

## ON GENERALIZED ALTERNATIVE STATIONARITY FOR MATHEMATICAL PROGRAMS WITH SWITCHING CONSTRAINTS

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**Abstract.** We study a mathematical program with switching constraints (MPSC). By introducing a suitable constraint qualification and an alternative stationarity concept, we derive necessary optimality conditions using  $\mathcal{D}$ -directional upper convexificators and  $\mathcal{D}$ -directional upper semi-regular convexificators, where  $\mathcal{D}$  is a nonempty closed cone within the set of continuity directions for functions that may lack continuity. An illustrative example is provided.

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### 1. INTRODUCTION

In recent years, researchers have focused on developing generalized subdifferentials that provide sharp optimality conditions and effective calculus rules for nonsmooth functions. In this context, convexificators were introduced to handle continuous functions [12]. Since lower semicontinuous functions may be discontinuous, their convexificators can be unbounded. To address this, Dempe and Pilecka [4] introduced directional convexificators, defined along directions where the function remains continuous. Gadhi *et al.* [9] later applied this concept to derive necessary optimality conditions for set-valued optimization problems. More recently, Gadhi [7] refined this notion to the  $\mathcal{D}$ -directional convexificator, where  $\mathcal{D}$  is a nonempty cone contained in the set of continuity directions of the involved (possibly discontinuous) functions, enabling mean value theorems and variational inequalities to be formulated in this generalized framework. In recent years, the notion of directional convexificators has attracted growing interest from the scientific community; see, for instance, [6, 8, 14, 19, 21, 22].

This paper studies the following mathematical program with switching constraints

$$(\text{MPSC}) : \begin{cases} \text{Minimize} & f(x) \\ \text{s.t.} & \begin{cases} g_i(x) \leq 0, & i \in I, \\ h_j(x) = 0, & j \in J, \\ G_s(x)H_s(x) = 0, & s \in \mathcal{L} \end{cases} \end{cases}$$

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*Keywords.*  $\mathcal{D}$ -directional upper convexificators,  $\mathcal{D}$ -directional upper semi-regular convexificators, constraint qualification, mathematical programs with switching constraints, optimality conditions.

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where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $i \in I$ ,  $h_j : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $j \in J$ ,  $G_s : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $H_s : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $s \in \mathcal{L}$ , are lower semicontinuous functions (possibly discontinuous), and  $I := \{1, \dots, m\}$ ,  $J := \{1, \dots, p\}$ ,  $\mathcal{L} := \{1, \dots, l\}$ , are finite index sets,  $n, m, p, l \in \mathbb{N}$ .

Mathematical programs with switching constraints (MPSCs) have attracted significant attention in recent literature, with various authors establishing optimality conditions under different assumptions [10, 11, 13, 15–18]. For example, Jennane and Kalmoun [10] derived necessary optimality conditions for nonsmooth, multiobjective, semi-infinite MPSCs with tangentially convex functions, while Jennane *et al.* [11] obtained analogous results for locally Lipschitz functions. In a separate development, Mehritz [18] proved M-stationarity for differentiable MPSCs and applied the results to discretized optimal control problems. Recently, Lv *et al.* [17] studied MPSCs with an objective given by the sum of a continuously differentiable function and a non-Lipschitz term, and analyzed various stationarity concepts under specific constraint qualifications.

The switching constraints, given by  $G_s(x)H_s(x) = 0$ ,  $s \in \mathcal{L}$ , require that at each feasible point  $x \in \mathbb{R}^n$ , at least one of  $G_s(x)$  or  $H_s(x)$  must vanish, introducing a combinatorial disjunction. This structure inherently violates standard constraint qualifications (LICQ, MFCQ), even in differentiable cases, and the possible discontinuity of the functions defining the MPSC adds further theoretical challenges.

In this work, we develop a novel constraint qualification using  $\mathcal{D}$ -directional upper convexificators, which enables the derivation of  $\mathcal{D}$ -generalized alternative stationarity conditions for the MPSC under study. Our approach provides the first optimality conditions that simultaneously handle the intrinsic combinatorial structure of switching constraints, accommodate potentially discontinuous objective and constraint functions, and operate without convexity or compactness requirements on continuity directions. Unlike previous work by Dempe and Pilecka [4] and Gadhi *et al.* [9], we do not require the set of continuity directions to be convex or compact. Examples 6 and 7 illustrate cases where  $\mathcal{D}$ -directional upper (semi-regular) convexificators diverge from classical convexificators, highlighting the necessity of our generalized framework. An example demonstrating both our findings and the limitations of certain prior studies is provided (see Example 16).

To our knowledge, this work makes significant theoretical advances by introducing the first  $\mathcal{D}$ -generalized alternative stationarity criteria that simultaneously eliminate continuity requirements for data functions and dispense with convexity or compactness assumptions on continuity directions.

The remainder of this paper is structured as follows: Section 2 introduces the necessary preliminary definitions and fundamental concepts. Section 3 presents our main theoretical results, including the derivation of optimality conditions for MPSC problems. Finally, Section 4 concludes with a summary of contributions and potential directions for future research.

## 2. PRELIMINARIES

Throughout,  $\mathbb{N}$  and  $\mathbb{R}^n$  stand for the set of positive integers and the  $n$ -dimensional Euclidean space, respectively. For a nonempty subset  $S \subseteq \mathbb{R}^n$ , its closure, convex hull, and convex cone (including the origin) are denoted by  $cl S$ ,  $conv S$  and  $pos S$ , respectively. The negative polar cone of  $S$  is defined by

$$S^- := \{v \in \mathbb{R}^n \mid \langle x, v \rangle \leq 0, \forall x \in S\}.$$

The contingent cone of  $S$  at  $x \in cl S$  is given by

$$T(S, x) = \{v \in \mathbb{R}^n : \exists t_n \rightarrow 0^+, \exists v_n \rightarrow v \text{ such that } x + t_n v_n \in S, \forall n\}.$$

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a given function and let  $x \in \mathbb{R}^n$  with  $f(x)$  finite. For a direction  $d \in \mathbb{R}^n$ , the lower and upper Dini directional derivatives of  $f$  at  $x$  in the direction  $d$  are defined by

$$f^-(x, d) = \liminf_{t \rightarrow 0^+} \frac{f(x + td) - f(x)}{t} \quad \text{and} \quad f^+(x, d) = \limsup_{t \rightarrow 0^+} \frac{f(x + td) - f(x)}{t}.$$

Note that if  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is locally Lipschitz, its lower and upper Dini directional derivatives exist and are finite [5].

