

ON THE DIFFERENTIAL IN CENTRAL OPERATOR $\mathcal{C}(G)$

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Abstract. The central operator $\mathcal{C}(G)$ is a structural transformation that combines edge subdivision with the complementation on the original vertex set. In this paper, we characterize the behavior of the differential $\partial(\mathcal{C}(G))$, an invariant that measures the maximum influence potential of a network. We establish sharp bounds for $\partial(\mathcal{C}(G))$ in terms of the order n and the maximum degree $\Delta(G)$ of the base graph, proving a guaranteed growth property: for any connected graph of order $n \geq 4$, $\partial(\mathcal{C}(G)) \geq \partial(\overline{G}) + 1$. Our structural analysis reveals a parsimony property of optimal sets. Specifically, Theorem 3.16 shows that minimum differential sets in $\mathcal{C}(G)$ are primarily supported on the original vertex set V , effectively reducing the search space for optimization algorithms. Furthermore, in Theorem 3.22 we establish a notable theoretical convergence by identifying conditions under which the central, subdivision, and $\mathcal{R}(G)$ operators yield the same differential value, namely $m + n - 4$. Finally, we provide exact evaluations for several fundamental graph families and derive Nordhaus–Gaddum type inequalities (Prop. 3.24 and Cor. 3.25) for the class of trees. These results clarify how the central operator enhances the diffusion capacity of networks, thereby bridging the gap between topological transformations and practical influence maximization.

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1. INTRODUCTION

Social networks such as TikTok, Instagram, Facebook, X, and WhatsApp have emerged as powerful tools for communication and the dissemination of information. Due to their massive popularity, these platforms have found widespread applications in areas such as viral marketing and political campaigning. Motivated by these real-world scenarios, several authors [26, 27, 30] have formulated a range of influential maximization problems, all revolving around a common algorithmic challenge in information diffusion: determining the most effective set of nodes to influence the rest of the network.

For instance, in [26], the authors approached this problem using submodular functions and designed an algorithm capable of achieving a solution within 63% of the optimal value under certain conditions. Later,

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in [27], the same group proposed a broader framework based on the decreasing cascade model, which allowed for more general approximations of the influence maximization problem.

Interestingly, as noted in [9], the concept of the differential $\partial(G)$ of a graph G is a parameter that quantifies the influence potential of a subset of nodes, and it can be naturally motivated by diffusion problems in social networks. This connection opens up new perspectives for the study of graph differentials through the lens of real-world applications. Several papers have studied different versions of the differential of a graph, such as [13, 14, 16, 17, 28, 30].

Various studies have analyzed the differential in the context of graph operators [4–7, 10, 36, 40]. Recall that a *graph operator* is a function $\mathcal{O} : \mathcal{G} \rightarrow \mathcal{G}$, where \mathcal{G} denotes the class of all graphs. Understanding how invariants transform under such operators is a classical topic in graph theory. Krausz formalized this notion in [29], introducing four fundamental operators: $Q(G)$, $\mathcal{R}(G)$, $\mathcal{S}(G)$, and $T(G)$. These operators have since been extensively studied due to their relevance in both theoretical and applied contexts. Accordingly, various types of these operators have been investigated in studies related to dynamic networks (see [24, 34, 35]), mathematical chemistry (see [12, 20, 21, 37, 38]), discrete geometry (see [18, 31, 32]), and domination theory (see [8, 15, 33, 39, 41]).

This line of research highlights the theoretical richness and practical relevance of the differential of a graph as a parameter. By examining its behavior under various structural modifications and graph operations, researchers gain deeper insights into how local changes in a network can influence global dynamics. Moreover, the study of its variants contributes to a better understanding of optimization problems and the modeling of real-world scenarios, such as information spread or resource allocation. These developments not only strengthen the mathematical foundations of the concept but also open the door to its application in diverse areas of science and technology. Motivated by these considerations, this article focuses on the differential under the central operator.

Throughout this paper, $G = (V(G), E(G))$ denotes a simple, finite graph of order $n \geq 2$ and size m , with vertex set $V(G)$ and edge set $E(G)$. We recall that the *complement graph* \overline{G} of G is defined on the same vertex set $V(G)$, where two distinct vertices are adjacent if and only if they are nonadjacent in G . If $v \in V(G)$, then $N_G(v)$ denotes the open neighborhood of v , that is, the set of all vertices adjacent to v in G and $N_G[v] := N_G(v) \cup \{v\}$ denotes the closed neighborhood. The number $|N_G(v)|$ is called the *degree* of v in G and is denoted by $\delta_G(v)$. As usual, the minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex v is called a *leaf* if $\delta_G(v) = 1$. If $v \in S \subseteq V(G)$, a vertex $y \in V(G) \setminus S$ is called an *S -external private neighbor* of v if $N_G(y) \cap S = \{v\}$. The set of all such external private neighbors of v is denoted by $epn[v, S]$.

The *distance* $d_G(u, v)$ between two vertices $u, v \in V(G)$ in G is the length of a shortest $u - v$ path in G . The *diameter* $diam(G)$ of G is the greatest distance between any pair of vertices; if G is not connected, then $diam(G) = \infty$. The length of the shortest cycle in G is called the *girth* of G , denoted by $g(G)$; if G does not contain a cycle, then $g(G) = \infty$. A *matching* in G is a set $M \subseteq E(G)$ such that no two edges in M have a vertex in common. The *matching number* $\beta(G)$ is the maximum cardinality of such a matching. We write $G \simeq H$ to indicate that graphs G and H are isomorphic.

Let S be a subset of $V(G)$. We recall that S is an *independent set* of G if any two distinct vertices in S are nonadjacent in G . The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent set of G . Conversely, S is a *clique set* of G if any two distinct vertices in S are adjacent in G . The *clique number* $\omega(G)$ of G is the maximum cardinality of a clique set in G . Note that $\alpha(G) = \omega(\overline{G})$, and similarly, $\omega(G) = \alpha(\overline{G})$. A set $S \subseteq V(G)$ is called a *dominating set* of G if every vertex in $V(G) \setminus S$ at least one neighbor in S . The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set in G . Similarly, S is called a *vertex cover* of G if every edge of G has at least one endvertex in S . The *vertex cover number* $\tau(G)$ of G is the minimum cardinality of a vertex cover in G .

For $S \subseteq V(G)$, we define $N_G(S) := \bigcup_{v \in S} N_G(v)$, $N_G[S] := N_G(S) \cup S$, and denote by $\langle S \rangle_G$ the subgraph of G induced by S . Similarly, we shall use $B_G(S)$ to denote the set of vertices in $V(G) \setminus S$ that have a neighbor in S , and by $C_G(S)$ the set $V(G) \setminus (S \cup B_G(S))$. Then S , $B_G(S)$, and $C_G(S)$ are pairwise disjoint and $V(G) = S \cup B_G(S) \cup C_G(S)$. The *differential* of S is defined as $\partial_G(S) := |B_G(S)| - |S|$, and the *differential* of G is defined as $\partial(G) := \max\{\partial_G(S) : S \subseteq V(G)\}$. A set S is a *differential set* of G if $\partial(G) = \partial_G(S)$. If such a set S

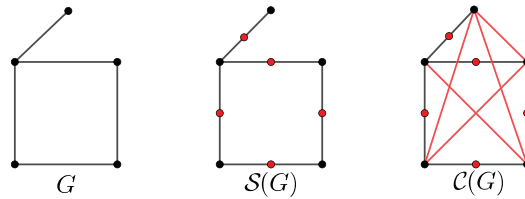


FIGURE 1. In the middle the subdivision graph $\mathcal{S}(G)$ of G , and on the right the central operator $\mathcal{C}(G)$ of G .

has the minimum (respectively, maximum) cardinality among all differential sets, then it is called a *minimum differential set* (respectively, *maximum differential set*).

If there is no ambiguity regarding the graph under consideration, we will omit the reference to G in notations such as $d_G(u, v)$, $\delta_G(v)$, $N_G(v)$, $\partial_G(S)$, $B_G(S)$, $C_G(S)$, $\langle S \rangle_G$, etc.

As usual, we denote by E_n , K_n , $K_{p,q}$, S_n , $S_{p,q}$, P_n , C_n , and W_n the graphs on $n = p + q$ vertices known respectively as the empty graph, complete graph, complete bipartite graph, star graph, double star graph, path graph, cycle graph, and wheel graph.

From the definition of $\partial(G)$, if G is disconnected, and G_1, \dots, G_k are its connected components, then:

$$\partial(G) = \partial(G_1) + \dots + \partial(G_k).$$

In view of this equality, and unless otherwise specified, all graphs considered in this paper will be assumed to be connected.

The *subdivision graph* $\mathcal{S}(G)$ of G is the graph obtained from G by subdividing every edge in $E(G)$ exactly once [19]. The *central operator* of G is obtained by subdividing every edge in $E(G)$ exactly once and connecting each pair of nonadjacent vertices in G , then $E(\mathcal{C}(G))$ is the disjoint union of $E(\mathcal{S}(G))$ and $E(\overline{G})$ (see [1, 2, 42, 43]). See Figure 1.

Due to the structure of $\mathcal{C}(G)$, we can partition the vertex set $V(\mathcal{C}(G))$ into two subsets: the vertex subset corresponding to the edges of G , denoted by J , and the set $V := V(\mathcal{C}(G)) \setminus J$. Then $|V| = |V(G)| = n$ and $|J| = |E(G)| = m$. If $x, y \in V$ are adjacent in G , we denote by v_{xy} the unique vertex in J that is adjacent to both x and y .

2. DOMINATION PARAMETERS OF CENTRAL GRAPHS

In this section, we conduct a systematic study of classical graph invariants, including the diameter and the girth. Moreover, we present and analyze several results concerning domination parameters in central graphs. To this end, we first derive a collection of structural properties of the graph $\mathcal{C}(G)$, which will serve as foundational tools for the development of the subsequent results.

