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A NOVEL FAMILY OF FRACTIONAL INTEGRAL BASED ARCHIMEDEAN T-NORM WITH APPLICATION TO CIRCULAR INTUITIONISTIC FUZZY MULTI-CRITERIA DECISION MAKING PROBLEM

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Abstract. This paper introduces a novel family of t-norm and t-conorm based on the Riemann Liouville fractional integral. These t-norm and t-conorm offer a generalized algebraic foundation that enhances the capacity of analyzing uncertain data. Leveraging the proposed family of t-norm and t-conorm, a set of advanced aggregation operators is developed for circular intuitionistic fuzzy sets (CIFSs). The proposed operators for CIFSs include the Heronian mean operator, the weighted average operator and the geometric mean operator. Each operator is specifically designed to preserve the structural properties of CIFSs while accurately capturing the uncertainty represented by hesitancy radius, membership and non-membership components. The applicability of the proposed approach is demonstrated through Multi-Criteria Decision Making (MCDM) under circular intuitionistic fuzzy environment. Numerical examples are presented to illustrate its effectiveness. The example focuses on the selection of a dominant farming strategy for small-scale farmers, highlighting the practical relevance of CIFSs. Comparative analyses shows that the method is more adaptable, efficient and effective than existing approaches. In addition, a higher-dimensional MCDM example is included to demonstrate the applicability of the method to complex decision problems.

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1. Introduction

Decision-making under uncertainty and imprecision remains a persistent challenge in many application areas, such as engineering systems, economic planning, environmental assessment and artificial intelligence. In practical situations, decision data are often affected by vagueness, hesitation and incomplete knowledge, which limits the effectiveness of traditional crisp decision models. To overcome these difficulties, Zadeh [61] introduced the theory of fuzzy sets, allowing uncertainty to be represented through membership degrees taking values in the interval $[0, 1]$. This concept provided a flexible mathematical tool for handling imprecise information. Later, Atanassov [4] proposed intuitionistic fuzzy sets (IFSs), which incorporate both membership(μ) and non-membership degrees (ν) such that their total is less than one. This enhancement provides a richer representation of ambiguous

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and imprecise information by allowing an additional degree of freedom for hesitation. Subsequent developments resulted in the creation of CIFSs [5], in which uncertainty is captured by the radius and membership and non-membership values. This circular structure allows uncertainty to be expressed as a circular region rather than a single fixed point in the (μ, ν) plane. Consequently, CIFSs are capable of capturing dispersion, hesitation, and inconsistency in expert judgments, making them particularly suitable for complex decision making environments. The geometric interpretation of CIFS is explained as follows.

Geometric interpretation and mathematical structure

A CIFS element $(\mu(x), \nu(x), \gamma(x))$ is geometrically represented by a circle centered at $(\mu(x), \nu(x))$ in the (μ, ν) plane with radius $\gamma(x) \in [0, \sqrt{2}]$. The corresponding set of admissible regions is defined as

$$\left\{(\mu', \nu') \;\middle|\; (\mu' - \mu(x))^2 + (\nu' - \nu(x))^2 \leq \gamma(x)^2,\ \mu' + \nu' \leq 1, \mu', \nu' \in [0, 1]\right\}.$$

The constraint $\mu' + \nu' \leq 1$ ensures that all regions remain within the intuitionistic fuzzy domain, while the radius $\gamma(x)$ quantifies the level of uncertainty, instability, or hesitation associated with the expert assessment. When $\gamma(x) = 0$, a CIFS reduces to a classical IFS.

There are various types of fuzzy sets. A comparison between CIFS and existing fuzzy sets is described as follows.

Comparison with existing fuzzy extensions

Various extensions of fuzzy set theory have been developed over time to more effectively model uncertainty. In IFSs, uncertainty is captured indirectly through the hesitation degree, which depends on the membership and non-membership values. Pythagorean fuzzy sets (PFSs) [58] extend this idea by allowing the condition $\mu^2 + \nu^2 \leq 1$, while q-rung orthopair fuzzy sets (q-ROFSs) [59] further generalize it to $\mu^q + \nu^q \leq 1$.

Although PFSs and q-ROFSs provide a larger feasible domain for membership and non-membership degrees, the evaluations are still represented as single points. As a result, these models fail to capture expert opinions that fluctuate or spread around a central assessment (μ, ν).

CIFSs address this limitation by keeping the classical intuitionistic condition $\mu + \nu \leq 1$ and introducing a hesitancy radius to represent uncertainty geometrically. Instead of a fixed point, the evaluation is allowed to vary within a bounded circular region. This makes CIFSs more suitable for situations where expert judgments are unstable or imprecise. Table 1 summarizes the key differences among IFSs, PFSs, q-ROFSs and CIFSs, which shows that CIFSs provide a clearer and more realistic representation of uncertainty.

In many decision making problems, expert opinions are not exact and usually involve uncertainty and hesitation. CIFSs are useful for representing such situations. To make this idea clearer, a simple illustrative example of CIFSs is given below.

Illustrative example for CIFSs

Consider an expert judgment characterized by a membership degree 0.65, a non-membership degree 0.25, and an associated uncertainty radius 0.7. This assessment is expressed as the CIFS $(0.65, 0.25, 0.7)$. The pair $(0.65, 0.25)$ represents the central judgment of the expert, while the radius $\gamma = 0.7$ reflects the uncertainty associated with this assessment. This means that the actual opinion of the expert is not fixed at a single point but may vary within a circular region around $(0.65, 0.25)$. Any point (μ', ν') lying inside this region and satisfying $\mu' + \nu' \leq 1$ is considered an acceptable realization of the expert's judgment, as illustrated in Figure 1. This example clearly illustrates the practical usefulness of circular CIFSs in real decision making situations, where expert opinions are uncertain, imprecise and not perfectly consistent.

(1) **Analysing uncertainty fluctuations clearly**: In many real problems, experts are unable to provide a single precise membership and non-membership value. Classical IFSs, PFS and q-ROFSs force experts to give point-valued assessments, which may hide hesitation or disagreement. CIFSs address this limitation

TABLE 1. Comparison of IFSs, PFSs, q-ROFSs, and CIFSs.

Aspect	IFS	PFS	q-ROFS	CIFS
Representation	(μ,ν)	(μ,ν)	(μ,ν)	(μ,ν,γ)
Admissible Constraint	$\mu+\nu\leq 1$	$\mu^2+\nu^2\leq 1$	$\mu^q+\nu^q\leq 1$	$\mu+\nu\leq 1$, $\gamma\in[0,\sqrt{2}]$ $(\mu'-\mu(x))^2+(\nu'-\nu(x))^2 \leq \gamma(x)^2, \mu'+\nu' \leq 1, \mu',\nu'\in[0,1]$
Geometric Interpretation	Line segment	Circular arc	q-power curve	Circular region
Uncertainty control	Implicit (via hesitation)	Implicit (via squared constraint)	Parameter-controlled (through q)	Expressed directly with the hesitancy radius γ
Limitation Limitation	Cannot represent dispersion	Point-based uncertainty	Point-based uncertainty	No
Practical Applicability	Low-complexity problems	Moderate uncertainty	High hesitation	Highly complex, real world uncertainty

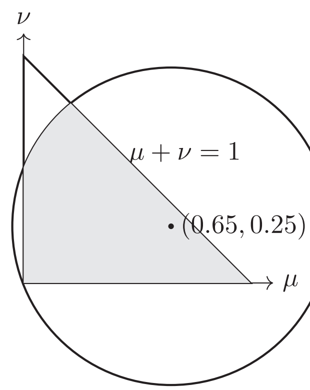


FIGURE 1. Geometric representation of a CIFS element $(0.65, 0.25; 0.7)$. The shaded region denotes the admissible uncertainty area within the intuitionistic fuzzy domain.

by allowing expert evaluations to be represented as a bounded circular region. This representation reflects not only the central assessment but also the possible variation around it, which leads to a more realistic description of uncertain and hesitant information.

(2) **Practical relevance in real world decision making**: The circular uncertainty representation offered by CIFSs is effective in practical applications, explained as follows:
   - **Selection of farming strategies for small-scale farmers**. Agricultural experts, extension workers and experienced farmers may assign slightly different evaluations to criteria such as expected yield, water demand, soil compatibility and risk exposure. These differences typically arise from heterogeneous field conditions, seasonal variability and individual experience. CIFSs capture this variability by allowing evaluations to lie within a controlled circular region, rather than forcing experts to agree on a single exact value.

MCDM problem and aggregation operator

MCDM refers to a systematic decision support process that assists in comparing and prioritizing alternatives when several evaluation criteria are involved simultaneously. Such problems frequently arise in areas like engineering, management, healthcare, supply chain planning and sustainability studies, where decisions depend on both numerical data and subjective judgements. In MCDM, aggregation operators play a central role because they combine the evaluations of different criteria into a single representative value [55]. Over time, many aggregation operators have been proposed to reflect different decision attitudes and aggregation behaviors. The simplest and most commonly used are the arithmetic and geometric averages [27,57]. These operators are easy to interpret and are limited when criteria are correlated or influence each other.

To overcome this limitation, more advanced aggregation operators have been developed. For example, the Bonferroni mean operator [38] allows the interaction between pairs of criteria to be taken into account. Among interaction-based aggregation operators, the Heronian mean has received significant attention due to its balanced aggregation structure and its ability to model interdependence amongst different criteria [10]. By introducing two adjustable parameters $p, q > 0$, the Heronian mean allows the decision maker to control the strength of interaction and compensation among criteria. This flexibility makes it suitable for complex MCDM problems where criteria are not independent and simple averaging is inadequate.

In crisp decision environment, aggregation operators are usually constructed using standard arithmetic operations such as addition and multiplication. However, in fuzzy decision making frameworks, these operators are generalized through triangular norms (t-norms) and t-conorms, which represent fuzzy logical conjunction and disjunction, respectively. Schweizer and Sklar [48] formally introduced the concept of t-norms and t-conorms within the framework of probabilistic metric spaces, providing a solid algebraic foundation for fuzzy aggregation. Since then, a variety of t-norms, including minimum, product, Hamacher, Einstein, Lukasiewicz, Dombi and drastic norms, have been widely applied in fuzzy aggregation and MCDM problems [37].

It is well known that aggregation operators in uncertain environment such as fuzzy sets, IFSs, PFSs, FFSs, q-ROFSs and CIFSs strongly depend on the choice of the underlying t-norm and t-conorm. Changing the selected norm directly affects the behavior and characteristics of the resulting aggregation operator. For instance,

- Hamacher t-norms lead to Hamacher-type aggregation operators,
- Dombi t-norms generate Dombi aggregation operators,
- Einstein and product norms produce aggregation structures with their own distinct properties.

Weighting criteria is a crucial step in MCDM problems since it indicates the influences the final decision. A wide range of weighting methods are proposed in the literature and they are mainly classified into two categories: subjective and objective [54].

Subjective weighting methods determine the importance of criteria based on expert judgments and decision maker preferences. Common examples include the Analytic Hierarchy Process (AHP) [50] and the Step-wise Weight Assessment Ratio Analysis (SWARA) [34]. In contrast, objective weighting methods compute criterion weights directly from the numerical information contained in the decision matrix. Approaches such as entropy-based weighting [60], the CRITIC method [20], and standard deviation-based techniques [51] rely on data dispersion rather than expert opinion.

AHP [12] is widely applied subjective weighting methods since it structures the decision problem into a clear hierarchy and employs pairwise comparisons to evaluate relative importance. However, classical AHP requires precise numerical judgments, which may be difficult to provide when decision makers face uncertainty or hesitation.

To overcome this limitation, fuzzy AHP is developed by integrating fuzzy set theory into the traditional AHP framework. In fuzzy AHP [52], pairwise comparisons are represented using fuzzy numbers, allowing decision makers to express their assessments more flexibly. By incorporating imprecision and hesitation into the weighting process, fuzzy AHP yields more stable and realistic criterion weights for complex MCDM problems. The list of abbreviations used in this paper is provided in Table 2.

2. Literature review

Over the past several decades, MCDM has attracted significant attention as a tool for analyzing decision problems involving multiple and often conflicting criteria. As a result, many decision methods have been pro-

TABLE 2. List of abbreviations.

Abbreviations	Representation
CIFS	Circular intuitionistic fuzzy set
IFS	Intuitionistic fuzzy set
MCDM	Multi-criteria decision making
CIFWAA	Circular intuitionistic fuzzy weighted average aggregation
CIFWGA	Circular intuitionistic fuzzy weighted geometric aggregation
CIFWHA	Circular intuitionistic fuzzy weighted heronian aggregation
$g_p(x)$	Proposed additive generator
$h_p(x)$	Dual of additive generator
$T_p^1(x,y)$	Proposed t-norm for case 1
$T_p^2(x,y)$	Proposed t-norm for case 2
$S_p^1(x,y)$	Proposed t-conorm for case 1
$S_p^2(x,y)$	Proposed t-conorm for case 2
CIFWHA^1	CIFWHA based on $T_p^1(x,y)$
CIFWHA^2	CIFWHA based on $T_p^2(x,y)$
CIFWAA^1	CIFWAA based on $T_p^1(x,y)$
CIFWAA^2	CIFWAA based on $T_p^2(x,y)$
CIFWGA^1	CIFWGA based on $T_p^1(x,y)$
CIFWGA^2	CIFWGA based on $T_p^2(x,y)$
EDAS	Evaluation based on distance from average solution
PF	Pythagorean fuzzy
IF	Intuitionistic fuzzy
FF	Fermatean fuzzy
CODAS	Combinative Distance based Assessment
CoCoSo	Combined Compromise Solution
DEMATEL	Decision Making Trial and Evaluation Laboratory
MARCOS	Measurement of Alternatives and Ranking according to Compromise Solution
MABAC	Multi Attributive Border Approximation Area Comparison
CRITIC	Criteria Importance Through Intercriteria Correlation
SWARA	Stepwise Weight Assessment Ratio Analysis
COPRAS	Complex Proportional Assessment
PROMETHEE	Preference Ranking Organization Method for Enrichment Evaluation

posed, including VIKOR [42], TOPSIS [11] and the Simple Weighted Averaging (SWA) approach [33]. These techniques are mainly designed to rank alternatives based on their overall performance across criteria.