**Definition 1** ([5]). The function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is said to have an upper convexificator  $\partial^u f(x)$  at  $x \in \mathbb{R}^n$  if  $\partial^u f(x) \subseteq \mathbb{R}^n$  is closed and, for each  $d \in \mathbb{R}^n$ ,

$$f^-(x, d) \leq \sup_{x^* \in \partial^u f(x)} \langle x^*, d \rangle.$$

The function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is said to have an upper semiregular convexificator  $\partial^{us} f(x)$  at  $x$  if  $\partial^{us} f(x)$  is an upper convexificator at  $x \in \mathbb{R}^n$  and, for each  $d \in \mathbb{R}^n$ ,

$$f^+(x, d) \leq \sup_{x^* \in \partial^{us} f(x)} \langle x^*, d \rangle.$$

We shall need the following definition.

**Definition 2** ([4]). Consider  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ . A vector  $d \in \mathbb{R}^n$  is a continuity direction of  $f$  at the point  $x \in \mathbb{R}^n$  if for all sequences  $\{t_k\} \subset \mathbb{R}$  with  $t_k \rightarrow 0^+$  we have

$$\lim_{k \rightarrow \infty} f(x + t_k d) = f(x).$$

The set of all continuity directions of  $f$  at  $x$  is denoted by  $D_f(x)$ .

We note that there exist lower semicontinuous functions for which the zero vector  $0_n \in \mathbb{R}^n$  is their only direction of continuity at certain points. The following definition adapts the concept of directional convexificators introduced in Definition 3 of [4] where the authors considered the complete set of continuity directions  $D_f(x)$ . In our framework, we work with a specified subset of these directions rather than the entire set.

**Definition 3.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be an extended-real-valued function and  $\mathcal{D} \subseteq \mathbb{R}^n$  a nonempty cone.

- The function  $f$  admits a  $\mathcal{D}$ -directional upper convexificator  $\partial_{\mathcal{D}}^u f(x) \subseteq \mathbb{R}^n$  at  $x \in \mathbb{R}^n$  if  $\mathcal{D} \subseteq D_f(x)$ , the set  $\partial_{\mathcal{D}}^u f(x)$  is closed and for each  $d \in \mathcal{D}$  we have:

$$f^-(x, d) \leq \sup_{x^* \in \partial_{\mathcal{D}}^u f(x)} \langle x^*, d \rangle.$$

- The function  $f$  admits a  $\mathcal{D}$ -directional upper semi-regular convexificator  $\partial_{\mathcal{D}}^{us} f(x) \subseteq \mathbb{R}^n$  at  $x \in \mathbb{R}^n$  if  $\mathcal{D} \subseteq D_f(x)$ , the set  $\partial_{\mathcal{D}}^{us} f(x)$  is closed and for each  $d \in \mathcal{D}$  we have:

$$f^+(x, d) \leq \sup_{x^* \in \partial_{\mathcal{D}}^{us} f(x)} \langle x^*, d \rangle.$$

It is worth noting that  $D_f(x) = \mathbb{R}^n$  when  $f$  is continuous at  $x \in \mathbb{R}^n$ . In addition, if  $\mathcal{D} = D_f(x)$ ,  $\mathcal{D}$ -directional upper convexificators (resp.  $\mathcal{D}$ -directional upper semi-regular convexificators) coincide with upper convexificators (resp. upper semi-regular convexificators).

Using the  $\mathcal{D}$ -directional upper semi-regular convexificator and the normal cone to  $\mathcal{D}$  at  $0_n$ , defined as

$$N_{\mathcal{D}}(0_n) := \{v \in \mathbb{R}^n : \langle v, d \rangle \leq 0, \forall d \in \mathcal{D}\},$$

we can construct an upper semi-regular convexificator for lower semicontinuous functions.

**Lemma 4.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a lower semicontinuous function, and let  $\bar{x} \in \mathbb{R}^n$ . Assume that  $f$  admits a bounded  $\mathcal{D}$ -directional upper semi-regular convexificator  $\partial_{\mathcal{D}}^{us} f(\bar{x})$  at  $\bar{x}$ , where  $\mathcal{D} \subseteq \mathbb{R}^n$  is a nonempty, convex, and closed cone. Then, the set

$$\partial^{us} f(\bar{x}) := \partial_{\mathcal{D}}^{us} f(\bar{x}) + N_{\mathcal{D}}(0_n)$$

forms an upper semi-regular convexificator of  $f$  at  $\bar{x}$ .

*Proof.* Let  $d \in \mathbb{R}^n$  be arbitrary. Since  $\partial_{\mathcal{D}}^{us} f(\bar{x})$  is compact and  $N_{\mathcal{D}}(0_n)$  is closed, their Minkowski sum  $\partial^{us} f(\bar{x})$  is closed [2].

– **Case 1:**  $d \in \mathcal{D}$ .

Since  $\partial_{\mathcal{D}}^{us} f(\bar{x})$  is a  $\mathcal{D}$ -directional upper semi-regular convexificator of  $f$  at  $\bar{x}$ , we have

$$f^+(\bar{x}, d) \leq \sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle x^*, d \rangle.$$

By construction,  $\partial_{\mathcal{D}}^{us} f(\bar{x}) \subseteq \partial^{us} f(\bar{x})$ , and therefore

$$f^+(\bar{x}, d) \leq \sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle x^*, d \rangle \leq \sup_{x^* \in \partial^{us} f(\bar{x})} \langle x^*, d \rangle.$$

– **Case 2:**  $d \notin \mathcal{D}$ .

By lower semicontinuity of  $f$ , we have  $f^+(\bar{x}, d) = +\infty$ . In this situation, we prove that:

$$\sup_{x^* \in \partial^{us} f(\bar{x})} \langle x^*, d \rangle = +\infty.$$

- On the one hand, since  $\mathcal{D}$  is closed and convex, we can find  $v^* \in N_{\mathcal{D}}(0_n)$  such that

$$\langle v^*, d \rangle > 0.$$

- On the other hand, we have:

$$\sup_{x^* \in \partial^{us} f(\bar{x})} \langle x^*, d \rangle = \sup_{a \in \partial_{\mathcal{D}}^{us} f(\bar{x}), v \in N_{\mathcal{D}}(0_n)} \langle a + v, d \rangle = \sup_{a \in \partial_{\mathcal{D}}^{us} f(\bar{x}), v \in N_{\mathcal{D}}(0_n)} (\langle a, d \rangle + \langle v, d \rangle).$$

Consequently, since  $\lambda v^* \in N_{\mathcal{D}}(0_n)$  for all  $\lambda \geq 0$ , we obtain

$$\sup_{x^* \in \partial^{us} f(\bar{x})} \langle x^*, d \rangle \geq \sup_{a \in \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle a, d \rangle + \langle \lambda v^*, d \rangle. \tag{1}$$

