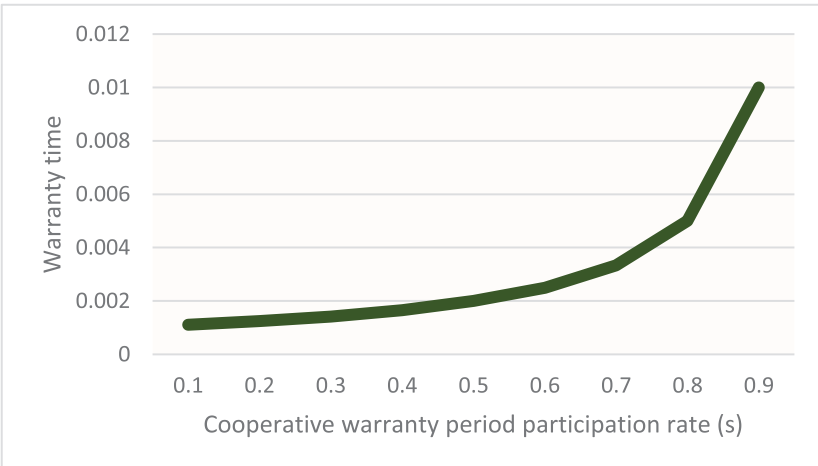

FIGURE 13. The effect of cooperative warranty participation rate (s) on warranty time (t) Nash (N) game.

if cooperative advertising participation rate (k) increases then the advertising cost, retailer price (p) and the manufacturer price (w) decrease as it is shown in Figure 12.

Figure 13 shows the impact of cooperative warranty participation rate (s) on warranty time (t) Nash (N) game. It is observed that cooperative warranty participation rate (s) increases then warranty time (t) increases too (Fig. 14).

In Figure 15 it is easy to see that when the quality level (q) increases then both the retailer price (p) and warranty (t) rise also. Figures 16 and 17 display the impact of cooperative advertising participation rate (k) and cooperative warranty participation rate (s) on the manufacturer price (w) in different games. On the one hand,

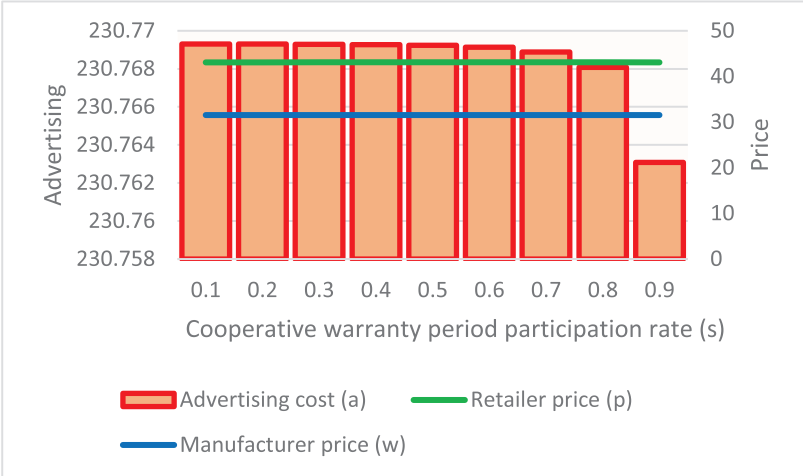

FIGURE 14. The effect of cooperative warranty participation rate (s) on advertising cost retailer price (p) and warranty (t) in Nash (N) game.

FIGURE 15. The effect of quality level (q) on the retailer price (p) and warranty (t) in Nash (N) game.

Figure 15 says that as the cooperative advertising participation rate (k) increases then the manufacturer price (w) decreases in all games. On the other hand, Figure 17 shows that the manufacturer price (w) permanence constant as the cooperative warranty participation rate (s) increases.

Analyzing the changes in the level of advertising cost is another important insight for decision makers. Table 2 demonstrates the amount of advertising cost in MS, RS and N game models in different participation rates. For example, decision makers by observing this table can found that, changing the value of cooperative advertising participation rate (k) is very significant for advertising cost in MS, RS and N games because this cost decreases, but changing cooperative warranty participation rate (s) is not very significant for all games because the advertising cost keep up constant. Table 2 reveals that advertising costs exhibit minimal sensitivity to changes in cooperative warranty participation rates (s), with variations occurring only in decimal places. This

