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THE INDEX OF UNBALANCED SIGNED COMPLETE GRAPHS WHOSE NEGATIVE-EDGE-INDUCED SUBGRAPH IS $K_{2,2}$-MINOR FREE

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Abstract. Let $\Gamma = (K_n, H^-)$ be a signed complete graph with the negative-edge-induced subgraph H. According to the properties of the negative-edge-induced subgraph, characterizing the extremum problem of the index of the signed complete graph is a concern in signed graphs. A graph G is called H-minor free if G has no minor which is isomorphic to H. In this paper, we characterize the extremal signed complete graphs that achieve the maximum and the second maximum index when H is a $K_{2,2}$-minor free spanning subgraph of K_n.

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1. Introduction

Let G be a simple graph with the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and the edge set $E(G) = \{e_1, \ldots, e_m\}$. The order and size of G are defined as $|V(G)|$ and $|E(G)|$, respectively. A subgraph H of G is called a spanning subgraph if $|V(H)| = |V(G)|$. The degree of a vertex v_i in G is denoted by $d_G(v_i)$ which is the number of edges incident with v_i. If $d_G(v_i) = 1$, then vertex v_i is called a pendant vertex. We denote the set of all neighbors of u in G by $N_G(u)$. Let K_n be the complete graph of order n. As usual, P_k, $K_{1,k}$ and C_k denote the path of order k, the star of order $k+1$ and the cycle of order k, respectively. An underlying graph G with a signature $\sigma : E(G) \to \{-1, +1\}$ makes up a signed graph $\Gamma = (G, \sigma)$. In signed graphs, edge signs are usually interpreted as ± 1. An edge e is positive (resp. negative) if $\sigma(e) = +1$ (resp. $\sigma(e) = -1$). A cycle in Γ is said to be positive if it contains an even number of negative edges, otherwise the cycle is negative. $\Gamma = (G, \sigma)$ is balanced if there are no negative cycles, otherwise it is unbalanced. For more details about the notion of signed graphs, we refer to [1]. Signed graphs were first introduced in works of Harary [11] and Cartwright and Harary [5], and the matroids of graphs were extended to matroids of signed graphs by Zaslavsky [26]. Chaiken [6] and Zaslavsky [26] obtained the Matrix-Tree Theorem for signed graphs independently. The theory of signed graphs is a special case of that of gain graphs and of biased graphs [27]. The adjacency matrix of Γ is defined as $A(\Gamma) = (a_{ij}^{\sigma})$. Then $a_{uv} = \sigma(uv)$ if $u \sim v$, otherwise, $a_{uv} = 0$. The eigenvalues of Γ are denoted by $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$ with decreasing order which are the eigenvalues of $A(\Gamma)$ and the index of Γ is $\lambda_1(\Gamma)$.

Extremal graph theory deals with the problem of determining extremal values and extremal graphs for a given graph invariant in a given set of graphs. It is worth noting that Brunetti and Stanić [4] studied the extremal

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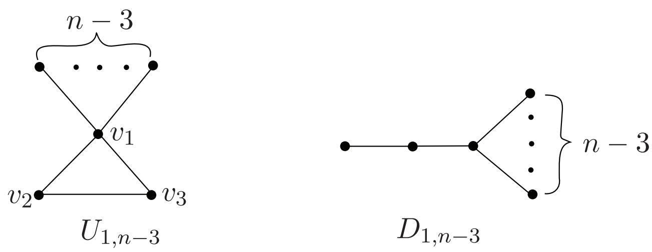

FIGURE 1. The graphs $U_{1,n-3}$ and $D_{1,n-3}$.

spectral radius among all unbalanced connected signed graphs. Let $\Gamma = (K_n, H^-)$ be a signed complete graph with the negative-edge-induced subgraph H. In 2023, Li, Lin and Meng [14] gave the maximum $\lambda_1(A(\Sigma))$ among graphs of type $\Sigma = (K_n, T^-)$, where T is a spanning tree of K_n. More results on the index of signed graphs can be found in [9, 10, 21]. Let H be a simple graph. A graph G is H-free if no subgraph of G is isomorphic to H. The classical spectral Turán problem is to determine the maximum spectral radius of an H-free graph, which is known as the spectral Turán number of H. This problem was originally proposed by Nikiforov [15]. Turán [17] raised and solved the extremal problem for K_r-free graphs with $r \geq 3$. For more on the spectral Turán problem for unsigned graphs, see [2, 24]. A graph H is a minor of a graph G if H can be obtained from G by deleting edges, deleting vertices and contracting edges. A graph G is called H-minor free if G has no minor which is isomorphic to H. In particular, Wagner [18] proved that a graph is planar if and only if it is K_5-minor and $K_{3,3}$-minor free. Similar to Wagner's result, Ding and Oporowski [8] showed that a graph is outerplanar if and only if it is K_4-minor and $K_{2,3}$-minor free. In spectral extremal graph theory, it is interesting to determine the maximum spectral radius of H-minor free graphs for some given H. For example, in 2019, Tait [16] determined the maximum spectral radius of K_r-minor free graph of order n and also presented a conjecture on the extremal graph that achieves the maximum spectral radius among $K_{s,t}$-minor free graphs. In 2022, Zhai and Lin [28] proved the conjecture for sufficiently large order.

In this paper, we focus on the spectral Turán problem in signed graphs. Let K_r^- and C_r^- be the sets of all unbalanced signed graphs with underlying graphs K_r and C_r, respectively. Given a set F of signed graphs, if a signed graph Γ contains no signed subgraph isomorphic to any one in F, then Γ is called F-free. Chen and Yuan [7] gave the spectral Turán number of K_4^-. For $r \geq 5$, the K_r^--free unbalanced signed graphs of fixed order n with maximum index (resp. spectral radius $r \leq \frac{n}{2}$) have been determined in [19, 25]. In 2022, Wang, Hou and Li [22] determined the spectral Turán number of C_3^-. The C_4^--free unbalanced signed graphs of fixed order with maximum index have been determined by Wang and Lin [20]. Moreover, the C_{2k+1}^--free unbalanced signed graphs of fixed order n with maximum index have been determined in [23], where $3 \leq k \leq n/10 - 1$. Motivated by these works, we are interested in determining the extremal graph that attains the maximum index in the class of signed graphs whose negative-edge-induced subgraphs are H-minor free. In this paper, we consider this problem in the unbalanced signed complete graphs whose negative-edge-induced subgraph is $K_{2,2}$-minor free. Let $U_{1,n-3}$ (Fig. 1) denote a triangle with all remaining vertices being pendant at the same vertex of the triangle, and $D_{1,n-3}$ (Fig. 1) be obtained by adding $n-3$ pendant vertices to one end vertex of P_3. We know that $H \cong U_{1,n-3}$ or $H \cong P_4$ or $H \cong 2P_2$ for signed complete graph $\Gamma = (K_4, H^-)$. Since $(K_4, U_{1,n-3}^-)$ is switching isomorphic to (K_4, P_4^-), $\lambda_1(A((K_4, U_{1,n-3}^-))) = \lambda_1(A((K_4, P_4^-))) > \lambda_1(A((K_4, 2P_2^-)))$ by direct calculation. Thus, we only need to consider $n \geq 5$, and the main result of this paper is as follows.

Theorem 1. *Let H be a $K_{2,2}$-minor free spanning subgraph of K_n for $n \geq 5$. If Γ is not switching isomorphic to $(K_n, U_{1,n-3}^-)$ or $(K_n, D_{1,n-3}^-)$, then $\lambda_1(A((K_n, U_{1,n-3}^-))) > \lambda_1(A((K_n, D_{1,n-3}^-))) > \lambda_1(A(\Gamma))$.*

The remainder of this paper is organized as follows. In Section 2, we will investigate lemmas that will be useful in proving Theorem 1. Furthermore, Section 3 will review some established results, investigate lemmas pertinent to the proof of Theorem 1, and present its proof.

2. Preliminaries

Let M be a real symmetric matrix with block form $M = [M_{ij}]$, and q_{ij} be the average row sum of M_{ij} and the matrix $Q = (q_{ij})$ be the quotient matrix of M. Furthermore, Q is referred to as an equitable quotient matrix if every block M_{ij} has a constant row sum. Let $\mathrm{Spec}(Q) = \{\lambda_1^{[t_1]}, \ldots, \lambda_k^{[t_k]}\}$ be the spectrum of Q with eigenvalue λ_i with multiplicity t_i for $1 \le i \le k$.

Lemma 1 ([3])**.** *There are two kinds of eigenvalues of the real symmetric matrix M.*

(i) *The eigenvalues that match the eigenvalues of Q.*
(ii) *The eigenvalues of M not in Spec(Q) that are unchanged when αJ is added to block M_{ij} for every $1 \le i, j \le m$, where α is any constant. Moreover, $\lambda_1(M) = \lambda_1(Q)$ when M is irreducible and nonnegative.*

Let $\Gamma = (G, \sigma)$ be a signed graph and $\varphi_\Gamma(\lambda) = \det(\lambda I - A(\Gamma))$ be the characteristic polynomial of Γ. The matrix $J_{r\times s}$ is all-one matrix of size $r \times s$, and when $r = s$ it is denoted by J_r. Also, we use $j_k = (1, \ldots, 1)^T \in R^k$.

Lemma 2. *Let $\Gamma = (K_n, D^-_{1,n-3})$ and $\Gamma' = (K_n, U^-_{1,n-3})$. Then $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$ for $n \ge 5$.*

Proof. By [14], the characteristic polynomial of Γ is

$$\varphi_\Gamma(\lambda) = (\lambda + 1)^{n-3}(\lambda^3 + (3 - n)\lambda^2 + (3 - 2n)\lambda + 7n - 23).$$

For $\Gamma' = (K_n, U^-_{1,n-3})$, the vertex partition is $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$ and $V_4 = \{v_4, \ldots, v_n\}$. Then $A(\Gamma')$ is given by the following

$$A(\Gamma') = \begin{bmatrix} 0 & -1 & -1 & -j^T_{n-3} \\ -1 & 0 & -1 & j^T_{n-3} \\ -1 & -1 & 0 & j^T_{n-3} \\ -j_{n-3} & j_{n-3} & j_{n-3} & (J - I)_{n-3} \end{bmatrix}.$$

Hence,

$$\lambda I_n - A(\Gamma') = \begin{bmatrix} \lambda & 1 & 1 & j^T_{n-3} \\ 1 & \lambda & 1 & -j^T_{n-3} \\ 1 & 1 & \lambda & -j^T_{n-3} \\ j_{n-3} & -j_{n-3} & -j_{n-3} & ((\lambda + 1)I - J)_{n-3} \end{bmatrix}.$$

Now, we apply finitely many elementary row and column operations on the matrix $\lambda I_n - A(\Gamma')$. First, subtracting the 4-th row from all the lower rows and adding the i-th column to the 4-th column for $i = n, \ldots, 5$. This leads to the following matrix:

$$\begin{bmatrix} \lambda & 1 & 1 & n-3 & * \\ 1 & \lambda & 1 & 3-n & * \\ 1 & 1 & \lambda & 3-n & * \\ 1 & -1 & -1 & \lambda - n + 4 & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\lambda + 1)\, I_{n-4} \end{bmatrix}.$$

Therefore, $\varphi_{\Gamma'}(\lambda) = (\lambda+1)^{n-3}(\lambda-1)(\lambda^2+(4-n)\lambda+7-3n)$. Let $f_1(\lambda) = (\lambda^3+(3-n)\lambda^2+(3-2n)\lambda+7n-23)$ and $f_2(\lambda) = (\lambda-1)(\lambda^2+(4-n)\lambda+7-3n)$. Note that $f_2(\lambda)-f_1(\lambda) = 16-4n < 0$ for $n \ge 5$, then $\varphi(\Gamma', \lambda_1(A(\Gamma))) < 0$. This implies that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$. $\square$

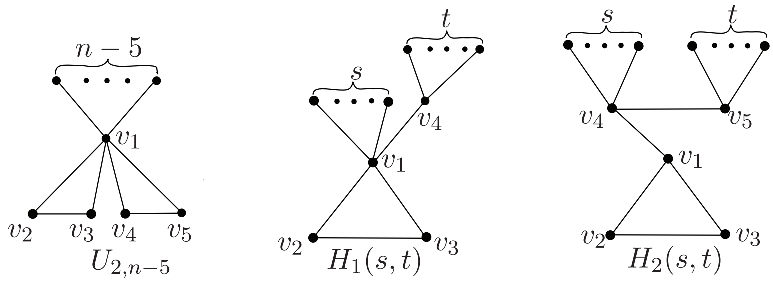

FIGURE 2. The graphs $U_{2,n-5}$, $H_1(s,t)$ and $H_2(s,t)$.