Since  $\partial_{\mathcal{D}}^{us} f(\bar{x})$  is bounded and closed, it is compact; and thus we can find  $a^* \in \partial_{\mathcal{D}}^{us} f(\bar{x})$  such that:

$$+\infty \neq \langle a^*, d \rangle = \sup_{a \in \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle a, d \rangle. \tag{2}$$

- Combining (1) and (2),

$$\sup_{x^* \in \partial^{us} f(\bar{x})} \langle x^*, d \rangle \geq \langle a^*, d \rangle + \lambda \langle v^*, d \rangle.$$

Letting  $\lambda \rightarrow +\infty$ , we obtain

$$\sup_{x^* \in \partial^{us} f(\bar{x})} \langle x^*, d \rangle \geq \lim_{\lambda \rightarrow +\infty} \langle a^*, d \rangle + \lambda \langle v^*, d \rangle.$$

Since  $\langle v^*, d \rangle > 0$ , we get

$$\sup_{x^* \in \partial^{us} f(\bar{x})} \langle x^*, d \rangle = +\infty.$$

□

Similarly, using the  $\mathcal{D}$ -directional upper convexificator together with the normal cone to the set  $\mathcal{D}$  at  $d = 0$ , we can construct the following upper convexificator of a lower semicontinuous function.

**Lemma 5.** *Assume that the lower semicontinuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  admits a bounded  $\mathcal{D}$ -directional upper convexificator  $\partial_{\mathcal{D}}^u f(\bar{x})$  at  $\bar{x} \in \mathbb{R}^n$ , where  $\mathcal{D} \subseteq \mathbb{R}^n$  is a nonempty, convex, and closed cone. Then, the set*

$$\partial^u f(\bar{x}) := \partial_{\mathcal{D}}^u f(\bar{x}) + N_{\mathcal{D}}(0_n)$$

*is an upper convexificator of the function  $f$  at  $\bar{x}$ .*

The following examples illustrate how to use Lemmas 5 and 4. Furthermore, they show that it is not possible to omit the closedness or convexity of the set  $\mathcal{D}$  in the assumptions of these latter.

**Example 6.** Let  $\bar{x} := 0_2$  and define

$$f(x_1, x_2) := \begin{cases} 3 - 3x_2 - x_1 - \sin \frac{1}{x_1} & \text{if } x_1 < 0 \text{ and } x_2 < 0 \\ -|x_1| + 3|x_2| & \text{otherwise.} \end{cases}$$

– The set of all continuity directions of  $f$  at  $\bar{x}$  is given by

$$D_f(\bar{x}) = \{d \in \mathbb{R}^2 : d_1 \geq 0\} \cup \{d \in \mathbb{R}^2 : d_2 \geq 0\},$$

which is not convex. Notice that  $\mathcal{D} := \mathbb{R} \times \mathbb{R}^+$  is a closed and convex cone contained in  $D_f(\bar{x})$ .

– The normal cone to the set  $\mathcal{D}$  is  $N_{\mathcal{D}}(0_2) = \{0\} \times \mathbb{R}^-$ , whereas the normal cone to  $D_f(\bar{x})$  is  $N_{D_f(\bar{x})}(0_2) = \{0_2\}$ .  
 •  $\partial_{\mathcal{D}}^{us} f(\bar{x}) := \{(1, 3), (-1, 3)\}$  is a  $\mathcal{D}$ -directional upper semi-regular convexificator of  $f$  at  $\bar{x}$ . Indeed,  $\partial_{\mathcal{D}}^{us} f(\bar{x})$  is closed, and for each  $d = (d_1, d_2) \in \mathcal{D}$ , we have

$$f^+(\bar{x}, d) = -|d_1| + 3|d_2| \leq -d_1 + 3d_2 \leq \max(-d_1 + 3d_2, d_1 + 3d_2) = \sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle x^*, d \rangle.$$

According to Lemma 4, the set

$$\partial^{us} f(\bar{x}) := \partial_{\mathcal{D}}^{us} f(\bar{x}) + N_{\mathcal{D}}(0_2) = (\{1\} \times ]-\infty, 3]) \cup (\{-1\} \times ]-\infty, 3])$$

is an upper semi-regular convexificator of  $f$  at  $\bar{x}$ .

• In light of the foregoing,  $\partial_{D_f(\bar{x})}^u f(\bar{x}) := \{(1, 3), (-1, 3)\}$  is a directional upper convexificator of  $f$  at  $\bar{x}$ . However, neither  $\partial_{D_f(\bar{x})}^u f(\bar{x}) + N_{D_f(\bar{x})}(0_2)$  nor  $\partial_{\mathcal{D}}^{us} f(\bar{x})$  constitutes an upper convexificator of  $f$  at  $\bar{x}$ . Indeed,

$$\partial_{D_f(\bar{x})}^u f(\bar{x}) + N_{D_f(\bar{x})}(0_2) = \partial_{\mathcal{D}}^{us} f(\bar{x}) = \{(1, 3), (-1, 3)\}.$$

Moreover, for  $\bar{d} = (0, -1)$ , we have

$$+\infty = f^-(\bar{x}, \bar{d}) > 3 = \sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle x^*, \bar{d} \rangle.$$

**Example 7.** Let  $\bar{x} = (0, 1)$  and define the function

$$f(x_1, x_2) := \begin{cases} 0 & \text{if } x_1^2 + x_2^2 \leq 1, \\ 1 & \text{if } x_1^2 + x_2^2 > 1. \end{cases}$$

The set of all continuity directions at  $\bar{x}$  is

$$D_f(\bar{x}) = \{d \in \mathbb{R}^2 : d_2 < 0\} \cup \{0_2\},$$

which is not closed. The normal cone to  $D_f(\bar{x})$  at the origin is

$$N_{D_f(\bar{x})}(0_2) = \{d \in \mathbb{R}^2 : d_2 \geq 0 \text{ and } d_1 = 0\}.$$

– The set  $\partial_{D_f(\bar{x})}^{us} f(\bar{x}) = \{0_2\}$  is a directional upper semi-regular convexificator of  $f$  at  $\bar{x}$ . However,

$$\partial_{D_f(\bar{x})}^{us} f(\bar{x}) + N_{D_f(\bar{x})}(0_2) = \{(x_1^*, x_2^*) \in \mathbb{R}^2 : x_2^* \geq 0 \text{ and } x_1^* = 0\}$$

is not an upper semi-regular convexificator. Indeed, for  $\bar{d} = (1, 0)$ , we have

$$f^+(\bar{x}, \bar{d}) = +\infty > 0 = \sup_{x^* \in \left(\partial_{D_f(\bar{x})}^{us} f(\bar{x}) + N_{D_f(\bar{x})}(0_2)\right)} \langle x^*, \bar{d} \rangle.$$