Remark 2.1. Let $G = (V(G), E(G))$ be a graph of order $n \geq 2$ and size m , and let V, J be as in previous paragraph. Then the following hold.

- (1) $|V(\mathcal{C}(G))| = n + m$.
- (2) $|E(\mathcal{C}(G))| = m + \frac{n(n-1)}{2}$.
- (3) $\mathcal{C}(G)$ is 2-regular if $n = 3$, and biregular if $n \neq 3$, i.e., if $v \in V$ and $u \in J$, then $\delta_{\mathcal{C}(G)}(v) = n - 1$, $\delta_{\mathcal{C}(G)}(u) = 2$, $\Delta(\mathcal{C}(G)) = n - 1$ and $\delta(\mathcal{C}(G)) = 2$.
- (4) J is a maximal independent set.
- (5) \overline{G} and $\mathcal{S}(G)$ are subgraphs of $\mathcal{C}(G)$.
- (6) If $G \in \{E_n, K_n\}$, then $\mathcal{S}(G) \simeq \mathcal{C}(G)$.

- (7) If $n \in \{2, 3\}$, then $G \in \{P_2, P_3, C_3\}$, and so
 - (7.1) $\partial(\mathcal{C}(P_2)) = \partial(\mathcal{C}(P_3)) = 1$ and $\partial(\mathcal{C}(C_3)) = 2$.
 - (7.2) $\alpha(\mathcal{C}(P_2)) = \alpha(\mathcal{C}(P_3)) = 2$ and $\alpha(\mathcal{C}(C_3)) = 3$.
 - (7.3) $\tau(\mathcal{C}(P_2)) = 1$, $\tau(\mathcal{C}(P_3)) = \tau(\mathcal{C}(C_3)) = 3$.
 - (7.4) $\gamma(\mathcal{C}(P_2)) = 1$ and $\gamma(\mathcal{C}(P_3)) = \gamma(\mathcal{C}(C_3)) = 2$.

In view of the equalities in Remark 2.1(7), unless otherwise stated, from now on we assume that $n = |V(G)| \geq 4$. When we need to refer to a general finite graph (*i.e.* not necessarily connected, simple, of order $n \geq 4$, etc.), we shall use H to denote it.

We recall that $G = (V(G), E(G))$ denotes a simple connected graph of order $n \geq 4$ and size m . Note that the connectivity of G implies $m \geq n - 1 \geq 3$.

Here we exhibit an interesting relationship between the diameter of $\mathcal{C}(G)$ and the clique number of G , and subsequently, we present a result involving $\partial(\mathcal{C}(G))$ and the diameter of $\mathcal{C}(G)$.

Proposition 2.2. $diam(\mathcal{C}(G)) \in \{2, 3, 4\}$.

Proof. By Remark 2.1(3) we know that $\Delta(\mathcal{C}(G)) = n - 1 \geq 3$ and $\delta(\mathcal{C}(G)) = 2$. Since complete graphs are the only graphs with diameter 1, then $\mathcal{C}(G)$ is not a complete graph, and so $diam(\mathcal{C}(G)) \geq 2$.

Let x and y be two vertices of $V(\mathcal{C}(G))$. We now show that $d(x, y) \leq 4$.

- (A1) If $x, y \in V$, then $d(x, y) \leq 2$. Indeed, if $xy \notin E(G)$, then x and y are adjacent in $\mathcal{C}(G)$ and we are done. Similarly, if $xy \in E(G)$, then the path $xv_{xy}y$ guarantees that $d_{\mathcal{C}(G)}(x, y) \leq 2$.
- (A2) If $x \in J$ and $y \in V$, then $d(x, y) \leq 3$. Indeed, let $w, z \in V$ be such that $x = v_{wz}$. Then $d(x, y) \leq d(x, z) + d(z, y) \leq 1 + 2$, by (A1).
- (A3) If $x, y \in J$, then $d(x, y) \leq 4$. Indeed, let $w, z \in V$ be such that $x = v_{wz}$. Then $d(x, y) \leq d(x, z) + d(z, y) \leq 1 + 3$, by (A2).

□

Corollary 2.3. *If $v \in V = V(\mathcal{C}(G)) \setminus J$, then $d_{\mathcal{C}(G)}(v, u) \leq 3$ for any $u \in V(\mathcal{C}(G))$.*

Theorem 2.4. *The following hold:*

- (1) $diam(\mathcal{C}(G)) = 4$ iff $\omega(G) \geq 4$,
- (2) $diam(\mathcal{C}(G)) = 2$ iff $G \simeq S_n$,
- (3) and otherwise $diam(\mathcal{C}(G)) = 3$.

Proof. We recall that $n \geq 4$ and $m = |E(G)| \geq n - 1 \geq 3$, by the connectivity of G .

- (1) Suppose that $diam(\mathcal{C}(G)) = 4$. Then $\mathcal{C}(G)$ has two vertices x, y such that the distance between x and y in $\mathcal{C}(G)$ is equal to 4. From Corollary 2.3 it follows that $x, y \in J$. Let $u_1, u_2, w_1, w_2 \in V$ be such that $x = v_{u_1u_2}$ and $y = v_{w_1w_2}$. Then $u_1u_2 \in E(G)$ and $w_1w_2 \in E(G)$. Since $d_{\mathcal{C}(G)}(x, y) = 4$, the pairs $\{u_1, u_2\}$ and $\{w_1, w_2\}$ must be disjoint. If there exist $u \in \{u_1, u_2\}$ and $w \in \{w_1, w_2\}$ such that $uw \notin E(G)$, then the path $xuwy$ in $\mathcal{C}(G)$ has length 3, contradicting the assumption. Hence $\langle \{u_1, u_2, w_1, w_2\} \rangle_G \simeq K_4$, as required. Conversely, suppose $\omega(G) \geq 4$. Then V has four distinct vertices, say u_1, u_2, w_1, w_2 , inducing a K_4 . It is not hard to see that if $x := v_{u_1u_2}$ and $y := v_{w_1w_2}$, then $d_{\mathcal{C}(G)}(x, y) \geq 4$. By Proposition 2.2, it follows that $diam(\mathcal{C}(G)) = 4$.
- (2) Suppose that $diam(\mathcal{C}(G)) = 2$. Since any path $xywz$ of length 3 in G implies that $d_{\mathcal{C}(G)}(v_{xy}, v_{wz}) \geq 3$, we may assume that G contains no paths of length 3. Taking this into account, together with the condition $n \geq 4$ and the connectivity of G , it follows that $G \simeq S_n$. On the other hand, it is straightforward to verify that $diam(\mathcal{C}(S_n)) = 2$.
- (3) It follows immediately from (1), (2), and Proposition 2.2.

□

The next lemma was proved in [11].

Lemma 2.5. *Let G be a graph of order $n \geq 3$, $\delta := \delta(G)$ and $\Delta := \Delta(G)$. The following hold:*

- (1) *If $\text{diam}(G) = 2$, then $n - 2\delta \leq \partial(G) \leq (\Delta - 1)^2$.*
- (2) *If $\text{diam}(G) = 3$, then $2(\delta - 1) \leq \partial(G) \leq (\Delta - 1)^3 + 1$.*

Proposition 2.6. *If G is a graph of order $n \geq 2$ and size m , then:*

- (1) *If $\text{diam}(\mathcal{C}(G)) = 2$, then $\partial(\mathcal{C}(G)) = 2n - 5$.*
- (2) *If $\text{diam}(\mathcal{C}(G)) = 3$, then $2 \leq \partial(\mathcal{C}(G)) \leq (n - 2)^3 + 1$.*

Proof. By Theorem 2.4(2) we know that $\text{diam}(\mathcal{C}(G)) = 2$ iff $G \simeq S_n$. From this, it follows that $\partial(\mathcal{C}(S_n)) = 2n - 5$. This proves (1). If $\text{diam}(\mathcal{C}(G)) = 3$, then $n \geq 4$ and the required result follows by combining Remark 2.1(3) and Lemma 2.5(2). □

Here we exhibit a close relationship between $g(\mathcal{C}(G))$ and the values of $\alpha(\mathcal{C}(G))$ and $\delta(G)$, and afterward, with the help of Lemma 2.10, we provide bounds for $\partial(\mathcal{C}(G))$.

Proposition 2.7. *Let H be a simple graph of order $n \geq 2$. The following hold:*

- (1) *$g(\mathcal{C}(H)) = \infty$ iff $n = 2$, and*
- (2) *$g(\mathcal{C}(H)) \in \{3, 4, 5, 6\}$ if $n \geq 3$.*

Proof. Since H is simple, then $\mathcal{C}(H)$ is simple. This implies that $g(\mathcal{C}(H)) \geq 3$.

If $n = 2$, then $H \in \{E_2, P_2\}$. Since $\mathcal{C}(P_2) = P_3$ and $\mathcal{C}(E_2) = P_2$, then $g(\mathcal{C}(H)) = \infty$, as claimed.

On the other hand, since for $n \geq 3$, we have that $\mathcal{C}(H)$ is a topological minor of K_n , then a triangle Δ of K_n corresponds to a cycle C_Δ of $\mathcal{C}(G)$ of length either 3, 4, 5 or 6. Then if $g(\mathcal{C}(H)) = \infty$, we must have $n = 2$, as required. □

Proposition 2.8. *Let H be a simple graph of order $n \geq 3$. If x and y are distinct vertices of $V(H)$ such that $xy \notin E(H)$, then $g(\mathcal{C}(H)) \leq 5$.*

Proof. Since $xy \notin E(H)$, then $xy \in E(\mathcal{C}(H))$. Let $z \in V \setminus \{x, y\}$. By (A1) in the proof of Proposition 2.2, we know that $d_{\mathcal{C}(H)}(x, z) \leq 2$ and $d_{\mathcal{C}(H)}(y, z) \leq 2$. Then x, y and z are contained in a cycle of $\mathcal{C}(H)$ of length at most 5, as claimed. □

Theorem 2.9. *Let H be a simple graph of order $n \geq 4$.*

- (1) *$g(\mathcal{C}(H)) = 3$ iff $\alpha(H) \geq 3$,*
- (2) *$g(\mathcal{C}(H)) = 4$ iff $\alpha(H) = 2$ and $\delta(H) \leq n - 3$,*
- (3) *$g(\mathcal{C}(H)) = 5$ iff $\alpha(H) = 2$ and $\delta(H) = n - 2$,*
- (4) *$g(\mathcal{C}(H)) = 6$ iff $\alpha(H) = 1$ iff $H \simeq K_n$.*

Proof. We start by noting that if C is a cycle of $\mathcal{C}(H)$ containing $v_{xy} \in J$, then C must contain also x and y . Since the existence of v_{xy} implies that xy cannot be an edge of $\mathcal{C}(H)$, then C must have length at least 4.