Fuzzy set theory and its extensions have been widely incorporated into MCDM frameworks. Several studies have combined IFSs with classical MCDM methods. For example, Joshi and Kumar [31] proposed IF TOPSIS approach for portfolio selection, while Liang *et al.* [39] developed IF extended MABAC method. Similarly, Alghazzawi *et al.* [2] introduced a t-IF TOPSIS model based on lift-distance measures, and Dağıstanlı [16] extended the VIKOR method to an interval-valued IF environment.

In parallel, a large number of aggregation operators based on different t-norms and t-conorms have been proposed to improve information aggregation under fuzzy environments [28, 49]. Xu and Yager [56] introduced geometric aggregation operators, while Zhao *et al.* [62] presented generalized aggregation operators for IFSs. Other important contributions include Einstein-based aggregation operators [53], as well as Heronian mean aggregation operators [40], which are effective in capturing interactions among decision criteria.

Many recent studies have focused on designing advanced aggregation operators under orthopair fuzzy environments, including PFSs, q-ROFSs and their generalized or complex extensions. The main objective of these studies is to improve aggregation flexibility, interaction modeling and priority handling in MCDM problems. Within this context, Asif *et al.* [7] introduced Hamacher aggregation operators for PFSs, which allow parametric

TABLE 3. Comparative summary of MCDM approaches under fuzzy environments.

Author	Environment	MCDM method	Operational laws	Operator-based modeling	Applications
Zhang [63]	IVIF	Aggregation-based approach	Frank t-norm	Averaging and geometric operators	Software package selection
Ayyildiz [8]	FF	SWARA-based model	Algebraic laws	Geometric operator	Sustainability assessment
Debnath *et al.* [17]	q-ROFS	SWARA–MABAC	Aczel–Alsina t-norm	Bonferroni mean operator	Transport based problems
Deveci *et al.* [19]	FF	SWARA	Algebraic operational laws	Averaging operator	Risk factors influencing sustainable mining
Garg *et al.* [24]	Triangular fuzzy	Fuzzy SWARA and CoCoSo	Classical arithmetic laws for triangular fuzzy	Bonferroni mean operator	Industrial robot selection in the automobile industry
Santonab *et al.* [15]	IF	CODAS and CoCoSo	Algebraic t-norm	Averaging operator	Healthcare supplier selection
Khan *et al.* [36]	IF	AHP-based PROMETHEE	Hamacher and Dombi t-norms	Averaging operators	Municipal solid waste management
Bajaj *et al.* [9]	PF	VIKOR	Algebraic t-norm		Transportation challenges
Sahoo *et al.* [47]	PF	DEMATEL, MARCOS, and MABAC	Algebraic operational laws	Fairly aggregation operators	Sustainable supplier selection
Ahemad *et al.* [1]	interval-valued q-ROFN	COPRAS	Algebraic operational laws	Weighted average aggregation operator	Solid waste management
Rani *et al.* [45]	Type-2 trapezoidal PF	CODAS and COPRAS	Algebraic t-norm	Arithmetic, geometric, and power operators	Healthcare decision support
Proposed model	CIFS	Circular intuitionistic fuzzy AHP	Proposed novel family of t-norm and t-conorm	Weighted Heronian mean, weighted averaging and geometric aggregation operators	Selection of a dominant farming strategy for small-scale farmers

control over the aggregation process. To further strengthen prioritization effects, Petchimuthu *et al.* [43] proposed generalized power prioritized Yager aggregation operators under q-ROFSs and demonstrated their applicability in sustainable urban development decision problems. Extending interaction modeling to more complex environments, Hussain *et al.* [29] developed Muirhead mean based aggregation operators under complex q-ROFSs. Moreover, Ijaz *et al.* [30] addressed the dynamic nature of real world decision problems by proposing aggregation operators that explicitly account for time varying information. More recent works have incorporated advanced operator families to further enhance aggregation performance. Debnath *et al.* [17] proposed an Aczél Alsina generalized weighted Bonferroni mean combined with the MABAC method under q-ROFSs to model attribute interrelationships more effectively. Similarly, Giri *et al.* [25] developed Frank-operations-based power aggregation operators for (p, q)-quasirung orthopair fuzzy sets, offering enhanced parametric flexibility. Furthermore, Debnath *et al.* [18] introduced a Dice Jaccard similarity based decision model under q-ROFSs to improve the accuracy of similarity measurement in MCDM problems.

Table 3 provides a brief comparison of selected studies that apply different fuzzy frameworks, operational rules, and aggregation operators to solve MCDM. These approaches successfully model membership and non-membership information, but they usually treat evaluations as point-based values. As a result, they do not represent circular or geometric uncertainty, which often arises in MCDM problems.

After the introduction of CIFSs, several studies focused on developing supporting tools and applications. Atanassov and Marinov [6] proposed distance measures for CIFSs. Later, Kahraman and Otay [32] and Alkan and Kahraman [3] combined CIFSs with classical MCDM methods such as VIKOR and TOPSIS, showing that CIFSs are effective in decision problems involving uncertainty and hesitation. Khan *et al.* [35] further extended CIFS based models by introducing divergence measures and applying them to medical diagnosis and pattern recognition problems. In recent years, aggregation operators for CIFSs have also received growing attention. Garg *et al.* [23] developed an Archimedean t-norm-based aggregation framework and integrated it with the

EDAS method. Rukhsar *et al.* [46] investigated power aggregation operators under the CIFS environment, while Fahmi *et al.* [22] proposed Hamacher t-norm based aggregation operators to enhance decision accuracy. In addition, several hybrid CIFS based decision models that integrate methods such as AHP, VIKOR and EDAS have been proposed in the literature [13, 14, 41].

3. Motivation of the study

Aggregation operators and t-norms form the basic mathematical tools of fuzzy decision making models. Over the years, many aggregation techniques have been developed to improve flexibility, reliability and interaction modeling under different fuzzy environments. Despite these developments, there are several important limitations to contemplate. These limitations provide the main motivation for the present study and are summarized below.

(1) **Limited aggregation capability of classical Archimedean t-norms**
Most classical Archimedean t-norms are constructed using logarithmic or power-type additive generators. Popular families such as Hamacher, Frank, Yager, Dombi and Aczél-Alsina introduced parametric flexibility by adjusting the aggregation strength [37]. However, these generators operate through point-wise transformations of membership values. For a given input x, the corresponding additive generator evaluates the aggregation behavior only at that fixed value, without incorporating information from the entire interval $[0, x]$. As a consequence, classical Archimedean t-norms exhibit a pointwise aggregation behavior, where the final result depends solely on the current input values.

(2) **Point-based uncertainty representation in fuzzy models**
These approaches effectively model membership and non-membership information and have been widely applied in MCDM problems as presented in Table 3. Consequently, such models are unable to capture circular or geometric uncertainty, which frequently arises in MCDM problems [7, 17, 18, 25, 29, 30, 43].

(3) **Limitations of existing CIFS based MCDM models**
Most existing CIFS based decision making models adopt aggregation mechanisms originally developed for classical fuzzy or intuitionistic fuzzy environments. Standard averaging operators, algebraic norms, Hamacher norms, and power aggregation operators are often directly extended to CIFSs considering their circular geometry. Consequently, basic CIFS properties like the radius closure property are not always maintained [22, 46].

- **Non-closure of traditional t-norms and t-conorms:** Classical operators such as Hamacher, Einstein, Dombi and Łukasiewicz t-norms do not inherently preserve the admissible radius range, $\gamma \in [0, \sqrt{2}]$. For instance, if two CIFSs satisfy $\gamma_1 = \gamma_2 = 1$, applying a traditional t-norm may yield a resulting radius value $\gamma = 2$, which does not belong to $[0, \sqrt{2}]$. This violates the CIFS structure. Hence, the aggregated result is no longer a valid CIFS.
- **Limitations in representing CIFS decision structures:** Certain CIFS aggregation operators, including those of Guo *et al.* [26], do not consistently satisfy essential properties like idempotency and closure of the uncertainty radius. This limitation may affect the reliability of CIFSs and restrict their applicability in practical decision making problems.
- **Lack of Parameterization and Flexibility:** Many existing aggregation methods are based on fixed arithmetic or geometric averaging operators (using algebraic t-norms) [23, 46].This lacks the adjustable parameters for controlling the aggregation process.

3.1. Contributions

The major contributions and novelties of this study are summarized as follows:

- **Riemann Liouville fractional integral based t-norm:**
This study constructs a new family of additive generators using the Riemann Liouville fractional integral. Unlike traditional generators that operate at a single point, the proposed generator aggregates information over the entire interval $[0, x]$.

- **Fractional Archimedean t-norms and t-conorms with parametric control:**
  This study introduces a new family of Archimedean t-norms and t-conorms based on the Riemann Liouville fractional integral. A Beta-type kernel is used to allow flexible control of the aggregation process, so that membership and non-membership information can be combined more smoothly.
- **CIFS aggregation framework preserving geometric interpretation and closure properties:**
  The proposed family of fractional t-norms and t-conorms is designed to be fully compatible with the circular structure of CIFSs. The operators are constructed such that the aggregation of the uncertain radius always satisfies the CIFS closure condition. As a result, membership, non-membership and radius information are aggregated consistently, without distorting the geometric interpretation of CIFSs.
- **Development of advanced CIFS aggregation operators:**
  Based on the proposed t-norm, several new CIFS aggregation operators are constructed, including:
  - Circular Intuitionistic Fuzzy Weighted Averaging Aggregation (CIFWAA).
  - Circular Intuitionistic Fuzzy Weighted Geometric Aggregation (CIFWGA).
  - Heronian mean-based Circular Intuitionistic Fuzzy Weighted Heronian Aggregation (CIFWHA).

  These operators effectively model both linear and nonlinear interactions among multiple attributes while simultaneously preserving membership, non-membership, and circular hesitancy information.
- **Interaction-aware aggregation via Heronian mean under CIFSs:**
  Unlike conventional averaging and geometric aggregation operators, the proposed CIFWHA operator explicitly captures inter-attribute dependencies through the combined additive and multiplicative structure of the Heronian mean. This enables more realistic information in MCDM problems where criteria interactions and mutual influence play a significant role.
- **Extension of fuzzy AHP under the CIFS framework for criteria weight determination:**
  The proposed aggregation framework is integrated with a CIFS based fuzzy AHP approach for determining criteria weights.
- **A novel and discriminative CIFS ranking function:** A new ranking function is introduced to compare CIFSs by jointly considering membership, non-membership and hesitancy radius information.
- **Construction and validation of a robust CIFS based MCDM framework:**
  In this study, a CIFS based MCDM framework is developed by combining the proposed fractional t-norms, aggregation operators and the ranking method. The usefulness of the framework is shown through numerical examples. The comparative and sensitivity analyses indicate that the proposed framework produces stable and meaningful ranking results and performs better than existing CIFS based and fuzzy based aggregation approaches.

In addition, a higher dimensional CIFS based MCDM example is presented to verify that the proposed approach can handle larger decision problems involving multiple alternatives and criteria, as commonly observed in real world agricultural and sustainable related evaluations.

The remainder of this paper is organized as follows. Section 4 introduces the fundamental preliminaries and formal definitions related to CIFSs. Section 5 presents the construction of a novel family of t-norm and t-conorm operators based on the Riemann Liouville fractional integral and their properties. In Section 6, a Heronian mean-based aggregation operator is developed under these new operators. Section 7 presents the proposed CIFWAA and CIFWGA operators and discusses their main properties. Section 8 outlines the proposed MCDM framework that integrates these operators into a decision making methodology. Section 9 demonstrates the applicability of the proposed approach through a farming strategy selection. A higher-dimensional MCDM example is also included in this section. Section 10 provides a comparative analysis between the proposed operators and the existing aggregation techniques. Finally, Section 11 concludes the paper and suggests directions for future research.

4. Preliminaries

This section provides a brief review of the fundamental ideas behind CIFSs, Archimedean t-norms and t-conorms.

Definition 4.1 ([5])**.** Let X be a universal set. A CIFS $\tilde{A}$ of radius γ over X is defined as:

$$\tilde{A} = \{(x, \mu(x), \nu(x), \gamma(x)) \mid x \in X\}.$$

where:

- $\mu(x) : X \to [0,1]$ is the membership function,
- $\nu(x) : X \to [0,1]$ is the non-membership function,
- $\gamma(x) : X \to [0, \sqrt{2}]$ is the radius of uncertainty.

These values satisfy the condition $\mu(x) + \nu(x) \leq 1, \quad \forall x \in X$.

The degree of hesitation is given by $\pi(x) = 1 - (\mu(x) + \nu(x))$.

Each element $(\mu(x), \nu(x), \gamma(x))$ represents a CIFS, which geometrically corresponds to a circle centered at $(\mu(x), \nu(x))$ with radius $\gamma(x) \in [0, \sqrt{2}]$.

Definition 4.2 ([5])**.** Let $\tilde{a} = (a, b, \gamma)$ and $\tilde{a}' = (a', b', \gamma')$ be two CIFS elements, where a and a' denote the membership degrees, b and b' denote the non-membership degrees and γ and γ' denote the radii of uncertainty, respectively. Then the following relations are satisfied:

- $\tilde{a} \leq \tilde{a}'$ if and only if $\quad a \leq a', \quad b \geq b', \quad \gamma \leq \gamma'$.
- $\tilde{a} \geq \tilde{a}'$ if and only if $\quad a \geq a', \quad b \leq b', \quad \gamma \geq \gamma'$.
- $\tilde{a} = \tilde{a}'$ if and only if $\quad a = a', \quad b = b', \quad \gamma = \gamma'$.

Definition 4.3 (Archimedean t-norm and t-conorm [37])**.** A function $T : [0,1]^2 \to [0,1]$ is a t-norm if it satisfies:

(1) $T(x, 1) = x$, $T(x, 0) = 0$
(2) $T(x, y) = T(y, x)$
(3) $T(x, T(y, z)) = T(T(x, y), z)$
(4) If $x \leq x'$, $y \leq y'$ then $T(x, y) \leq T(x', y')$.