Lemma 3. *Let $U_{2,n-5}$ be the graph depicted in Figure 2. If $n \geq 9$, then*

$$\lambda_1(A((K_n, D^-_{1,n-3}))) > \lambda_1(A((K_n, U^-_{2,n-5}))).$$

Proof. Notice that the characteristic polynomial of $A((K_n, D^-_{1,n-3}))$ is $(\lambda+1)^{n-3}(\lambda^3+(3-n)\lambda^2+(3-2n)\lambda+7n-23)$ by Li *et al.* [14]. For $(K_n, U^-_{2,n-5})$, the vertex partition is $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$, $V_5 = \{v_5\}$ and $V_6 = \{v_6, \dots, v_n\}$. Then $A((K_n, U^-_{2,n-5}))$ is given by the following

$$A((K_n, U^-_{2,n-5})) = \begin{bmatrix} 0 & -1 & -1 & -1 & -1 & -j^T_{n-5} \\ -1 & 0 & -1 & 1 & 1 & j^T_{n-5} \\ -1 & -1 & 0 & 1 & 1 & j^T_{n-5} \\ -1 & 1 & 1 & 0 & -1 & j^T_{n-5} \\ -1 & 1 & 1 & -1 & 0 & j^T_{n-5} \\ -j_{n-5} & j_{n-5} & j_{n-5} & j_{n-5} & j_{n-5} & (J-I)_{n-5} \end{bmatrix}.$$

Thus,

$$\lambda I_n - A((K_n, U^-_{2,n-5})) = \begin{bmatrix} \lambda & 1 & 1 & 1 & 1 & j^T_{n-5} \\ 1 & \lambda & 1 & -1 & -1 & -j^T_{n-5} \\ 1 & 1 & \lambda & -1 & -1 & -j^T_{n-5} \\ 1 & -1 & -1 & \lambda & 1 & -j^T_{n-5} \\ 1 & -1 & -1 & 1 & \lambda & -j^T_{n-5} \\ j_{n-5} & -j_{n-5} & -j_{n-5} & -j_{n-5} & -j_{n-5} & ((\lambda+1)I-J)_{n-5} \end{bmatrix}.$$

Now, we apply finitely many elementary row and column operations on the matrix $\lambda I_n - A((K_n, U^-_{2,n-5}))$. First, subtracting the 6-th row from all the lower rows and adding the i-th column to the 6-th column, for $i = n, \dots, 7$. This leads to the following matrix:

$$\begin{bmatrix} \lambda & 1 & 1 & 1 & 1 & n-5 & * \\ 1 & \lambda & 1 & -1 & -1 & 5-n & * \\ 1 & 1 & \lambda & -1 & -1 & 5-n & * \\ 1 & -1 & -1 & \lambda & 1 & 5-n & * \\ 1 & -1 & -1 & 1 & \lambda & 5-n & * \\ 1 & -1 & -1 & -1 & -1 & \lambda+6-n & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\lambda+1)I_{n-6} \end{bmatrix}.$$

Hence, $\varphi((K_n,U^-_{2,n-5}),\lambda)=(\lambda+1)^{n-5}(\lambda-1)^2(\lambda+3)(\lambda^2+(4-n)\lambda+11-3n)$. Let $f_1(\lambda)=(\lambda-1)^2(\lambda+3)(\lambda^2+(4-n)\lambda+11-3n)$ and $f_2(\lambda)=(\lambda+1)^2(\lambda^3+(3-n)\lambda^2+(3-2n)\lambda+7n-23)$. Note that

$$h(\lambda)=f_1(\lambda)-f_2(\lambda)=8\lambda^2-16n+56.$$

The maximal solution of $h(\lambda)=0$ is $\sqrt{2n-7}$ for $n\geq 9$. And

$$f_1\left(\frac{n}{2}-1\right)=\left(\frac{n-4}{2}\right)^2\left(\frac{n+4}{2}\right)\left(-\frac{1}{4}n^2-n+8\right).$$

Now, let $g_1(n)=-\frac{1}{4}n^2-n+8$. Notice that $g_1'(n)=-\frac{1}{2}n-1<0$ for $n\geq 9$. Hence, $g_1(n)\leq g_1(9)=-\frac{85}{4}<0$ and $f_1(\frac{n}{2}-1)<0$ for $n\geq 9$. Thus, $\lambda_1(A(K_n,U^-_{2,n-5}))>\frac{n}{2}-1>\sqrt{2n-7}$ for $n\geq 9$. Let $\lambda_1=\lambda_1(A((K_n,U^-_{2,n-5})))$, then

$$-f_2(\lambda_1)=f_1(\lambda_1)-f_2(\lambda_1)>0.$$

This indicates that $\lambda_1(A((K_n,D^-_{1,n-3})))>\lambda_1(A((K_n,U^-_{2,n-5})))$. $\square$

Let the matrix Q be the quotient matrix of a real symmetric graph matrix M, and let $P_Q(\lambda)=\det(\lambda I-Q)$ be the characteristic polynomial of Q.

Lemma 4. *Let s and t be non-negative integers such that $s+t=n-4\geq 5$. If $H_1(s,t)\not\cong U_{1,n-3}$, where $H_1(s,t)$ is the graph depicted in Figure 2, then*

$$\lambda_1(A((K_n,D^-_{1,n-3})))>\lambda_1(A((K_n,H_1(s,t)^-))).$$

Proof. Assume that $s,t\geq 1$. We present the matrix $A((K_n,H_1(s,t)^-))$ and its corresponding quotient matrix $Q_3(s,t)$ by the vertex partition $V_1=\{v_1\}$, $V_2=\{v_2\}$, $V_3=\{v_3\}$, $V_4=\{v_4\}$, $V_5=N_{H_1(s,t)}(v_1)\backslash\{v_2,v_3,v_4\}$ and $V_6=N_{H_1(s,t)}(v_4)\backslash\{v_1\}$ as follows

$$A((K_n,H_1(s,t)^-))=\begin{bmatrix}0&-1&-1&-1&-j_s^T&j_t^T\\-1&0&-1&1&j_s^T&j_t^T\\-1&-1&0&1&j_s^T&j_t^T\\-1&1&1&0&j_s^T&-j_t^T\\-j_s&j_s&j_s&j_s&(J-I)_s&J_{s\times t}\\j_t&j_t&j_t&-j_t&J_{t\times s}&(J-I)_t\end{bmatrix},$$

and

$$Q_3(s,t)=\begin{bmatrix}0&-1&-1&-1&-s&t\\-1&0&-1&1&s&t\\-1&-1&0&1&s&t\\-1&1&1&0&s&-t\\-1&1&1&1&s-1&t\\1&1&1&-1&s&t-1\end{bmatrix}.$$

Note that the characteristic polynomial of $Q_3(s,t)$ is

$$P_{Q_3(s,t)}(\lambda)=(\lambda-1)(\lambda+1)(\lambda^4-(n-6)\lambda^3-(5n-16)\lambda^2+(8st-7s+9t-10)\lambda+24st-3s+13t-5).$$

Now, we add αJ to each block of $A((K_n, H_1(s,t)^-))$ for a constant α. Then $A((K_n, H_1(s,t)^-))$ becomes

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{0^T} & \mathbf{0^T} \\ 0 & 0 & 0 & 0 & \mathbf{0^T} & \mathbf{0^T} \\ 0 & 0 & 0 & 0 & \mathbf{0^T} & \mathbf{0^T} \\ 0 & 0 & 0 & 0 & \mathbf{0^T} & \mathbf{0^T} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_t \end{bmatrix}.$$

Since $\lambda_1(Q_3(s,t)) > 0$ and Spec(A_3)= $\left\{-1^{[n-4]}, 0^{[4]}\right\}$, we have $\lambda_1(A((K_n, H_1(s,t)^-))) = \lambda_1(Q_3(s,t))$.

Notice that

$$P_{Q_3(s,t)}(\lambda) - P_{Q_3(s+1,t-1)}(\lambda) = 8(s-t+3)\lambda + 24(s-t) + 40.$$

Next, we will discuss in two cases.

Case 1. $s \geq t$.

Note that $P_{Q_3(s,t)}(\lambda) - P_{Q_3(s+1,t-1)}(\lambda) > 0$ for $\lambda > 0$. Then $P_{Q_3(s+1,t-1)}(\lambda_1(Q_3(s,t))) < 0$, which implies that $\lambda_1(Q_3(s,t)) < \lambda_1(Q_3(s+1,t-1))$. Thus, $\lambda_1(A((K_n, H_1(s,t)^-))) < \lambda_1(A((K_n, H_1\ (s+1,t-1)^-)))$. By repeatedly using this operation, we can obtain that

$$\lambda_1(A((K_n, H_1(s,t)^-))) \leq \lambda_1(A((K_n, H_1(n-5,1)^-))),$$

the equality holds if and only if $s = n-5$, $t = 1$.

Next, we will show that $\lambda_1(A((K_n, U_2^-))) > \lambda_1(A((K_n, H_1(n-5,1)^-)))$. Notice that $P_{Q_3(n-5,1)}\ (\lambda) = (\lambda-1)(\lambda+1)(\lambda^4 - (n-6)\lambda^3 - (5n-16)\lambda^2 + (n-6)\lambda + 21n - 97)$ and $\lambda_1(A((K_n, H_1(n-5,1)^-)))$ is the largest root of $P_{Q_3(n-5,1)}(\lambda) = 0$. Clearly, $\varphi((K_n, U_2^-), \lambda) = (\lambda+1)^{n-5}(\lambda-1)^2(\lambda+3)(\lambda^2 + (4-n)\lambda + 11 - 3n)$. Let $f_3(\lambda) = (\lambda-1)^2(\lambda+1)(\lambda+3)(\lambda^2 + (4-n)\lambda + 11 - 3n)$. Hence,

$$h(\lambda) = P_{Q_3(n-5,1)}(\lambda) - f_3(\lambda) = 4(\lambda+1)(\lambda-1)((n-4)\lambda + 3n - 16).$$

The maximal solution of $h(\lambda) = 0$ is 1 for $n \geq 9$. And

$$P_{Q_3(n-5,1)}\left(\frac{n}{2} - 1\right) = \left(\frac{n-4}{2}\right)\left(\frac{n}{2}\right)\left(-\frac{1}{16}n^4 - \frac{1}{4}n^3 + 5n^2 + 4n - 80\right).$$

Set $f(n) = -\frac{1}{16}n^4 - \frac{1}{4}n^3 + 5n^2 + 4n - 80$. It is evident that $f'(n) = -\frac{1}{4}n^3 - \frac{3}{4}n^2 + 10n + 4$, $f''(n) = -\frac{3}{4}n^2 - \frac{3}{2}n + 10$ and $f'''(n) = -\frac{3}{2}n - \frac{3}{2}$. We observe that $f'''(n) = -\frac{3}{2}n - \frac{3}{2} < 0$ for $n \geq 9$. Hence, $f''(n) \leq f''(9) = -\frac{257}{4} < 0$, and thus $f'(n) \leq f'(9) = -149 < 0$. This means that $f(n)$ is a monotone decreasing for $n \geq 9$ such that $f(n) \leq f(9) = -\frac{3701}{16} < 0$, and then $P_{Q_3(n-5,1)}(\frac{n}{2} - 1) < 0$. Thus, $\lambda_1(A((K_n, H_1(n-5,1)^-))) > \frac{n}{2} - 1 > 1$ for $n \geq 9$. Let $\lambda_1 = \lambda_1(A((K_n, H_1(n-5,1)^-)))$, then

$$-f_3(\lambda_1) = P_{Q_3(n-5,1)}(\lambda_1) - f_3(\lambda_1) > 0.$$

This indicates that $\lambda_1(A((K_n, U_{2,n-5}^-))) > \lambda_1(A((K_n, H_1(n-5,1)^-)))$. Thus, $\lambda_1(A((K_n,\ D_{1,n-3}^-))) > \lambda_1(A((K_n, H_1(s,t)^-)))$ by Lemma 3.