– Let

$$\mathcal{D} := \{d \in \mathbb{R}^2 : d_1 + d_2 \leq 0 \text{ and } d_2 - d_1 \leq 0\}.$$

In this case,  $\mathcal{D}$  is a closed and convex cone included in  $D_f(\bar{x})$ . Moreover, the normal cone to  $\mathcal{D}$  at the origin is

$$N_{\mathcal{D}}(0_2) = \{d \in \mathbb{R}^2 : d_1 + d_2 \geq 0 \text{ and } d_1 - d_2 \geq 0\}.$$

- Since  $\partial_{\mathcal{D}}^{us} f(\bar{x}) = \{0_2\}$  is a  $\mathcal{D}$ -directional upper semi-regular convexificator of  $f$  at  $\bar{x}$ , according to Lemma 4, the set

$$\partial^{us} f(\bar{x}) := \partial_{\mathcal{D}}^{us} f(\bar{x}) + N_{\mathcal{D}}(0_2) = \{(x_1^*, x_2^*) \in \mathbb{R}^2 : x_1^* + x_2^* \geq 0 \text{ and } x_1^* - x_2^* \geq 0\}$$

is an upper semi-regular convexificator of  $f$  at  $\bar{x}$ .

- However,  $\partial_{\mathcal{D}}^{us} f(\bar{x}) = \{0_2\}$  is not upper semi-regular convexificator of  $f$  at  $\bar{x}$ . Indeed, for  $\bar{d} = (1, 0)$ , one obtains

$$+\infty = f^+(\bar{x}, \bar{d}) > 0 = \sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle x^*, \bar{d} \rangle.$$

The following result is crucial for the rest of our investigations.

**Lemma 8.** *Let  $A$  and  $B$  be two disjoint, nonempty convex sets. Suppose that  $A$  is a cone,  $B + A$  is closed, and*

$$\sup_{v \in B} \langle v, d \rangle \geq 0, \quad \text{for all } d \in A^-. \tag{3}$$

Then  $0 \in B + A$ .

*Proof.* Assume to the contrary that  $0 \notin B + A$ . Then the sets  $-A$  and  $B$  are disjoint. By the separation theorem ([3], Prop. 1.5.3 (1)) and the closedness of  $B + A$ , there exist a nonzero vector  $\pi^* \in \mathbb{R}^n$  and a real number  $\alpha \in \mathbb{R}$  such that:

$$\langle \pi^*, b \rangle < \alpha < \langle \pi^*, a \rangle, \quad \text{for all } b \in B, a \in -A. \tag{4}$$

– We claim that  $\pi^* \in A^-$ . Indeed, since  $-A$  is a cone, the right-hand inequality above implies

$$\alpha < \langle \pi^*, na \rangle, \quad \text{for all } a \in -A, n \in \mathbb{N}.$$

Dividing by  $n$  gives

$$\frac{1}{n} \alpha < \langle \pi^*, a \rangle, \quad \text{for all } a \in -A, n \in \mathbb{N}.$$

Letting  $n \rightarrow +\infty$ , we obtain

$$0 \leq \langle \pi^*, a \rangle, \quad \text{for all } a \in -A,$$

and thus  $\pi^* \in A^-$ .

– Now, since  $-A$  is cone, setting  $a = 0$  in the separation inequality (4) yields

$$\langle \pi^*, b \rangle < \alpha < 0, \quad \text{for all } b \in B.$$

Hence,

$$\sup_{b \in B} \langle \pi^*, b \rangle \leq \alpha < 0.$$

Because  $\pi^* \in A^-$ , this contradicts (3). The proof is therefore complete. □

### 3. OPTIMALITY CONDITIONS

Let  $\Omega$  denote the feasible set of the MPSC problem, defined by

$$\Omega := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I, h_j(x) = 0, j \in J, G_s(x)H_s(x) = 0, s \in \mathcal{L}\}.$$

For a given  $\bar{x} \in \Omega$ , define the active inequality index set  $I(\bar{x}) := \{i \in I : g_i(\bar{x}) = 0\}$ . Consider the sets

$$A := \{s \in \mathcal{L} : G_s(\bar{x}) = 0, H_s(\bar{x}) \neq 0\}, B := \{s \in \mathcal{L} : G_s(\bar{x}) = 0, H_s(\bar{x}) = 0\},$$

and

$$C := \{s \in \mathcal{L} : G_s(\bar{x}) \neq 0, H_s(\bar{x}) = 0\}.$$

Note that  $\{A, B, C\}$  forms a disjoint partition of  $\mathcal{L}$ . According to Lemma 4.1 of [18], if  $B$  is nonempty, both the MFCQ and LICQ fail at any feasible solution  $\bar{x} \in \Omega$ . On the other hand, if  $B$  is empty, these standard constraint qualifications may hold under weaker regularity assumptions, in which case the KKT conditions provide necessary optimality conditions.

Throughout the remainder of this paper, we focus on the case  $B \neq \emptyset$ . A partition of  $B$ , is of the form  $(B_1, B_2)$  where  $B = B_1 \cup B_2$  and  $B_1 \cap B_2 = \emptyset$ . We denote the set of all partitions of  $B$  by  $P(B)$ . We now recall the nonlinear program  $\text{MPSC}(B_1, B_2)$  as introduced by Mehlitz [18], with respect to a partition  $(B_1, B_2)$  of  $B$ , defined by

$$\text{MPSC}(B_1, B_2) : \begin{cases} \text{Minimize} & f(x) \\ \text{s.t.} & \begin{cases} g_i(x) \leq 0, & i \in I, \\ h_j(x) = 0, & j \in J, \\ G_s(x) = 0, & s \in A \cup B_2, \\ H_s(x) = 0, & s \in C \cup B_1. \end{cases} \end{cases}$$

Note that the feasible set of  $\text{MPSC}(B_1, B_2)$  is contained in the feasible set of the original MPSC problem. Moreover, locally around  $\bar{x}$ , the union of all feasible sets of  $\text{MPSC}(B_1, B_2)$  over  $(B_1, B_2) \in P(B)$  coincides with the feasible set of MPSC.