- (1) Suppose that $g(\mathcal{C}(H)) = 3$. Then $\mathcal{C}(H)$ has a triangle, say xyz . From previous paragraph it follows that $\{x, y, z\} \cap J = \emptyset$, and so $\{x, y, z\}$ must form an independent set of H , as required. The converse implication follows immediately from the involved definitions and Proposition 2.7(2).
- (4) Since $\alpha(H) = 1$ iff $H \simeq K_n$ is trivial, we only show that $g(\mathcal{C}(H)) = 6$ iff $H \simeq K_n$. From $n \geq 4$ and Proposition 2.7(1) we know that $g(\mathcal{C}(H)) \in \{3, 4, 5, 6\}$. Suppose that $g(\mathcal{C}(H)) = 6$. Then each pair of distinct vertices in $V(H)$ must be adjacent in H by Proposition 2.8, and so $H \simeq K_n$. Conversely, suppose that $H \not\simeq K_n$. Then H has distinct vertices x and y such that $xy \notin E(H)$. This last and Proposition 2.8 imply that $g(\mathcal{C}(H)) \neq 6$, as required.

- (2) Suppose that $g(\mathcal{C}(H)) = 4$. Let $C := xywz$ be a cycle of $\mathcal{C}(H)$ of length 4. It is not hard to see that if $|\{x, y, w, z\} \cap J| \geq 2$, then C must have length at least 5. Then we can assume that $|\{x, y, w, z\} \cap J| \leq 1$, and so C has at least 2 consecutive edges f, g such that $f, g \notin E(H)$. Assume w.l.o.g. that x is the common vertex of f and g . Then $\delta(H) \leq \delta_H(x) \leq n - 3$. On the other hand, from (1) we know that $\alpha(H) \leq 2$. Since $\alpha(H) = 1$ implies $H \simeq K_n$, then we must have $\alpha(H) = 2$ in view of (4).
 Suppose now that $\delta(H) \leq n - 3$ and $\alpha(H) = 2$. From (1), (4) and Proposition 2.7 we know that $g(\mathcal{C}(H)) \in \{4, 5\}$. Let $z \in V(H)$ be such that $\delta_H(z) \leq n - 3$. Then $V(H) \setminus \{z\}$ has two distinct vertices x and y such that $xz, yz \notin E(H)$. Since $\alpha(H) < 3$, we must have that $xy \in E(H)$, and so $xv_{xy}yz$ forms a cycle of $\mathcal{C}(H)$ of length 4, as required.
- (3) It follows immediately from (1), (2), (4) and Proposition 2.7.

□

The following result appears in [3].

Lemma 2.10. *Let G be a graph with finite girth $g(G)$ and order n , then*

$$\left\lfloor \frac{g(G)}{3} \right\rfloor \leq \partial(G) \leq n - 2 \left\lfloor \frac{g(G)}{3} \right\rfloor.$$

Proposition 2.11. *Let G be a graph of order $n \geq 2$ and size m . Then $n - 2 \leq \partial(\mathcal{C}(G)) \leq n + m - 4$.*

Proof. For the lower bound, observe that $\partial(\mathcal{C}(G)) \geq \Delta(\mathcal{C}(G)) - 1$, and recall that $\Delta(\mathcal{C}(G)) = n - 1$ (see Rem. 2.1(3)). Therefore, it follows that $\partial(\mathcal{C}(G)) \geq n - 2$. The upper bound is a combination of Remark 2.1(1), Proposition 2.7 and Lemma 2.10. □

It is well-known there is a close relationship between the independence number of a graph and its cover number. Now, we explore this relationship and establish a structural property of the maximum/minimum independent/cover sets of $\mathcal{C}(G)$. A weak version of the following result appears in [25].

Theorem 2.12. *Let G be a connected graph of order $n \geq 4$. If I is a maximum independent set of $\mathcal{C}(G)$, then $|I \setminus J| \leq 2$. In particular, J is a maximum independent set of $\mathcal{C}(G)$ and $\alpha(\mathcal{C}(G)) = m$.*

Proof. Clearly, J is a maximal independent set of $\mathcal{C}(G)$ of size m , and hence $\alpha(\mathcal{C}(G)) \geq m$. Now suppose that I is a maximum independent set of $\mathcal{C}(G)$ that is distinct from J . For brevity, let $V' := I \setminus J$. Then $V' \subseteq V$. In particular, if x and y are distinct vertices of V' , the choice of I guarantees that $xy \in E(G)$.

Seeking a contradiction, suppose that $|V'| \geq 3$. Let x, y and z be three distinct vertices of V' . By previous paragraph, we know that $K_3 \simeq \langle \{x, y, z\} \rangle_G$. Since $n \geq 4$ and G is connected, we can assume that x has a third neighbor in G , say u .

Since if $u \notin V'$, then $I' := (I \setminus \{x, y, z\}) \cup \{v_{xy}, v_{yz}, v_{xz}, v_{xu}\}$ is an independent set of $\mathcal{C}(G)$ bigger than I , we can assume that $u \in V'$. Again, by the conclusion in the first paragraph of this proof and $x, y, z, u \in V'$, it follows that $K_4 \simeq \langle \{x, y, z, u\} \rangle_G$, and hence $I'' := (I \setminus \{x, y, z, u\}) \cup \{v_{xy}, v_{yz}, v_{xz}, v_{xu}, v_{yu}, v_{zu}\}$ is an independent set bigger than I . Since in both cases the maximality of I is contradicted, we can conclude that $|V'| \leq 2$.

- Suppose that $I \setminus J = \{x\}$. Since $n \geq 4$ and G is connected, then x has at least one neighbor in G , say y . Since $y \notin I \setminus J$, then $X := (I \setminus \{x\}) \cup \{v_{xy}\} \subseteq J$ is an independent set of the same size as I , and so we must have that $J = X$, as required.
- Suppose finally that $I \setminus J = \{x, y\}$. Then $xy \in E(G)$. From the connectivity of G and $n \geq 4$, we can assume that y has a second neighbor z in G such that $z \notin I \setminus J$. Then, $Z := (I \setminus \{y\}) \cup \{v_{yz}\}$ is an independent set of the same size as I with $Z \setminus J = \{x\}$. By applying the argument in the previous case to $Z \setminus J = \{x\}$, we can conclude that $|J| = |Z| = |I|$, as required.

□

Remark 2.13. The following hold:

- If $G \simeq P_2$, then V is the unique maximum independent set of $\mathcal{C}(P_2)$.
- If $G \simeq P_3$ and $P_3 = xyz$, then $\mathcal{C}(P_3)$ has three maximum independent sets: J , $\{x, y\}$ and $\{y, z\}$.
- If $G \simeq C_3$, then V and J are the only maximum independent sets of $\mathcal{C}(C_3)$.

The following Gallai-type result was established in [23].

Lemma 2.14. $|V(G)| = \alpha(G) + \tau(G)$.

Theorem 2.15. *Let G be a connected graph of order $n \geq 4$. If U is a minimum vertex cover of $\mathcal{C}(G)$, then $|V \setminus U| \leq 2$. In particular, V is a minimum vertex cover of $\mathcal{C}(G)$ and $\tau(\mathcal{C}(G)) = n$.*

Proof. From Theorem 2.12 and Lemma 2.14 we know that $\tau(\mathcal{C}(G)) = n$. Since V is a vertex cover of $\mathcal{C}(G)$ with $|V| = n$, then V is a minimum vertex cover. Now suppose that $U \subseteq V(\mathcal{C}(G))$ is a minimum vertex cover of $\mathcal{C}(G)$ distinct from V . We need to show that $|V \setminus U| \leq 2$.

Seeking a contradiction, suppose that $|V \setminus U| \geq 3$. We note that if $u \in V \setminus U$, then $N_{\mathcal{C}(G)}(u) \subseteq U$, as any edge of $\mathcal{C}(G)$ that is incident with u must have an endvertex in U . We recall that $|N_{\mathcal{C}(G)}(u)| = n - 1$ by Remark 2.1(3).

Suppose that x and y are distinct vertices of $V \setminus U$. Since if $xy \notin E(G)$, then $xy \in E(\mathcal{C}(G))$ and U could not cover xy in $\mathcal{C}(G)$, we must have that $xy \in E(G)$.

Let x, y and z be three distinct vertices of $V \setminus U$. By previous paragraph, we can assume that xy, xz and yz are edges of G (forming a 3-cycle). Since $n \geq 4$ and G is connected, we can assume that x has a third neighbor, say u . Then $v_{xu}, v_{xz} \in U$. Since $N_{\mathcal{C}(G)}(y)$ and $\{v_{xu}, v_{xz}\}$ are disjoint, we have $|U| \geq (n - 1) + 2$, a contradiction. \square

Remark 2.16. The following hold:

- If $G \simeq P_2$, then J is the unique minimum vertex cover of $\mathcal{C}(P_2)$.
- If $G \simeq P_3$ and $P_3 = xyz$, then $\mathcal{C}(P_3)$ has four minimum vertex covers:

$$V, \{x, y, u_{yz}\}, \{x, u_{xy}, u_{yz}\}, \{u_{xy}, u_{yz}, z\}.$$

- If $G \simeq C_3$, then V and J are the only minimum vertex covers of $\mathcal{C}(C_3)$.