A function $S : [0,1]^2 \to [0,1]$ is a t-conorm if:

(1) $S(x, 0) = x$, $S(x, 1) = 1$
(2) $S(x, y) = S(y, x)$
(3) $S(x, S(y, z)) = S(S(x, y), z)$
(4) If $x \leq x'$, $y \leq y'$ then $S(x, y) \leq S(x', y')$.

They are called Archimedean if they are continuous and satisfies the following:

$$T(x, x) < x \quad \text{and} \quad S(x, x) > x \quad \text{for all } x \in (0, 1).$$

For a t-norm T, the *additive generator* is a strictly decreasing function $g\colon [0,1] \to [0, \infty]$ such that $g(1) = 0$. Dombi [21] proposed an additive generator, leading to the expression $T(x, y) = g^{-1}(g(x) + g(y))$. Similarly, the additive generator of the dual t-conorm S is defined using the function $h(t) = g(1 - t)$, with inverse $h^{-1}(x) = 1 - g^{-1}(x)$. Thus, the corresponding t-conorm is given by $S(x, y) = h^{-1}(h(x) + h(y))$. These generator functions g and h provide a unified framework for constructing both t-norms and their dual t-conorms.

In Table 4, several existing Archimedean t-norms and their corresponding t-conorms [37] are presented to illustrate the underlying relationships. The construction of the proposed family of t-norms is fundamentally grounded in the concept of an additive generator, which provides a systematic approach for defining t-norms through strictly decreasing functions, as formalized below.

TABLE 4. Existing Archimedean t-norms and t-conorms.

Operator	Type	Formula	Generator
Algebraic	t-norm	$T(x,y)=xy$	$g(x)=-\ln x$
	t-conorm	$S(x,y)=x+y-xy$	$h(x)=-\ln(1-x)$
Einstein	t-norm	$T(x,y)=\dfrac{xy}{1+(1-x)(1-y)}$	$g(x)=\ln\dfrac{2-x}{x}$
	t-conorm	$S(x,y)=\dfrac{x+y}{1+xy}$	$h(x)=\ln\dfrac{1+x}{1-x}$
Hamacher	t-norm	$T(x,y)=\dfrac{xy}{\gamma+(1-\gamma)(x+y-xy)}$	$\phi(x)=\ln\dfrac{\gamma+(1-\gamma)x}{x}$
	t-conorm	$S(x,y)=1-\dfrac{(1-x)(1-y)}{\gamma+(1-\gamma)(1-xy)}$	$h(x)=\ln\dfrac{\gamma+(1-\gamma)(1-x)}{1-x}$
Frank	t-norm	$T(x,y)=\log_\gamma\left(1+\dfrac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1}\right)$	$g(x)=-\ln\dfrac{\gamma-1}{\gamma^x-1}$
	t-conorm	$S(x,y)=\log_\gamma\left(1+\dfrac{(\gamma^{(1-x)}-1)(\gamma^{(1-y)}-1)}{\gamma-1}\right)$	$h(x)=-\ln\dfrac{\gamma-1}{\gamma^{(1-x)}-1}$

5. Proposed Archimedean t-norm and t-conorm based on the Riemann Liouville fractional integral

In CIFS environment, uncertain and hesitant information must be combined through suitable aggregation mechanisms. Most existing Archimedean t-norms and t-conorms, including well-known families such as Hamacher, Frank, Yager, Dombi and Aczél-Alsina are constructed using logarithmic or power-based additive generators that operates on point wise approach. In such constructions, the aggregation result depends only on the current input values and does not incorporate information from the entire input interval.

To address this limitation, this study introduces a new family of Archimedean t-norms and t-conorms based on fractional integrals. Specifically, additive generators are constructed using the Riemann Liouville fractional integral. Unlike traditional generators that evaluate aggregation behavior at a single point, the proposed fractional generator uses information from the entire interval $[0,x]$.

Definition 5.1 (Riemann Liouville fractional integral)**.** Let $f:[0,1]\to\mathbb{R}$ be a continuous function. The Riemann Liouville fractional integral of order $\alpha>0$ is defined as

$$I^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,\mathrm{d}t,$$

where the kernel function is chosen as $f(t)=t^k(1-t)^m$, with $k,m>-1$.

The Beta type kernel is selected because it is flexible, well-defined on $[0,1]$ and easy to interpret. The parameter k influences the concentration of information toward higher membership values, whereas m influences the behavior near non-membership values. Together, the parameters (k,m) regulate the contribution of membership and non-membership information during aggregation. By adjusting these parameters, the relative influence of higher and lower membership values can be systematically regulated.

Construction and interpretation of the additive generator

Based on the fractional integral, the proposed additive generator is defined as

$$g_p(x)=\log\left(\frac{I^\alpha f(1)}{I^\alpha f(x)}\right).$$

In classical t-norms, the additive generator $g(x) = -\log x$ evaluates the aggregation behavior at a fixed value x. The proposed generator extends this idea by incorporating fractional integration. Here, $I^\alpha f(x)$ represents the overall information up to level x, while $I^\alpha f(1)$ denotes the maximum possible information. Consequently, the ratio $\frac{I^\alpha f(1)}{I^\alpha f(x)}$ provides a measure of relative uncertainty based on overall information [0,x]. The parameter α uses the information based on intervals. Smaller α emphasizes pointwise information. Larger α emphasizes information from the interval [0,x].

Properties of the proposed generator

The proposed additive generator $g_p(x)$ satisfies all standard requirements of an Archimedean generator:

(1) **Normalization:** $g_p(1) = 0$.
(2) **Monotonicity:** $g_p(x)$ decreases continuously on $(0, 1)$.
(3) **Continuity:** Inherited from the continuity of f and the integral.
(4) **Archimedean property:** $\lim_{x\to 0^+} g_p(x) = +\infty$.

Hence, $g_p(x)$ is a valid Archimedean additive generator.

By applying the inverse form of the proposed Archimedean generator, the associated t-norm can be obtained as follows:

$$T_p(x, y) = g_p^{-1}\big(g_p(x) + g_p(y)\big),$$

Since $g_p(x)$ is continuous and strictly monotone on $(0, 1]$, its inverse g_p^{-1} exists. Using the dual generator $h_p(x) = g_p(1 - x)$, the associated t-conorm is defined as

$$S_p(x, y) = 1 - g_p^{-1}\big(g_p(1 - x) + g_p(1 - y)\big),$$

which maintains the standard duality between aggregation operators.

Table 5 presents the proposed Archimedean t-norm and t-conorm families obtained by choosing specific values of the kernel parameters (k, m) and the fractional order α. For convenience, the proposed operators are grouped into three cases. Case 1 and Case 2 corresponds to new fractional Archimedean t-norms, denoted by $T_p^1(x, y)$ and $T_p^2(x, y)$, along with their associated t-conorms $S_p^1(x, y)$ and $S_p^2(x, y)$. In these cases, the parameters (k, m, α) allow the aggregation behavior to be adjusted in a flexible manner. Case 3 reduces to the classical algebraic t-norm and t-conorm, confirming that the proposed framework generalizes existing operators rather than replacing them. This classification highlights the flexibility, generality, and theoretical coherence of the proposed aggregation framework.

5.1. Properties of the proposed Archimedean t-norm and t-conorm based on the fractional integral

This subsection establishes the Archimedean properties of the proposed t-norm and t-conorm derived using the Riemann Liouville fractional integral.

Theorem 5.1. *Let $k \in (-1, 0]$ and $x, y \in [0, 1]$. Then, the functions defined by*

$$T_p^1(x, y) = \frac{1 - \sqrt{1 - \frac{(1+k)}{1-k} x(2 - (k + 1)x)y(2 - (k + 1)y)}}{k + 1} \quad \text{and}$$

$$S_p^1(x, y) = \frac{k + \sqrt{1 - \frac{(1+k)}{1-k}(1 - x)(2 - (k + 1)(1 - x))(1 - y)(2 - (k + 1)(1 - y))}}{k + 1}$$

presents Archimedean t-norm and t-conorm, respectively.

Proof. **(1) $T_p^1(x, y)$ is a t-norm:** The following properties are proved to show that $T_p^1(x, y)$ is a t-norm:

TABLE 5. Proposed families of Archimedean t-norms derived from kernel $f(t) = t^k(1-t)^m$ and Riemann Liouville integral of order α.

Case	k	m	α	$f(t)$	Generator $g_p(x)$ and $h_p(x)$	t-Norm $T_p(x,y)$ and t-conorm $S_p(x,y)$
1	k	1	α	$t^k(1-t)$	$g_p(x) = \log\left(\frac{1-\frac{k+1}{k+\alpha+1}}{x^{\alpha+k}(1-x\frac{k+1}{k+\alpha+1})}\right)$	Let $\alpha+k=1,\ -1<k\le 0$
					$h_p(x) =$ $\log\left(\frac{1-\frac{k+1}{k+\alpha+1}}{(1-x)^{\alpha+k}(1-(1-x)\frac{k+1}{k+\alpha+1})}\right)$	$T_p^1(x,y) = \frac{1-\sqrt{1-\frac{(1+k)}{1-k}x(2-(k+1)x)y(2-(k+1)y)}}{k+1}$ $S_p^1(x,y) =$ $\frac{k+\sqrt{1-\frac{(1+k)}{1-k}(1-x)(2-(k+1)(1-x))(1-y)(2-(k+1)(1-y))}}{k+1}$
1(a)	0	1	1	$1-t$	$g_p(x) = \log\left(\frac{1}{x(2-x)}\right)$ $h_p(x) = \log\left(\frac{1}{1-x^2}\right)$	$T_p^1(x,y) = 1-\sqrt{1-x(2-x)y(2-y)}$ $S_p^1(x,y) = \sqrt{1-(1-x^2)(1-y^2)}$
1(b)	-0.5	1	1.5	$t^{\frac{-1}{2}}(1-t)$	$g_p(x) = \log\left(\frac{3}{x(4-x)}\right)$ $h_p(x) = \log\left(\frac{3}{(1-x)(3+x)}\right)$	$T_p^1(x,y) = 2-\sqrt{4-\frac{1}{3}x(4-x)y(4-y)}$ $S_p^1(x,y) =$ $-1+\sqrt{4-\frac{1}{3}(1-x)(3+x)(1-y)(3+y)}$
2	0	m	1	$(1-t)^m$	$g_p(x) = \log\left(\frac{1}{1-(1-x)^{m+1}}\right)$ $h_p(x) = \log\left(\frac{1}{1-x^{m+1}}\right)$	$T_p^2(x,y) =$ $1-\left[1-(1-(1-x)^{m+1})(1-(1-y)^{m+1})\right]^{\frac{1}{m+1}}$ $S_p^2(x,y) = \left[1-(1-x^{m+1})(1-y^{m+1})\right]^{\frac{1}{m+1}}$
3	k	0	α	t^k	$g_p(x) = (\alpha+k)\log\left(\frac{1}{x}\right)$ $h_p(x) = (\alpha+k)\log\left(\frac{1}{1-x}\right)$	$T_p^3(x,y) = xy$ $S_p^3(x,y) = x+y-xy$
4	k	m	α	$t^k(1-t)^m$	$g_p(x) = \log\left(\frac{I^\alpha f(1)}{I^\alpha f(x)}\right)$	Parametric family of flexible t-norms

– *Boundary Conditions:*

$$T_p^1(x,1) = \frac{1-\sqrt{1-\frac{(1+k)}{1-k}\cdot x\cdot(2-(k+1)x)\cdot(2-(k+1))}}{k+1} = x,$$

$$T_p^1(x,0) = \frac{1-\sqrt{1-0}}{k+1} = 0.$$

– *Commutativity:* The function is symmetric in x and y, so $T_p^1(x,y) = T_p^1(y,x)$.

– *Associativity:*

$$\begin{aligned} T_p^1(T_p^1(x,y),z) &= \frac{1-\sqrt{1-\frac{(1+k)}{1-k}x(2-(k+1)x)y(2-(k+1)y)x(2-(k+1)x)}}{k+1} \\ &= T_p^1(x,T_p^1(y,z)). \end{aligned}$$

– *Monotonicity:* If $x \le x'$ and $y \le y'$, then the operator T_p^1 satisfies the monotonic property:

$$T_p^1(x,y) \le T_p^1(x',y').$$

(2) $S_p^1(x,y)$ **is a t-conorm:** The following properties are proved to show that $S_p^1(x,y)$ is a t-conorm:

– *Boundary Conditions:*

$$S_p^1(x,0) = \frac{k+\sqrt{1-\frac{(1+k)}{1-k}(1-x)(2-(k+1)(1-x))(2-(k+1))}}{k+1} = x,$$

$$S_p^1(x,1) = \frac{k+\sqrt{1-0}}{k+1} = 1.$$

– *Commutativity:* Clearly, the function is symmetric in x and y, so $S_p^1(x,y) = S_p^1(y,x)$.

– *Associativity:*

$$S_p^1(S_p^1(x,y),z) = \frac{k + \sqrt{\begin{array}{c}1 - \frac{(1+k)}{1-k}(1-x)(2-(k+1)(1-x))(1-y)(2-(k+1)(1-y)) \\ (1-z)(2-(k+1)(1-z))\end{array}}}{k+1}$$
$$= S_p^1(x, S_p^1(y,z)).$$

– *Monotonicity:* If $x \le x'$ and $y \le y'$, then the operator S_p^1 satisfies the monotonic property:

$$S_p^1(x,y) \le S_p^1(x',y').$$

(3) Archimedean property: The Archimedean nature of $T_p^1(x,y)$ and $S_p^1(x,y)$ is established via the following properties:

– For all $x \in (0,1)$,

$$T_p^1(x,x) = \frac{1 - \sqrt{1 - \frac{(1+k)}{1-k}[x(2-(k+1)x)]^2}}{k+1} < x.$$

– Similarly,

$$S_p^1(x,x) = \frac{k + \sqrt{1 - \frac{(1+k)}{1-k}(1-x)^2(2-(k+1)(1-x))^2}}{k+1} > x.$$

□

Theorem 5.2. *Let $m > 0$and $x, y \in [0,1]$. Then, the functions defined by*

$$T_p^2(x,y) = 1 - \left[1 - (1-(1-x)^{m+1})(1-(1-y)^{m+1})\right]^{\frac{1}{m+1}} \quad \text{and}$$
$$S_p^2(x,y) = \left[1 - (1-x^{m+1})(1-y^{m+1})\right]^{\frac{1}{m+1}}$$

presents Archimedean t-norm and t-conorm, respectively.