Let  $\mathcal{D}$  be a nonempty closed cone of  $\mathbb{R}^n$ , not necessarily convex, such that the functions  $f, g_i$  for  $i \in I(\bar{x}), h_j$  for  $j \in J, G_s$  for  $s \in A \cup B_2$ , and  $H_s$  for  $s \in C \cup B_1$ , are continuous on  $\{\bar{x}\} + \mathcal{D}$ . A feasible point  $\bar{x} \in \Omega$  is called a local  $\mathcal{D}$ -optimal solution of MPSC if and only if there exists a neighborhood  $\mathcal{V}$  of  $\bar{x}$  such that

$$f(x) \geq f(\bar{x}), \quad \forall x \in \Omega_{\mathcal{D}} \cap \mathcal{V},$$

where

$$\Omega_{\mathcal{D}} := \Omega \cap (\{\bar{x}\} + \mathcal{D}) = \{x \in \Omega \text{ such that } \exists d \in \mathcal{D}, x = \bar{x} + d\}.$$

When the constraint functions are continuous and  $\mathcal{D}$  is taken as  $\mathbb{R}^n$ , the concept of  $\mathcal{D}$ -optimality coincides with the classical notion of local optimality, since in this case the set of continuity directions equals  $\mathbb{R}^n$ . Moreover,  $\bar{x} \in \Omega$  is a local  $\mathcal{D}$ -optimal solution of MPSC if and only if it is a local  $\mathcal{D}$ -optimal solution of  $\text{MPSC}(B_1, B_2)$  for all  $(B_1, B_2) \in P(B)$ .

For the remainder of this paper, we adopt the following assumptions.

**Assumption 9.** – The function  $f$  admits a bounded  $\mathcal{D}$ -directional upper semi-regular convexificator  $\partial_{\mathcal{D}}^{us} f(\bar{x})$  at  $\bar{x} \in \Omega$ .

– The function  $f$  is locally  $\mathcal{D}$ -Lipshitz; that is, there exist a constant  $k > 0$  and a neighborhood  $U$  of  $\bar{x}$  such that

$$|f(x) - f(y)| \leq k\|x - y\|, \quad \forall x, y \in (\bar{x} + \mathcal{D}) \cap U.$$

**Assumption 10.** The functions  $g_i$  for  $i \in I(\bar{x}), h_j$  for  $j \in J, G_s$  for  $s \in A \cup B_2$ , and  $H_s$  for  $s \in C \cup B_1$  admit  $\mathcal{D}$ -directional upper convexificators, denoted respectively by  $\partial_{\mathcal{D}}^u g_i(\bar{x}), \partial_{\mathcal{D}}^u h_j(\bar{x}), \partial_{\mathcal{D}}^u G_s(\bar{x})$  and  $\partial_{\mathcal{D}}^u H_s(\bar{x})$ .

**Assumption 11.** The functions  $(-h_j)$  for  $j \in J$ ,  $(-G_s)$  for  $s \in A \cup B_2$ , and  $(-H_s)$  for  $s \in C \cup B_1$  admit  $\mathcal{D}$ -directional upper convexifiers, denoted respectively by  $\partial_{\mathcal{D}}^u(-h_j)(\bar{x})$ ,  $\partial_{\mathcal{D}}^u(-G_s)(\bar{x})$ , and  $\partial_{\mathcal{D}}^u(-H_s)(\bar{x})$ .

Now, assuming that Assumptions 10 and 11 hold, we introduce the following notations:

$$\begin{aligned} \Delta_g &:= \bigcup_{i \in I(\bar{x})} \text{conv } \partial_{\mathcal{D}}^u g_i(\bar{x}), \\ \Upsilon_h &:= \bigcup_{j \in J} (\text{conv } \partial_{\mathcal{D}}^u h_j(\bar{x}) \cup \text{conv } \partial_{\mathcal{D}}^u (-h_j)(\bar{x})), \\ \Upsilon_H^{C \cup B_1} &:= \bigcup_{s \in D \cup B_1} (\text{conv } \partial_{\mathcal{D}}^u H_s(\bar{x}) \cup \text{conv } \partial_{\mathcal{D}}^u (-H_s)(\bar{x})), \\ \Upsilon_G^{A \cup B_2} &:= \bigcup_{s \in A \cup B_2} (\text{conv } \partial_{\mathcal{D}}^u G_s(\bar{x}) \cup \text{conv } \partial_{\mathcal{D}}^u (-G_s)(\bar{x})), \end{aligned}$$

and

$$\Xi(\bar{x}) := \Delta_g \cup \Upsilon_h \cup \Upsilon_G^{A \cup B_2} \cup \Upsilon_H^{C \cup B_1}.$$

Using the concept of  $\mathcal{D}$ -directional upper convexifiers, we now introduce the following nonsmooth constraint qualification.

**Definition 12.** Let  $\bar{x} \in \Omega$  and let  $(B_1, B_2) \in P(B)$ . Suppose that Assumptions 10 and 11 hold. We say that the Abadie constraint qualification  $\partial_{\mathcal{D}}^u - ACQ(B_1, B_2)$  holds at  $\bar{x}$  if

$$(\Xi(\bar{x}) + N_{\mathcal{D}}(0_n))^- \subseteq \text{cl conv } T(\Omega_{\mathcal{D}}, \bar{x}). \tag{5}$$

In the following definition, we introduce a generalized alternative stationarity concept formulated in terms of  $\mathcal{D}$ -directional upper convexifiers.

**Definition 13.** A feasible point  $\bar{x}$  of MPSC is called a  $\mathcal{D}$ -generalized alternative stationary point if there exists a vector  $(\lambda^g, \lambda^h, \mu^h, \lambda^G, \lambda^H, \mu^G, \mu^H) \in \mathbb{R}^m \times \mathbb{R}^{2p} \times \mathbb{R}^{2l} \times \mathbb{R}^{2l}$  such that

$$0 \in \left[ \begin{aligned} &\text{conv } \partial_{\mathcal{D}}^{u_s} f(\bar{x}) + \sum_{i=1}^m \lambda_i^g \text{conv } \partial_{\mathcal{D}}^u g_i(\bar{x}) + \sum_{j \in J} \mu_j^h \text{conv } \partial_{\mathcal{D}}^u h_j(\bar{x}) + \sum_{j \in J} \lambda_j^h \text{conv } \partial_{\mathcal{D}}^u (-h_j)(\bar{x}) \\ &+ \sum_{s=1}^l \lambda_s^G \text{conv } \partial_{\mathcal{D}}^u (-G_s)(\bar{x}) + \sum_{s=1}^l \lambda_s^H \text{conv } \partial_{\mathcal{D}}^u (-H_s)(\bar{x}) \\ &+ \sum_{s=1}^l \mu_s^G \text{conv } \partial_{\mathcal{D}}^u G_s(\bar{x}) + \sum_{s=1}^l \mu_s^H \text{conv } \partial_{\mathcal{D}}^u H_s(\bar{x}) + N_{\mathcal{D}}(0_n) \end{aligned} \right] \tag{6}$$

with the complementarity conditions

$$\lambda_i^g g_i(\bar{x}) = 0, \quad \forall i \in I \tag{7}$$

and

$$\begin{cases} \mu_s^G = \lambda_s^G = 0 \text{ or } \mu_s^H = \lambda_s^H = 0, & \forall s \in B, \\ \lambda_s^G = \mu_s^G = 0, & \forall s \in C, \\ \lambda_s^H = \mu_s^H = 0, & \forall s \in A, \\ \lambda_s^G, \lambda_s^H, \mu_s^G, \mu_s^H \geq 0, & \forall s \in \mathcal{L}, \\ \lambda_i^g \geq 0, \forall i \in I, \text{ and } \lambda_j^h \geq 0, \mu_j^h \geq 0, & \forall j \in J. \end{cases} \tag{8}$$