We emphasize that Theorems 2.12 and 2.15 are best possible. Indeed, suppose that G is the path graph $P_n := x_1, x_2, \dots, x_n$ and that, as usual, $v_{x_i x_{i+1}} \in J$ is the subdividing vertex of the edge $v_i v_{i+1}$. It is not difficult to see that $I = \{x_1, x_2, v_{x_3 x_4}, v_{x_4 x_5}, \dots, v_{x_{n-1} x_n}\}$ is a maximum independent set of $\mathcal{C}(P_n)$, and that $U = \{v_{x_1 x_2}, v_{x_2 x_3}, x_3, x_4, \dots, x_n\}$ is a minimum vertex cover of $\mathcal{C}(P_n)$.

Now, we present some interesting results concerning $\gamma(\mathcal{C}(G))$, which reflect structural properties of the operator within specific classes of graphs. The proof of the following is straightforward.

Observation 2.17. Let $n \geq 4$ be an integer. Then $\gamma(\mathcal{C}(S_n)) = 2$, $\gamma(\mathcal{C}(K_n)) = n - 1$ and $\gamma(\mathcal{C}(S_{p,q})) = 2$, where $p, q \geq 2$ and $n = p + q$.

Proposition 2.18. *Let G be a simple connected graph of order $n \geq 4$. Then*

$$2 \leq \beta(G) \leq \gamma(\mathcal{C}(G)) \leq n - 1.$$

Moreover, these bounds are tight.

Proof. By Observation 2.17 we know that these bounds are tight. On the other hand, let $k := \beta(G)$, and let $x_1 y_1, \dots, x_k y_k$ be a maximum matching of G . We know that any dominating set of $\mathcal{C}(G)$ must include at least one vertex of $\{x_i, v_{x_i y_i}, y_i\}$ for each $i \in \{1, \dots, k\}$, then $n \geq 4$ implies $\gamma(\mathcal{C}(G)) \geq k \geq 2$, as required.

Since the result holds for $G \simeq K_n$, by Observation 2.17, we may assume that there are distinct vertices $x, y \in V$ such that $xy \notin E(G)$. It can be observed that $V \setminus \{x\}$ is a dominating set of $\mathcal{C}(G)$, and therefore $\gamma(\mathcal{C}(G)) \leq n - 1$. \square

Theorem 2.19. *Let G be a connected graph of order $n \geq 4$. Then $\mathcal{C}(G)$ contains a minimum dominating set D such that $|D \cap J| \leq 1$.*

Proof. Let D be a minimum dominating set of $\mathcal{C}(G)$ which minimizes $|D \cap J|$. Since $|D \cap J| \leq 1$ implies there is nothing to prove, we may assume that $v_{xy}, v_{wz} \in D \cap J$ for distinct edges xy and wz of G . Since if $x \in D$, then $D \setminus \{v_{xy}\} \cup \{y\}$ contradicts the choice of D , we can assume that $x \notin D$. Similarly, we can conclude that y, w, z are not in D .

If there is $u \in D \setminus \{v_{xy}\}$ such that $xu \in E(\mathcal{C}(G))$, then $D_y := (D \setminus \{v_{xy}\}) \cup \{y\}$ contradicts the choice of D . Then v_{xy} is the only vertex of D that is adjacent to x . Similarly, we can deduce that v_{xy} is the only vertex of D that is adjacent to y , and that v_{wz} is the only vertex of D that is adjacent to w (respectively, z) in $\mathcal{C}(G)$. The last two conclusions and $v_{xy}, v_{wz} \in D$ imply that $\{x, y\} \cap \{w, z\} = \emptyset$.

We note that if xz is an edge of G , then v_{xz} must be in D because D is a dominating set of $\mathcal{C}(G)$ and $x, z \notin D$. Since $v_{xz} \in D$ contradicts that v_{xy} is the only vertex of D that is adjacent to x , we must have that $xz \notin E(G)$. Similarly, we can deduce that xw, yw, yz are not edges of G . Then xw and yz are edges of $\mathcal{C}(G)$ and $xv_{xy}yzv_{wz}wx$ forms a 6-cycle in $\mathcal{C}(G)$. Observe that $D_{y,w} := (D \setminus \{v_{xy}, v_{wz}\}) \cup \{y, w\}$ is a dominating set of $\mathcal{C}(G)$ such that $|D_{y,w}| = |D|$ and $|D_{y,w} \cap J| < |D \cap J|$, contradicting the choice of D . \square

We remark that Theorem 2.19 is best possible. Indeed, let $p \geq 3$ be an integer, and let x, y, w, z be the four vertices of K_4 . Let G be the graph that results by adding p pendant edges to K_4 in each of x and y . It can be verified that $D = \{x, y, v_{wz}\}$ is the unique minimum dominating set of $\mathcal{C}(G)$. Another example of this tightness is given by the class of complete graphs.

Proposition 2.20. *Let G be a connected graph of order $n \geq 4$ and size m . Then $\gamma(\mathcal{C}(G)) = 2$ iff $G \in \{S_n, S_{p,q}\}$, where $p, q \geq 2$ and $n = p + q$.*

Proof. By Observation 2.17 we need only show the forward implication. Suppose that $\gamma(\mathcal{C}(G)) = 2$. By Theorem 2.19 we know that $\mathcal{C}(G)$ has a minimum dominating set $D = \{z, z'\}$ such that $|D \cap J| \leq 1$. If $z' \in J$, then $z' = v_{xy}$ for some edge xy of G and so $z \in V$. The connectivity of G and $n \geq 4$ imply the existence of a vertex $u \in V \setminus \{x, y, z\}$ that is adjacent in G to at least one of $\{x, y, z\}$. Since if $uz \in E(G)$, then u cannot be dominated by D , we must have that $uw \in E(G)$ for some $w \in \{x, y\}$. But then v_{uw} cannot be dominated by D . Thus we can assume that $z, z' \in V$.

Let P be a shortest $z - z'$ path in G , and let $\mathcal{C}(P)$ be the $z - z'$ subgraph of $\mathcal{C}(G)$ corresponding to P . Since D must dominate all the vertices in $J \cap V(\mathcal{C}(P))$, then zz' must be an edge of G . Similarly, since D dominates J , then any edge of G is incident with at least one of z or z' . Hence, due to the connectivity of G and the requirement that every edge is incident to either z or z' , the graph must be isomorphic to either S_n or $S_{p,q}$. \square

In [22], the authors provide a classification of graphs based on the domination number of central operator and the vertex cover of the original graphs. In this work, we include new proofs of these relations for the sake of completeness.

Corollary 2.21. *Let G be a simple connected graph of order $n \geq 4$. Then*

$$\tau(G) \leq \gamma(\mathcal{C}(G)) \leq \tau(G) + 1.$$

Proof. By Theorem 2.19 there exists a minimum dominating set D of $\mathcal{C}(G)$ such that $|D \cap J| \leq 1$. If $v_{xy} \in D$ for some edge $xy \in E(G)$, then $D' := (D \setminus \{v_{xy}\}) \cup \{x\}$ is a vertex cover of G . Similarly, if $D \subseteq V$, then D dominates J and so D is a vertex cover of G . Since $|D'| = |D| = \gamma(\mathcal{C}(G))$, we conclude that $\gamma(\mathcal{C}(G)) \geq \tau(G)$.

Let X be a minimum vertex cover of G , and let $Y := V \setminus X$. If $|Y| = 0$, then $V = X$ is clearly a dominating set of $\mathcal{C}(G)$, and so $\gamma(\mathcal{C}(G)) \leq |X| = \tau(G)$, as required. Suppose then that $Y = \{y_1, \dots, y_r\}$ for some integer $r \geq 1$. From the choice of X , we know that any edge incident with a vertex of Y has its other endvertex in X . Then Y is an independent set of G , and so $X' := X \cup \{y_1\}$ is a dominating set of $\mathcal{C}(G)$. From this last it follows that $\gamma(\mathcal{C}(G)) \leq |X'| = \tau(G) + 1$, as claimed. \square

Proposition 2.22. *Let G be a simple connected graph of order $n \geq 4$ with minimum degree $\delta(G)$. Then*

$$\gamma(\mathcal{C}(G)) \leq \left\lfloor \frac{n + \delta(G) + 1}{2} \right\rfloor.$$

Moreover, this bound is tight.

Proof. Let $u \in V$ be a vertex of minimum degree in G , and let $\{x_1y_1, \dots, x_ky_k\}$ be the set of edges with both endvertices in $C_G(\{u\})$. The application of the operator $\mathcal{C}(G)$ to the graph G gives rise to the following statements:

- $B_G(\{u\})$ dominates $J \setminus \{v_{x_1y_1}, \dots, v_{x_ky_k}\}$.
- $\{u\}$ dominates $C_G(\{u\})$.
- $\{v_{x_1y_1}, \dots, v_{x_ky_k}\}$ is dominated by $\{x_1, \dots, x_k\}$.

Therefore, the set $\{u\} \cup B_G(\{u\}) \cup \{x_1, \dots, x_k\}$ forms a dominating set in $\mathcal{C}(G)$, and hence

$$\gamma(\mathcal{C}(G)) \leq 1 + \delta(G) + \frac{n - (\delta(G) + 1)}{2} = \left\lfloor \frac{n + \delta(G) + 1}{2} \right\rfloor.$$

Finally, a straightforward calculation shows that $\gamma(\mathcal{C}(P_{2k+1})) = \left\lfloor \frac{(2k+1) + \delta(G) + 1}{2} \right\rfloor$. □

3. THE DIFFERENTIAL ON THE CENTRAL GRAPH $\mathcal{C}(G)$

In this section, we investigate the differential of the central graph $\mathcal{C}(G)$, focusing on its structural behavior and extremal properties. We derive general bounds for the differential in terms of fundamental parameters of the base graph and establish conditions under which the differential exhibits guaranteed growth under the central graph transformation. These results provide a deeper understanding of how the central operator influences domination-related invariants.