Proof. **(1)** $T_p(x,y)$ **is a t-norm:** The following properties are proved to show that $T_p^2(x,y)$ is a t-norm:

– *Boundary Conditions:*

$$T_p^2(x,1) = 1 - \left[1 - (1-(1-x)^{m+1})(1-0)\right]^{\frac{1}{m+1}} = x,$$
$$T_p^2(x,0) = 1 - (1-0)^{\frac{1}{m+1}} = 0.$$

– *Commutativity:* Clearly, the function is symmetric in x and y, so $T_p^2(x,y) = T_p^2(y,x)$.

– *Associativity:*

$$T_p^2(x, T_p^2(y,z)) = 1 - \left[1 - (1-(1-x)^{m+1})(1-(1-y)^{m+1})(1-(1-z)^{m+1})\right]^{\frac{1}{m+1}}$$
$$= T_p^2(T_p^2(x,y),z).$$

– *Monotonicity:* If $x \le x'$ and $y \le y'$, then the operator T_p^2 satisfies the monotonic property:

$$T_p^2(x,y) \le T_p^2(x',y').$$

(2) $S_p^2(x,y)$ **is a t-conorm:** The following properties are proved to show that $S_p^2(x,y)$ is a t-conorm:

– *Boundary Conditions:*

$$S_p^2(x,0) = \left[1-(1-x^{m+1})(1-0^{m+1})\right]^{\frac{1}{m+1}} = x,$$
$$S_p^2(x,1) = \left[1-(1-x^{m+1})\cdot 0\right]^{\frac{1}{m+1}} = 1.$$

– *Commutativity:* Clearly, the function is symmetric in x and y, so $S_p^2(x,y) = S_p^2(y,x)$.

– *Associativity:*

$$\begin{aligned} S_p^2(x, S_p^2(y,z)) &= \left[1-(1-x^{m+1})(1-y^{m+1})(1-z^{m+1})\right]^{\frac{1}{m+1}} \\ &= S_p^2(S_p^2(x,y),z). \end{aligned}$$

If $x \le x'$ and $y \le y'$, then the operator S_p^2 satisfies the monotonic property:

$$S_p^2(x,y) \le S_p^2(x',y').$$

(3) Archimedean property: The Archimedean nature of $T_p^1(x,y)$ and $S_p^1(x,y)$ is established via the following properties:

– For $x \in (0,1)$, $T_p^2(x,x) = 1 - \left[1-(1-(1-x)^{m+1})^2\right]^{1/(m+1)} < x$.

– Similarly, $S_p^2(x,x) = \left[1-(1-x^{m+1})^2\right]^{1/(m+1)} > x$.

□

5.2. Extension of families of t-norm and t-conorm for radius with domain $[0, \sqrt{2}]$

The extended t-norm and t-conorm for radius of uncertainty are denoted by $T_p^*(\gamma_1,\gamma_2)$ and $S_p^*(\gamma_1,\gamma_2)$ and they are defined as follows:

$$\begin{aligned} T_p^*(\gamma_1,\gamma_2) &= \sqrt{2}\cdot T_p\left(\frac{\gamma_1}{\sqrt{2}}, \frac{\gamma_2}{\sqrt{2}}\right), \\ S_p^*(\gamma_1,\gamma_2) &= \sqrt{2}\cdot S_p\left(\frac{\gamma_1}{\sqrt{2}}, \frac{\gamma_2}{\sqrt{2}}\right) \qquad \forall \gamma_1,\gamma_2 \in [0,\sqrt{2}]. \end{aligned}$$

This enables the application of T_p^* and S_p^* to the radius component of CIFSs in a consistent Archimedean framework.

5.3. Visualization of the proposed t-norm and t-conorm

Figure 2 illustrates three-dimensional surface plots of the proposed parametric t-norm and t-conorm, namely $T_p^1(x,y)$, $S_p^1(x,y)$, $T_p^2(x,y)$ and $S_p^2(x,y)$, defined over the unit square $[0,1]^2$. These plots provide a clear geometric view of the proposed operators and the changes in aggregation behavior under different parameter settings.

The operators $T_p^1(x,y)$ and $S_p^1(x,y)$ illustrated in subfigures (A) to (F) are influenced by the parameter k. When $k=0$, the aggregation surfaces are smooth and nearly symmetric, showing behavior close to standard Archimedean operators. As k takes negative values (for example, $k=-0.5$ and $k=-0.99$), the t-norm output decreases more quickly when one of the input values (x,y) becomes small. This indicates that smaller input values play a stronger role in the aggregation result. At the same time, the corresponding t-conorm surfaces increase more rapidly for larger input values.

Subfigures (G) to (L) present the aggregation surfaces of $T_p^2(x,y)$ and $S_p^2(x,y)$ for different values of the parameter m. For moderate positive values of m (such as $m=1$ and $m=10$), the surfaces change gradually over the unit square, indicating balanced aggregation behavior. When m takes negative values (for example, $m=-0.9$), the surfaces become steeper, and small changes in the inputs (x,y) lead to larger changes in the output. This shows that the parameter m controls the sensitivity of the aggregation result to variations in the input values.

All the plotted surfaces satisfy the basic properties required for t-norms and t-conorms. In particular, the t-norm values increase when the input values increase, while the t-conorm values behave accordingly. The boundary conditions $T_p(x,1)=x$, $T_p(x,0)=0$, $S_p(x,0)=x$, and $S_p(x,1)=1$ are also clearly observed in all cases, which represents the graphical view of the proposed operators.

5.4. Operational laws of circular intuitionistic fuzzy sets using proposed t-norm and t-conorm

Let $\tilde{a}_1=(a_1,b_1,\gamma_1)$ and $\tilde{a}_2=(a_2,b_2,\gamma_2)$ be two CIFNs, where a_i and b_i denote the membership and non-membership degrees, respectively, and $\gamma_i \in [0,\sqrt{2}]$ represents the corresponding radius of uncertainty. Using the proposed Archimedean t-norm T_p, t-conorm S_p, and their corresponding generators $g_p(x)$ and $h_p(x)$, the arithmetic operations are defined as follows:

$$\begin{aligned}
\tilde{a}_1 \oplus_{\min} \tilde{a}_2 &= \big(S_p(a_1,a_2),\, T_p(b_1,b_2),\, S_p^*(\gamma_1,\gamma_2)\big),\\
\tilde{a}_1 \oplus_{\max} \tilde{a}_2 &= \big(S_p(a_1,a_2),\, T_p(b_1,b_2),\, T_p^*(\gamma_1,\gamma_2)\big),\\
\tilde{a}_1 \otimes_{\min} \tilde{a}_2 &= \big(T_p(a_1,a_2),\, S_p(b_1,b_2),\, T_p^*(\gamma_1,\gamma_2)\big),\\
\tilde{a}_1 \otimes_{\max} \tilde{a}_2 &= \big(T_p(a_1,a_2),\, S_p(b_1,b_2),\, S_p^*(\gamma_1,\gamma_2)\big),\\
\tilde{a}^{\lambda_{\min}} &= \left(g_p^{-1}(\lambda g_p(a)),\, h_p^{-1}(\lambda h_p(b)),\, \sqrt{2}\cdot g_p^{-1}\left(\lambda g_p\left(\tfrac{\gamma}{\sqrt{2}}\right)\right)\right),\\
\tilde{a}^{\lambda_{\max}} &= \left(g_p^{-1}(\lambda g_p(a)),\, h_p^{-1}(\lambda h_p(b)),\, \sqrt{2}\cdot h_p^{-1}\left(\lambda h_p\left(\tfrac{\gamma}{\sqrt{2}}\right)\right)\right),\\
\lambda \odot_{\min} \tilde{a} &= \left(h_p^{-1}(\lambda h_p(a)),\, g_p^{-1}(\lambda g_p(b)),\, \sqrt{2}\cdot h_p^{-1}\left(\lambda h_p\left(\tfrac{\gamma}{\sqrt{2}}\right)\right)\right),\\
\lambda \odot_{\max} \tilde{a} &= \left(h_p^{-1}(\lambda h_p(a)),\, g_p^{-1}(\lambda g_p(b)),\, \sqrt{2}\cdot g_p^{-1}\left(\lambda g_p\left(\tfrac{\gamma}{\sqrt{2}}\right)\right)\right).
\end{aligned}$$

These operations exhibits that the resulting hesitation radius remains within the interval $[0,\sqrt{2}]$, thereby preserving consistency with the structure of CIFSs.

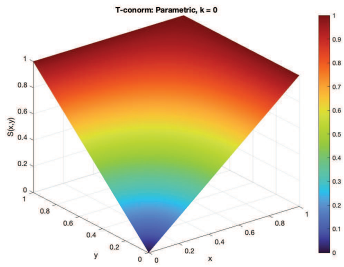
Interpretation of the min- and max-based operational laws

Two variants of the proposed operational laws are considered, namely the min-type and the max-type. These variants represent two different ways of handling the uncertainty radius in CIFS based operations. For the membership and non-membership components, operations follow standard intuitionistic fuzzy principles. Membership values are combined using the proposed t-norm T_p, while non-membership values are combined using the corresponding t-conorm S_p. The uncertainty radius γ is treated separately because it indicates the spread of expert opinions around the central assessment (μ,ν).

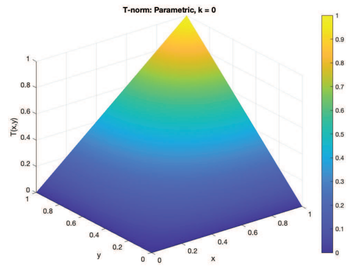
In the min-type operators, the smaller uncertainty radius is selected. Therefore, the overall uncertainty is reduced or remains unchanged compared to the input values. This variant is suitable when expert evaluations are close to each other and a more precise assessment is desired. In this case, same type of operation (either a t-norm or a t-conorm) used for aggregating the membership component is applied to the uncertainty radius. The max-type operators select the larger uncertainty radius. Here, uncertainty is preserved rather than reduced, which is appropriate when expert evaluations differ noticeably. Under this variant, the uncertainty radius is treated in the same manner as the non membership component. Accordingly, when non-membership values are aggregated using a t-conorm, the same operation is applied to the uncertainty radius.

Overall, the choice between the min-type and max-type operators depends on the decision environment. The min-type operator is suitable when a more precise and confident evaluation is desired, whereas the max-type operator is appropriate when uncertainty should be preserved. Therefore, the decision maker can select the operator according to the nature of the problem and their preference toward risk and caution.

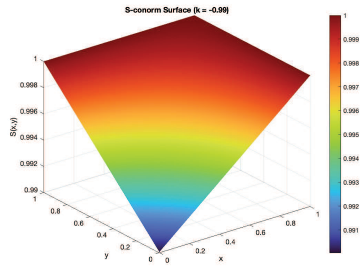
Based on the proposed families of t-norm and t-conorm operators, $T_p^1(x,y), S_p^1(x,y), T_p^2(x,y), S_p^2(x,y)$, the corresponding arithmetic operations are defined as follows:



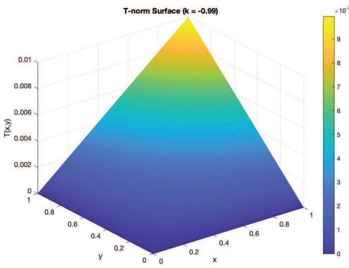
(A) $S_p^1(x, y)$ for $k = 0$



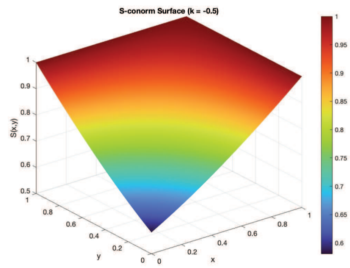
(B) $T_p^1(x, y)$ for $k = 0$



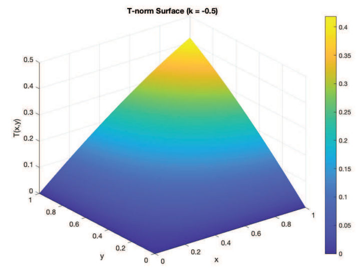
(C) $S_p^1(x, y)$ for $k = -0.99$



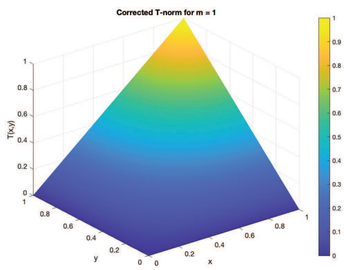
(D) $T_p^1(x, y)$ for $k = -0.99$



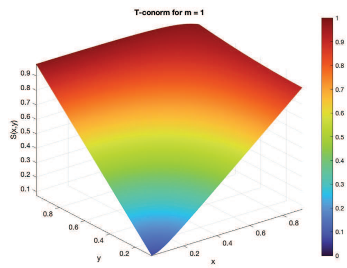
(E) $S_p^1(x, y)$ for $k = -0.5$



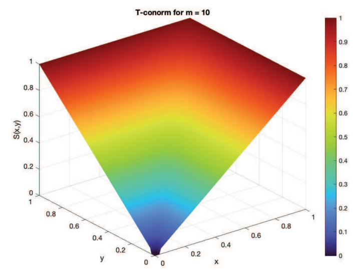
(F) $T_p^1(x, y)$ for $k = -0.5$



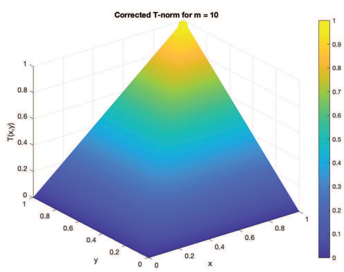
(G) $T_p^2(x, y)$ for $m = 1$



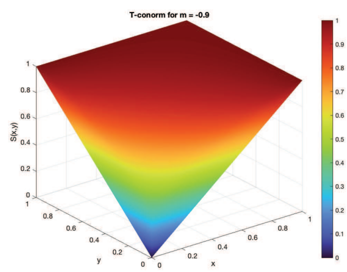
(H) $S_p^2(x, y)$ for $m = 1$



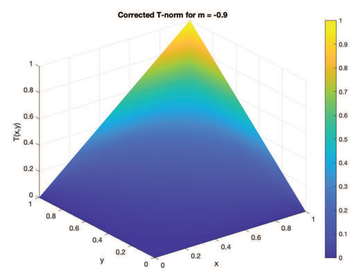
(I) $S_p^2(x, y)$ for $m = 10$



(J) $T_p^2(x, y)$ for $m = 10$



(K) $S_p^2(x, y)$ for $m = -0.9$



(L) $T_p^2(x, y)$ for $m = -0.9$

FIGURE 2. Proposed t-norm and t-conorm for different values of parameters k and m.