$\mathcal{D}$ -directional upper convexifiers reduce to the standard upper convexifiers whenever the functions are continuous and  $\mathcal{D} = D_f(x)$ . In this case, since  $\mathcal{D} = \mathbb{R}^n$  and  $N_{\mathcal{D}}(0_n) = \{0_n\}$ , both the subscript  $\mathcal{D}$  and the normal cone  $N_{\mathcal{D}}(0_n)$  can be omitted. Accordingly, (5) reduces to

$$\Xi(\bar{x})^- \subseteq \text{cl conv } T(\Omega, \bar{x}). \tag{9}$$

**Remark 14.** Notice that if all the functions are differentiable and the  $\mathcal{D}$ -directional upper convexifiers in the above stationary concept are replaced by the standard gradients, then this notion reduces to the M-stationarity condition introduced by Mehrlitz ([18], Def. 4.1(2)).

**Theorem 15.** *Let  $\bar{x}$  be a local  $\mathcal{D}$ -optimal solution of MPSC. Assume that  $B$  is nonempty, that Assumptions 9–11 hold, that  $\mathcal{A}(\bar{x}) := \text{pos } \Xi(\bar{x}) + N_{\mathcal{D}}(0_n)$  is closed and that there exists a partition  $(B_1, B_2)$  of  $B$  such that  $\partial_{\mathcal{D}}^{\mu} - ACQ(B_1, B_2)$  holds at  $\bar{x}$ . Then,  $\bar{x}$  is a  $\mathcal{D}$ -generalized alternative stationary point.*

*Proof.* Since  $\bar{x}$  is a local  $\mathcal{D}$ -optimal solution of MPSC, there exists a neighborhood  $\mathcal{V}$  of  $\bar{x}$  such that

$$f(x) \geq f(\bar{x}), \quad \forall x \in \Omega_{\mathcal{D}} \cap \mathcal{V}. \tag{10}$$

– We claim that

$$\sup_{\eta \in \partial_{\mathcal{D}}^{\mu} f(\bar{x})} \langle \eta, v \rangle \geq 0, \quad \forall v \in \text{cl conv } T(\Omega_{\mathcal{D}}, \bar{x}). \tag{11}$$

Indeed, let  $v \in \text{cl conv } T(\Omega_{\mathcal{D}}, \bar{x})$  be arbitrary. Then, there exists a sequence  $v_k \rightarrow v$ , as  $k \rightarrow \infty$ , such that

$$v_k \in \text{conv } T(\Omega_{\mathcal{D}}, \bar{x}) \quad \text{for all } k.$$

By Carathéodory’s Theorem ([20], Thm. 2.1), for each  $k$  we can find vectors  $v_k^s \in T(\Omega_{\mathcal{D}}, \bar{x})$  and scalars  $\lambda_k^s \in [0, 1]$  with  $s \in \{1, \dots, n + 1\}$ , satisfying  $\sum_{s=1}^{n+1} \lambda_k^s = 1$  and

$$v_k = \sum_{s=1}^{n+1} \lambda_k^s v_k^s, \quad \text{for all } k \in \mathbb{N}. \tag{12}$$

Since  $v_k^s \in T(\Omega_{\mathcal{D}}, \bar{x})$ , we can find  $v_{kj}^s \rightarrow v_k^s$  and  $t_{kj}^s \rightarrow 0^+$  such that

$$\bar{x} + t_{kj}^s v_{kj}^s \in \Omega_{\mathcal{D}}, \quad \text{for all } j \in \mathbb{N}.$$

Hence, for sufficiently large  $j$  and  $k$ , we have  $\bar{x} + t_{kj}^s v_{kj}^s \in \Omega \cap \mathcal{V}$  and  $t_{kj}^s v_{kj}^s \in \mathcal{D}$ . Therefore, by (10), it follows that

$$f(\bar{x} + t_{kj}^s v_{kj}^s) - f(\bar{x}) \geq 0,$$

and consequently

$$\frac{f(\bar{x} + t_{kj}^s v_{kj}^s) - f(\bar{x})}{t_{kj}^s} \geq 0.$$

Taking into account that

$$\frac{f(\bar{x} + t_{kj}^s v_{kj}^s) - f(\bar{x})}{t_{kj}^s} = \frac{f(\bar{x} + t_{kj}^s v_{kj}^s) - f(\bar{x} + t_{kj}^s v_k^s)}{t_{kj}^s} + \frac{f(\bar{x} + t_{kj}^s v_k^s) - f(\bar{x})}{t_{kj}^s}$$

we get

$$\frac{f(\bar{x} + t_{kj}^s v_{kj}^s) - f(\bar{x} + t_{kj}^s v_k^s)}{t_{kj}^s} + \frac{f(\bar{x} + t_{kj}^s v_k^s) - f(\bar{x})}{t_{kj}^s} \geq 0. \tag{13}$$

Thus,

$$\limsup_{k \rightarrow \infty} \frac{f(\bar{x} + t_{kj}^s v_{kj}^s) - f(\bar{x} + t_{kj}^s v_k^s)}{t_{kj}^s} + \frac{f(\bar{x} + t_{kj}^s v_k^s) - f(\bar{x})}{t_{kj}^s} \geq 0.$$

On the one hand, since  $\mathcal{D}$  is a cone, and given that  $t_{kj}^s v_{kj}^s \in \mathcal{D}$  with  $t_{kj}^s > 0$ , it follows that  $v_{kj}^s \in \mathcal{D}$ . By the closedness of  $\mathcal{D}$ , we also have  $v_k^s \in \mathcal{D}$ . Moreover, since  $f$  satisfies the  $\mathcal{D}$ -Lipschitz property at  $\bar{x}$ , we obtain

$$\lim_{k \rightarrow \infty} \frac{f(\bar{x} + t_{kj}^s v_{kj}^s) - f(\bar{x} + t_{kj}^s v_k^s)}{t_{kj}^s} = 0.$$

Using (13), we have

$$\limsup_{t_{kj}^s \rightarrow 0^+} \frac{f(\bar{x} + t_{kj}^s v_k^s) - f(\bar{x})}{t_{kj}^s} \geq 0, \quad \text{for all } s \in \{1, \dots, n+1\}.$$