The following result appears in [3].

Lemma 3.1. *For any graph G , $\partial(G) + \gamma(G) \geq \alpha(G)$.*

Proposition 3.2. *Let G be a connected graph of order $n \geq 4$ and size m . Then $\partial(\mathcal{C}(G)) \geq m - n + 1$.*

Proof. By Theorem 2.12 and Proposition 2.18 we know that $\alpha(\mathcal{C}(G)) = m$ and $\gamma(\mathcal{C}(G)) \leq n - 1$, respectively. Then Lemma 3.1 implies the required result. □

The following result is well known (see [3, 30]).

Lemma 3.3. *Let G be a graph of order n with maximum degree $\Delta(G)$. Then:*

$$n - 2\gamma(G) \leq \partial(G) \leq \gamma(G)(\Delta(G) - 1).$$

For the following result, μ is defined as $\min\{\lfloor \frac{n + \delta(G) + 1}{2} \rfloor, n - 1\}$.

Proposition 3.4. *Let G be a graph of order n , size m , and minimum degree $\delta(G)$. Then:*

$$n + m - 2\mu \leq \partial(\mathcal{C}(G)) \leq \mu(n - 2).$$

Proof. Recall that $\Delta(\mathcal{C}(G)) = n - 1$ (see Rem. 2.1(3)). By Propositions 2.18 and 2.22, we know that $\gamma(\mathcal{C}(G)) \leq n - 1$ and $\gamma(\mathcal{C}(G)) \leq \lfloor \frac{n + \delta(G) + 1}{2} \rfloor$, respectively. Therefore, Lemma 3.3 implies the desired result. □

Proposition 3.5. *Let G be a simple connected graph of order $n \geq 2$, then:*

- (1) $\partial(\mathcal{C}(G)) = 1$ iff $G = P_2, P_3$,

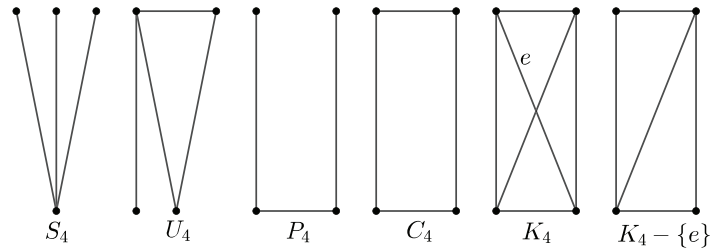


FIGURE 2. All connected graphs of order 4.

- (2) $\partial(\mathcal{C}(G)) = 2$ iff $G = C_3$,
- (3) $\partial(\mathcal{C}(G)) = 3$ iff G has order 4 and $G \neq K_4$.

Proof. (1) Clearly, $\partial(\mathcal{C}(P_2)) = 2 = \partial(\mathcal{C}(P_3))$. Suppose now that $\partial(\mathcal{C}(G)) = 1$. Then the order of G must be either 2 or 3 by $\partial(\mathcal{C}(G)) \geq n - 2$. Since P_2, P_3 and C_3 are the only connected graphs of order $n \in \{2, 3\}$ and $\partial(\mathcal{C}(C_3)) = 2$, we are done.

(2) Clearly, $\partial(\mathcal{C}(C_3)) = 2$. Suppose now that $\partial(\mathcal{C}(G)) = 2$. Then the order of G must be either 3 or 4, since $\partial(\mathcal{C}(G)) \geq n - 2$. If $n = 4$, then G is one of the graphs in Figure 2, and so $\partial(\mathcal{C}(G)) \geq 3$. Then, we can assume that $n = 3$. Since P_3 and C_3 are the only connected graphs of order 3, and $\partial(\mathcal{C}(P_3)) = 1$ by previous case, we must have $G \simeq C_3$, as claimed.

(3) Let $F := \{S_4, U_4, P_4, C_4, K_4 \setminus \{e\}\}$ be the graphs in Figure 2. We note that F contains every connected graph of order 4, except for K_4 . A routine calculation shows that if $G \in F$, then $\partial(\mathcal{C}(G)) = 4$. Suppose now that $\partial(\mathcal{C}(G)) = 4$. Since P_2, P_3 and C_3 are the only connected graphs of order $n \in \{2, 3\}$, we can assume that $n \geq 4$. Similarly, since $\partial(\mathcal{C}(G)) \geq n - 2$, we know that G has order at most 5. Since $\partial(\mathcal{C}(K_4)) = 4$, it is enough to show that if $n = 5$, then $\partial(\mathcal{C}(G)) \neq 3$.

Suppose that $n = 5$. Let x be a maximum degree vertex of G and let x_1, \dots, x_k be its neighbors. The connectivity of G implies that $k \in \{2, 3, 4\}$. If $x_i x_j \in E(G)$, then $\partial(\mathcal{C}(G)) \geq \partial_{\mathcal{C}(G)}(\{x, v_{x_i, x_j}\}) \geq 4$. Then we can assume that $\{x_1, \dots, x_k\}$ is an independent set of G . If $k \geq 3$, then $\partial(\mathcal{C}(G)) \geq \partial_{\mathcal{C}(G)}(\{x, x_1\}) \geq 4$. Then we can assume that $k = 2$, and so G is either P_5 or C_5 . In both cases, it can be observed that $\partial(\mathcal{C}(G)) = 4$. \square

Proposition 3.6. *If G is a simple connected graph of order $n \geq 4$, then $\partial(\mathcal{C}(G)) \geq n + \Delta(G) - 4$.*

Proof. Let x be a maximum degree vertex of G , and let $k := \Delta(G)$. Let y_1, \dots, y_k be the neighbors of x in G . Then, for every $i \in \{2, \dots, k\}$, the vertex y_1 is not adjacent to v_{xy_i} in $\mathcal{C}(G)$. On the other hand, by Remark 2.1(3) we know that $\delta_{\mathcal{C}(G)}(y_1) = n - 1$. Then,

$$\partial(\mathcal{C}(G)) \geq \partial_{\mathcal{C}(G)}(\{x, y_1\}) \geq (k - 1) + (n - 1) - 2 = n + \Delta(G) - 4.$$

\square

Corollary 3.7. *If G is a connected graph of order $n \geq 4$, then $\partial(\mathcal{C}(G)) \geq n - 1$.*

Proof. Since G is connected and $n \geq 4$, it follows that $\Delta(G) \geq 2$. If $\Delta(G) \geq 3$, then $\partial(\mathcal{C}(G)) \geq n + \Delta(G) - 4 \geq n - 1$.

Suppose then that $\Delta(G) = 2$. Thus, $G \in \{P_n, C_n\}$. Let $x_1 x_2, x_2 x_3$ and $x_3 x_4$ be three consecutive edges of G . Since $B_{\mathcal{C}(G)}(\{x_2, x_3\}) = (V \setminus \{x_2, x_3\}) \cup \{v_{x_1 x_2}, v_{x_2 x_3}, v_{x_3 x_4}\}$, then $\partial_{\mathcal{C}(G)}(\{x_2, x_3\}) = n - 1$. \square

Corollary 3.8. *If G is a connected graph of order $n \geq 4$, then $\partial(\mathcal{C}(G)) \geq \partial(\overline{G}) + 1$.*

Proof. Let \overline{D} be a differential set of \overline{G} . If $\overline{D} = \emptyset$, then $\partial(\overline{G}) = 0$, and the result follows since $\partial(\mathcal{C}(G)) \geq n - 2$. If $\overline{D} \neq \emptyset$, then $\partial(\overline{G}) \leq n - 2|\overline{D}| \leq n - 2$. The required inequality follows from Corollary 3.7. \square

The following results appear in [3, 11], respectively.

Lemma 3.9. *The following hold:*

- (1) *If D is a differential set of G , then $|D| \leq \gamma(G)$.*
- (2) *If D is a minimum differential set of G , then $|epn[x, D]| \geq 2$ for any $x \in D$.*

Proposition 3.10. *Let G be a connected graph of order $n \geq 4$ and let D be a differential set of $\mathcal{C}(G)$. Then:*

- (1) $2 \leq |D| \leq n - 1$.
- (2) $D \cap V \neq \emptyset$.

Proof. (1) By Lemma 3.9(1) and Proposition 2.18 we know that $|D| \leq \gamma(\mathcal{C}(G)) \leq n - 1$. Since $\partial(\{x\}) \leq n - 2$ for each $x \in V(\mathcal{C}(G))$ and $\partial(D) \geq n - 1$ by Corollary 3.7, then $|D| \geq 2$.

(2) If $D \subseteq J$, then $B(D) \subseteq V$ and, as a consequence, $\partial(D) \leq n - |D| \leq n - 2$, contradicting Corollary 3.7. \square

If G is a simple finite graph of order $n \geq 2$, then $\mathcal{S}(G)$ is the graph that results by subdividing (exactly) once each edge of G . The next lemma was proved in [6].