Case 2. $s < t$.

Subcase 2.1. $t - s = 1$, *i.e.*, $H_1(s,t) \cong H_1(\frac{n-5}{2}, \frac{n-3}{2})$. Note that $P_{Q_3(s,t)}(\lambda) - P_{Q_3(s+1,t-1)}\ (\lambda) = 8(s-t+3)\lambda + 24(s-t) + 40 > 0$ for $\lambda > 0$. Let $\lambda_1 = \lambda_1(A((K_n, H_1(s,t)^-)))$, then $P_{Q_3(s+1,t-1)}(\lambda_1) < 0$.

Thus, $\lambda_1(A((K_n, H_1(\frac{n-5}{2}, \frac{n-3}{2})^-))) < \lambda_1(A((K_n, H_1(\frac{n-3}{2}, \frac{n-5}{2})^-)))$. Note that $\lambda_1(A((K_n, H_1(\frac{n-3}{2}, \frac{n-5}{2})^-))) < \lambda_1(A((K_n, D^-_{1,n-3})))$ by Case 1. Hence, $\lambda_1(A((K_n, H_1(\frac{n-5}{2}, \frac{n-3}{2})^-))) < \lambda_1(A((K_n, D^-_{1,n-3})))$.

Subcase 2.2. $t - s = 2$, *i.e.*, $H_1(s,t) \cong H_1(\frac{n-6}{2}, \frac{n-2}{2})$. Note that $P_{Q_{3(s,t)}}(\lambda) - P_{Q_{3(s+1,t-1)}}(\lambda) = 8(s - t + 3)\lambda + 24(s-t) + 40 > 0$ for $\lambda > 1$. Let $\lambda_1 = \lambda_1(A((K_n, H_1(s,t)^-)))$, then $P_{Q_{3(s+1,t-1)}}(\lambda_1) < 0$. Thus, $\lambda_1(A((K_n, H_1(\frac{n-6}{2}, \frac{n-2}{2})^-))) < \lambda_1(A((K_n, H_1(\frac{n-4}{2}, \frac{n-4}{2})^-)))$. Since $\lambda_1(A((K_n, H_1(\frac{n-4}{2}, \frac{n-4}{2})^-))) < \lambda_1(A((K_n, D^-_{1,n-3})))$ by Case 1, $\lambda_1(A((K_n, H_1(\frac{n-6}{2}, \frac{n-2}{2})^-))) < \lambda_1(A((K_n, D^-_{1,n-3})))$.

Subcase 2.3. $t - s \geq 3$. Similarly, notice that $P_{Q_{3(s,t)}}(\lambda) - P_{Q_{3(s-1,t+1)}}(\lambda) = (8t - 8s - 8)\lambda + 24(t-s) + 8 > 0$ for $\lambda > 0$ and $P_{Q_{3(s-1,t+1)}}(\lambda_1(Q_3(s,t))) < 0$. Hence, $\lambda_1(Q_3(s,t)) < \lambda_1(Q_3(s-1,t+1))$, and then $\lambda_1(A((K_n, H_1(s,t)^-))) < \lambda_1(A((K_n, H_1(s-1,t+1)^-)))$. By repeatedly using this operation, we can get that

$$\lambda_1(A((K_n, H_1(s,t)^-))) \leq \lambda_1(A((K_n, H_1(1,n-5)^-))),$$

the equality holds if and only if $s = 1$, $t = n-5$.

Next, we will assert that $\lambda_1(A((K_n, H_1(1,n-5)^-))) < \lambda_1(A((K_n, H_1(0,n-4)^-)))$. Note that

$$P_{Q_{3(1,n-5)}}(\lambda) = (\lambda-1)(\lambda+1)(\lambda^4 - (n-6)\lambda^3 - (5n-16)\lambda^2 + (17n-102)\lambda + 37n - 193).$$

By appropriately marking the vertices of $(K_n, H_1(0,n-4)^-)$, one can have

$$A((K_n, H_1(0,n-4)^-)) = \begin{bmatrix} 0 & -1 & -1 & -1 & j^T_{n-4} \\ -1 & 0 & -1 & 1 & j^T_{n-4} \\ -1 & -1 & 0 & 1 & j^T_{n-4} \\ -1 & 1 & 1 & 0 & -j^T_{n-4} \\ j_{n-4} & j_{n-4} & j_{n-4} & -j_{n-4} & (J-I)_{n-4} \end{bmatrix}.$$

Then

$$\lambda I_n - A((K_n, H_1(0,n-4)^-)) = \begin{bmatrix} \lambda & 1 & 1 & 1 & -j^T_{n-4} \\ 1 & \lambda & 1 & -1 & -j^T_{n-4} \\ 1 & 1 & \lambda & -1 & -j^T_{n-4} \\ 1 & -1 & -1 & \lambda & j^T_{n-4} \\ -j_{n-4} & -j_{n-4} & -j_{n-4} & j_{n-4} & ((\lambda+1)I - J)_{n-4} \end{bmatrix}.$$

Now, we apply finitely many elementary row and column operations on the matrix $\lambda I_n - A((K_n, H_1\ (0, n-4)^-))$. First, subtracting the 5-th row from all the lower rows and adding the i-th column to the 5-th column, for $i = n, \ldots, 6$. This leads to the following matrix:

$$\begin{bmatrix} \lambda & 1 & 1 & 1 & 4-n & * \\ 1 & \lambda & 1 & -1 & 4-n & * \\ 1 & 1 & \lambda & -1 & 4-n & * \\ 1 & -1 & -1 & \lambda & n-4 & * \\ -1 & -1 & -1 & 1 & \lambda - n + 5 & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\lambda+1)I_{n-5} \end{bmatrix}.$$

Therefore, $\varphi((K_n, H_1(0,n-4)^-), \lambda) = (\lambda+1)^{n-4}(\lambda-1)(\lambda^3 + (5-n)\lambda^2 + (11-4n)\lambda + 13n - 57)$. Let $f_2(\lambda) = (\lambda+1)^2(\lambda-1)(\lambda^3 + (5-n)\lambda^2 + (11-4n)\lambda + 13n - 57)$. Set

$$h(\lambda) = P_{Q_{3(1,n-5)}}(\lambda) - f_2(\lambda) = 8(\lambda+1)(\lambda-1)((n-7)\lambda + 3n - 17),$$

and the maximal solution of $h(\lambda) = 0$ is 1 for $n \geq 9$. And

$$P_{Q_{3(1,n-5)}}\left(\frac{n}{2} - 1\right) = \left(\frac{n-4}{2}\right)\left(\frac{n}{2}\right)\left(-\frac{1}{16}n^4 - \frac{1}{4}n^3 + 13n^2 - 44n - 80\right).$$

Set $g(n) = -\frac{1}{16}n^4 - \frac{1}{4}n^3 + 13n^2 - 44n - 80$. It is easy to see that $g'(n) = -\frac{1}{4}n^3 - \frac{3}{4}n^2 + 26n - 44$, $g''(n) = -\frac{3}{4}n^2 - \frac{3}{2}n + 26$ and $g'''(n) = -\frac{3}{2}n - \frac{3}{2}$. Note that $g'''(n) < 0$ for $n \geq 9$, then $g''(n) \leq g''(9) = -\frac{193}{4} < 0$, and thus $g'(n) \leq g'(9) = -53 < 0$. Hence, $g(n)$ is a monotone decreasing function for $n \geq 9$ and $g(n) \leq g(9) = -\frac{245}{16} < 0$, and then $P_{Q_3(1,n-5)}(\frac{n}{2} - 1) < 0$. Hence, $\lambda_1(Q_3(1, n-5)) > \frac{n}{2} - 1 > 1$ for $n \geq 9$. Let $\lambda_1 = \lambda_1(Q_3(1, n-5))$, so

$$-f_2(\lambda_1) = P_{Q_3(1,n-5)}(\lambda_1) - f_2(\lambda_1) > 0.$$

This indicates that $\lambda_1(A((K_n, H_1(1, n-5)^-))) < \lambda_1(A((K_n, H_1(0, n-4)^-)))$.

Finally, we will assert that $\lambda_1(A((K_n, H_1(n-5, 1)^-))) > \lambda_1(A((K_n, H_1(0, n-4)^-)))$. Let $f_4(\lambda) = (\lambda+1)^2(\lambda-1)(\lambda^3 + (5-n)\lambda^2 + (11-4n)\lambda + 13n - 57)$. Note that

$$h(\lambda) = f_4(\lambda) - P_{Q_3(n-5,1)}(\lambda) = 8(n-5)(\lambda-1)^2(\lambda+1).$$

The maximal solution of $h(\lambda) = 0$ is 1 for $n \geq 9$, and $\varphi((K_n, H_1(0, n-4)^-), \frac{n}{2} - 1) = (\frac{n}{2})^{n-4}(\frac{n-4}{2})\ (-\frac{1}{8}n^3 - \frac{1}{2}n^2 + 18n - 64)$. Similarly, we obtain that $f(n) = -\frac{1}{8}n^3 - \frac{1}{2}n^2 + 18n - 64 < 0$ for $n \geq 9$. Then $\varphi((K_n, H_1(0, n-4)^-), \frac{n}{2} - 1) < 0$. Hence, $\lambda_1(A((K_n, H_1(0, n-4)^-))) > \frac{n}{2} - 1 > 1$ for $n \geq 9$. Let $\lambda_1 = \lambda_1(A((K_n, H_1(0, n-4)^-)))$, then

$$-P_{Q_3(n-5,1)}(\lambda_1) = f_4(\lambda_1) - P_{Q_3(n-5,1)}(\lambda_1) > 0.$$

This indicates that $\lambda_1(A((K_n, H_1(n-5, 1)^-))) > \lambda_1(A((K_n, H_1(0, n-4)^-)))$. Thus, $\lambda_1(A((K_n,\ D^-_{1,n-3}))) > \lambda_1(A((K_n, H_1(s, t)^-)))$ by Case 1.

Hence, $\lambda_1(A((K_n,\ D^-_{1,n-3}))) > \lambda_1(A((K_n, H_1(s, t)^-)))$. The proof is completed. □

Lemma 5. *Let s and t be non-negative integers such that $s + t = n - 5 \geq 4$, and $H_2(s, t)$ be the graph depicted in Figure 2. Then*

$$\lambda_1(A((K_n, D^-_{1,n-3}))) > \lambda_1(A((K_n, H_2(s, t)^-))).$$

Proof. Assume that $s, t \geq 1$. We present the matrix $A((K_n, H_2(s, t)^-))$ and its corresponding quotient matrix $Q_4(s, t)$ by the vertex partition $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$, $V_5 = \{v_5\}$, $V_6 = N_{H_{2(s,t)}}(v_4)\backslash\{v_1, v_5\}$ and $V_7 = N_{H_2(s,t)}(v_5)\backslash\{v_4\}$ as follows

$$A((K_n, H_2(s,t)^-)) = \begin{bmatrix} 0 & -1 & -1 & -1 & 1 & j_s^T & j_t^T \\ -1 & 0 & -1 & 1 & 1 & j_s^T & j_t^T \\ -1 & -1 & 0 & 1 & 1 & j_s^T & j_t^T \\ -1 & 1 & 1 & 0 & -1 & -j_s^T & j_t^T \\ 1 & 1 & 1 & -1 & 0 & j_s^T & -j_t^T \\ j_s & j_s & j_s & -j_s & j_s & (J-I)_s & J_{s\times t} \\ j_t & j_t & j_t & j_t & -j_t & J_{t\times s} & (J-I)_t \end{bmatrix},$$

and

$$Q_4(s,t) = \begin{bmatrix} 0 & -1 & -1 & -1 & 1 & s & t \\ -1 & 0 & -1 & 1 & 1 & s & t \\ -1 & -1 & 0 & 1 & 1 & s & t \\ -1 & 1 & 1 & 0 & -1 & -s & t \\ 1 & 1 & 1 & -1 & 0 & s & -t \\ 1 & 1 & 1 & -1 & 1 & s-1 & t \\ 1 & 1 & 1 & 1 & -1 & s & t-1 \end{bmatrix}.$$

Notice that the characteristic polynomial of $Q_4(s, t)$ is

$$\begin{aligned} P_{Q_{4(s,t)}}(\lambda) = (\lambda-1)(\lambda+1)(\lambda^5 + (7-n)\lambda^4 - (6n-22)\lambda^3 + (8st + 4n + 8t - 30)\lambda^2 \\ + (32st + 22n - 103)\lambda + 13n - 72st - 40t - 57. \end{aligned}$$

Now, we add αJ to each block of $A((K_n, H_2(s,t)^-))$ for a constant α. Then $A((K_n, H_2(s,t)^-))$ becomes

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{0^T} & \mathbf{0^T} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0^T} & \mathbf{0^T} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0^T} & \mathbf{0^T} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0^T} & \mathbf{0^T} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0^T} & \mathbf{0^T} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_t \end{bmatrix}.$$

Since $\lambda_1(Q_4(s,t)) > 0$ and Spec(A_4)= $\{-1^{[n-5]}, 0^{[5]}\}$, $\lambda_1(A((K_n, H_2(s,t)^-))) = \lambda_1(Q_4(s,t))$.