Consequently,

$$f^+(\bar{x}, v_k^s) \geq 0, \quad \text{for all } s \in \{1, \dots, n+1\}.$$

By Theorem 2.8 (a) of [1], since  $\lambda_k^s \geq 0$ , it follows that

$$f^+(\bar{x}, \lambda_k^s v_k^s) \geq 0, \quad \text{for all } s \in \{1, \dots, n+1\}.$$

Since  $\partial_{\mathcal{D}}^{us} f(\bar{x})$  is a  $\mathcal{D}$ -directional upper semi-regular convexificator of  $f$  at  $\bar{x}$ , and since  $\lambda_k^s v_k^s \in \mathcal{D}$ , we deduce that

$$\sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle x^*, \lambda_k^s v_k^s \rangle \geq 0, \quad \text{for all } s \in \{1, \dots, n+1\}.$$

Hence,

$$\sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\bar{x})} \left\langle x^*, \sum_{s=1}^{n+1} \lambda_k^s v_k^s \right\rangle \geq 0.$$

From (12), we have  $v_k = \sum_{s=1}^{n+1} \lambda_k^s v_k^s$ , and therefore  $\sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle x^*, v_k \rangle \geq 0$ . Letting  $k \rightarrow \infty$ , we obtain

$$\sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle x^*, v \rangle \geq 0.$$

Therefore,

$$\sup_{x^* \in \text{conv } \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle x^*, v \rangle \geq 0. \tag{14}$$

Since  $v$  was chosen arbitrarily, the desired inequality follows directly from (14).

– From (11), since  $\partial_{\mathcal{D}}^u - ACQ(B_1, B_2)$  holds at  $\bar{x}$ , we obtain

$$\sup_{\eta \in \text{conv } \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle \eta, v \rangle \geq 0, \quad \forall v \in (\Xi(\bar{x}) + N_{\mathcal{D}}(0_n))^-.$$

- Since  $\Xi(\bar{x}) \subseteq \text{pos } \Xi(\bar{x})$ , it follows that

$$\sup_{\eta \in \text{conv } \partial_{\mathcal{D}}^{us} f(\bar{x})} \langle \eta, v \rangle \geq 0, \quad \forall v \in (\text{pos } \Xi(\bar{x}) + N_{\mathcal{D}}(0_n))^-.$$

Thus,

$$\sup_{\eta \in \text{conv } \partial_{\mathcal{D}}^{us} f(\bar{x}) + N_{\mathcal{D}}(0_n)} \langle \eta, v \rangle \geq 0, \quad \forall v \in (\text{pos } \Xi(\bar{x}) + N_{\mathcal{D}}(0_n))^-.$$

- On the one hand, since both  $N_{\mathcal{D}}(0_n)$  and  $\text{pos } \Xi(\bar{x})$  are convex cones, their sum  $\text{pos } \Xi(\bar{x}) + N_{\mathcal{D}}(0_n)$  is also a convex cone. On the other hand, using the closedness of  $\text{pos } \Xi(\bar{x}) + N_{\mathcal{D}}(0_n)$  and  $N_{\mathcal{D}}(0_n)$ , together with the compactness of  $\text{conv } \partial_{\mathcal{D}}^{us} f(\bar{x})$ , one deduces that

$$(\text{conv } \partial_{\mathcal{D}}^{us} f(\bar{x}) + N_{\mathcal{D}}(0_n)) + (\text{pos } \Xi(\bar{x}) + N_{\mathcal{D}}(0_n)) = \text{conv } \partial_{\mathcal{D}}^{us} f(\bar{x}) + \text{pos } \Xi(\bar{x}) + N_{\mathcal{D}}(0_n)$$

is closed. Consequently, by Lemma 8, we obtain

$$0 \in \text{conv } \partial_{\mathcal{D}}^{us} f(\bar{x}) + \text{pos } \Xi(\bar{x}) + N_{\mathcal{D}}(0_n).$$

Therefore, there exist multipliers  $\lambda_i^g \geq 0, i \in I(\bar{x}), \mu_j^h \geq 0, \lambda_j^h \geq 0, j \in J, \mu_s^G \geq 0, s \in A \cup B_2, \lambda_s^G \geq 0, s \in A \cup B_2, \mu_s^H \geq 0, s \in C \cup B_1,$  and  $\lambda_s^H \geq 0, s \in C \cup B_1,$  such that

$$0 \in \left[ \begin{aligned} & \text{conv } \partial_{\mathcal{D}}^{us} f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i^g \text{conv } \partial_{\mathcal{D}}^u g_i(\bar{x}) \\ & + \sum_{j \in J} \mu_j^h \text{conv } \partial_{\mathcal{D}}^u h_j(\bar{x}) + \sum_{j \in J} \lambda_j^h \text{conv } \partial_{\mathcal{D}}^u (-h_j)(\bar{x}) \\ & + \sum_{s \in A \cup B_2} \mu_s^G \text{conv } \partial_{\mathcal{D}}^u G_s(\bar{x}) + \sum_{s \in A \cup B_2} \lambda_s^G \text{conv } \partial_{\mathcal{D}}^u (-G_s)(\bar{x}) \\ & + \sum_{s \in C \cup B_1} \mu_s^H \text{conv } \partial_{\mathcal{D}}^u H_s(\bar{x}) + \sum_{s \in C \cup B_1} \lambda_s^H \text{conv } \partial_{\mathcal{D}}^u (-H_s)(\bar{x}) + N_{\mathcal{D}}(0_n). \end{aligned} \right].$$

- By setting

$$\begin{cases} \mu_s^G = 0, \lambda_s^G = 0, & \forall s \in C \cup B_1, \\ \mu_s^H = 0, \lambda_s^H = 0, & \forall s \in A \cup B_2, \end{cases}$$

we obtain (6), (7) and (8). This completes the proof. □

The following example serves two purposes: it not only illustrates Theorem 15 but also highlights the advantages of our method over existing approaches. In particular, our technique offers effective alternatives to Mehlitz’s method ([18], Thm. 5.1) for nonsmooth functions, Jennane and Kalmoun’s approach ([10], Thm. 3.1) for non-tangentially convex functions, and the framework proposed by Jennane ([11], Thm. 3.1) for non-Lipschitz cases. Crucially, because the function  $f$  is both discontinuous and non-tangentially convex, none of these prior results can be applied to our setting. Moreover, since  $G$  is not continuously differentiable and  $f$  cannot be expressed as the sum of a continuously differentiable function and a quasi-norm to the power  $p$  (which would necessarily be continuous), the results of Lv *et al.* [17] are likewise not applicable.