Lemma 3.11. *If $n \geq 2$, then $\partial(\mathcal{S}(K_n)) = \frac{n(n-1)}{2} - n + 2$.*

Proposition 3.12. *If H is a subgraph of G , then $\partial(\mathcal{C}(H)) \leq \partial(\mathcal{C}(G))$.*

Proof. Let D be a differential set of $\mathcal{C}(H)$. In order to show that $\partial(\mathcal{C}(H)) \leq \partial(\mathcal{C}(G))$, it is enough to show that $|B_{\mathcal{C}(H)}(D)| \leq |B_{\mathcal{C}(G)}(D)|$. Clearly, $D \subseteq V(\mathcal{C}(G))$. Let $x \in X := B_{\mathcal{C}(H)}(D) \setminus B_{\mathcal{C}(G)}(D)$. Then there is $x' \in D$ such that $x'x \in E(\mathcal{C}(H))$ but $x'x \notin E(\mathcal{C}(G))$. This implies that $x'x \notin E(H)$ and $x'x \in E(G)$. Then $v_{x'x} \in B_{\mathcal{C}(G)}(D) \setminus B_{\mathcal{C}(H)}(D)$. It is straightforward to verify that the mapping $x \rightarrow v_{x'x}$ defines an injective function from X to $B_{\mathcal{C}(G)}(D) \setminus B_{\mathcal{C}(H)}(D)$, and so we are done. \square

Corollary 3.13. *If G is a graph of order $n \geq 2$, then $\partial(\mathcal{C}(G)) \leq \partial(\mathcal{C}(K_n)) = \frac{n(n-1)}{2} - n + 2$.*

Proof. From the definitions of $\mathcal{S}(K_n)$ and $\mathcal{C}(K_n)$, it can be deduced that $\mathcal{S}(K_n) \simeq \mathcal{C}(K_n)$. The result follows from Lemma 3.11 and Proposition 3.12. \square

Proposition 3.14. *Let $E_n, P_n, C_n, S_n, S_{p,q}, K_{p,q}$ and K_n be the graph without edges, path, cycle, star, double star, complete bipartite and complete graphs of order $n \geq 4$, respectively. Then the following hold.*

- (1) $\partial(\mathcal{C}(E_n)) = n - 2$.
- (2) $\partial(\mathcal{C}(P_n)) = n - 1$ if $n \in \{4, 5\}$ and $\partial(\mathcal{C}(P_n)) = n$ for $n \geq 6$.
- (3) $\partial(\mathcal{C}(C_n)) = n - 1$ if $n \in \{4, 5\}$ and $\partial(\mathcal{C}(C_n)) = n$ for $n \geq 6$.
- (4) $\partial(\mathcal{C}(S_n)) = 2n - 5$.
- (5) $\partial(\mathcal{C}(S_{p,q})) = 2n - 5$, where $2 \leq p \leq q$ and $p + q = n$.
- (6) $\partial(\mathcal{C}(K_{p,q})) = pq + q - p - 2$, where $2 \leq p \leq q$ and $p + q = n \geq 5$.
- (7) $\partial(\mathcal{C}(K_n)) = \frac{n(n-1)}{2} - n + 2$.

Proof. We proceed in the given order.

- (1) Since $G \simeq E_n$ has no edges, then $\mathcal{C}(G) \simeq K_n$, and so $\partial(\mathcal{C}(G)) = \partial(K_n) = n - 2$.

- (2) A direct computation shows that $\partial(\mathcal{C}(P_4)) = 3$ and $\partial(\mathcal{C}(P_5)) = 4$. For $n \geq 6$, let $P_n := x_1, x_2, \dots, x_n$. Observe that $B_{\mathcal{C}(P_n)}(\{x_2, x_{n-1}\}) = \{v_{x_1x_2}, v_{x_2x_3}, x_4, \dots, x_n\} \cup \{x_1, x_3, v_{x_{n-2}x_{n-1}}, v_{x_{n-1}x_n}\}$, and so $\partial(\mathcal{C}(P_n)) \geq (n - 2) + 4 - 2 = n$. Let D be a minimum differential set of $\mathcal{C}(P_n)$. We claim that $|D \cap V| \leq 2$. Indeed, if $x_i, x_j, x_k \in D$ with $i < j < k$, then Lemma 3.9(2) implies that $epn[x_i, D] = \{v_{x_{i-1}x_i}, v_{x_ix_{i+1}}\}$, and hence $\partial(D \setminus \{x_i\}) = (|B(D)| - 2 + 1) - (|D| - 1) = \partial(D)$, contradicting the minimality of D .
 By Proposition 3.10(2) we know that $D \cap V \neq \emptyset$. Let $x_i \in D \cap V$. Since the existence of x_i prevents $|epn[v_{x_jx_{j+1}}, D]| = 2$ for any $j \in \{1, \dots, n-1\}$, then $D \subseteq V$ and $|D| = 2$. Thus, $\partial(D) \leq (n - 2) + 4 - 2 = n$, as required.
- (3) As in (2), a direct computation shows that $\partial(\mathcal{C}(C_4)) = 3$ and $\partial(\mathcal{C}(C_5)) = 4$. For $n \geq 6$, let $C_n := x_1, x_2, \dots, x_n$. From the previous case and Proposition 3.12 we know that $\partial(\mathcal{C}(C_n)) \geq n$. Let D be a minimum differential set of $\mathcal{C}(C_n)$. By an entirely analogous argument to the previous paragraph, we can conclude that $D \subseteq V(C_n)$ and $|D| = 2$. Thus, $\partial(D) \leq (n - 2) + 4 - 2 = n$, as required.
- (4) By Proposition 3.6 we know that $\partial(\mathcal{C}(S_n)) \geq 2n - 5$, and by Proposition 2.11 we have that $\partial(\mathcal{C}(S_n)) \leq (n) + (n - 1) - 4$, as required.
- (5) Let x and y be the vertices of $S_{p,q}$ such that $\delta_{\mathcal{C}(S_{p,q})}(x) = p - 1$ and $\delta_{\mathcal{C}(S_{p,q})}(y) = q - 1$. It is straightforward to check that $\partial(\{x, y\}) = 2n - 5$. Then $\partial(\mathcal{C}(S_{p,q})) \geq 2n - 5$. On the other hand, Proposition 2.11 guarantees that $\partial(\mathcal{C}(S_{p,q})) \leq (n) + (n - 1) - 4$, as required.
- (6) Let $\{P, Q\}$ be the bipartition of $V(K_{p,q})$ with $2 \leq p = |P| \leq |Q| \leq q$. Observe that $q \geq 3$ because $n \geq 5$. Let $D \subseteq V(\mathcal{C}(G))$ be a minimum differential set of $\mathcal{C}(G)$. The following assertions are straightforward to verify:
 - (a) If $D \subseteq P$, then $\partial_{\mathcal{C}(K_{p,q})}(D) \leq pq - p$.
 - (b) If $D \subseteq Q$, then $\partial_{\mathcal{C}(K_{p,q})}(D) \leq pq - q$.
 - (c) If $D = P \cup \{x\}$ for all $x \in Q$, then $\partial_{\mathcal{C}(K_{p,q})}(D) = pq + q - p - 2$.
 By Proposition 3.10(2) we know that $D \cap V \neq \emptyset$. On the other hand, we note that if $v_{xy} \in D \cap J$, then the minimality of D implies that $epn[v_{xy}, D] = \{x, y\}$. Since these two facts are incompatible, we must have that $D \subseteq P \cup Q$. Let $D_P = D \cap P$ and $D_Q = D \cap Q$.
 By (a), (b), and (c) we have that $p' := |P \setminus D_P|$ and $q' := |Q \setminus D_Q|$ satisfy $1 \leq p' \leq p$ and $1 \leq q' \leq q$. Then $|B_{\mathcal{C}(K_{p,q})}(D)| \leq (p' + q') + (pq - (p'q'))$ and $|D| = (p - p') + (q - q')$. Then we have $\partial_{\mathcal{C}(K_{p,q})}(D) \leq p' + q' + pq - p'q' - (p - p' + q - q') = pq - p - q - p'q' + 2p' + 2q' \leq (pq - p + q) - p'q' + 2p' + 2q' - 2q \leq pq + q - p - 2$, as required.
- (7) It follows immediately from Corollary 3.13.

□

Proposition 3.15. *Let T be a tree of order $n \geq 4$. Then $\partial(\mathcal{C}(T)) \leq 2n - 5$, with equality iff $\Delta(T) \geq n - 2$.*

Proof. By Proposition 2.11 we have that $\partial(\mathcal{C}(T)) \leq n + (n - 1) - 4 = 2n - 5$. Suppose that $\Delta(T) \in \{n - 1, n - 2\}$. If $\Delta(T) = n - 1$, then $T \simeq S_n$ and $\partial(\mathcal{C}(T)) = 2n - 5$ by Proposition 3.14(4). If $\Delta(T) = n - 2$, we choose $v \in V(T)$ such that $\delta_T(v) = n - 2$. Then there exists a unique vertex $u \in V(T) \setminus \{v\}$ such that $\delta_T(u) = 2$. Observe that $\{u, v\}$ is a dominating set in $\mathcal{C}(T)$. Then $\partial(\mathcal{C}(T)) \geq \partial_{\mathcal{C}(T)}(\{u, v\}) = 2n - 5$. Finally, we note that if $D \subseteq V(\mathcal{C}(T))$ and $|D| \geq 3$, then $\partial_{\mathcal{C}(T)}(D) \leq (2n - 1) - 2|D| \leq 2n - 7$. □

From an optimization perspective, it is essential to evaluate the cost of influence selection with respect to the types of vertices involved. The following result, establishes a structural restriction on minimum differential sets of $\mathcal{C}(G)$, demonstrating a parsimony property whereby the edge-vertex set J has only a marginal impact on the optimal influence potential. This insight significantly reduces the effective search space and supports the design of efficient discrete optimization algorithms.

Theorem 3.16. *Let G be a connected graph of order $n \geq 2$. If D is a minimum differential set of $\mathcal{C}(G)$, then $|D \cap J| \leq 1$.*

Proof. The statement is straightforward to verify for $n = 2, 3$. For $n \geq 4$, let D be a minimum differential set which minimizes $|D \cap J|$. Seeking a contradiction, suppose that $v_{x_1y_1}$ and $v_{x_2y_2}$ are distinct vertices in $D \cap J$. By the minimality of D and Lemma 3.9(2) for each $i \in \{1, 2\}$, x_i and y_i are private neighbors of $v_{x_iy_i}$ with respect to D . In particular, x_1, y_1, x_2 and y_2 must be pairwise distinct.