Note that

$$P_{Q_{4(s,t)}}(\lambda) - P_{Q_{4(s+1,t-1)}}(\lambda) = 8((s-t+2)\lambda^2 + (4s-4t+4)\lambda + 9t - 9s - 14).$$

Next, we will discuss in two cases.

Case 1. $s \geq t$.

Let $f(\lambda) = 8((s-t+2)\lambda^2 + (4s-4t+4)\lambda + 9t - 9s - 14)$. Then $f'(\lambda) = 16(s-t+2)\lambda + 32(s-t+1) > 0$ for $\lambda > 0$, and $f(\lambda)$ is a monotone increasing function for $\lambda > 0$. We can obtain that $P_{Q_{4(s,t)}}(\frac{n}{2}-1) - P_{Q_{4(s+1,t-1)}}(\frac{n}{2}-1) > 0$. And

$$P_{Q_{4(s,t)}}\left(\frac{n}{2}-1\right) = \left(\frac{n}{2}\right)\left(\frac{n-4}{2}\right)\left(-\frac{1}{32}n^5 - \frac{1}{8}n^4 + \frac{9}{2}n^3 + (2st+2t-16)n^2 + (8st-8t)n - 96st - 32t\right).$$

Let $h(n) = (-\frac{1}{32}n^5 - \frac{1}{8}n^4 + \frac{9}{2}n^3 + (2st+2t-16)n^2 + (8st-8t)n - 96st - 32t)$. Next, we will show $P_{Q_{4(s,t)}}(\frac{n}{2}-1) < 0$. Thus, we just need to prove $h(n) < 0$. Note that

$$\begin{aligned} h'(n) &= -\frac{5}{32}n^4 - \frac{1}{2}n^3 + \frac{27}{2}n^2 + 2(2st+2t-16)n + 8st - 8t, \\ h''(n) &= -\frac{5}{8}n^3 - \frac{3}{2}n^2 + 27n + 2(2st+2t-16), \\ h'''(n) &= -\frac{15}{8}n^2 - 3n + 27, \\ h^{(4)}(n) &= -\frac{15}{4}n - 3 < 0. \end{aligned}$$

Recall that $s, t \geq 1$ such that $s + t = n - 5$, then

$$n - 6 \leq st \leq \begin{cases} (\frac{n-5}{2})^2 & \text{if } n \textit{ is odd}, \\ (\frac{n-4}{2})(\frac{n-6}{2}) & \text{if } n \text{ is even}. \end{cases}$$

Then $h'''(n)$ is a monotone decreasing for $n \geq 9$ and $h'''(n) \leq h'''(9) = -\frac{1215}{8} < 0$, and then $h''(n)$ is a monotone decreasing function for $n \geq 9$. One can have

$$h''(n) \leq \begin{cases} -\frac{5}{8}n^3 - \frac{1}{2}n^2 + 19n - 17 & \text{if } n \text{ is odd}, \\ -\frac{5}{8}n^3 - \frac{1}{2}n^2 + 19n - 20 & \text{if } n \text{ is even}. \end{cases}$$

Set $h_1(n) = -\frac{5}{8}n^3 - \frac{1}{2}n^2 + 19n - 17$. Its derivative is $h_1'(n) = -\frac{15}{8}n^2 - n + 19$ and $h_1''(n) = -\frac{15}{4}n - 1$. Obviously, $h_1''(n) < 0$ for $n \geq 9$ and $h_1'(n) \leq h_1'(9) = -\frac{1135}{8} < 0$. Thus, $h_1(n)$ is a monotone decreasing for $n \geq 9$ and

$h_1(n) \le h_1(9) = -\frac{2737}{8} < 0$. Note that $-\frac{5}{8}n^3 - \frac{1}{2}n^2 + 19n - 20 < h_1(n) < 0$. So, $h'(n)$ is monotone decreasing for $n \ge 9$ and

$$h'(n) \le \begin{cases} -\frac{5}{32}n^4 + \frac{1}{2}n^3 + \frac{15}{2}n^2 - 41n + 70 & \text{if } n \text{ is odd,} \\ -\frac{5}{32}n^4 + \frac{1}{2}n^3 + \frac{15}{2}n^2 - 44n + 72 & \text{if } n \text{ is even.} \end{cases}$$

Set $h_2(n) = -\frac{5}{32}n^4 + \frac{1}{2}n^3 + \frac{15}{2}n^2 - 41n + 70$. Clearly, $h_2'(n) = -\frac{5}{8}n^3 + \frac{3}{2}n^2 + 15n - 41$, $h_2''(n) = -\frac{15}{8}n^2 + 3n + 15$ and $h_2'''(n) = -\frac{15}{4}n + 3$. Obviously, $h_2'''(n) < 0$ for $n \ge 9$. Thus, $h_2''(n) \le h_2''(9) = -\frac{879}{8} < 0$ and $h_2'(n) \le h_2'(9) = -\frac{1921}{8} < 0$. Hence, $h_2(n)$ is a monotone decreasing for $n \ge 9$ and $h_2(n) \le h_2(9) = -\frac{11269}{32} < 0$. Note that $-\frac{5}{32}n^4 + \frac{1}{2}n^3 + \frac{15}{2}n^2 - 44n + 7 < h_2(n) < 0$. This implies that $h(n)$ is monotone decreasing for $n \ge 9$ and

$$h(n) \le \begin{cases} -\frac{1}{32}n^5 + \frac{3}{8}n^4 + \frac{5}{2}n^3 - \frac{113}{2}n^2 + 294n - 520 & \text{if } n \text{ is odd,} \\ -\frac{1}{32}n^5 + \frac{3}{8}n^4 + \frac{5}{2}n^3 - 58n^2 + 296n - 480 & \text{if } n \text{ is even.} \end{cases}$$

Set $h_3(n) = -\frac{1}{32}n^5 + \frac{3}{8}n^4 + \frac{5}{2}n^3 - \frac{113}{2}n^2 + 294n - 520$. Similarly, we obtain that $h_3(n) < 0$. Note that $-\frac{1}{32}n^5 + \frac{3}{8}n^4 + \frac{5}{2}n^3 - 58n^2 + 296n - 480 < h_3(n) < 0$. This means that $\lambda_1(A((K_n, H_2(s,t)^-))) > \frac{n}{2} - 1$. Let $\lambda_1 = \lambda_1(A((K_n, H_2(s,t)^-)))$, thus

$$-P_{Q_{4(s+1,t-1)}}(\lambda_1) = P_{Q_{4(s,t)}}(\lambda_1) - P_{Q_{4(s+1,t-1)}}(\lambda_1) > 0.$$

Then $\lambda_1(A((K_n, H_2(s,t)^-))) < \lambda_1(A((K_n, H_2(s+1,t-1)^-)))$. By repeatedly using this operation, we can get that

$$\lambda_1(A((K_n, H_2(s,t)^-))) \le \lambda_1(A((K_n, H_2(n-6,1)^-))),$$

the equality holds if and only if $s = n-6$, $t = 1$. Note that

$$P_{Q_4(n-6,1)}(\lambda) = (\lambda-1)(\lambda+1)(\lambda^5 + (7-n)\lambda^4 - (6n-22)\lambda^3 + (12n-70)\lambda^2 + (54n-295)\lambda + 335 - 59n).$$

Clearly, $H_2(n-5,0) \cong H_1(0,n-4)$. Then $\varphi((K_n, H_2(n-5,0)^-), \lambda) = (\lambda+1)^{n-4}(\lambda-1)(\lambda^3 + (5-n)\lambda^2 + (11-4n)\lambda + 13n - 57)$, and set $f_1(\lambda) = (\lambda+1)^3(\lambda-1)(\lambda^3 + (5-n)\lambda^2 + (11-4n)\lambda + 13n - 57)$ by Lemma 4. Hence

$$P_{Q_4(n-6,1)}(\lambda) - f_1(\lambda) = 8(\lambda+1)(\lambda-1)((n-5)\lambda^2 + 4(n-6)\lambda + 49 - 9n).$$

Let $f(\lambda) = (n-5)\lambda^2 + 4(n-6)\lambda + 49 - 9n$. It is evident that $f'(\lambda) = 2(n-5)\lambda + 4(n-6) > 0$ for $n \ge 9$ and $\lambda > 0$. Thus, $f(\lambda)$ is a monotone increasing function for $n \ge 9$ and $\lambda > 0$, and $f(\frac{n}{2} - 1) > 0$ for $n \ge 9$. Notice that

$$P_{Q_4(n-6,1)}\left(\frac{n}{2} - 1\right) = \left(\frac{n}{2}\right)\left(\frac{n-4}{2}\right)\left(-\frac{1}{32}n^5 - \frac{1}{8}n^4 + \frac{13}{2}n^3 - 18n^2 - 152n + 544\right).$$

Let $w(n) = -\frac{1}{32}n^5 - \frac{1}{8}n^4 + \frac{13}{2}n^3 - 18n^2 - 152n + 544$. Similarly, we obtain that $w(n) < 0$ for $n \ge 9$. Then $P_{Q_4(n-6,1)}(\frac{n}{2} - 1) < 0$ and $\lambda_1(A((K_n, H_2(n-6,1)^-))) > \frac{n}{2} - 1$. Let $\lambda_1 = \lambda_1(A((K_n, H_2(n-6,1)^-)))$, then $f_1(\lambda_1) < 0$. thus

$$\varphi((K_n, H_2(n-5,0)^-), \lambda_1) < 0.$$

This indicates that $\lambda_1(A((K_n, H_2(n-6,1)^-))) < \lambda_1(A((K_n, H_2(n-5,0)^-)))$. Since $H_2(n-5,0) \cong H_1(0,n-4)$, $\lambda_1(A((K_n, H_2(s,t)^-))) < \lambda_1(A((K_n, D_{1,n-3}^-)))$ by Lemma 4.

Case 2. $s < t$.

Subcase 2.1. $t - s = 1$, *i.e.*, $H_2(s,t) \cong H_2(\frac{n-5}{2}, \frac{n-3}{2})$ and $P_{Q_{4(s,t)}}(\lambda) - P_{Q_{4(s+1,t-1)}}(\lambda) = 8(\lambda^2 - 5) > 0$ for $\lambda > \sqrt{5}$. Let $\lambda_1 = \lambda_1(A((K_n, H_2(s,t)^-)))$. Note that $P_{Q_{4(s,t)}}(\frac{n}{2} - 1) < 0$ by Case 1, then $\lambda_1 > \frac{n}{2} - 1 > \sqrt{5}$ for $n \ge 9$. Hence, $P_{Q_{4(s+1,t-1)}}(\lambda_1) < 0$ and $\lambda_1(A((K_n, H_2(\frac{n-5}{2}, \frac{n-3}{2})^-))) < \lambda_1(A((K_n, H_2(\frac{n-3}{2}, \frac{n-5}{2})^-)))$. Since

$\lambda_1(A((K_n,H_2(\frac{n-3}{2},\ \frac{n-5}{2})^-))) < \lambda_1(A((K_n,D^-_{1,n-3})))$ by Case 1, $\lambda_1(A((K_n,H_2(\frac{n-5}{2},\frac{n-3}{2})^-))) < \lambda_1(A((K_n, D^-_{1,n-3})))$.