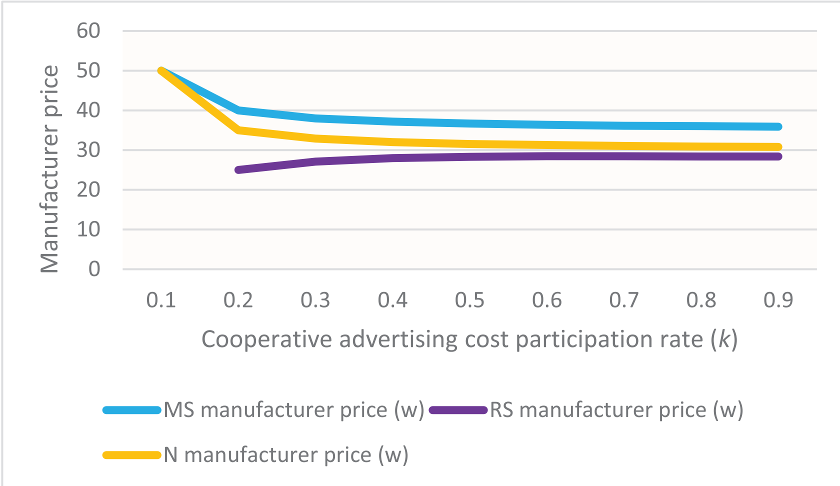

FIGURE 16. Comparing the effect of cooperative advertising participation rate (k) on manufacturer price (w) in different games.

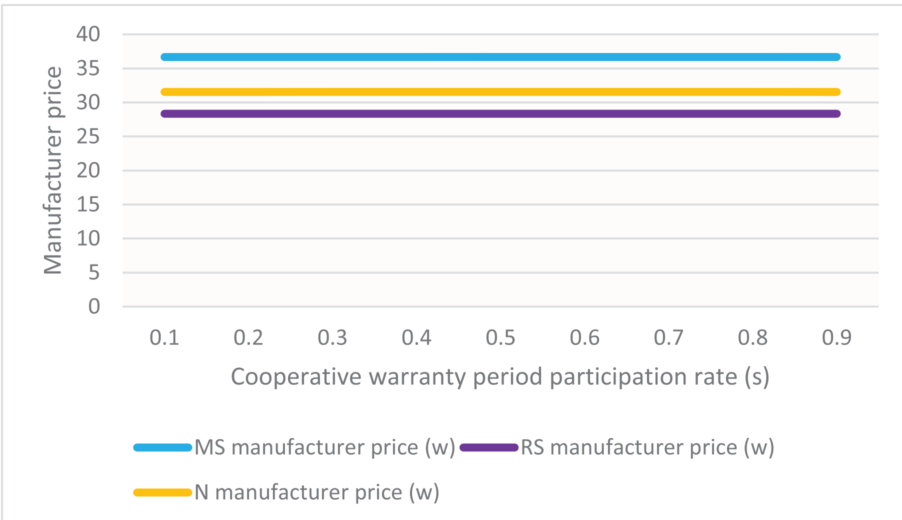

FIGURE 17. Comparing the effect of cooperative warranty participation rate (s) on manufacturer price (w) in different games.

finding indicates strategic independence between advertising and warranty cooperation programs. The minimal variation occurs because advertising and warranty serve distinct demand functions: advertising influences demand through awareness and preference formation, while warranty affects demand through risk reduction and quality signaling. The optimization process demonstrates that these mechanisms operate independently, allowing optimal advertising levels to be determined primarily by demand responsiveness and cost structure rather than warranty cost-sharing arrangements.

From a managerial perspective, this independence provides operational flexibility. Supply chain partners can adjust warranty participation rates without requiring simultaneous recalibration of advertising investments,

TABLE 2. Comparing the effect of cooperative advertising (k) and cooperative warranty participation rate (s) on advertising cost in the different games models.

	MS advertising cost	RS advertising cost	N advertising cost
	($S = 0.1$)		
$K = 0.1$	1500	–	3000.001
$K = 0.2$	500.0002	250.0001	750.0002
$K = 0.3$	300.0001	236.8422	428.5716
$K = 0.4$	214.2858	200.0001	300.0001
$K = 0.5$	166.6667	166.6667	230.7693
$K = 0.6$	136.3637	140.625	187.5001
$K = 0.7$	115.3847	120.6897	157.8948
$K = 0.8$	100	105.2632	136.3637
$K = 0.9$	88.23532	93.10348	120
	($K = 0.5$)		
$s = 0.1$	166.6667215	166.6665125	230.7693067
$s = 0.2$	166.6667188	166.666525	230.7693029
$s = 0.3$	166.666712	166.6665375	230.7692936
$s = 0.4$	166.6666975	166.66655	230.7692735
$s = 0.5$	166.6666667	166.6665625	230.7692308
$s = 0.6$	166.6665972	166.666575	230.7691346
$s = 0.7$	166.6664198	166.6665875	230.7688889
$s = 0.8$	166.6658333	166.666725	230.7680769
$s = 0.9$	166.6622222	166.6667375	230.7630769

simplifying decision-making and allowing specialized focus on each cooperative program. For contract design, this suggests that advertising and warranty cooperation can be structured as separate modules within partnership agreements. This result holds within the examined parameter range ($s = 0.1$ to 0.9) and the specific model structure. The strategic independence finding supports the theoretical coherence of treating cooperative advertising and warranty as parallel rather than interdependent strategies.