Let $H := \langle \{x_1, y_1, x_2, y_2\} \rangle_G$. Then $v_{x_1y_1}$ and $v_{x_2y_2}$ are vertices of degree 2 in $\mathcal{C}(H)$, $\mathcal{C}(H)$ is a subgraph of $\mathcal{C}(G)$, $\mathcal{C}(H)$ is a subdivision of K_4 and each vertex in $\{x_1, y_1, x_2, y_2\}$ has degree 3 in $\mathcal{C}(H)$. Since for $i \in \{1, 2\}$, x_i and y_i are private neighbors of $v_{x_iy_i}$ with respect to D , then $u \notin B_{\mathcal{C}(G)}(D \setminus \{v_{x_1y_1}, v_{x_2y_2}\})$ for any $u \in V(\mathcal{C}(H))$, and therefore:

$$\partial((D \setminus \{v_{x_1y_1}, v_{x_2y_2}\}) \cup \{x_1\}) \geq (|B_{\mathcal{C}(G)}(D)| - 4 + 3) - (|D| - 2 + 1) = \partial_{\mathcal{C}(G)}(D).$$

Then $D' := (D \setminus \{v_{x_1y_1}, v_{x_2y_2}\}) \cup \{x_1\}$ must be a differential set of $\mathcal{C}(G)$ with $|D' \cap J| < |D \cap J|$, contradicting the choice of D . \square

For the following result, ϕ is defined as $|\bigcap_{x \in D} N_G(x)|$.

Proposition 3.17. *Let G be a graph of order $n \geq 4$, and let D be an independent differential set of G with $|D| \geq 2$, then*

$$\partial(\mathcal{C}(G)) \geq \left(\sum_{x \in D} \delta_G(x) \right) + n - 2\gamma(G) - \phi.$$

Moreover, this bound is sharp.

Proof. We note that $|B_{\mathcal{C}(G)}(D)| \geq |B_{\mathcal{C}(G)}(D) \cap V| + |B_{\mathcal{C}(G)}(D) \cap J|$. Since D is an independent set of G , then $|B_{\mathcal{C}(G)}(D) \cap J| = \sum_{x \in D} \delta_G(x)$. On the other hand, it is not difficult to see that $|B_{\mathcal{C}(G)}(D) \cap V| \geq n - |D| - \phi$. By Lemma 3.9(1), we have

$$\partial(\mathcal{C}(G)) \geq \partial_{\mathcal{C}(G)}(D) \geq \left(\sum_{x \in D} \delta_G(x) \right) + n - 2\gamma(G) - \phi.$$

Finally, for $k \in \{2, 3, \dots\}$ we note that any differential set D of $G \in \{P_{3k}, C_{3k}\}$ satisfies all conditions of Proposition 3.17. Since $\gamma(G) = |D|$, then

$$\left(\sum_{x \in D} \delta_G(x) \right) + n - 2\gamma(G) - \phi = 2|D| + n - 2|D| - 0 = n.$$

\square

To understand the incremental influence provided by the central operator, we analyze the additive properties of the differential. The next Theorem establishes a recursive framework, relating the differential of the central graph to the internal structure of the original differential sets in G .

Theorem 3.18. *Let G be a graph of order $n \geq 3$, and let D be a differential set of G . Then:*

$$\partial(\mathcal{C}(G)) \geq \partial(G) + |E(\langle D \rangle_G)| + \partial(\mathcal{C}(\langle V \setminus D \rangle_G)).$$

Proof. Let $D' := D \cup (N_{\mathcal{C}(G)}(D) \cap J)$. Then $\langle D' \rangle_{\mathcal{C}(G)} \cup \mathcal{C}(\langle V \setminus D \rangle_G)$ is a disconnected subgraph of $\mathcal{C}(G)$, and $\partial(\langle D' \rangle_{\mathcal{C}(G)}) \geq \partial(G) + |E(\langle D \rangle_G)|$. Therefore:

$$\begin{aligned} \partial(\mathcal{C}(G)) &\geq \partial(\langle D' \rangle_{\mathcal{C}(G)} \cup \mathcal{C}(\langle V \setminus D \rangle_G)) \\ &= \partial(\langle D' \rangle_{\mathcal{C}(G)}) + \partial(\mathcal{C}(\langle V \setminus D \rangle_G)) \\ &\geq \partial(G) + |E(\langle D \rangle_G)| + \partial(\mathcal{C}(\langle V \setminus D \rangle_G)). \end{aligned}$$

\square

Proposition 3.19. *Let G be a graph of order $n \geq 4$. If $D \subseteq V$, then*

$$\partial_{\mathcal{C}(G)}(D) = \partial_{\mathcal{S}(G)}(D) + \partial_{\overline{G}}(D) + |D|.$$

Proof. Let E_D be the set of edges in G that have at least one endvertex in D , and let $J_D := \{v_{xy} : xy \in E_D\}$. Since $xy \rightarrow v_{xy}$ defines a bijection between E_D and J_D , then $|E_D| = |J_D|$. We let V_D denote the set of vertices in G such that $y \in V_D$ iff $xy \in E(G)$ for some $x \in D$. The following equalities are straightforward consequences of $D \subseteq V$:

- (1) $B_{\mathcal{S}(G)}(D) = J_D$ and $B_{\overline{G}}(D) = (V \setminus D) \setminus V_D$.
- (2) $B_{\mathcal{C}(G)}(D)$ is disjoint union of $B_{\mathcal{S}(G)}(D)$ and $B_{\overline{G}}(D)$.
- (3) $\partial_{\mathcal{S}(G)}(D) = |J_D| - |D|$.
- (4) $\partial_{\overline{G}}(D) = (n - |D| - |V_D|) - |D|$.

By combining these four facts we obtain the following:

$$\begin{aligned} \partial_{\mathcal{C}(G)}(D) - |D| &= |B_{\mathcal{C}(G)}(D)| - 2|D| \\ &= |B_{\mathcal{S}(G)}(D)| + |B_{\overline{G}}(D)| - 2|D| \\ &= |J_D| - |D| + |(V \setminus D) \setminus V_D| - |D| \\ &= |J_D| - |D| + n - |D| - |V_D| - |D| \\ &= \partial_{\mathcal{S}(G)}(D) + \partial_{\overline{G}}(D). \end{aligned}$$

□

Proposition 3.20. *Let G be a graph of order $n \geq 4$. If D is a differential set of \overline{G} , then D is not a differential set of $\mathcal{S}(G)$.*

Proof. Since D is a differential set of \overline{G} , then $D \subseteq V$. Seeking a contradiction, suppose that D is a differential set also for $\mathcal{S}(G)$. We note that $B_{\mathcal{S}(G)}(D) \subseteq J$ and $B_{\overline{G}}(D) \subseteq V \setminus D$.

Since $n \geq 4$, it can be deduced that at least one of $\partial(\overline{G})$ or $\partial(\mathcal{S}(G))$ is positive. Then $D \notin \{\emptyset, V\}$. Since $\partial(\overline{G})$ and $\partial(\mathcal{S}(G))$ are nonnegative, then $B_{\overline{G}}(D)$ and $B_{\mathcal{S}(G)}(D)$ are nonempty. We note that if there are $x, y \in V \setminus D$ such that $xy \in E(G)$, then $\partial_{\mathcal{S}(G)}(D \cup \{v_{xy}\}) = \partial_{\mathcal{S}(G)}(D) + 1$. Since this is impossible, $V \setminus D$ must be an independent set of G , and so $V \setminus D$ is a clique of \overline{G} .

Let $w \in B_{\overline{G}}(D)$ and let $u \in D$ be such that $uw \in E(\overline{G})$. Then $|B_{\overline{G}}(D)| - |D| = \partial_{\overline{G}}(D) \geq \partial_{\overline{G}}(\{w\}) \geq |V \setminus D| - 1 = n - |D| - 1$. Then $|B_{\overline{G}}(D)| = n - 1$ and $|D| = 1$, because $D \neq \emptyset$ and $D \cup B_{\overline{G}}(D) \subseteq V$. These two equalities and the fact that $V \setminus D = V \setminus \{u\}$ is a clique of \overline{G} imply that $\overline{G} \simeq K_n$. Then we have $B_{\mathcal{S}(G)}(D) = \emptyset$, the required contradiction. □

Theorem 3.21. *Let G be a graph of order $n \geq 4$ and let D be a minimum differential set of $\mathcal{C}(G)$. Let D' and \overline{D} be differential sets of $\mathcal{S}(G)$ and \overline{G} , respectively. If*

$$F := \{uv \in E(\mathcal{C}(G)) : u \in D' \setminus \overline{D} \text{ and } v \in \overline{D} \setminus D'\},$$

then $\partial(\mathcal{S}(G)) + \partial(\overline{G}) - \partial_{\mathcal{C}(G)}(D' \cap \overline{D}) - |F| \leq \partial(\mathcal{C}(G)) \leq \partial(\mathcal{S}(G)) + \partial(\overline{G}) + |D| - 1$.

Proof. Clearly,

$$\begin{aligned} \partial(\mathcal{C}(G)) &\geq \partial_{\mathcal{C}(G)}(D' \cup \overline{D}) = |B_{\mathcal{C}(G)}(D' \cup \overline{D})| - |D' \cup \overline{D}| \\ &= |B_{\mathcal{C}(G)}(D')| + |B_{\mathcal{C}(G)}(\overline{D})| - |B_{\mathcal{C}(G)}(D' \cap \overline{D})| \\ &\quad - |F| - |D'| - |\overline{D}| + |D' \cap \overline{D}| \\ &= \partial_{\mathcal{C}(G)}(D') + \partial_{\mathcal{C}(G)}(\overline{D}) - \partial_{\mathcal{C}(G)}(D' \cap \overline{D}) - |F| \\ &\geq \partial_{\mathcal{S}(G)}(D') + \partial_{\overline{G}}(\overline{D}) - \partial_{\mathcal{C}(G)}(D' \cap \overline{D}) - |F| \\ &= \partial(\mathcal{S}(G)) + \partial(\overline{G}) - \partial_{\mathcal{C}(G)}(D' \cap \overline{D}) - |F|. \end{aligned}$$

We now show the upper bound. By the minimality of D and Theorem 3.16, we have $|D \cap J| \leq 1$.