Subcase 2.2. $t-s=2$, *i.e.*, $H_2(s,t)\cong H_2(\frac{n-7}{2},\frac{n-3}{2})$. Set $h(\lambda)=P_{Q_{4(s,t)}}(\lambda)-P_{Q_{4(s-1,t+1)}}(\lambda)=8(2\lambda^2+12\lambda-22)$, and the maximal solution of $h(\lambda)=0$ is $-3+2\sqrt{5}$. Let $\lambda_1=\lambda_1(A((K_n,H_2(s,t\ )^-)))$. Note that $P_{Q_{4(s,t)}}(\frac{n}{2}-1)<0$ by Case 1, then $\lambda_1>\frac{n}{2}-1>-3+2\sqrt{5}$ for $n\geq 9$. This means that $P_{Q_{4(s-1,t+1)}}(\lambda_1)<0$ and $\lambda_1(A((K_n,H_2(s,t)^-)))<\lambda_1(A((K_n,H_2(s-1,t+1)^-)))$. By repeatedly using this operation, we can get that

$$\lambda_1(A((K_n,H_2(s,t)^-)))\leq\lambda_1(A((K_n,H_2(1,n-6)^-))),$$

the equality holds if and only if $s=1$, $t=n-6$.

Subcase 2.3. $t-s\geq 3$. Note that $P_{Q_{4(s,t)}}(\lambda)-P_{Q_{4(s-1,t+1)}}(\lambda)=8((t-s)\lambda^2+4(t-s+1)\lambda+9s-9t-4)$. Set $f(\lambda)=8((t-s)\lambda^2+4(t-s+1)\lambda+9s-9t-4)$, and then $f'(\lambda)=16(t-s)\lambda+32(t-s+1)>0$ for $\lambda>0$. Hence, $f(\lambda)$ is a monotone increasing function for $\lambda>0$ and $f(2)=8(3t-3s+4)>0$. Let $\lambda_1=\lambda_1(A((K_n,H_2(s,t)^-)))$. Note that $P_{Q_{4(s,t)}}(\frac{n}{2}-1)<0$ by Case 1, then $\lambda_1>\frac{n}{2}-1>2$ for $n\geq 9$. This indicates that $P_{Q_{4(s-1,t+1)}}(\lambda_1)<0$ and $\lambda_1(A((K_n,H_2(s,t)^-)))<\lambda_1(A((K_n,H_2(s-1,t+1)^-)))$. By repeatedly using this operation, one can have

$$\lambda_1(A((K_n,H_2(s,t)^-)))\leq\lambda_1(A((K_n,H_2(1,n-6)^-))),$$

the equality holds if and only if $s=1$, $t=n-6$.

Next, we will give that $\lambda_1(A((K_n,H_2(1,n-6)^-)))<\lambda_1(A((K_n,H_2(0,n-5)^-)))$. Note that

$$P_{Q_4(1,n-6)}(\lambda)=(\lambda-1)(\lambda+1)(\lambda^5+(7-n)\lambda^4-(6n-22)\lambda^3+(20n-126)\lambda^2+(54n-295)\lambda+615-99n).$$

By appropriately marking the vertices of $(K_n,H_2(0,n-5)^-)$, we can get that

$$A((K_n,H_2(0,n-5)^-))=\begin{bmatrix} 0 & -1 & -1 & -1 & 1 & j^T_{n-5}\\ -1 & 0 & -1 & 1 & 1 & j^T_{n-5}\\ -1 & -1 & 0 & 1 & 1 & j^T_{n-5}\\ -1 & 1 & 1 & 0 & -1 & j^T_{n-5}\\ 1 & 1 & 1 & -1 & 0 & -j^T_{n-5}\\ j_{n-5} & j_{n-5} & j_{n-5} & j_{n-5} & -j_{n-5} & (J-I)_{n-5}\end{bmatrix}.$$

Then

$$\lambda I_n-A((K_n,H_2(0,n-5)^-))=\begin{bmatrix} \lambda & 1 & 1 & 1 & -1 & -j^T_{n-5}\\ 1 & \lambda & 1 & -1 & -1 & -j^T_{n-5}\\ 1 & 1 & \lambda & -1 & -1 & -j^T_{n-5}\\ 1 & -1 & -1 & \lambda & 1 & -j^T_{n-5}\\ -1 & -1 & -1 & 1 & \lambda & j^T_{n-5}\\ -j_{n-5} & -j_{n-5} & -j_{n-5} & -j_{n-5} & j_{n-5} & ((\lambda+1)\,I-J)_{n-5}\end{bmatrix}.$$

Now, we apply finitely many elementary row and column operations on the matrix $\lambda I_n-A((K_n,H_2\ (0,n-5)^-))$. First, subtracting the 6-th row from all the lower rows and adding the i-th column to the 6-th column, for $i=n,\ldots,7$. This leads to the following matrix:

$$\begin{bmatrix} \lambda & 1 & 1 & 1 & -1 & 5-n & *\\ 1 & \lambda & 1 & -1 & -1 & 5-n & *\\ 1 & 1 & \lambda & -1 & -1 & 5-n & *\\ 1 & -1 & -1 & \lambda & 1 & 5-n & *\\ -1 & -1 & -1 & 1 & \lambda & n-5 & *\\ -1 & -1 & -1 & -1 & 1 & \lambda+6-n & *\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\lambda+1)I_{n-6}\end{bmatrix}.$$

Thus, the characteristic polynomial of $A((K_n, H_2(0, n-5)^-))$ is

$$\varphi((K_n, H_2(0, n-5)^-), \lambda) = (\lambda+1)^{n-6}(\lambda-1)^2(\lambda^4 + (8-n)\lambda^3 + (30-7n)\lambda^2 + (5n-40)\lambda + 27n - 143).$$

Let $f_2(\lambda) = (\lambda+1)(\lambda-1)^2(\lambda^4 + (8-n)\lambda^3 + (30-7n)\lambda^2 + (5n-40)\lambda + 27n-143)$. Notice that

$$P_{Q_4(1,n-6)}(\lambda) - f_2(\lambda) = 8(\lambda-1)(\lambda+1)((n-7)\lambda^2 + 4(n-6)\lambda + 59 - 9n).$$

Let $g(\lambda) = (n-7)\lambda^2 + 4(n-6)\lambda + 59 - 9n$. Then $g'(\lambda) = 2(n-7)\lambda + 4(n-6) > 0$ for $n \geq 9$, $\lambda > 0$ and $g(\lambda)$ is a monotone increasing function for $n \geq 9$ and $\lambda > 0$. Note that $g(\frac{n}{2}-1) = \frac{1}{4}n^3 - \frac{3}{4}n^2 - 17n + 76$, thus $g'(\frac{n}{2}-1) = \frac{3}{4}n^2 - \frac{3}{2}n - 17$ and $g''(\frac{n}{2}-1) = \frac{3}{2}n - \frac{3}{2} > 0$ for $n \geq 9$. Hence, $g'(\frac{n}{2}-1) \geq \frac{121}{4} > 0$, and $g(\frac{n}{2}-1) \geq \frac{89}{2} > 0$. Notice that

$$P_{Q_4(1,n-6)}\left(\frac{n}{2}-1\right) = \left(\frac{n}{2}\right)\left(\frac{n-4}{2}\right)\left(-\frac{1}{32}n^5 - \frac{1}{8}n^4 + \frac{17}{2}n^3 - 40n^2 - 128n + 768\right).$$

Let $z(n) = -\frac{1}{32}n^5 - \frac{1}{8}n^4 + \frac{17}{2}n^3 - 40n^2 - 128n + 768$. It is easy to see that $z'(n) = -\frac{5}{32}n^4 - \frac{1}{2}n^3 + \frac{51}{2}n^2 - 80n - 128$, $z''(n) = -\frac{5}{8}n^3 - \frac{3}{2}n^2 + 51n - 80$, $z'''(n) = -\frac{15}{8}n^2 - 3n + 51$ and $z^{(4)}(n) = -\frac{15}{4}n - 3 < 0$ for $n \geq 9$. Hence, $z'''(n) \leq z'''(9) = -\frac{1023}{8} < 0$ and $z''(n) \leq z''(9) = -\frac{1585}{8} < 0$. So, $z'(n) \leq z'(9) = -\frac{5509}{32} < 0$ and $z(n)$ is a monotone decreasing for $n \geq 9$, and then $z(n) \leq z(9) = -\frac{2973}{32} < 0$. Since $P_{Q_4(1,n-6)}(\frac{n}{2}-1) < 0$, $\lambda_1(A((K_n, H_2(1, n-6)^-))) > \frac{n}{2} - 1$. Let $\lambda_1 = \lambda_1(A((K_n, H_2(1, n-6)^-)))$. Hence, $f_2(\lambda_1) < 0$ and $\lambda_1(A((K_n, H_2(1, n-6)^-))) < \lambda_1(A((K_n, H_2(0, n-5)^-)))$.

Finally, we assert that $\lambda_1(A((K_n, H_2(0, n-5)^-))) < \lambda_1(A((K_n, H_2(n-5, 0)^-)))$. Let $f_3(\lambda) = (\lambda+1)^2(\lambda-1)(\lambda^3 + (5-n)\lambda^2 + (11-4n)\lambda + 13n - 57)$ and $f_4(\lambda) = (\lambda-1)^2(\lambda^4 + (8-n)\lambda^3 + (30-7n)\lambda^2 + (5n-40)\lambda + 27n - 143)$. Set

$$h(\lambda) = f_4(\lambda) - f_3(\lambda) = 8(n-5)(\lambda-1)(\lambda^2-5),$$

and the maximal solution of $h(\lambda) = 0$ is $\sqrt{5}$. And

$$f_4\left(\frac{n}{2}-1\right) = \left(\frac{n-4}{2}\right)^2\left(-\frac{1}{16}n^4 - \frac{1}{2}n^3 + 11n^2 - 24n - 80\right).$$

Let $y(n) = -\frac{1}{16}n^4 - \frac{1}{2}n^3 + 11n^2 - 24n - 80$. Similarly, we obtain that $y(n) < 0$ for $n \geq 9$. Then $f_4(\frac{n}{2}-1) < 0$ and $\lambda_1(A((K_n, H_2(0, n-5)^-))) > \frac{n}{2} - 1 > \sqrt{5}$ for $n \geq 9$. Let $\lambda_1 = \lambda_1(A((K_n, H_2(0, n-5)^-)))$, then $f_3(\lambda_1) < 0$. This induces that $\lambda_1(A((K_n, H_2(0, n-5)^-))) < \lambda_1(A((K_n, H_2(n-5, 0)^-)))$.

Obviously, $\lambda_1(A((K_n, H_2(n-5, 0)^-))) = \lambda_1(A((K_n, H_1(0, n-4)^-))) < \lambda_1(A((K_n, D^-_{1,n-3})))$ by Lemma 4. Hence, the proof is completed. □

3. Proof of Theorem 1

Lemma 6 ([14]). *Let T be a spanning tree of K_n and $n \geq 6$. If $\Gamma = (K_n, T^-)$ is an unbalanced signed complete graph with maximum index, then $T \cong D_{1,n-3}$.*

Lemma 7 ([13]). *Let r, s, t and u be distinct vertices of a signed graph Γ and $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to $\lambda_1(A(\Gamma))$. Then*

(i) *Let Γ' be obtained from Γ by reversing the sign of the positive edge rs and the negative edge rt. If*

$$\begin{cases} x_r \geq 0, x_s \leq x_t & \text{or} \\ x_r \leq 0, x_s \geq x_t, \end{cases}$$

then $\lambda_1(A(\Gamma)) \leq \lambda_1(A(\Gamma'))$. If at least one inequality for the entries of X is strict, then $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$.

(ii) *Let Γ' be obtained from Γ by reversing the sign of the positive edge rs and the negative edge tu. If $x_r x_s \leq x_t x_u$, then $\lambda_1(A(\Gamma)) \leq \lambda_1(A(\Gamma'))$. If at least one of the entries x_r, x_s, x_t, x_u is distinct from zero, then $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$.*

Let $R(r, s, t)$ stand for relocation (i) in Lemma 7. Kafai, Heydari, Rad and Maghasedi [12] showed that among all signed complete graphs of order $n > 5$ whose negative-edge-induce a unicyclic graph of order k and maximize the index, the negative edges induce a triangle with all remaining vertices being pendant at the same vertex of the triangle.