Figure 18, compares the effect of quality level (q) on warranty time (t) in the three different game models. In all of these game models, the quality level (q) has a direct relationship with warranty time (t).

5.4. Managerial implications

The findings of this study provide several actionable insights for supply chain managers considering cooperative advertising and warranty programs. The practical significance of these results extends beyond theoretical contributions to offer concrete guidance for strategic decision-making.

5.4.1. Strategic leadership and game structure selection

The choice of leadership structure significantly impacts both individual firm profitability and overall supply chain efficiency. Key Finding: Retailer Stackelberg (RS) games generate the highest retailer profits (\$959 211) while maintaining reasonable manufacturer profits (\$302 909), whereas Manufacturer Stackelberg (MS) games heavily favor manufacturers (\$1 215 001 *vs.* \$162 000). Managerial Guidance: Supply chain partners should carefully negotiate leadership roles based on their strategic objectives. Retailers with strong market position and customer relationships should pursue leadership in cooperative programs, as this structure maximizes their returns while maintaining supply chain viability. Conversely, manufacturers with significant brand power or technological advantages may prefer the MS structure despite its potential negative impact on retailer profitability and long-term relationship sustainability.

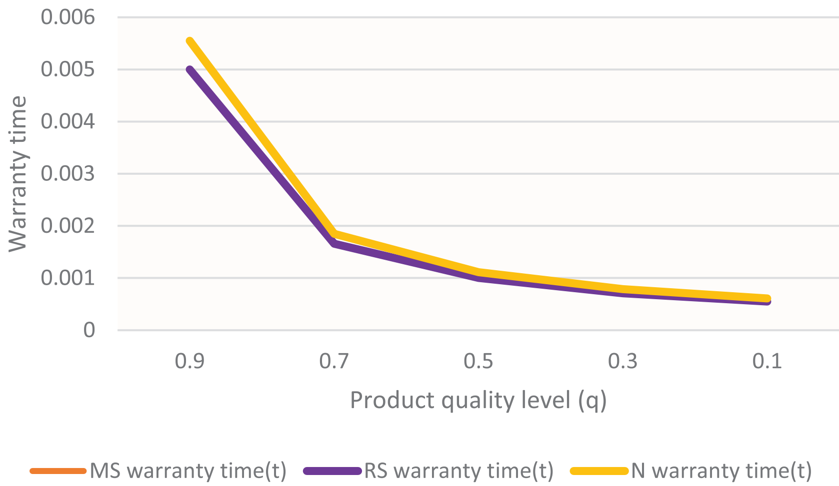

FIGURE 18. Comparing the effect of quality level (q) on warranty time (t) in different games models.

5.4.2. Pricing strategy optimization

The results reveal substantial differences in optimal pricing strategies across game structures. Key Finding: The RS game enables lower manufacturer prices ($31.32) combined with competitive retail prices ($92.89), while the MS game results in higher manufacturer prices ($64.00) but similar retail prices ($91.00). Managerial Guidance: Managers should align their pricing strategies with their chosen cooperative structure. In retailer-led arrangements, manufacturers should focus on competitive wholesale pricing to support retailer margins, while in manufacturer-led structures, higher wholesale prices can be sustained. The Nash game's intermediate pricing ($48.57 manufacturer, $87.14 retail) suggests that balanced power structures may offer compromise solutions when neither party can establish clear leadership.

5.4.3. Advertising investment decisions

Advertising cost optimization varies significantly across game structures. Key Finding: Nash games result in the highest advertising costs ($1285.72), representing 20–80% higher investments than Stackelberg games (MS: $900.00, RS: $710.53). Managerial Guidance: Supply chain partners should avoid simultaneous decision-making scenarios that lead to Nash outcomes, as these result in advertising over-investment without proportional returns. Establishing clear leadership in advertising decisions reduces costs and improves efficiency. The RS structure offers the lowest advertising costs while maintaining competitive market presence, making it attractive for cost-conscious partnerships.