If $D \cap J = \emptyset$, then Propositions 3.19 and 3.20 imply that

$$\partial(\mathcal{C}(G)) - |D| = \partial_{\mathcal{C}(G)}(D) - |D| = \partial_{\mathcal{S}(G)}(D) + \partial_{\overline{G}}(D) \leq \partial(\mathcal{S}(G)) + \partial(\overline{G}) - 1,$$

as required.

Suppose now that $D \cap J = \{v_{xy}\}$ for some $xy \in E(G)$. Let $D' := D \cap V$. From the minimality of D and Lemma 3.9(2) we know that v_{xy} contributes +1 to $\partial(\mathcal{C}(G))$, and so $\partial_{\mathcal{C}(G)}(D) = \partial_{\mathcal{C}(G)}(D') + 1$. From this fact and Propositions 3.19 and 3.20 it follows that

$$\begin{aligned} \partial_{\mathcal{C}(G)}(D) - |D| &= (\partial_{\mathcal{C}(G)}(D') + 1) - (|D'| + 1) \\ &= \partial_{\mathcal{S}(G)}(D') + \partial_{\overline{G}}(D') \\ &\leq \partial(\mathcal{S}(G)) + \partial(\overline{G}) - 1. \end{aligned}$$

□

We remark that the bounds in Theorem 3.21 are best possible. Indeed, for the lower bound consider the star graph $G := S_n$. Let x be the apex vertex of S_n . It is not hard to see that $D' := \{x\}$ is a differential set of $\mathcal{S}(S_n)$ and $\partial(\mathcal{S}(S_n)) = n - 2$. Similarly, it can be observed that if v is a leaf of S_n , then $\overline{D} := \{v\}$ is a differential set for $\overline{S_n}$ and $\partial(\overline{S_n}) = n - 3$. If F is defined in terms of D' and \overline{D} as above, then

$$\partial(\mathcal{S}(G)) + \partial(\overline{G}) - \partial_{\mathcal{C}(G)}(D' \cap \overline{D}) - |F| = (n - 2) + (n - 3) - 0 - 0 = 2n - 5 = \partial(\mathcal{C}(S_n)).$$

For the upper bound consider the double star graph $S_{p,q}$ with $p, q \geq 2$ and $p + q = n$. It is not hard to see that if x and y are two vertices of $S_{p,q}$ that are not leaves, then $D := \{x, y\}$ is the unique minimum differential set of $\mathcal{C}(S_{p,q})$. Then

$$\partial(\mathcal{C}(S_{p,q})) = 2n - 5 = (n - 3) + (n - 3) + 2 - 1 = \partial(\overline{S_{p,q}}) + \partial(\mathcal{S}(S_{p,q})) + |D| - 1.$$

Recall that the graph $\mathcal{R}(G)$ is defined as the graph obtained from G by adding a new vertex v_{xy} for each edge $xy \in E(G)$ and by joining v_{xy} to both vertices x and y .

The following result presents a remarkable theoretical convergence. We identify a broad class of graphs in which the central operator $\mathcal{C}(G)$, the subdivision graph $\mathcal{S}(G)$, and the operator $\mathcal{R}(G)$ produce identical differential values. This result suggests a topological equivalence from the point of view of maximum information diffusion capacity.

Theorem 3.22. *Let G be a graph of order $n \geq 3$ and size $m \geq 1$. Then $\partial(\overline{\mathcal{O}(G)}) = m + n - 4$ for every $\mathcal{O}(G) \in \{\mathcal{C}(G), \mathcal{S}(G), \mathcal{R}(G)\}$.*

Proof. Suppose that $\mathcal{O}(G) \in \{\mathcal{C}(G), \mathcal{S}(G), \mathcal{R}(G)\}$. Then $V(\mathcal{O}(G)) = V(\overline{\mathcal{O}(G)}) = V \cup J$ with $V := V(G)$ and $J := \{v_{xy} : xy \in E(G)\}$. Let $E[J, V] := \{v_{xy}z : v_{xy} \in J \text{ and } z \in V \setminus \{x, y\}\}$ and $E[J, J] := \{v_{xy}v_{x'y'} : v_{xy}, v_{x'y'} \in J \text{ with } v_{xy} \neq v_{x'y'}\}$.

We now describe the sets $E(\mathcal{O}(G))$ and $E(\overline{\mathcal{O}(G)})$ for each instance of $\mathcal{O}(G)$.

- If $\mathcal{O}(G) = \mathcal{S}(G)$, then $E(\mathcal{S}(G)) = \{xv_{xy}, v_{xy}y : xy \in E(G)\}$ and $E(\overline{\mathcal{S}(G)}) = E(G) \cup E(\overline{G}) \cup E[J, V] \cup E[J, J]$.
- If $\mathcal{O}(G) = \mathcal{C}(G)$, then $E(\mathcal{C}(G)) = E(\overline{G}) \cup E(\mathcal{S}(G))$ and $E(\overline{\mathcal{C}(G)}) = E(G) \cup E[J, V] \cup E[J, J]$.
- If $\mathcal{O}(G) = \mathcal{R}(G)$, then $E(\mathcal{R}(G)) = E(G) \cup E(\mathcal{S}(G))$ and $E(\overline{\mathcal{R}(G)}) = E(\overline{G}) \cup E[J, V] \cup E[J, J]$.

Let $v_{xy} \in J$ and $z \in V \setminus \{x, y\}$. The following assertions are straightforward to verify:

- (1) $\Delta(\overline{\mathcal{O}(G)}) = m + n - 3$ and v_{xy} is a vertex of maximum degree in $\overline{\mathcal{O}(G)}$.
- (2) $\partial_{\overline{\mathcal{O}(G)}}(\{v_{xy}\}) = m + n - 4$.

By (2) we have that $\partial(\overline{\mathcal{C}(G)}) \geq m + n - 4$. Let D be a differential set of $\overline{\mathcal{C}(G)}$. Since $|D| \geq 2$ implies $\partial(\overline{\mathcal{C}(G)}) \leq m + n - 2|D| \leq m + n - 4$, we obtain the required equality. Thus, we can assume that $|D| = 1$, and we are done by (1). \square

The following result is an immediate consequence of Theorem 3.22.

Corollary 3.23. *Let K_n , S_n , $S_{p,q}$, $K_{p,q}$, P_n , C_n , W_n and T denote the complete, star, double star, complete bipartite, path, cycle, wheel, and tree graphs, respectively, of order $n \geq 3$. Then the following hold:*

- (1) $\partial(\overline{\mathcal{C}(K_n)}) = \frac{n(n-1)}{2} + n - 4$.
- (2) $\partial(\overline{\mathcal{C}(S_n)}) = 2n - 5$.
- (3) $\partial(\overline{\mathcal{C}(S_{p,q})}) = 2n - 5$.
- (4) $\partial(\overline{\mathcal{C}(K_{p,q})}) = pq + n - 4$.
- (5) $\partial(\overline{\mathcal{C}(P_n)}) = 2n - 5$.
- (6) $\partial(\overline{\mathcal{C}(C_n)}) = 2n - 4$.
- (7) $\partial(\overline{\mathcal{C}(W_n)}) = 3(n - 2)$.
- (8) $\partial(\overline{\mathcal{C}(T)}) = 2n - 5$.

Finally, we investigate the relationship between the differential of the central operator and its behavior on the complement graph. The following results, establish Nordhaus–Gaddum type inequalities that characterize the combined influence potential of $\mathcal{C}(G)$ and $\overline{\mathcal{C}(G)}$, providing sharp extremal bounds for the class of trees.

Proposition 3.24. *Let G be a graph of order $n \geq 4$. Then*

$$m + 2n - 5 \leq \partial(\mathcal{C}(G)) + \partial(\overline{\mathcal{C}(G)}) \leq m + \frac{n(n-1)}{2} - 2.$$

Proof. The result follows by combining Corollaries 3.7, 3.13, and Theorem 3.22. \square

The following result is a consequence of Proposition 3.24.

Corollary 3.25. *Let T be a tree of order $n \geq 4$, then*

$$3n - 6 \leq \partial(\mathcal{C}(T)) + \partial(\overline{\mathcal{C}(T)}) \leq \frac{(n-2)(n+3)}{2}.$$

4. CONCLUSIONS

In this paper, we have provided a definitive characterization of the differential $\partial(\mathcal{C}(G))$ under the central operator, a topic with significant implications for influence maximization and network optimization. Our research bridges the gap between abstract graph operators and their practical behavior, revealing that the central operator acts as a consistent enhancer of a network's diffusion capacity.

The main contributions of this study are summarized as follows:

- **Structural parsimony:** Theorem 3.16 proves that minimum differential sets in $\mathcal{C}(G)$ are primarily rooted in the original vertex set V , with minimal intersection with the edge-vertex set J . This finding is critical for reducing the search space in algorithmic applications.
- **Operator convergence:** We established a notable identity in Theorem 3.22, where $\mathcal{C}(G)$, $\mathcal{S}(G)$, and $\mathcal{R}(G)$ yield the same differential value ($m + n - 4$). This result unifies distinct structural transformations under a common combinatorial framework.
- **Monotonicity and growth:** Our results demonstrate a guaranteed growth of the differential, as shown in Corollary 3.8 ($\partial(\mathcal{C}(G)) \geq \partial(\overline{G}) + 1$), confirming that the central operator is a robust strategy for increasing the influence potential of any connected topology.

- **Extremal bounds:** Through Nordhaus–Gaddum type inequalities (Prop. 3.24 and Cor. 3.25), we characterized the total influence capacity of the operator and its complement, providing sharp bounds specifically for the class of trees.

Future work could focus on investigating the computational complexity of the differential in these transformed networks, particularly for classes of graphs where the domination problem remains NP-hard. Additionally, exploring the behavior of the k -differential variant under $\mathcal{C}(G)$ would provide further insights into multi-stage information diffusion models in complex systems.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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