Lemma 8. *Let H be a connected $K_{2,2}$-minor free spanning subgraph of K_n for $n \geq 9$. If $\Gamma = (K_n, H^-)$ has the maximum index, then $H \cong U_{1,n-3}$.*

Proof. It is clear that H is either a tree or a connected graph that only contains 3-length cycles. According to Lemmas 2 and 6, H is a connected graph which contains 3-length cycles. Without loss of generality, suppose that $C_3 = v_1v_2v_3v_1$ is a 3-length cycle of H. Let $X = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector associated with $\lambda_1(A(\Gamma))$. Note that $-X$ must be a unit eigenvector of Γ if X is a unit eigenvector.

Claim 1. There exists an integer i such that $x_i \neq 0$ for $1 \leq i \leq 3$.

Otherwise, $x_1 = x_2 = x_3 = 0$. Since H is a connected graph for $n \geq 9$, there is a vertex $v_4 \in V(H)\setminus V(C_3)$. Without loss of generality, assume that $v_3v_4 \in E(H)$. Firstly, let $x_4 \neq 0$. If v_4 is not contained in any 3-length cycle, then the relocation $R(v_4, v_3, v_1)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. If there is another 3-length cycle C_3' such that $v_4 \in V(C_3')$, then we will consider three cases. If $V(C_3) \cap V(C_3') = \phi$, then the relocation $R(v_4, v_3, v_1)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. If $V(C_3) \cap V(C_3') = \{v_3\}$, then we can construct a new unbalanced signed complete graph Γ' whose negative-edge-induced connected subgraph is still $K_{2,2}$-minor free by reversing the sign of the negative edge v_3v_4 such that $\lambda_1(A(\Gamma')) > \lambda_1(A(\Gamma))$, a contradiction. If $|V(C_3) \cap V(C_3')| = 2$, then H contains a $K_{2,2}$-minor, a contradiction. Hence, $x_4 = 0$. By repeatedly conducting similar discussion on the vertices in $V(H)\setminus\{v_1, v_2, v_3, v_4\}$, we have $X = \mathbf{0}$, a contradiction.

Claim 2. H contains only one 3-length cycle.

Otherwise, let $V(C_3') = v_4v_5v_6v_4$ be another 3-length cycle of H. Then we divide the proof of Claim 2 into the following three cases.

Case 1. There exists two vertices $v_i, v_j \in V(C_3')$ such that $x_ix_j > 0$.

Without loss of generality, assume that $x_4x_5 > 0$. Then we can construct a new unbalanced signed complete graph Γ' whose negative-edge-induced connected subgraph is still $K_{2,2}$-minor free by reversing the sign of the negative edge v_4v_5 such that

$$\lambda_1(A(\Gamma')) - \lambda_1(A(\Gamma)) \geq X^T(A(\Gamma') - A(\Gamma))X = 4x_4x_5 > 0,$$

that is, $\lambda_1(A(\Gamma')) > \lambda_1(A(\Gamma))$, a contradiction.

Case 2. There exists only one vertex $v_i \in V(C_3')$ such that $x_i = 0$.

Without loss of generality, assume that $x_4 = 0$. By Case 1, $x_5x_6 < 0$. Then we can construct a new unbalanced signed complete graph Γ' whose negative-edge-induced connected subgraph is still $K_{2,2}$-minor free by reversing the sign of the negative edge v_4v_5 such that

$$\lambda_1(A(\Gamma')) - \lambda_1(A(\Gamma)) \geq X^T(A(\Gamma') - A(\Gamma))X = 4x_4x_5 = 0.$$

If $\lambda_1(A(\Gamma')) = \lambda_1(A(\Gamma))$, then X is also a unit eigenvector of $A(\Gamma')$ corresponding to $\lambda_1(A(\Gamma'))$. However,

$$\lambda_1(A(\Gamma'))x_4 - \lambda_1(A(\Gamma))x_4 = 2x_5 \neq 0,$$

a contradiction.

Case 3. There exists two vertices $v_i, v_j \in V(C_3')$ such that $x_i = x_j = 0$.

Without loss of generality, assume that $x_4 = x_5 = 0$. By Claim 1, $x_6 \neq 0$. Then we can construct a new unbalanced signed complete graph Γ' whose negative-edge-induced connected subgraph is still $K_{2,2}$-minor free by reversing the sign of the negative edge v_4v_6 such that

$$\lambda_1(A(\Gamma')) - \lambda_1(A(\Gamma)) \geq X^T(A(\Gamma') - A(\Gamma))X = 4x_4x_6 = 0.$$

If $\lambda_1(A(\Gamma')) = \lambda_1(A(\Gamma))$, then X is also a unit eigenvector of $A(\Gamma')$ corresponding to $\lambda_1(A(\Gamma'))$. However,

$$\lambda_1(A(\Gamma'))x_4 - \lambda_1(A(\Gamma))x_4 = 2x_6 \neq 0.$$

a contradiction.

Claim 2 means that H contains only one 3-length cycle $C_3 = v_1v_2v_3v_1$. Then $H \cong U_{1,n-3}$ by Kafai *et al.* [12]. So, the proof is completed. □

Lemma 9. *Let H be a connected $K_{2,2}$-minor free spanning subgraph of K_n for $n \geq 9$. If $\Gamma = (K_n, H^-)$ is not switching isomorphic to $(K_n, U_{1,n-3}^-)$ and has the maximum index, then $H \cong D_{1,n-3}$.*

Proof. Recall that H is either a tree or a connected graph that only contains 3-length cycles. Next, we will assert that H must be a tree. Otherwise, let $C_3 = v_1v_2v_3v_1$ be a 3-length cycle of H and $X = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector associated with $\lambda_1(A(\Gamma))$. Note that $-X$ must be a unit eigenvector of Γ if X is a unit eigenvector. Then we divide the proof into the following two cases. Firstly, we consider that H contains only one 3-length cycle $C_3 = v_1v_2v_3v_1$.

Claim 1. C_3 has only one vertex v_i such that $d_H(v_i) \geq 3$ for $1 \leq i \leq 3$.

Otherwise, without loss of generality, assume that $d_H(v_1) \geq 3$ and $d_H(v_2) \geq 3$. Let $v_p \in N_H(v_2)\backslash\{v_1, v_3\}$ and $v_{p'} \in N_H(v_1)\backslash\{v_2, v_3\}$. Then we will divide into the following five cases.

Case 1. $x_1 = x_2 = 0$.

We first assert that $x_{p'} = 0$. Otherwise, the relocation $R(v_{p'}, v_2, v_1)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Secondly, we assert that $x_3 = 0$. Otherwise, we can construct a new unbalanced signed graph Γ' whose negative-edge-induced connected subgraph is still $K_{2,2}$-minor free by reversing the sign of the positive edge $v_3v_{p'}$ and the negative edge v_1v_2 such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$ by (ii) of Lemma 7, a contradiction. However, similar to Claim 1 of Lemma 8, this will lead to $X = \mathbf{0}$, a contradiction.

Case 2. $x_1 = x_2 \neq 0$.

Without loss of generality, assume that $x_1 = x_2 > 0$. At first, we assert that $x_p = 0$. Otherwise, the relocation $R(v_p, v_1, v_2)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Next, we claim that $x_3 < 0$. Otherwise, the relocation $R(v_3, v_p, v_1)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Hence, we can construct a new unbalanced signed complete graph Γ' whose negative-edge-induced connected subgraph is still $K_{2,2}$-minor free by reversing the sign of the positive edge v_pv_3 and the negative edge v_1v_2 such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$ by (ii) of Lemma 7, a contradiction.

Case 3. $x_1 \neq x_2$ and $x_1x_2 < 0$.

Without loss of generality, assume that $x_1 > 0$, $x_2 < 0$. Firstly, we assert that $x_p > 0$. Otherwise, the relocation $R(v_p, v_1, v_2)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. If $x_{p'} \leq x_2$, then the relocation $R(v_p, v_{p'}, v_2)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. So, $x_{p'} > x_2$. Secondly, we claim that $x_3 > 0$. Otherwise, the relocation $R(v_3, v_{p'}, v_2)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Next, we assert that $x_{p'} < 0$. Otherwise, we can construct a new unbalanced signed complete graph Γ' whose negative-edge-induced connected subgraph is still $K_{2,2}$-minor free by reversing the sign of the positive edge $v_2v_{p'}$ and the negative edge v_1v_3 such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$ by (ii) of Lemma 7, a contradiction. Finally, we assert that $x_p < x_1$. Otherwise, the relocation $R(v_{p'}, v_p, v_1)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. However, the relocation $R(v_3, v_p, v_1)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7.

Case 4. $x_1 \neq x_2$ and $x_1x_2 = 0$.

Without loss of generality, assume that $x_1 > 0$, $x_2 = 0$. At first, we assert that $x_p > 0$. Otherwise, the relocation $R(v_p, v_1, v_2)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Next, we claim that $x_3 > 0$. Otherwise, the relocation $R(v_p, v_3, v_2)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. However, we can construct a new unbalanced signed complete graph Γ' whose negative-edge-induced connected subgraph is still $K_{2,2}$-minor free by reversing the sign of the positive edge $v_2v_{p'}$ and the negative edge v_1v_3 such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$ by (ii) of Lemma 7, a contradiction.

Case 5. $x_1 \neq x_2$ and $x_1x_2 > 0$.

Without loss of generality, assume that $x_1 > x_2 > 0$. Firstly, we claim that $x_p > 0$. Otherwise, the relocation $R(v_p, v_1, v_2)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Secondly, we assert that $x_3 > x_2$. Otherwise, the relocation $R(v_p, v_3, v_2)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Finally, we claim that $x_{p'} > x_2$. Otherwise, the relocation $R(v_3, v_{p'}, v_2)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. However, the relocation $R(v_{p'}, v_2, v_1)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7.

By Claim 1, without loss of generality, it can be assume that $d_H(v_1) \geq 3$, $d_H(v_2) = d_H(v_3) = 2$. Since $H \ncong U_{1,n-3}$, there is a vertex $v_s \in N_H(v_1)\backslash\{v_2, v_3\}$ such that $d_H(v_s) \geq 2$.

Claim 2. $H \cong D_{1,n-3}$.

Without loss of generality, assume that $v_t \in N_H(v_s)\backslash \{v_1\}$. Then we divide the proof into the following two cases.

Case 1. $x_t = 0$.

Firstly, we assert that $x_1 = x_s$. Otherwise, the relocation $R(v_t, v_1, v_s)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Similarly, we have $x_2 = x_3 = x_s$. Next, we claim that $x_1 = x_2 = x_3 = x_s = 0$. Otherwise, we can construct a new unbalanced signed complete graph Γ' whose negative-edge-induced connected subgraph is still $K_{2,2}$-minor free by reversing the sign of the positive edge v_tv_1 and the negative edge v_2v_3 such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$ by (ii) of Lemma 7, a contradiction. However, similar to Claim 1 of Lemma 8, this will lead to $X = \mathbf{0}$, a contradiction.

Case 2. $x_t \neq 0$.

Without loss of generality, assume that $x_t > 0$. At first, we assert that $x_1 > x_s$. Otherwise, the relocation $R(v_t, v_1, v_s)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Similarly, we have $x_3 > x_s$ and $x_2 > x_s$. If $x_3 \geq 0$, then the relocation $R(v_3, v_s, v_2)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. So, $x_3 < 0$. Similarly, $x_2 < 0$. We also claim that $x_1 > 0$. Otherwise, the relocation $R(v_1, v_t, v_2)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Finally, we assert that $x_s < 0$. Otherwise, the relocation $R(v_s, v_2, v_1)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7.

After the above preparations, we will further discuss in two subcases.

Subcase 2.1. $d_H(v_1) > 3$.