5.4.4. Cooperative program independence

The analysis demonstrates that advertising and warranty cooperation operate independently. Key Finding: Changes in warranty participation rates have minimal impact on optimal advertising strategies and costs. Managerial Guidance: This independence provides operational flexibility for managers. Companies can negotiate and modify warranty cooperation agreements without requiring simultaneous renegotiation of advertising arrangements. This modular approach simplifies contract management, allows specialized teams to focus on each program, and enables responsive adjustments to market conditions without disrupting the entire cooperative framework.

Based on these findings, we recommend the following practical steps:

(1) Partnership Structure Design: Establish clear leadership roles rather than pursuing symmetric power arrangements. Consider the RS structure for balanced profit distribution and cost efficiency.
(2) Contract Architecture: Structure advertising and warranty cooperation as separate contract modules that can be independently negotiated and adjusted.
(3) Performance Monitoring: Focus on game structure efficiency metrics rather than only individual profit maximization to ensure long-term partnership sustainability.
(4) Strategic Flexibility: Leverage the independence between cooperative programs to respond quickly to market changes in one area without disrupting the entire partnership.

These insights enable supply chain managers to make informed decisions about cooperative strategies, optimize their partnership structures, and achieve sustainable competitive advantages through effective collaboration.

6. Conclusion

Cooperative supply chain has many applications in real world, for example, one of the most common supply chain for this kind of modeling is manufacturer–retailer supply chain. In this supply chain proper communication and cooperation among members is very important. Previously, many of the papers in the field examined warranty period and its effect on improving supply chain profit. However, there is paucity of research in the literature about cooperative warranty and cooperative advertisement simultaneously. This paper approaches this gap in the literature. For this purpose, a bi-level supply chain with one manufacturer and one retailer is developed. The existence of two participation rates increase the importance of cooperation between manufacturer and retailer. It is obvious that if the participation rate in each cooperative program increases then the members' profit function decreases. It was found that the manufacturer price, retailer price and advertisement cost are affected by cooperation participation rates. And, different games create diverse conditions and consequently, produce dissimilar effects. The main purpose of this paper is to help manufactures or retailers to make more rational and appropriate decisions. Participation rate is a matter of cooperation between the manufacturer and retailer. In future, it is desirable to have companies that use more of cooperative programs. And cooperative warranty time is one the best cooperative programs for companies and retailers. Within these cooperative programs, the local advertisements and warranty period costs are shared among supply chain members and there is an improvement in retailer and manufacturer's profit. In other words, even with lower costs, retailers and manufacturers can benefit from local advertisement and warranty period and have an increase in the number of customers. Additionally, future research could extend the present framework by incorporating three-dimensional analyses to explicitly explore the trade-offs between players' profits as specific parameters vary. Such an approach would provide a richer understanding of the strategic interactions between players in game-theoretic contexts. Future research can be built on these findings and develop supply chains and distribution channels with more than one cooperative program.

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Appendix A.

Proof of the optimality for the Manufacturer Stackelberg (MS) game

The concavity of retailer profit function should proof. For this reason, the Hessian matrix is formed as follows:

$$\text{Retailer profit function Hessian matrix} = \begin{pmatrix} \frac{\partial^2 \pi_s}{\partial p^2} & \frac{\partial^2 \pi_s}{\partial p \partial t} \\ \frac{\partial^2 \pi_s}{\partial t \partial p} & \frac{\partial^2 \pi_s}{\partial t^2} \end{pmatrix}.$$

$$\frac{\partial^2 \pi_r}{\partial p^2} = -2\beta \tag{A.1}$$

$$\frac{\partial^2 \pi_r}{\partial t^2} = -2c_2(s-1)(q-1)(a\gamma - \beta p + 3\lambda t + \alpha) \tag{A.2}$$

$$\frac{\partial^2 \pi_r}{\partial t \partial p} = \frac{\partial^2 \pi_r}{\partial p \partial t} = \lambda + 2c_2 t\beta(1-s)(1-q) \tag{A.3}$$

$$|H| = -(4(s-1))c_2(c_2(s-1)(q-1)t^2 + p)(q-1)\beta^2 + 4c_2(s-1)(q-1)(a\gamma + 2\lambda t + \alpha)\beta - \lambda^2. \tag{A.4}$$