Operational Laws of CIFSs Using Proposed $T_p^1(x,y)$ *and* $S_p^1(x,y)$

$$\tilde{a}_1 \oplus_{\min} \tilde{a}_2 = \begin{pmatrix} \dfrac{k+\sqrt{1-\dfrac{(1+k)}{1-k}(1-a_1)(2-(k+1)(1-a_1))(1-a_2)(2-(k+1)(1-a_2))}}{k+1}, \\ \dfrac{1-\sqrt{1-\dfrac{(1+k)}{1-k}b_1(2-(k+1)b_1)b_2(2-(k+1)b_2)}}{k+1}, \\ \sqrt{2}\cdot\dfrac{k+\sqrt{\begin{array}{c}1-\dfrac{(1+k)}{1-k}\left(1-\dfrac{\gamma_1}{\sqrt{2}}\right)\left(2-(k+1)\left(1-\dfrac{\gamma_1}{\sqrt{2}}\right)\right)\\ \left(1-\dfrac{\gamma_2}{\sqrt{2}}\right)\left(2-(k+1)\left(1-\dfrac{\gamma_2}{\sqrt{2}}\right)\right)\end{array}}}{k+1} \end{pmatrix}$$

$$\tilde{a}_1 \oplus_{\max} \tilde{a}_2 = \begin{pmatrix} \dfrac{k+\sqrt{1-\dfrac{(1+k)}{1-k}(1-a_1)(2-(k+1)(1-a_1))(1-a_2)(2-(k+1)(1-a_2))}}{k+1}, \\ \dfrac{1-\sqrt{1-\dfrac{(1+k)}{1-k}b_1(2-(k+1)b_1)b_2(2-(k+1)b_2)}}{k+1}, \\ \sqrt{2}\cdot\dfrac{1-\sqrt{1-\dfrac{(1+k)}{1-k}\dfrac{\gamma_1}{\sqrt{2}}\left(2-(k+1)\dfrac{\gamma_1}{\sqrt{2}}\right)\dfrac{\gamma_2}{\sqrt{2}}\left(2-(k+1)\dfrac{\gamma_2}{\sqrt{2}}\right)}}{k+1} \end{pmatrix}$$

$$\tilde{a}_1 \otimes_{\min} \tilde{a}_2 = \begin{pmatrix} \dfrac{1-\sqrt{1-\dfrac{(1+k)}{1-k}a_1(2-(k+1)a_1)a_2(2-(k+1)a_2)}}{k+1}, \\ \dfrac{k+\sqrt{1-\dfrac{(1+k)}{1-k}(1-b_1)(2-(k+1)(1-b_1))(1-b_2)(2-(k+1)(1-b_2))}}{k+1}, \\ \sqrt{2}\cdot\dfrac{1-\sqrt{1-\dfrac{(1+k)}{1-k}\dfrac{\gamma_1}{\sqrt{2}}\left(2-(k+1)\dfrac{\gamma_1}{\sqrt{2}}\right)\dfrac{\gamma_2}{\sqrt{2}}\left(2-(k+1)\dfrac{\gamma_2}{\sqrt{2}}\right)}}{k+1} \end{pmatrix}$$

$$\tilde{a}_1 \otimes_{\max} \tilde{a}_2 = \begin{pmatrix} \dfrac{1-\sqrt{1-\dfrac{(1+k)}{1-k}a_1(2-(k+1)a_1)a_2(2-(k+1)a_2)}}{k+1}, \\ \dfrac{k+\sqrt{1-\dfrac{(1+k)}{1-k}(1-b_1)(2-(k+1)(1-b_1))(1-b_2)(2-(k+1)(1-b_2))}}{k+1}, \\ \sqrt{2}\cdot\dfrac{k+\sqrt{\begin{array}{c}1-\dfrac{(1+k)}{1-k}\left(1-\dfrac{\gamma_1}{\sqrt{2}}\right)\left(2-(k+1)\left(1-\dfrac{\gamma_1}{\sqrt{2}}\right)\right)\\ \left(1-\dfrac{\gamma_2}{\sqrt{2}}\right)\left(2-(k+1)\left(1-\dfrac{\gamma_2}{\sqrt{2}}\right)\right)\end{array}}}{k+1} \end{pmatrix}$$

$$\tilde{a}^{\lambda_{\min}} = \left(\begin{array}{l} \dfrac{1-\sqrt{1-(1-k^2)\left(\frac{a(2-(k+1)a)}{1-k}\right)^{\lambda}}}{k+1}, \quad \dfrac{k+\sqrt{1-(1-k^2)\left(\frac{(1-b)(2-(k+1)(1-b))}{1-k}\right)^{\lambda}}}{k+1}, \\ \sqrt{2}\dfrac{1-\sqrt{1-(1-k^2)\left(\frac{\frac{\gamma}{\sqrt{2}}\left(2-(k+1)\frac{\gamma}{\sqrt{2}}\right)}{1-k}\right)^{\lambda}}}{k+1} \end{array} \right)$$

$$\tilde{a}^{\lambda_{\max}} = \left(\begin{array}{l} \dfrac{1-\sqrt{1-(1-k^2)\left(\frac{a(2-(k+1)a)}{1-k}\right)^{\lambda}}}{k+1}, \quad \dfrac{k+\sqrt{1-(1-k^2)\left(\frac{(1-b)(2-(k+1)(1-b))}{1-k}\right)^{\lambda}}}{k+1}, \\ \sqrt{2}\dfrac{k+\sqrt{1-(1-k^2)\left(\frac{(1-\frac{\gamma}{\sqrt{2}})\left(2-(k+1)(1-\frac{\gamma}{\sqrt{2}})\right)}{1-k}\right)^{\lambda}}}{k+1} \end{array} \right)$$

$$\lambda \odot_{\min} \tilde{a} = \left(\begin{array}{l} \dfrac{k+\sqrt{1-(1-k^2)\left(\frac{(1-a)(2-(k+1)(1-a))}{1-k}\right)^{\lambda}}}{k+1}, \quad \dfrac{1-\sqrt{1-(1-k^2)\left(\frac{b(2-(k+1)b)}{1-k}\right)^{\lambda}}}{k+1}, \\ \sqrt{2}\dfrac{k+\sqrt{1-(1-k^2)\left(\frac{(1-\frac{\gamma}{\sqrt{2}})(2-(k+1)(1-\frac{\gamma}{\sqrt{2}}))}{1-k}\right)^{\lambda}}}{k+1} \end{array} \right)$$

$$\lambda \odot_{\max} \tilde{a} = \left(\begin{array}{l} \dfrac{k+\sqrt{1-(1-k^2)\left(\frac{(1-a)(2-(k+1)(1-a))}{1-k}\right)^{\lambda}}}{k+1}, \dfrac{1-\sqrt{1-(1-k^2)\left(\frac{b(2-(k+1)b)}{1-k}\right)^{\lambda}}}{k+1}, \\ \sqrt{2}\dfrac{1-\sqrt{1-(1-k^2)\left(\frac{\frac{\gamma}{\sqrt{2}}\left(2-(k+1)\frac{\gamma}{\sqrt{2}}\right)}{1-k}\right)^{\lambda}}}{k+1} \end{array} \right).$$

Operational Laws of CIFSs Using Proposed $T_p^2(x,y)$ and $S_p^2(x,y)$

$$\tilde{a}_1 \oplus_{\min} \tilde{a}_2 = \left(\begin{array}{l} \left[1-(1-a_1^{m+1})(1-a_2^{m+1})\right]^{\frac{1}{m+1}}, \\ 1-\left[1-(1-(1-b_1)^{m+1})(1-(1-b_2)^{m+1})\right]^{\frac{1}{m+1}}, \\ \sqrt{2}\left[1-\left(1-\frac{\gamma_1}{\sqrt{2}}^{m+1}\right)\left(1-\frac{\gamma_2}{\sqrt{2}}^{m+1}\right)\right]^{\frac{1}{m+1}} \end{array} \right)$$

$$\tilde{a}_1 \oplus_{\max} \tilde{a}_2 = \left(\begin{array}{l} \left[1-(1-a_1^{m+1})(1-a_2^{m+1})\right]^{\frac{1}{m+1}}, \\ 1-\left[1-(1-(1-b_1)^{m+1})(1-(1-b_2)^{m+1})\right]^{\frac{1}{m+1}}, \\ \sqrt{2}\left(1-\left[1-\left(1-\left(1-\frac{\gamma_1}{\sqrt{2}}\right)^{m+1}\right)\left(1-\left(1-\frac{\gamma_2}{\sqrt{2}}\right)^{m+1}\right)\right]^{\frac{1}{m+1}}\right) \end{array} \right)$$

$$\tilde{a}_1 \otimes_{\min} \tilde{a}_2 = \begin{pmatrix} 1 - \left[1 - (1 - (1 - a_1)^{m+1})(1 - (1 - a_2)^{m+1})\right]^{\frac{1}{m+1}}, \\ \left[1 - (1 - b_1^{m+1})(1 - b_2^{m+1})\right]^{\frac{1}{m+1}}, \\ \sqrt{2}\left(1 - \left[1 - \left(1 - \left(1 - \frac{\gamma_1}{\sqrt{2}}\right)^{m+1}\right)\left(1 - \left(1 - \frac{\gamma_2}{\sqrt{2}}\right)^{m+1}\right)\right]^{\frac{1}{m+1}}\right) \end{pmatrix}$$

$$\tilde{a}_1 \otimes_{\max} \tilde{a}_2 = \begin{pmatrix} 1 - \left[1 - (1 - (1 - a_1)^{m+1})(1 - (1 - a_2)^{m+1})\right]^{\frac{1}{m+1}}, \\ \left[1 - (1 - b_1^{m+1})(1 - b_2^{m+1})\right]^{\frac{1}{m+1}}, \\ \sqrt{2}\left[1 - \left(1 - \frac{\gamma_1}{\sqrt{2}}^{m+1}\right)\left(1 - \frac{\gamma_2}{\sqrt{2}}^{m+1}\right)\right]^{\frac{1}{m+1}} \end{pmatrix}$$

$$\tilde{a}^{\lambda_{\min}} = \begin{pmatrix} 1 - \left[1 - (1 - (1 - a)^{m+1})^{\lambda}\right]^{\frac{1}{m+1}}, \left[1 - (1 - b^{m+1})^{\lambda}\right]^{\frac{1}{m+1}}, \\ \sqrt{2}\left(1 - \left[1 - (1 - (1 - \frac{\gamma}{\sqrt{2}})^{m+1})^{\lambda}\right]^{\frac{1}{m+1}}\right) \end{pmatrix}$$

$$\tilde{a}^{\lambda_{\max}} = \begin{pmatrix} 1 - \left[1 - (1 - (1 - a)^{m+1})^{\lambda}\right]^{\frac{1}{m+1}}, \left[1 - (1 - b^{m+1})^{\lambda}\right]^{\frac{1}{m+1}}, \\ \sqrt{2}\left[1 - (1 - (\frac{\gamma}{\sqrt{2}})^{m+1})^{\lambda}\right]^{\frac{1}{m+1}} \end{pmatrix}$$

$$\lambda \odot_{\min} \tilde{a} = \begin{pmatrix} \left[1 - (1 - a^{m+1})^{\lambda}\right]^{\frac{1}{m+1}}, 1 - \left[1 - (1 - (1 - b)^{m+1})^{\lambda}\right]^{\frac{1}{m+1}}, \\ \sqrt{2}\left[1 - (1 - (\frac{\gamma}{\sqrt{2}})^{m+1})^{\lambda}\right]^{\frac{1}{m+1}} \end{pmatrix}$$

$$\lambda \odot_{\max} \tilde{a} = \begin{pmatrix} \left[1 - (1 - a^{m+1})^{\lambda}\right]^{\frac{1}{m+1}}, 1 - \left[1 - (1 - (1 - b)^{m+1})^{\lambda}\right]^{\frac{1}{m+1}}, \\ \sqrt{2}\left(1 - \left[1 - (1 - (1 - \frac{\gamma}{\sqrt{2}})^{m+1})^{\lambda}\right]^{\frac{1}{m+1}}\right) \end{pmatrix}.$$

Algebraic properties of CIFS operations: Let $\tilde{a}_1 = (a_1, b_1, \gamma_1)$, $\tilde{a}_2 = (a_2, b_2, \gamma_2)$, and $\tilde{a}_3 = (a_3, b_3, \gamma_3)$ be CIFSs, where $a_1, a_2, a_3, b_1, b_2, b_3 \in [0, 1]$ and $\gamma_1, \gamma_2, \gamma_3 \in [0, \sqrt{2}]$. Let $\lambda, \lambda_1, \lambda_2 > 0$. Then, the following properties hold:

(1) **Commutativity**:

$$\tilde{a}_1 \oplus_{\min} \tilde{a}_2 = \tilde{a}_2 \oplus_{\min} \tilde{a}_1, \quad \tilde{a}_1 \oplus_{\max} \tilde{a}_2 = \tilde{a}_2 \oplus_{\max} \tilde{a}_1,$$
$$\tilde{a}_1 \otimes_{\min} \tilde{a}_2 = \tilde{a}_2 \otimes_{\min} \tilde{a}_1, \quad \tilde{a}_1 \otimes_{\max} \tilde{a}_2 = \tilde{a}_2 \otimes_{\max} \tilde{a}_1.$$

(2) **Distributivity over scalar multiplication**:

$$\lambda(\tilde{a}_1 \oplus_{\min} \tilde{a}_2) = \lambda\tilde{a}_1 \oplus_{\min} \lambda\tilde{a}_2, \quad \lambda(\tilde{a}_1 \oplus_{\max} \tilde{a}_2) = \lambda\tilde{a}_1 \oplus_{\max} \lambda\tilde{a}_2.$$

(3) **Additivity under scalars (Max)**:

$$(\lambda_1 + \lambda_2)(\tilde{a}_1 \oplus_{\max} \tilde{a}_2) = (\lambda_1 + \lambda_2)\tilde{a}_1 \oplus_{\max} (\lambda_1 + \lambda_2)\tilde{a}_2.$$

(4) **Power distributivity over multiplication**:

$$(\tilde{a}_1 \otimes_{\min} \tilde{a}_2)^{\lambda} = \tilde{a}_1^{\lambda} \otimes_{\min} \tilde{a}_2^{\lambda}, \quad (\tilde{a}_1 \otimes_{\max} \tilde{a}_2)^{\lambda} = \tilde{a}_1^{\lambda} \otimes_{\max} \tilde{a}_2^{\lambda}.$$

The above properties follow from the symmetry, monotonicity, and associativity of the proposed aggregation operators T_p, S_p, T_p^*, S_p^*, which are derived using Archimedean t-norms and t-conorms based on fractional integral generators. The consistency of scalar operations and power forms is established using the generator functions $g_p(x)$, $h_p(x)$ and their respective inverses. The results are obtained through component-wise application of the corresponding operational definitions.