**Example 16.** Let  $\bar{x} := 0_2$ . Consider the following nonsmooth optimization problem:

$$(\text{MPSC}) : \begin{cases} \text{Minimize } f(x_1, x_2) \\ \text{s.t. } \begin{cases} g(x_1, x_2) \leq 0, h(x_1, x_2) = 0, \\ G(x_1, x_2)H(x_1, x_2) = 0, \end{cases} \end{cases}$$

where  $g(x_1, x_2) = -x_1 + x_2, h(x_1, x_2) = 0, H(x_1, x_2) = 2x_2$

$$G(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 < 0 \text{ and } x_2 < 0 \\ x_2 & \text{elsewhere} \end{cases}$$

and

$$f(x_1, x_2) := \begin{cases} \sin\left(\frac{1}{x_2}\right) + x_1 + x_2 & \text{if } x_1 < 0 \text{ and } x_2 < 0 \\ x_1 + x_2^2 + 1 & \text{elsewhere.} \end{cases}$$

– On the one hand, the sets of continuity directions of the functions  $G$ ,  $f$ ,  $g$ ,  $h$  and  $H$  at  $\bar{x}$  are given by

$$D_G(\bar{x}) = D_f(\bar{x}) = \{d \in \mathbb{R}^2 : d_1 \geq 0\} \cup \{d \in \mathbb{R}^2 : d_2 \geq 0\}$$

and

$$D_g(\bar{x}) = D_h(\bar{x}) = D_H(\bar{x}) = \mathbb{R} \times \mathbb{R}.$$

Therefore, we may choose

$$\mathcal{D} := \{d \in \mathbb{R}^2 : d_1 \geq 0\} \cup \{d \in \mathbb{R}^2 : d_2 \geq 0\}.$$

Consequently,  $N_{\mathcal{D}}(0_2) = \{0_2\}$ . Notice that  $\mathcal{D}$  is closed but nonconvex.

– On the other hand,  $\bar{x}$  is a  $\mathcal{D}$ -optimal solution of MPSC. Moreover, we have  $A = C = \emptyset$ ,  $B = \{1\} \neq \emptyset$ ,  $I = \{1, 2\}$ ,  $I(\bar{x}) = \{1, 2\}$ ,  $J = \{1\}$ ,  $\Omega = \mathbb{R}^+ \times \{0\}$  and  $T(\Omega, \bar{x}) = \mathbb{R}^+ \times \{0\}$ . Notice that  $\Omega_{\mathcal{D}} = \mathbb{R}^+ \times \{0\}$  is closed and that  $T(\Omega_{\mathcal{D}}, \bar{x}) = \mathbb{R}^+ \times \{0\}$ .

- $\partial_{\mathcal{D}}^u - ACQ(B_1, B_2)$  holds at  $\bar{x}$ .
  - $\partial_{\mathcal{D}}^{us} f(\bar{x}) := \{(1, 0)\}$  is a bounded  $\mathcal{D}$ -directional upper semi-regular convexificator of  $f$  at  $\bar{x}$ .
  - $\partial_{\mathcal{D}}^u g(\bar{x}) = \{(-1, 1)\}$ ,  $\partial_{\mathcal{D}}^u h(\bar{x}) = \{(0, 0)\}$ ,  $\partial_{\mathcal{D}}^u G(\bar{x}) = \{(0, 1)\}$ ,  $\partial_{\mathcal{D}}^u (-G)(\bar{x}) = \{(0, -1)\}$ ,  $\partial_{\mathcal{D}}^u H(\bar{x}) = \{(0, 1)\}$  and  $\partial_{\mathcal{D}}^u (-H)(\bar{x}) = \{(0, -1)\}$  are  $\mathcal{D}$ -directional upper convexificator of  $g$ ,  $h$ ,  $G$ ,  $-G$ ,  $H$  and  $-H$  at  $\bar{x}$ .
  - For  $B_1 = \{1\}$  and  $B_2 = \emptyset$ , we have

$$\Xi(\bar{x}) = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\};$$

and thus

$$\Xi(\bar{x})^- = \mathbb{R}^+ \times \{0\} \subseteq cl \ conv \ T(\Omega \cap (\bar{x} + \mathcal{D}), \bar{x}).$$

- $\mathcal{A}(\bar{x})$  is closed. Indeed,

$$\mathcal{A}(\bar{x}) = pos \ \Xi(\bar{x}) = \mathbb{R}^- \times \mathbb{R}.$$

– By taking  $\lambda^g = \lambda^h = 1$ ,  $\lambda^H = 2$ ,  $\mu^H = \mu^h = \lambda^G = 1$  and  $\mu^G = 1$ , since  $0_2 \in N_{\mathcal{D}}(0_2)$ , we get

$$0 \in \left[ \begin{array}{l} conv \ \partial_{\mathcal{D}}^{us} f(\bar{x}) + \lambda^g \ conv \ \partial_{\mathcal{D}}^u g(\bar{x}) \\ + \mu^h \ conv \ \partial_{\mathcal{D}}^u h(\bar{x}) + \lambda^h \ conv \ \partial_{\mathcal{D}}^u (-h)(\bar{x}) \\ + \lambda^G \ conv \ \partial_{\mathcal{D}}^u (-G)(\bar{x}) + \lambda^H \ conv \ \partial_{\mathcal{D}}^u (-H)(\bar{x}) \\ + \mu^G \ conv \ \partial_{\mathcal{D}}^u G(\bar{x}) + \mu^H \ conv \ \partial_{\mathcal{D}}^u H(\bar{x}) + N_{\mathcal{D}}(0_2). \end{array} \right]$$

**Remark 17.** When the data functions are locally Lipschitz and their Clarke subdifferentials are used as  $\mathcal{D}$ -directional upper convexificators with  $\mathcal{D} = \mathbb{R}^n$ , Theorem 15 recovers ([23], Thm. 4) as a corollary. In the continuously differentiable case, using the gradients as  $\mathcal{D}$ -directional upper convexificators, Theorem 15 yields the smooth version of necessary stationary conditions presented in [18].

### 4. CONCLUSION

In this work, we investigated Mathematical Programs with Switching Constraints (MPSCs) in the nonsmooth setting, where the involved functions are lower semicontinuous and may lack continuity. Using  $\mathcal{D}$ -directional upper convexificators and  $\mathcal{D}$ -directional upper semi-regular convexificators, we established necessary optimality conditions under a tailored constraint qualification and a generalized alternative stationary concept. We provide an illustrative example that both demonstrates our theoretical findings and exposes limitations in certain prior published results.

Future research could explore sufficient optimality conditions, or duality theory for this challenging class of problems. Another intriguing extension of this work would be to investigate necessary optimality conditions for multiobjective optimization problems involving infinite switching constraints.

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## CONFLICTS OF INTEREST

The authors declare that they have no conflict of interest.

## AUTHOR CONTRIBUTION STATEMENT

All authors contributed equally to the preparation and writing of this manuscript.

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