According to concavity conditions these inequalities are stated as $\frac{\partial^2 \pi_r}{\partial p^2} < 0$ and $\frac{\partial^2 \pi_r}{\partial t^2} < 0$. The parameter β is defined between 0 and 1. So $-2\beta < 0$. Assume that $\theta = c_2(s-1)(q-1)$. So $\frac{\partial^2 \pi_r}{\partial t^2} = -2\theta(a\gamma - \beta p + 3\lambda t + \alpha)$ and α is a big number so $a\gamma - \beta p + 3\lambda t + \alpha$ is greater than zero, therefore $\frac{\partial^2 \pi_r}{\partial t^2} < 0$. Consequently, the Hessian matrix determinant is greater than zero if

$$p < \frac{(8a\beta\gamma\lambda t\theta - 4\beta^2 t^2 \theta^2 + 4\alpha\beta\theta - \lambda^2)}{4(\theta\beta^2)}.$$

The concavity of manufacturer profit function should proof. For this reason, the Hessian matrix is built as follows:

$$\text{Manufacturer profit function Hessian matrix} = \begin{pmatrix} \frac{\partial^2 \pi_m}{\partial w^2} & \frac{\partial^2 \pi_m}{\partial w \partial a} \\ \frac{\partial^2 \pi_m}{\partial a \partial w} & \frac{\partial^2 \pi_m}{\partial a^2} \end{pmatrix}$$

$$\frac{\partial^2 \pi_m}{\partial w^2} = -2\beta \tag{A.5}$$

$$\frac{\partial^2 \pi_m}{\partial a^2} = -2k \tag{A.6}$$

$$\frac{\partial^2 \pi_m}{\partial a \partial w} = \frac{\partial^2 \pi_m}{\partial w \partial a} = \gamma \tag{A.7}$$

$$|H| = 4\beta k - \gamma^2. \tag{A.8}$$

Equations (A.5) and (A.6) are negative because β and k are positive parameters. The Hessian matrix determinant is greater than zero if $\beta k > \frac{1}{4}\gamma^2$.

Appendix B.

Proof of the optimality for the Retailer Stackelberg (RS) game

The concavity of manufacturer profit function should proof. For this reason, the Hessian matrix is constructed as follows:

$$\text{Manufacturer profit function Hessian matrix} = \begin{pmatrix} \frac{\partial^2 \pi_m}{\partial w^2} & \frac{\partial^2 \pi_m}{\partial w \partial a} \\ \frac{\partial^2 \pi_m}{\partial a \partial w} & \frac{\partial^2 \pi_m}{\partial a^2} \end{pmatrix}.$$

$$\frac{\partial^2 \pi_m}{\partial w^2} = -2\beta \tag{B.9}$$
$$\frac{\partial^2 \pi_m}{\partial a^2} = -2k \tag{B.10}$$
$$\frac{\partial^2 \pi_m}{\partial a \partial w} = \frac{\partial^2 \pi_m}{\partial w \partial a} = \gamma \tag{B.11}$$
$$|H| = 4\beta k - \gamma^2. \tag{B.12}$$

According to concavity conditions, these inequalities are established as $\frac{\partial^2 \pi_m}{\partial w^2} < 0$ and $\frac{\partial^2 \pi_m}{\partial a^2} < 0$. It is obvious that -2β and $-2k$ are less than zero because both β and k are positive. And if $4\beta k > \gamma^2$ the Hessian matrix determinant is greater than zero The concavity of retailer profit function should proof. For this reason, the Hessian matrix is given below.

$$\text{Retailer profit function Hessian matrix} = \begin{pmatrix} \frac{\partial^2 \pi_s}{\partial m^2} & \frac{\partial^2 \pi_s}{\partial m \partial t} \\ \frac{\partial^2 \pi_s}{\partial t \partial m} & \frac{\partial^2 \pi_s}{\partial t^2} \end{pmatrix}.$$

$$\frac{\partial^2 \pi_r}{\partial m^2} = -2\beta \tag{B.13}$$
$$\frac{\partial^2 \pi_r}{\partial t^2} = -2c_2(s-1)(q-1)(a\gamma - \beta p + 3\lambda t + \alpha) \tag{B.14}$$
$$\frac{\partial^2 \pi_r}{\partial t \partial m} = \frac{\partial^2 \pi_r}{\partial m \partial t} = \lambda + 2c_2 t\beta(1-s)(1-q) \tag{B.15}$$
$$|H| = -(4(s-1))c_2(c_2(s-1)(q-1)t^2 + p)(q-1)\beta^2 + 4c_2(s-1)(q-1)(a\gamma + 2\lambda t + \alpha)\beta - \lambda^2. \tag{B.16}$$

The Hessian matrix determinant is greater than zero.

Appendix C.

Proof of the optimality for the Retailer Stackelberg (RS) game

This proof can be done in the similar manner as Appendices A and B.