6. CIFS Heronian Aggregation operator based on the proposed t-norm and t-conorm

In this section, the CIFWHA operator is introduced based on the operational rules of the proposed Archimedean t-norm and t-conorm for CIFSs. Furthermore, the fundamental properties of this operator are analyzed, and several special cases are examined to illustrate its applicability.

Definition 6.1. Let $\tilde{a}_j = (a_j, b_j, \gamma_j)$ $(j = 1, 2, \ldots, n)$ be a collection of CIFSs, and let $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the corresponding weight vector such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. For given parameters $p, q > 0$, the CIFWHA operator is defined as

$$\mathrm{CIFWHA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\frac{2}{n(n+1)} \odot \left(\bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} [(n\omega_i \odot \tilde{a}_i)^p \otimes (n\omega_j \odot \tilde{a}_j)^q] \right) \right)^{\frac{1}{p+q}},$$

where:

- $\odot$ denotes the scalar multiplication of a CIFS defined via the proposed Archimedean additive generator;
- $\otimes$ and $\oplus$ represent the CIFS multiplication and addition induced by the proposed t-norm T_p and t-conorm S_p, respectively;
- the power operation $(\cdot)^r$ $(r > 0)$ is defined component-wise using the generator-based operational laws.

Now, the Heronian operator for CIFSs based on the proposed t-norm and t-conorm families $T_p^1(x, y), S_p^1(x, y)$, $T_p^2(x, y), S_p^2(x, y)$ is explained as follows:

Heronian operator for CIFSs using proposed t-norm and t-conorm $\boldsymbol{T_p^1(x, y), S_p^1(x, y)}$

Theorem 6.1 (Closure property of the CIFWHA^1 operator)**.** *Let Ω denote the set of all CIFSs. Let $\tilde{a}_i = (a_i, b_i, \gamma_i) \in \Omega, \quad i = 1, 2, \ldots, n$, where $0 \le a_i, b_i \le 1$, $a_i + b_i \le 1$ and $0 < \gamma_i \le \sqrt{2}$. Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be a weight vector such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Let $p, q > 0$. Then, the* CIFWHA^1 *operator satisfies the closure property, i.e.,* CIFWHA$^1(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \in \Omega$.

Proof. To prove that the proposed CIFWHA^1 operator is well-defined on CIFSs, it is first constructed based on the operational laws of CIFSs using the proposed $T_p^1(x, y)$ and $S_p^1(x, y)$. The validity of CIFWHA^1 as a CIFS is then verified through the following steps.

Step 1. Construction of the operator: The CIFWHA^1 operator is formulated using the minimum-based operational laws. Specifically, the aggregation utilizes the proposed $T_p^1(x, y)$ and $S_p^1(x, y)$, as well as their associated operational laws defined under the framework of CIFSs.

$$(n\omega_i \odot_{\min} \tilde{a}_i)^{p_{\min}} = \left(\frac{1-\sqrt{1-(1-k^2)\left[\frac{1-\left(1-k-\sqrt{1-(1-k^2)\left(\frac{(1-a_i)(2-(k+1)(1-a_i))}{1-k}\right)^{n\omega_i}}\right)^2}{1-k^2}\right]^p}}{k+1}, \right.$$

$$\frac{k+\sqrt{1-(1-k^2)\left[\frac{1-\left(1-k-\sqrt{1-(1-k^2)\left(\frac{b_i(2-(k+1)b_i)}{1-k}\right)^{n\omega_i}}\right)^2}{1-k^2}\right]^p}}{k+1},$$

$$\left. \frac{1-\sqrt{1-(1-k^2)\left[\frac{1-\left(1-k-\sqrt{1-(1-k^2)\left(\frac{\left(1-\frac{\gamma_i}{\sqrt{2}}\right)\left(2-(k+1)\left(1-\frac{\gamma_i}{\sqrt{2}}\right)\right)}{1-k}\right)^{n\omega_i}}\right)^2}{1-k^2}\right]^p}}{k+1} \right).$$

Let

$$A_i = 1-\left(1-k-\sqrt{1-(1-k^2)\left(\frac{(1-a_i)(2-(k+1)(1-a_i))}{1-k}\right)^{n\omega_i}}\right)^2. \tag{1}$$

$$B_i = 1-\left(1-k-\sqrt{1-(1-k^2)\left(\frac{b_i(2-(k+1)b_i)}{1-k}\right)^{n\omega_i}}\right)^2. \tag{2}$$

$$\gamma_i = 1-\left(1-k-\sqrt{1-(1-k^2)\left(\frac{\left(1-\frac{\gamma_i}{\sqrt{2}}\right)\left(2-(k+1)\left(1-\frac{\gamma_i}{\sqrt{2}}\right)\right)}{1-k}\right)^{n\omega_i}}\right)^2. \tag{3}$$

Step 2: Pairwise aggregation: The pairwise aggregation using the $\otimes_{\min}$ operator is:

$$(n\omega_i \odot_{\min} \tilde{a}_i)^{p_{\min}} \otimes_{\min} (n\omega_j \odot_{\min} \tilde{a}_j)^{q_{\min}} = \left(\frac{1-\sqrt{1-(1-k^2)\left[\left(\frac{A_i}{1-k^2}\right)^p\left(\frac{A_j}{1-k^2}\right)^q\right]}}{k+1}, \right.$$

$$\frac{k+\sqrt{1-(1-k^2)\left[\left(\frac{B_i}{1-k^2}\right)^p\left(\frac{B_j}{1-k^2}\right)^q\right]}}{k+1},$$

$$\sqrt{2}\frac{1-\sqrt{1-(1-k^2)\left[\left(\frac{\gamma_i}{1-k^2}\right)^p\left(\frac{R_j}{1-k^2}\right)^q\right]}}{k+1}\Bigg).$$

Step 3: Summation over all pairs: Applying the summation over all pairs (i,j) where $i \leq j$:

$$\oplus_{i=1}^{n} \oplus_{j=i}^{n}\big((n\omega_i \odot_{\min} \tilde{a}_i)^{p_{\min}} \otimes_{\min} (n\omega_j \odot_{\min} \tilde{a}_j)^{q_{\min}}\big) =$$

$$\left(\frac{k+\sqrt{1-\frac{1}{(1-k^2)^{\frac{n(n+1)}{2}-1}}\prod_{i=1}^{n}\prod_{j=i}^{n}\left[1-\left(1-k-\sqrt{1-(1-k^2)\Big(\Big(\frac{A_i}{1-k^2}\Big)^p\Big(\frac{A_j}{1-k^2}\Big)^q\Big)}\right)^2\right]}}{k+1},\right.$$

$$\frac{1-\sqrt{1-\frac{1}{(1-k^2)^{\frac{n(n+1)}{2}-1}}\prod_{i=1}^{n}\prod_{j=i}^{n}\left[1-\left(1-k-\sqrt{1-(1-k^2)\Big(\Big(\frac{B_i}{1-k^2}\Big)^p\Big(\frac{B_j}{1-k^2}\Big)^q\Big)}\right)^2\right]}}{k+1},$$

$$\left.\sqrt{2}\,\frac{k+\sqrt{1-\frac{1}{(1-k^2)^{\frac{n(n+1)}{2}-1}}\prod_{i=1}^{n}\prod_{j=i}^{n}\left[1-\left(1-k-\sqrt{1-(1-k^2)\Big(\Big(\frac{\gamma_i}{1-k^2}\Big)^p\Big(\frac{R_j}{1-k^2}\Big)^q\Big)}\right)^2\right]}}{k+1}\right).$$

Let

$$A_{ij} = \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-k-\sqrt{1-(1-k^2)\left[\left(\frac{A_i}{1-k^2}\right)^p\left(\frac{A_j}{1-k^2}\right)^q\right]}\right)^2\right) \tag{4}$$

$$B_{ij} = \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-k-\sqrt{1-(1-k^2)\left[\left(\frac{B_i}{1-k^2}\right)^p\left(\frac{B_j}{1-k^2}\right)^q\right]}\right)^2\right) \tag{5}$$

$$R_{ij} = \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-k-\sqrt{1-(1-k^2)\left[\left(\frac{R_i}{1-k^2}\right)^p\left(\frac{R_j}{1-k^2}\right)^q\right]}\right)^2\right). \tag{6}$$

Step 4: Normalization: The normalized aggregation is:

$$\frac{2}{n(n+1)} \odot_{\min} \left(\oplus_{i=1}^{n} \oplus_{j=i}^{n} (n\omega_i \odot_{\min} \tilde{a}_i)^{p_{\min}} \otimes_{\min} (n\omega_j \odot_{\min} \tilde{a}_j)^{q_{\min}}\right) =$$

$$\left(\frac{k+\sqrt{1-(1-k^2)\left(\frac{A_{ij}}{(1-k^2)^{\frac{n(n+1)}{2}}}\right)^{\frac{2}{n(n+1)}}}}{k+1}, \right.$$

$$\frac{1-\sqrt{1-(1-k^2)\left(\frac{B_{ij}}{(1-k^2)^{\frac{n(n+1)}{2}}}\right)^{\frac{2}{n(n+1)}}}}{k+1},$$

$$\left. \frac{k+\sqrt{1-(1-k^2)\left(\frac{R_{ij}}{(1-k^2)^{\frac{n(n+1)}{2}}}\right)^{\frac{2}{n(n+1)}}}}{k+1} \right).$$

Final CIFWHA^1 expression

$$\mathrm{CIFWHA}^{1}_{\min}(\tilde{a}_1, \ldots, \tilde{a}_n) = \left(\begin{array}{c} \dfrac{1-\sqrt{1-(1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(A_{ij})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1}, \\ \dfrac{k+\sqrt{1-(1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(B_{ij})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1}, \\ \sqrt{2}\,\dfrac{1-\sqrt{1-(1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(R_{ij})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1} \end{array} \right).$$

Similarly, using the maximum-based operational laws, the operator is defined as follows.

$$\mathrm{CIFWHA}^1_{\max}(\tilde{a}_1,\ldots,\tilde{a}_n)=\left(\begin{array}{c}\dfrac{1-\sqrt{1-(1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(A_{ij})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1},\\ \dfrac{k+\sqrt{1-(1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(B_{ij})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1},\\ \sqrt{2}\dfrac{1-\sqrt{1-(1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(R_{ij\max})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1}\end{array}\right)$$

where

$$R_{ij\max}=\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-k-\sqrt{1-(1-k^2)\left[\left(\frac{R_i}{1-k^2}\right)^p\left(\frac{R_j}{1-k^2}\right)^q\right]}\right)^2\right),$$

$$R_{i\max}=1-\left(1-k-\sqrt{1-(1-k^2)\left(\frac{\left(\frac{\gamma_i}{\sqrt{2}}\right)\left(2-(k+1)\frac{\gamma_i}{\sqrt{2}}\right)}{1-k}\right)^{n\omega_i}}\right)^2.$$

Step 5: Verification of CIFS conditions: To verify that the output of the proposed CIFWHA^1 operator is a valid CIFS, assume that $\mathrm{CIFWHA}^1_{\min}(\tilde{a}_1,\ldots,\tilde{a}_n)=(\alpha,\beta,\delta)$. where α,β and δ, denote the membership degree, non-membership degree, and radius of uncertainty, respectively, produced by the proposed $\mathrm{CIFWHA}^1_{\min}(\tilde{a}_1,\ldots,\tilde{a}_n)$. The tuple (α,β,δ) must satisfy the properties of a CIFS. It is explained as follows:

- **Boundedness and intuitionistic condition:** It must satisfy the conditions: $0\le\alpha,\beta\le 1$, $0\le\delta<\sqrt{2}$, and $\alpha+\beta\le 1$.

(i) Membership value α For each $a_i\in[0,1]$:

$$A_i=1-\left(\sqrt{1-(1-k^2)\left(\frac{a_i(2-(k+1)a_i)}{1-k}\right)^{n\omega_i}}-(1-k)\right)^2\in[0,1].$$

Hence,

$$\alpha=\frac{1-\sqrt{1-(1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(A_{ij})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1}\in[0,1].$$

(ii) Non-membership value β For each $b_i \in [0,1]$:

$$B_i = 1 - \left(\sqrt{1 - (1-k^2)\left(\frac{(1-b_i)(2-(k+1)(1-b_i))}{1-k}\right)^{n\omega_i}} - (1-k) \right)^2 \in [0,1].$$

Hence,

$$\beta = \frac{k + \sqrt{1 - (1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(B_{ij})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1} \in [0,1].$$

(iii) Hesitation radius δ For each $\gamma_i \in [0,\sqrt{2}]$, define:

$$R_i = 1 - \left(\sqrt{1 - (1-k^2)\left(\frac{\frac{\gamma_i}{\sqrt{2}}\left(2-(k+1)\frac{\gamma_i}{\sqrt{2}}\right)}{1-k}\right)^{n\omega_i}} - (1-k) \right)^2 \in [0,1].$$

Then,

$$\delta = \sqrt{2} \cdot \frac{1 - \sqrt{1 - (1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(R_{ij})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1} \in [0,\sqrt{2}].$$

(iv) Intuitionistic condition $\alpha + \beta \leq 1$ This condition holds because:

- The transformation preserves the bounds within $[0,1]$.
- The proposed aggregation operators are monotonic.
- The initial CIFS condition $a_i + b_i \leq 1$ is preserved through the aggregation process.