Without loss of generality, assume that $v_w \in N_H(v_1)\backslash\{v_2, v_3, v_s\}$. At first, we assert that $x_w < 0$. Otherwise, the relocation $R(v_w, v_2, v_1)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. If $x_w \leq x_s$, then the relocation $R(v_t, v_w, v_s)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. So, $x_w > x_s$. We also claim that $x_t < x_1$. Otherwise, the relocation $R(v_w, v_t, v_1)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Next, we assert that $d_H(v_t) = 1$. Otherwise, Let $v_a \in N_H(v_t) \setminus \{v_s\}$. If $x_a \leq 0$, then the relocation $R(v_a, v_1, v_t)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. If $x_a > 0$, then the relocation $R(v_a, v_s, v_t)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Similarly, $d_H(u) = 1$ for any $u \in N_H(v_s) \setminus \{v_1\}$. Finally, we claim that $d_H(v_w) = 1$. Otherwise, let $v_b \in N_H(v_w) \setminus \{v_1\}$. If $x_b \leq 0$, then the relocation $R(v_b, v_1, v_w)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. If $x_b > 0$, then the relocation $R(v_b, v_s, v_w)$ contradicts the maximality of $\lambda_1(A(\Gamma))$

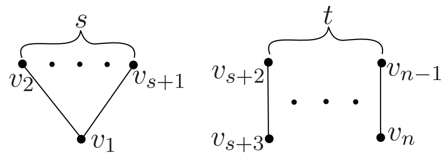

FIGURE 3. The graph $K_{1,s} \cup tP_2$.

by Lemma 7. In fact, $d_H(u) = 1$ for all $u \in N_H(v_1) \setminus \{v_2, v_3, v_s\}$. Thus, (K_n, H^-) is switching isomorphic to $(K_n, H_1(s,t)^-)$ for $t \geq 1$. However, $\lambda_1(A((K_n, H_1(s,t)^-))) < \lambda_1(A((K_n, D^-_{1,n-3})))$ for $t \geq 1$ by the proof of Lemma 4.

Subcase 2.2. $d_H(v_1) = 3$.

Firstly, we assert that the length of the longest path passing through v_s and v_t starting v_1 is at most 3. Otherwise, let $v_{t_1} \in N_H(v_t) \setminus \{v_s\}$ and $v_{t_2} \in N_H(v_{t_1}) \setminus \{v_t\}$. If $x_s \geq x_{t_1}$, then the relocation $R(v_1, v_{t_1}, v_s)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. So, $x_s < x_{t_1}$. If $x_{t_1} \geq 0$, then the relocation $R(v_{t_1}, v_s, v_t)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. So, $x_{t_1} < 0$. If $x_{t_2} \leq 0$, then the relocation $R(v_{t_2}, v_t, v_{t_1})$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Thus, $x_{t_2} > 0$. However, the relocation $R(v_{t_2}, v_s, v_{t_1})$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7, a contradiction. This implies that in the case where exists $u \in N_H(v_t) \setminus \{v_s\}$, we have $d_H(u) = 1$. Similar to the above discussion, we will next consider $d_H(v_s)$.

At first, we assume that $d_H(v_s) \geq 3$. If $d_H(u) = 1$ for any $u \in N_H(v_s) \setminus \{v_1\}$, then (K_n, H^-) is switching isomorphic to $(K_n, H_2(n-5, 0)^-)$. However, $\lambda_1(A((K_n, H_2(s,t)^-))) < \lambda_1(A((K_n, D^-_{1,n-3})))$ by the proof of Lemma 5. If there exists at least two vertices in $N_H(v_s) \setminus \{v_1\}$ whose degree is greater than or equal to 2. Without loss of generality, assume that $d_H(v_t) \geq 2$ and $d_H(v_b) \geq 2$ for $v_b \in N_H(v_s) \setminus \{v_1\}$. Then let $v_a \in N_H(v_t) \setminus \{v_s\}$ and $v_c \in N_H(v_b) \setminus \{v_s\}$. Firstly, we assert that $x_a < 0$. Otherwise, the relocation $R(v_a, v_s, v_t)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. Secondly, we claim that $x_b > 0$. Otherwise, the relocation $R(v_b, v_t, v_s)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. If $x_c \geq 0$, then the relocation $R(v_c, v_s, v_b)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. So, $x_c < 0$. Finally, we also assert that $x_t < x_b$. Otherwise, the relocation $R(v_c, v_t, v_b)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. However, the relocation $R(v_a, v_b, v_t)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7. This implies that there is only one vertex in $N_H(v_s) \setminus \{v_1\}$ whose degree is greater than or equal to 2. Without loss of generality, assume that $d_H(v_t) \geq 2$. Then $d_H(u) = 1$ for all $u \in N_H(v_t) \setminus \{v_s\}$ through the previous discussion. Thus, (K_n, H^-) is switching isomorphic to $(K_n, H_2(s,t)^-)$. However, $\lambda_1(A((K_n, H_2(s,t)^-))) < \lambda_1(A((K_n, D^-_{1,n-3})))$ by the proof of Lemma 5. Secondly, assume that $d_H(v_s) = 2$. Then (K_n, H^-) is switching isomorphic to $(K_n, H_2(0, n-5)^-)$. However, $\lambda_1(A((K_n, H_2(0, n-5)^-))) < \lambda_1(A((K_n, D^-_{1,n-3})))$ by the proof of Lemma 5.

Next, we consider that H contains at least two 3-length cycles. By similar arguments as in the proof of Claim 1 of Lemma 8, there exists a vertex u of 3-length cycle such that $x_u \neq 0$. Now, we assert that H contains at most two 3-length cycles. Otherwise, we can construct a new unbalanced signed complete graph $\Gamma' = (K_n, H_1^-)$ such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$ by similar arguments as in the proof of Claim 2 of Lemma 8, where H_1 is a $K_{2,2}$-minor free graph and contains two 3-length cycles. This also contradicts the maximality of $\lambda_1(A(\Gamma))$. Thus, H contains exactly two 3-length cycles. If $H \ncong U_{2,n-5}$, then we can construct a new unbalanced signed complete graph $\Gamma'' = (K_n, H_2^-)$ such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma''))$ by the proof of Claim 2 of Lemma 8, where H_2 is a $K_{2,2}$-minor free graph, $H_2 \ncong U_{1,n-3}$ and contains only one 3-length cycle. This also contradicts the maximality of $\lambda_1(A(\Gamma))$. If $H \cong U_{2,n-5}$, then $\lambda_1(A((K_n, D^-_{1,n-3}))) > \lambda_1(A((K_n, U^-_{2,n-5})))$ by Lemma 3, a contradiction.

Thus, H must be a tree and $H \cong D_{1,n-3}$ by Lemma 6 (Fig. 3). $\square$

Lemma 10. *Let s and t be positive integers such that $2t + s + 1 = n$. Then* $\lambda_1(A((K_n, D^-_{1,n-3}))) > \lambda_1(A((K_n, K^-_{1,n-3} \cup P_2^-))) \geq \lambda_1(A((K_n, K^-_{1,s} \cup tP_2^-)))$ *with the second equality holding if and only if* $s = n-3$, $t = 1$.

Proof. We present the matrix $A((K_n, K_{1,s}^- \cup tP_2^-))$ and its corresponding quotient matrix $Q_5(s,t)$ by the vertex partition $V_1 = \{v_1\}$, $V_2 = \{v_2, \ldots, v_{s+1}\}$, $V_3 = \{v_{s+2}, v_{s+3}\}$, $\ldots$, $V_{t+2} = \{v_{n-1}, v_n\}$ as follows

$$A((K_n, K_{1,s}^- \cup tP_2^-)) = \begin{bmatrix} 0 & -j_s^T & j_{n-s-1}^T \\ -j_s & \left(J-I\right)_s & J_{s\times(n-s-1)} \\ j_{n-s-1} & J_{s\times(n-s-1)}^T & B_{n-s-1} \end{bmatrix}$$

and

$$Q_{5(s,t)} = \begin{bmatrix} 0 & -s & 2j_t^T \\ -1 & s-1 & 2j_t^T \\ j_t & sj_t & (2J-3I)_t \end{bmatrix},$$

where

$$B_{n-s-1} = \begin{bmatrix} (I-J)_2 & J_{2\times2} & \cdots & J_{2\times2} \\ J_{2\times2} & (I-J)_2 & & \vdots \\ \vdots & & \ddots & J_{2\times2} \\ J_{2\times2} & \cdots & J_{2\times2} & (I-J)_2 \end{bmatrix}.$$

Thus,

$$\lambda I_{t+2} - Q_5(s,t) = \begin{bmatrix} \lambda & s & -2j_t^T \\ 1 & \lambda+1-s & -2j_t^T \\ -j_t & -sj_t & \left((\lambda+3)\, I - 2J\right)_t \end{bmatrix}.$$

Now, we apply finitely many elementary row and column operations on the matrix $\lambda I_{t+2} - Q_5(s,t)$. First, subtracting the 3-th row from all the lower rows and adding the i-th column to the 3-th column for $i = t{+}2, \ldots, 4$. This leads to the following matrix:

$$\begin{bmatrix} \lambda & s & -2t & * \\ 1 & \lambda+1-s & -2t & * \\ -1 & -s & \lambda+3-2t & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\lambda+3)I_{t-1} \end{bmatrix}.$$

Note that the characteristic polynomial of $Q_5(s,t)$ is

$$P_{Q_{5(s,t)}}(\lambda) = (\lambda+3)^{t-1}(\lambda^3 + (4-2t-s)\lambda^2 + (3-4t-4s)\lambda + 8st - 3s - 2t).$$

Now, we add αJ to each block of $A((K_n, K_{1,s}^- \cup tP_2^-))$ for a constant α. Then $A((K_n, K_{1,s}^- \cup tP_2^-))$ becomes

$$A_5 = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -I_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{2t} \end{bmatrix}.$$

Since $\lambda_1(Q_5(s,t)) > 1$ and $\mathrm{Spec}(A_5) = \left\{-1^{[s]}, 0, 1^{[2t]}\right\}$, $\lambda_1(A((K_n, K_{1,s}^- \cup tP_2^-))) = \lambda_1(Q_5(s,t))$. Let $f_{(s,t)}(\lambda) = (\lambda^3 + (4-2t-s)\lambda^2 + (3-4t-4s)\lambda + 8st - 3s - 2t)$, then $f_{(s,t)}(\lambda) - f_{(s+2,t-1)}(\lambda) = 4\lambda + 8s - 16t + 20$ and $f_{(s-2,t+1)}(\lambda) - f_{(s,t)}(\lambda) = 4\lambda + 8s - 16t - 12$. Assume that λ_1 is the largest root of $f_{(s,t)}(\lambda) = 0$. Next, we will discuss in two cases.

Case 1. $4\lambda_1 + 8s - 16t \geq 0$.

Then $f_{(s,t)}(\lambda_1) - f_{(s+2,t-1)}(\lambda_1) > 0$, which means that $f_{(s+2,t-1)}(\lambda_1) < 0$ and $\lambda_1(Q_5(s,t)) < \lambda_1(Q_5(s+2,t-1))$. Thus, $\lambda_1(A((K_n, K_{1,s}^- \cup tP_2^-))) < \lambda_1(A((K_n, K_{1,s+2}^- \cup (t-1)P_2^-)))$. By repeatedly using this operation, we can obtain that

$$\lambda_1(A((K_n, K_{1,s}^- \cup tP_2^-))) \leq \lambda_1(A((K_n, K_{1,n-3}^- \cup P_2^-))),$$

the equality holds if and only if $s = n-3$, $t = 1$.

Case 2. $4\lambda_1 + 8s - 16t < 0$.

Then $f_{(s-2,t+1)}(\lambda_1) - f_{(s,t)}(\lambda_1) < 0$, it implies that $f_{(s-2,t+1)}(\lambda_1) < 0$ and $\lambda_1(Q_5(s,t)) < \lambda_1(Q_5\ (s-2,t+1))$. Hence, $\lambda_1(A((K_n, K_{1,s}^- \cup tP_2^-))) < \lambda_1(A((K_n, K_{1,s-2}^- \cup (t+1)P_2^-)))$. By repeatedly using this operation, we have $\lambda_1(A((K_n, K_{1,s}^- \cup tP_2^-))) \leq \lambda_1(A((K_n, P_3^- \cup \frac{n-3}{2}P_2^-)))$ if n is odd, with the equality holding if and only if $s = 2$, $t = \frac{n-3}{2}$. And $\lambda_1(A((K_n, K_{1,s}^- \cup tP_2^-))) \leq \lambda_1(A((K_n, \frac{n}{2}P_2^-)))$ if n is even, with the equality holding if and only if $s = 1$, $t = \frac{n}{2} - 1$.