Therefore, $\alpha + \beta \leq 1$.

Hence, the output $\mathrm{CIFWHA}^1_{\min}(\tilde{a}_1,\ldots,\tilde{a}_n) = (\alpha,\beta,\delta)$ satisfies all conditions of a valid CIFS. Similarly, the result of $\mathrm{CIFWHA}^1_{\max}(\tilde{a}_1,\ldots,\tilde{a}_n)$ also satisfies the conditions for a CIFS, proving $\mathrm{CIFWHA}^1(\tilde{a}_1,\ldots,\tilde{a}_n) \in \Omega$. □

Theorem 6.2 (Idempotency). *Let $\tilde{a} = (a,b,\gamma)$ be a CIFS, where $0 \leq a,b \leq 1$, $a+b \leq 1$ and $0 \leq \gamma \leq \sqrt{2}$. Then, for any integer $n \geq 1$ and for all $p,q > 0$, the proposed CIFWHA operators satisfy:*

$$\mathrm{CIFWHA}^1_{\min}(\underbrace{\tilde{a},\ldots,\tilde{a}}_{n \text{ times}}) = \tilde{a}, \quad \mathrm{CIFWHA}^1_{\max}(\underbrace{\tilde{a},\ldots,\tilde{a}}_{n \text{ times}}) = \tilde{a}.$$

Proof. Assume all inputs are identical: $\tilde{a}_i = (a,b,\gamma)$ for each $i = 1,\ldots,n$, with weights $\omega_i = \frac{1}{n}$, so that $n\omega_i = 1$. The intermediate values of $CIFWHA^1_{\min}$ simplify to:

$$A_i = (k+1)a(2-(k+1)a), \quad B_i = (k+1)(1-b)(2-(k+1)(1-b)),$$
$$R_i = (k+1)\left(\frac{\gamma}{\sqrt{2}}\right)\left(2-(k+1)\frac{\gamma}{\sqrt{2}}\right).$$

Therefore,

$$A_{ij} = \left(1 - \left(1 - k - \sqrt{1 - (1-k^2)\left[\left(\frac{A_i}{1-k^2}\right)^{p+q}\right]}\right)^2\right)^{\frac{n(n+1)}{2}}$$

$$B_{ij} = \left(1 - \left(1 - k - \sqrt{1 - (1-k^2)\left[\left(\frac{B_i}{1-k^2}\right)^{p+q}\right]}\right)^2\right)^{\frac{n(n+1)}{2}}$$

$$R_{ij} = \left(1 - \left(1 - k - \sqrt{1 - (1-k^2)\left[\left(\frac{R_i}{1-k^2}\right)^{p+q}\right]}\right)^2\right)^{\frac{n(n+1)}{2}}.$$

Substituting the values of A_{ij}, B_{ij}, R_{ij} into the operator $\mathrm{CIFWHA}^1_{\min}(\tilde{a}_1, \ldots, \tilde{a}_n) = (\alpha, \beta, \delta)$:

$$\alpha = \frac{1 - \sqrt{1 - (1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(A_{ij})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1} = a,$$

$$\beta = \frac{k + \sqrt{1 - (1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(B_{ij})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1} = b,$$

$$\delta = \sqrt{2} \cdot \frac{1 - \sqrt{1 - (1-k^2)\left\{\frac{1-\left(1-k-\sqrt{1-(R_{ij})^{\frac{2}{n(n+1)}}}\right)^2}{1-k^2}\right\}^{\frac{1}{p+q}}}}{k+1} = \gamma.$$

Hence, the result of applying the CIFWHA operator to n identical CIFSs returns the original input:

$$\mathrm{CIFWHA}^1_{\min}(\tilde{a}, \ldots, \tilde{a}) = \tilde{a}.$$

In the same way $\mathrm{CIFWHA}^1_{\max}(\tilde{a}, \ldots, \tilde{a}) = \tilde{a}$. □

Theorem 6.3 (Monotonicity). *Let $\tilde{a}_i = (a_i, b_i, \gamma_i)$ and $\tilde{a}'_i = (a'_i, b'_i, \gamma'_i)$ be two collections of CIFS elements such that $\tilde{a}_i \leq \tilde{a}'_i$ for all $i = 1, 2, \ldots, n$, i.e., $a_i \leq a'_i$, $b_i \geq b'_i$, $\gamma_i \leq \gamma'_i$. Then the proposed operator $\mathrm{CIFWHA}^1_{\min}$ satisfies $\mathrm{CIFWHA}^1_{\min}(\tilde{a}_1, \ldots, \tilde{a}_n) \leq \mathrm{CIFWHA}^1_{\min}(\tilde{a}'_1, \ldots, \tilde{a}'_n)$, i.e., $\alpha \leq \alpha'$, $\beta \geq \beta'$, $\delta \leq \delta'$. Where $\mathrm{CIFWHA}^1_{\min}(\tilde{a}_1, \ldots, \tilde{a}_n) = (\alpha, \beta, \delta)$ and $\mathrm{CIFWHA}^1_{\min}(\tilde{a}'_1, \ldots, \tilde{a}'_n) = (\alpha', \beta', \delta')$.*

Proof. Using equations (1)–(3), it follows that A_i and γ_i are strictly increasing with respect to a_i and γ_i, respectively, whereas B_i is strictly decreasing with respect to b_i. That is

$$a_i \leq a'_i \Rightarrow A_i \leq A'_i, \quad b_i \geq b'_i \Rightarrow B_i \leq B'_i, \quad \gamma_i \leq \gamma'_i \Rightarrow \gamma_i \leq \gamma'_i.$$

It is observed from equations (3)–(4) that A_{ij}, B_{ij} and R_{ij} are decreasing functions of A_i, B_i and γ_i, respectively. therefore,

$$A'_{ij} \leq A_{ij}, B'_{ij} \leq B_{ij}, R'_{ij} \leq R_{ij}.$$

Thus the final outputs (α, β, δ) are strictly decreasing with respect to A_{ij} and R_{ij}, and increasing with respect to B_{ij}. It follows that: $\alpha \leq \alpha', \quad \beta \geq \beta', \quad \delta \leq \delta'$.

Hence, the operator $\mathrm{CIFWHA}^1_{\min}$ preserves monotonicity with respect to the partial order defined on CIFS tuples. Similarly, it can be shown that the operator $\mathrm{CIFWHA}^1_{\max}$ preserves monotonicity under the same ordering. □

Theorem 6.4 (Boundedness). *Let $\tilde{a}_i = (a_i, b_i, \gamma_i)$ be CIFSs, where $0 \leq a_i, b_i \leq 1$, $a_i + b_i \leq 1$, and $0 \leq \gamma_i \leq \sqrt{2}$. Then, the output of the operator $\mathrm{CIFWHA}^1_{\min}(\tilde{a}_1, \ldots, \tilde{a}_n) = (\alpha, \beta, \delta)$ satisfies the conditions: $0 \leq \alpha, \beta \leq 1, \quad \alpha + \beta \leq 1, \quad 0 \leq \delta \leq \sqrt{2}$.*

Proof. **Preservation of bounds.** The construction of the operator $\mathrm{CIFWHA}^1_{\min}$ is based on the proposed $T^1_p(x, y)$ and $S^1_p(x, y)$ functions, which are bounded within $[0, 1]$ for the membership and non-membership components and within $[0, \sqrt{2}]$ for the hesitation radius. Therefore, $0 \leq \alpha, \beta \leq 1, 0 \leq \delta \leq \sqrt{2}$ and $\alpha + \beta \leq 1$. Thus, the output (α, β, δ) satisfies all the conditions of a CIFS. In the same way, it can be shown that the output of $\mathrm{CIFWHA}^1_{\max}$ also satisfies these bounds and is a valid CIFS. □

Heronian operator for CIFSs by proposed t-norm and t-conorm $T^2_p(x, y), S^2_p(x, y)$

Theorem 6.5 (Closure property of the CIFWHA^2 operator). *Let Ω be the class of CIFSs. Let $\tilde{a}_i = (a_i, b_i, \gamma_i) \in \Omega$ for $i = 1, 2, \ldots, n$, where $0 \leq a_i, b_i \leq 1$, $0 < \gamma_i \leq \sqrt{2}$ and $a_i + b_i \leq 1$. Let $p, q > 0$ and weights ω_i such that $\sum_{i=1}^{n} \omega_i = 1$. Then, the output of the proposed CIFWHA operator: $CIFWHA^2(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ is also a CIFS, i.e., it belongs to Ω.*

Proof. To prove that the proposed CIFWHA^2 operator is a CIFS, the operator is first constructed using the operational laws of CIFSs defined by the proposed $T^2_p(x, y)$ and $S^2_p(x, y)$. The validity of the CIFWHA^2 operator as a CIFS is then established through the following steps.

Step 1. Construction of the operator: The CIFWHA operator is formulated using the minimum-based operational laws. Specifically, the aggregation utilizes the proposed t-norm and t-conorm $T^2_p(x, y)$, $S^2_p(x, y)$, and their associated scalar multiplication and power operations defined under the framework of CIFSs.

$$\begin{aligned}(n\omega_i \odot_{\min} \tilde{a}_i)^{p_{\min}} = \Bigg(&\Bigg(1 - \left[1 - \left\{1 - \left(1 - \left[1 - (1 - a_i^{m+1})^{n\omega_i}\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}^p\right]^{\frac{1}{m+1}}\Bigg), \\ &\left[1 - \left\{1 - \left(1 - \left[1 - (1 - (1 - b_i)^{m+1})^{n\omega_i}\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}^p\right]^{\frac{1}{m+1}}, \\ &\sqrt{2}\left(1 - \left[1 - \left\{1 - \left(1 - \left[1 - (1 - (\tfrac{\gamma_i}{\sqrt{2}})^{m+1})^{n\omega_i}\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}^p\right]^{\frac{1}{m+1}}\right)\Bigg).\end{aligned}$$

For simplicity, let

$$X_i = 1 - \left\{1 - \left[1 - (1 - a_i^{m+1})^{n\omega_i}\right]^{\frac{1}{m+1}}\right\}^{m+1} \tag{7}$$

$$Y_i = 1 - \left\{1 - \left[1 - (1 - (1 - b_i)^{m+1})^{n\omega_i}\right]^{\frac{1}{m+1}}\right\}^{m+1} \tag{8}$$

$$Z_i = 1 - \left\{1 - \left[1 - \left(1 - \left(\frac{\gamma_i}{\sqrt{2}}\right)^{m+1}\right)^{n\omega_i}\right]^{\frac{1}{m+1}}\right\}^{m+1}. \tag{9}$$

Step 2: Pairwise aggregation: The pairwise aggregation using the $\otimes_{\min}$ operator is:

$$\begin{aligned}&(n\omega_i \odot_{\min} \tilde{a}_i)^{p_{\min}} \otimes_{\min} (n\omega_j \odot_{\min} \tilde{a}_j)^{q_{\min}}\\ &\qquad = \left(1 - \left[1 - X_i^p X_j^q\right]^{\frac{1}{m+1}}, \left[1 - Y_i^p Y_j^q\right]^{\frac{1}{m+1}}, \sqrt{2}\left(1 - \left[1 - Z_i^p Z_j^q\right]^{\frac{1}{m+1}}\right)\right).\end{aligned}$$

Step 3: Summation over all pairs: Applying the summation over all pairs (i, j) where $i \leq j$:

$$\begin{aligned}&\oplus_{i=1}^{n} \oplus_{j=i}^{n} (n\omega_i \odot_{\min} \tilde{a}_i)^{p_{\min}} \otimes_{\min} (n\omega_j \cdot_{\min} \tilde{a}_j)^{q_{\min}}\\ &= \Bigg(\left[1 - \prod_{i=1}^{n}\prod_{j=i}^{n}\left\{1 - \left(1 - \left[1 - X_i^p X_j^q\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}\right]^{\frac{1}{m+1}},\\ &\quad 1 - \left[1 - \prod_{i=1}^{n}\prod_{j=i}^{n}\left\{1 - \left(1 - \left[1 - Y_i^p Y_j^q\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}\right]^{\frac{1}{m+1}},\\ &\quad \sqrt{2}\left[1 - \prod_{i=1}^{n}\prod_{j=i}^{n}\left\{1 - \left(1 - \left[1 - Z_i^p Z_j^q\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}\right]^{\frac{1}{m+1}}\Bigg).\end{aligned}$$

Let

$$X_{ij} = \prod_{i=1}^{n}\prod_{j=i}^{n}\left\{1 - \left(1 - \left[1 - X_i^p X_j^q\right]^{\frac{1}{m+1}}\right)^{m+1}\right\} \tag{10}$$

$$Y_{ij} = \prod_{i=1}^{n}\prod_{j=i}^{n}\left\{1 - \left(1 - \left[1 - Y_i^p Y_j^q\right]^{\frac{1}{m+1}}\right)^{m+1}\right\} \tag{11}$$

$$Z_{ij} = \prod_{i=1}^{n}\prod_{j=i}^{n}\left\{1 - \left(1 - \left[1 - Z_i^p Z_j^q\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}. \tag{12}$$