Note that $f_{(n-3,1)}(\lambda) = \lambda^3 + (5-n)\lambda^2 + (11-4n)\lambda + 5n - 17$, $f_{(2,\frac{n-3}{2})}(\lambda) = \lambda^3 + (5-n)\lambda^2 + (1-2n)\lambda + 7n - 27$ and $f_{(1,\frac{n}{2}-1)}(\lambda) = \lambda^3 + (5-n)\lambda^2 + (3-2n)\lambda + 3n - 9$. It is worth noting that

$$f_{(1,\frac{n}{2}-1)}(\lambda) - f_{(n-3,1)}(\lambda) = (2n-8)(\lambda - 1).$$

Note that $f_{(1,\frac{n}{2}-1)}(2) = 25 - 5n < 0$ for $n \geq 9$. Let λ_1 be the largest root of $f_{(1,\frac{n}{2}-1)}(\lambda) = 0$. Then $\lambda_1 > 1$ and $f_{(n-3,1)}(\lambda_1) < 0$ for $n \geq 9$. This means that $\lambda_1(Q_5(1, \frac{n}{2} - 1)) < \lambda_1(Q_5(n-3, 1))$. Thus, $\lambda_1(A((K_n, \frac{n}{2}P_2^-))) < \lambda_1(A((K_n, K_{1,n-3}^- \cup P_2^-)))$. As well as, note that

$$f_{(2,\frac{n-3}{2})}(\lambda) - f_{(n-3,1)}(\lambda) = (2n-10)(\lambda + 1).$$

Let λ_1 be the largest root of $f_{(2,\frac{n-3}{2})}(\lambda) = 0$. Then $f_{(n-3,1)}(\lambda_1) < 0$ for $n \geq 9$. This means that $\lambda_1(Q_5(2, \frac{n-3}{2})) < \lambda_1(Q_5(n-3, 1))$. Hence, $\lambda_1(A((K_n, P_3^- \cup \frac{n-3}{2}P_2^-))) < \lambda_1(A((K_n, K_{1,n-3}^- \cup P_2^-)))$.

Finally, we will show that $\lambda_1(A((K_n, K_{1,n-3}^- \cup P_2^-))) < \lambda_1(A((K_n, D_{1,n-3}^-)))$. By Li *et al.* [14], the characteristic polynomial of $A((K_n, D_{1,n-3}^-))$ is $(\lambda+1)^{n-3}(\lambda^3 + (3-n)\lambda^2 + (3-2n)\lambda + 7n - 23)$, and set $g(\lambda) = \lambda^3 + (3-n)\lambda^2 + (3-2n)\lambda + 7n - 23$. Note that $f_{(n-3,1)}(\lambda) = \lambda^3 + (5-n)\lambda^2 + (11-4n)\lambda + 5n - 17$ and $f_{(n-3,1)}(n-3) = -2(n-4)^2 < 0$ for $n \geq 9$. Let λ_1 be the largest root of $f_{(n-3,1)}(\lambda) = 0$, then $\lambda_1 > n - 3$. Set

$$h(\lambda) = g(\lambda) - f_{(n-3,1)}(\lambda) = -2\lambda^2 + (2n-8)\lambda + 2n - 6,$$

and the maximal solution of $h(\lambda) = 0$ is $n - 3$ for $n \geq 9$. This means that $g(\lambda_1) < 0$, then $\lambda_1(A((K_n, K_{1,n-3}^- \cup P_2^-))) < \lambda_1(A((K_n, D_{1,n-3}^-)))$.

So, the proof is completed. □

Lemma 11. *Let H be a disconnected $K_{2,2}$-minor free spanning subgraph of K_n for $n \geq 9$. If (K_n, H^-) has the maximum index, then $\lambda_1(A((K_n, D_{1,n-3}^-))) > \lambda_1(A((K_n, H^-)))$.*

Proof. Let $\Gamma = (K_n, H^-)$, and let $G_1, \ldots, G_k$ be k connected components of H for $k \geq 2$. Clearly, G_i is either a tree or a connected graph that only contains 3-length cycles for $1 \leq i \leq k$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector associated with $\lambda_1(A(\Gamma))$. Note that $-X$ must be a unit eigenvector of Γ if X is a unit eigenvector. Next, we will divide the proof into the following two cases.

Case 1. There exists a nonnegative eigenvector.

Subcase 1.1. $x_i > 0$ for any $1 \leq i \leq n$. Firstly, we assert that G_j is either a star graph or a P_2 for $1 \leq j \leq k$. Otherwise, we can find a 3-length cycle or a P_4 as a subgraph of H. If $C_3 = v_1v_2v_3v_1$ is a 3-length cycle of H, then we can construct a new unbalanced signed complete graph Γ' whose negative-edge-induced subgraph is still $K_{2,2}$-minor free by reversing the sign of the negative edge v_1v_2 such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$, a contradiction.

If $P_4 = v_4v_5v_6v_7$ is a subgraph of H, then we can construct a new unbalanced signed complete graph Γ' whose negative-edge-induced subgraph is still $K_{2,2}$-minor free by reversing the sign of the negative edge v_5v_6 such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$, a contradiction. First, we consider that H contains star subgraphs. Then we assert that H contains only one star subgraph. Otherwise, without loss of generality, assume that F_1 and F_2 are star subgraphs of H with $v_1$and v_3 as their central vertices, respectively. Let $v_1v_2 \in E(F_1)$ and $v_3v_4 \in E(F_2)$. If $x_1 \geq x_3$, then the relocation $R(v_2, v_3, v_1)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7, a contradiction. If $x_1 < x_3$, then the relocation $R(v_4, v_1, v_3)$ contradicts the maximality of $\lambda_1(A(\Gamma))$ by Lemma 7, a contradiction. This means that $H \cong K^-_{1,s} \cup tP^-_2$, where $s \geq 2$ and $t \geq 1$. Thus, $\lambda_1(A((K_n, D^-_{1,n-3}))) > \lambda_1(A((K_n, K^-_{1,s} \cup tP^-_2)))$ by Lemma 10. If H does not contain star subgraphs, then $H \cong \frac{n}{2}P_2$. Hence, $\lambda_1(A((K_n, D^-_{1,n-3}))) > \lambda_1(A((K_n, \frac{n}{2}P^-_2)))$ by the proof of Case 2 of Lemma 10.

Subcase 1.2. there exists at least one vertex $v_i \in V(H)$ such that $x_i = 0$. Without loss of generality, assume that $u \in V(G_1)$ such that $x_u = 0$. Next, we assert that $k = 2$. Otherwise, if there exists one vertex $v \in V(H)\backslash V(G_1)$ such that $x_v > 0$, then we can construct a new unbalanced signed complete graph $\Gamma' = (K_n, H^-_1)$ by reversing the sign of the positive edge uv such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$, where H_1 is a disconnected $K_{2,2}$-minor free graph. This contradicts the maximality of $\lambda_1(A(\Gamma))$. If $x_v = 0$ for any $v \in V(H)\backslash V(G_1)$, then there exists at least one vertex $w \in V(G_1)$ such that $x_w > 0$ since $X \neq \mathbf{0}$. Let $v \in V(H)\backslash V(G_1)$, then we can construct a new unbalanced signed complete graph $\Gamma'' = (K_n, H^-_2)$ by reversing the sign of the positive edge wv such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma''))$, where H_2 is a disconnected $K_{2,2}$-minor free graph. This also contradicts the maximality of $\lambda_1(A(\Gamma))$. Hence, $k = 2$. That is, $H = G_1 \cup G_2$. If there exists one vertex $v \in V(G_2)$ such that $x_v > 0$, then we can construct a new unbalanced signed complete graph $\Gamma_1 = (K_n, H^-_1)$ by reversing the sign of the positive edge uv such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma_1))$, where H_1 is a connected $K_{2,2}$-minor free graph and $H_1 \ncong U_{1,n-3}$. However, $\lambda_1(A((K_n, D^-_{1,n-3}))) \geq \lambda_1(A(\Gamma_1))$ with the equality holding if and only if $H_1 \cong D_{1,n-3}$ by Lemma 9. Thus, $\lambda_1(A((K_n, D^-_{1,n-3}))) > \lambda_1(A((K_n, H^-)))$. If $x_i = 0$ for any $v_i \in V(G_2)$, then there exists at least one vertex $w \in V(G_1)$ such that $x_w > 0$ since $X \neq \mathbf{0}$. Let $v \in V(G_2)$, then we can construct a new unbalanced signed complete graph $\Gamma_2 = (K_n, H^-_2)$ by reversing the sign of the positive edge wv such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma_2))$, where H_2 is a connected $K_{2,2}$-minor free graph and $H_2 \ncong U_{1,n-3}$. However, $\lambda_1(A((K_n, D^-_{1,n-3}))) \geq \lambda_1(A(\Gamma_2))$ with the equality holding if and only if $H_2 \cong D_{1,n-3}$ by Lemma 9. Thus, $\lambda_1(A((K_n, D^-_{1,n-3}))) > \lambda_1(A((K_n, H^-)))$.

Case 2. There are no nonnegative eigenvectors.

Without loss of generality, assume that u, $v \in V(H)$ such that $x_u \leq 0$, $x_v > 0$. Then we assert that $k = 2$. Otherwise, we first assume that vertices u and v are in the same connected component, without loss of generality, we assume that $u, v \in V(G_1)$ and $w \in V(G_2)$. If $x_w > 0$, then we can construct a new unbalanced signed complete graph $\Gamma' = (K_n, H^-_1)$ by reversing the sign of the positive edge uw such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'))$, where H_1 is a disconnected $K_{2,2}$-minor free graph. This contradicts the maximality of $\lambda_1(A(\Gamma))$. If $x_w \leq 0$, then we can construct a new unbalanced signed complete graph $\Gamma'' = (K_n, H^-_2)$ by reversing the sign of the positive edge vw such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma''))$, where H_2 is a disconnected $K_{2,2}$-minor free graph. This also contradicts the maximality of $\lambda_1(A(\Gamma))$. So, u and v are not in the same connected component, then we can construct a new unbalanced signed complete graph $\Gamma''' = (K_n, H^-_3)$ by reversing the sign of the positive edge uv such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma'''))$, where H_3 is a disconnected $K_{2,2}$-minor free graph. This also contradicts the maximality of $\lambda_1(A(\Gamma))$. Thus, $k = 2$. That is, $H = G_1 \cup G_2$. Let u, $v \in V(G_1)$ and $w \in V(G_2)$. If $x_w > 0$, then we can construct a new unbalanced signed complete graph $\Gamma_1 = (K_n, H^-_1)$ by reversing the sign of the positive edge uw such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma_1))$, where H_1 is a connected $K_{2,2}$-minor free graph and $H_1 \ncong U_{1,n-3}$. However, $\lambda_1(A((K_n, D^-_{1,n-3}))) \geq \lambda_1(A(\Gamma_1))$ with the equality holding if and only if $H_1 \cong D_{1,n-3}$ by Lemma 9. Thus, $\lambda_1(A((K_n, D^-_{1,n-3}))) > \lambda_1(A((K_n, H^-)))$. If $x_w \leq 0$, then we can construct a new unbalanced signed complete graph $\Gamma_2 = (K_n, H^-_2)$ by reversing the sign of the positive edge vw such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma_2))$, where H_2 is a connected $K_{2,2}$-minor free graph and $H_2 \ncong U_{1,n-3}$. However, $\lambda_1(A((K_n, D^-_{1,n-3}))) \geq \lambda_1(A(\Gamma_2))$ with the equality holding if and only if $H_2 \cong D_{1,n-3}$ by Lemma 9. So, $\lambda_1(A((K_n, D^-_{1,n-3}))) > \lambda_1(A((K_n, H^-)))$. If

$u \in V(G_1)$ and $v \in V(G_2)$, then we can construct a new unbalanced signed complete graph $\Gamma_3 = (K_n, H_3^-)$ by reversing the sign of the positive edge uv such that $\lambda_1(A(\Gamma)) < \lambda_1(A(\Gamma_3))$, where H_3 is a connected $K_{2,2}$-minor free graph and $H_3 \ncong U_{1,n-3}$. However, $\lambda_1(A((K_n, D_{1,n-3}^-))) \geq \lambda_1(A(\Gamma_3))$ with the equality holding if and only if $H_3 \cong D_{1,n-3}$ by Lemma 9. Hence, $\lambda_1(A((K_n, D_{1,n-3}^-))) > \lambda_1(A((K_n, H^-)))$.

This completes the proof. □

Proof of Theorem 1. For $5 \leq n \leq 8$, we can conclude that Theorem 1 holds through direct calculation. For $n \geq 9$, by Lemmas 8, 9 and 11, Theorem 1 holds directly. □

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Conflicts of interest

The authors declare that they have no conflicts of interest.

Author contribution statement

All authors jointly worked on the results and they read and approved the final manuscript.

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