Step 4: Normalization: The normalized aggregation is:

$$\begin{aligned}&\frac{2}{n(n+1)} \odot_{\min} \left(\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n} (n\omega_i \odot_{\min} \tilde{a}_i)^{p_{\min}} \otimes_{\min} (n\omega_j \odot_{\min} \tilde{a}_j)^{q_{\min}}\right)\\ &\quad = \left(\left[1 - (X_{ij})^{\frac{2}{n(n+1)}}\right]^{\frac{1}{m+1}}, 1 - \left[1 - (Y_{ij})^{\frac{2}{n(n+1)}}\right]^{\frac{1}{m+1}}, \sqrt{2}\left[1 - (Z_{ij})^{\frac{2}{n(n+1)}}\right]^{\frac{1}{m+1}}\right).\end{aligned}$$

Final CIFWHA^2 expression:

$$\mathrm{CIFWHA}^2_{\min} = \left(1 - \left[1 - \left\{1 - \left(1 - \left[1 - (X_{ij})^{\frac{2}{n(n+1)}}\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}^{\frac{1}{p+q}}\right]^{\frac{1}{m+1}},\right.$$

$$\left[1 - \left\{1 - \left(1 - \left[1 - (Y_{ij})^{\frac{2}{n(n+1)}}\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}^{\frac{1}{p+q}}\right]^{\frac{1}{m+1}},$$

$$\left.\sqrt{2}\left(\left[1 - \left\{1 - \left(1 - \left[1 - (Z_{ij})^{\frac{2}{n(n+1)}}\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}^{\frac{1}{p+q}}\right]^{\frac{1}{m+1}}\right)\right).$$

Similarly, using the maximum-based operational laws, the operator is defined as follows.

$$\mathrm{CIFWHA}^2_{\max} = \left(1 - \left[1 - \left\{1 - \left(1 - \left[1 - (X_{ij})^{\frac{2}{n(n+1)}}\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}^{\frac{1}{p+q}}\right]^{\frac{1}{m+1}},\right.$$

$$\left[1 - \left\{1 - \left(1 - \left[1 - (Y_{ij})^{\frac{2}{n(n+1)}}\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}^{\frac{1}{p+q}}\right]^{\frac{1}{m+1}},$$

$$\left.\sqrt{2}\left[1 - \left\{1 - \left(1 - \left[1 - (Z_{ij\,\max})^{\frac{2}{n(n+1)}}\right]^{\frac{1}{m+1}}\right)^{m+1}\right\}^{\frac{1}{p+q}}\right]^{\frac{1}{m+1}}\right),$$

where

$$Z_{ij\,\max} = \prod_{i=1}^{n}\prod_{j=i}^{n}\left\{1 - \left(1 - \left[1 - Z^p_{i\,\max} Z^q_{j\,\max}\right]^{\frac{1}{m+1}}\right)^{m+1}\right\},$$

$$Z_{i\,\max} = 1 - \left\{1 - \left[1 - \left(1 - \left(1 - \frac{\gamma_i}{\sqrt{2}}\right)^{m+1}\right)^{n\omega_i}\right]^{\frac{1}{m+1}}\right\}^{m+1}.$$

Step 5: Verification of CIFS conditions: To verify that the outputs of the proposed $\mathrm{CIFWHA}^2_{\min}$ and $\mathrm{CIFWHA}^2_{\max}$ operators are CIFSs, assume that $\mathrm{CIFWHA}^2_{\min}(\tilde{a}_1, \ldots, \tilde{a}_n) = (\alpha_1, \beta_1, \delta_1)$. Where α_1, β_1 and δ_1 denote the membership, non-membership and radius of uncertainty, respectively.

Preservation of bounds: The construction of the operator $\mathrm{CIFWHA}^2_{\min}$ is based on the proposed $T^2_p(x, y)$ and $S^2_p(x, y)$ functions, which are bounded within $[0, 1]$ for the membership and non-membership components, and within $[0, \sqrt{2}]$ for the hesitation radius. Therefore, $0 \le \alpha_1, \beta_1 \le 1, \quad 0 \le \delta_1 \le \sqrt{2} \quad \text{and} \quad \alpha_1 + \beta_1 \le 1.$ Thus, the output $(\alpha_1, \beta_1, \delta_1)$ satisfies all the conditions of a CIFS. In the same way, it can be shown that the output of $\mathrm{CIFWHA}^2_{\max}$ also satisfies all the conditions of a valid CIFS. □

Theorem 6.6 (Idempotency). *Let $\tilde{a} = (a, b, \gamma)$ be a CIFS, where $0 \le a, b \le 1$, $a + b \le 1$, and $0 \le \gamma \le \sqrt{2}$. Then, for any integer $n \ge 1$ and any $p, q > 0$, the operator* CIFWHA^2 *satisfies:* $\mathrm{CIFWHA}^2(\underbrace{\tilde{a}, \tilde{a}, \ldots, \tilde{a}}_{n \text{ times}}) = \tilde{a}$.

Proof. Assume $\tilde{a}_i = (a, b, \gamma)$ for all $i = 1, \ldots, n$ and uniform weights $\omega_i = \frac{1}{n} \Rightarrow n\omega_i = 1$. Then,

$$X_i = 1-(1-a)^{m+1}, \quad Y_i = 1-b^{m+1},$$
$$Z_i = 1-\left(1-\frac{\gamma}{\sqrt{2}}\right)^{m+1}, \quad Z_{i\max} = 1-\left(\frac{\gamma}{\sqrt{2}}\right)^{m+1}.$$

Thus,

$$\mathrm{CIFWHA}^2_{\min}(\tilde{a},\ldots,\tilde{a}) = \tilde{a}, \quad \mathrm{CIFWHA}^2_{\max}(\tilde{a},\ldots,\tilde{a}) = \tilde{a}.$$

□

Theorem 6.7 (Monotonicity). *Let $\tilde{a}_i = (a_i, b_i, \gamma_i)$ and $\tilde{a}'_i = (a'_i, b'_i, \gamma'_i)$ be two collections of CIFS elements such that$\tilde{a}_i \le \tilde{a}_i{}'$ for all $i = 1,\ldots,n$, i.e., $a_i \le a'_i$, $\quad b_i \ge b'_i$, $\quad \gamma_i \le \gamma'_i$. Then, the proposed operator $\mathrm{CIFWHA}^2_{\min}$ satisfies $\mathrm{CIFWHA}^2_{\min}(\tilde{a}_1,\ldots,\tilde{a}_n) \le \mathrm{CIFWHA}^2_{\min}(\tilde{a}'_1,\ldots,\tilde{a}'_n)$, i.e., $\alpha \le \alpha'$, $\beta \ge \beta'$, $\delta \le \delta'$. Where $\mathrm{CIFWHA}^2_{\min}(\tilde{a}_1,\ldots,\tilde{a}_n) = (\alpha_1, \beta_1, \delta_1)$ and $\mathrm{CIFWHA}^2_{\min}(\tilde{a}'_1,\ldots,\tilde{a}'_n) = (\alpha'_1, \beta'_1, \delta'_1)$.*

Proof. In the operator $\mathrm{CIFWHA}^2_{\min}$, using equations (7)–(9), it follows that X_i and Z_i are strictly increasing with respect to a_i and γ_i, and Y_i strictly decreasing with respect to b_i. That is,

$$a_i \le a'_i \Rightarrow X_i \le X'_i, \quad b_i \ge b'_i \Rightarrow Y_i \le Y'_i, \quad \gamma_i \le \gamma'_i \Rightarrow Z_i \le Z'_i.$$

It is observed from equations (10)–(12) that X_{ij}, Y_{ij}, and Z_{ij} are decreasing functions of X_i, Y_i, and Z_i, respectively. therefore,

$$X_{ij} \le X'_{ij}, \quad Y_{ij} \le Y'_{ij}, \quad Z_{ij} \le Z'_{ij}.$$

The final outputs $(\alpha_1, \beta_1, \delta_1)$ are strictly increasing with respect to A_{ij} and R_{ij} and decreasing with respect to B_{ij}. Thus, it follows that: $\alpha_1 \le \alpha'_1$, $\quad \beta_1 \ge \beta'_1$, $\quad \delta_1 \le \delta'_1$.

Hence, the operator CIFWHA^2min preserves monotonicity with respect to the partial order defined on CIFS tuples. Similarly, the operator CIFWHA^2max also preserves monotonicity under the same ordering. □

Theorem 6.8 (Boundedness). *Let $\tilde{a}_i = (a_i, b_i, \gamma_i)$ be CIFSs, where $0 \le a_i, b_i \le 1$, $a_i + b_i \le 1$ and $0 \le \gamma_i \le \sqrt{2}$. Then the output of the operator $\mathrm{CIFWHA}^2_{\min}(\tilde{a}_1,\ldots,\tilde{a}_n) = (\alpha_1, \beta_1, \delta_1)$ satisfies the conditions: $0 \le \alpha_1, \beta_1 \le 1$, $\quad \alpha_1 + \beta_1 \le 1$, $\quad 0 \le \delta_1 \le \sqrt{2}$.*

Proof. As established in Theorem 6.8, the operators $\mathrm{CIFWHA}^2_{\min}$ preserve the structural properties of CIFSs. Specifically, they ensure that the resulting membership α_1, non-membership β_1 and hesitation degree δ_1 remain within the required bounds. Hence, the output of the operator satisfies the boundedness conditions of CIFSs. Similarly, the operator CIFWHA^2max also preserves boundedness conditions under the same ordering. □

7. Proposed weighted averaging and geometric aggregation operators for CIFSs

This section introduces the CIFWAA and CIFWGA operators based on the proposed Archimedean t-norms and t-conorms. These operators are defined to aggregate CIFSs while preserving their structural constraints.

7.1. Weighted averaging aggregation operators for CIFSs

Definition 7.1. Let $\tilde{a}_i = (a_i, b_i, \gamma_i)$ be a CIFS, where $0 \le a_i, b_i \le 1$, $a_i + b_i \le 1$, and $0 \le \gamma_i \le \sqrt{2}$ for all $i = 1, 2, \ldots, n$. Let ω_i represent the weight assigned to $\tilde{a}_i$, where $0 \le \omega_i \le 1$ and $\sum_{i=1}^{n} \omega_i = 1$. The CIFWAA operators are defined as $\mathrm{CIFWAA}(\tilde{a}_1,\ldots,\tilde{a}_n) = \oplus_{i=1}^{n}(\tilde{a}_i \odot \omega_i)$.

Using the Archimedean t-norm $T_p^1(x,y)$ and its corresponding t-conorm $S_p^1(x,y)$, the CIFWAA^1 operators are expressed as:

$$\mathrm{CIFWAA}^1_{\min} = \left(\frac{k + \sqrt{1-(1-k^2)\cdot\prod_{i=1}^n \left(\frac{(1-a_i)(2-(k+1)(1-a_i))}{1-k}\right)^{\omega_i}}}{k+1}, \frac{1-\sqrt{1-(1-k^2)\cdot\prod_{i=1}^n\left(\frac{b_i(2-(k+1)b_i)}{1-k}\right)^{\omega_i}}}{k+1}, \sqrt{2}\cdot\frac{k+\sqrt{1-(1-k^2)\cdot\prod_{i=1}^n\left(\frac{\left(1-\frac{\gamma_i}{\sqrt{2}}\right)\left(2-(k+1)\left(1-\frac{\gamma_i}{\sqrt{2}}\right)\right)}{1-k}\right)^{\omega_i}}}{k+1} \right).$$

$$\mathrm{CIFWAA}^1_{\max} = \left(\frac{k + \sqrt{1-(1-k^2)\cdot\prod_{i=1}^n \left(\frac{(1-a_i)(2-(k+1)(1-a_i))}{1-k}\right)^{\omega_i}}}{k+1}, \frac{1-\sqrt{1-(1-k^2)\cdot\prod_{i=1}^n\left(\frac{b_i(2-(k+1)b_i)}{1-k}\right)^{\omega_i}}}{k+1}, \sqrt{2}\cdot\frac{1-\sqrt{1-(1-k^2)\cdot\prod_{i=1}^n\left(\frac{\frac{\gamma_i}{\sqrt{2}}\left(2-(k+1)\frac{\gamma_i}{\sqrt{2}}\right)}{1-k}\right)^{\omega_i}}}{k+1} \right).$$

The following expressions are derived using the power-based t-norm $T_p^2(x,y)$ and the corresponding t-conorm $S_p^2(x,y)$:

$$\mathrm{CIFWAA}^2_{\min} = \left(\left[1-\prod_{i=1}^n (1-a_i^{m+1})^{\omega_i}\right]^{\frac{1}{m+1}}, \quad 1-\left[1-\prod_{i=1}^n(1-(1-b_i)^{m+1})^{\omega_i}\right]^{\frac{1}{m+1}}, \sqrt{2}\cdot\left[1-\prod_{i=1}^n\left(1-\left(\frac{\gamma_i}{\sqrt{2}}\right)^{m+1}\right)^{\omega_i}\right]^{\frac{1}{m+1}} \right).$$

$$\mathrm{CIFWAA}^2_{\max} = \left(\left[1-\prod_{i=1}^n (1-a_i^{m+1})^{\omega_i}\right]^{\frac{1}{m+1}}, \quad 1-\left[1-\prod_{i=1}^n(1-(1-b_i)^{m+1})^{\omega_i}\right]^{\frac{1}{m+1}}, \sqrt{2}\cdot\left(1-\left[1-\prod_{i=1}^n\left(1-\left(1-\frac{\gamma_i}{\sqrt{2}}\right)^{m+1}\right)^{\omega_i}\right]^{\frac{1}{m+1}}\right) \right).$$

7.2. CIF weighted geometric aggregation (CIFWGA) operators

Definition 7.2. Let $\tilde{a}_i = (a_i, b_i, \gamma_i)$ be a set of CIFSs with corresponding weights ω_i such that $\sum_{i=1}^n \omega_i = 1$. The CIFWGA operators are defined as CIFWGA $= \bigotimes_{i=1}^n ((\tilde{a}_i)^{\omega_i})$.

CIFWGA using $T_p^1(x,y)$ and $S_p^1(x,y)$ is defined as follows:

$$\mathrm{CIFWGA}^1_{\min} = \left(\frac{1-\sqrt{1-(1-k^2)\prod_{i=1}^{n}\left(\frac{a_i(2-(k+1)a_i)}{1-k}\right)^{\omega_i}}}{k+1}, \right.$$

$$\frac{k+\sqrt{1-(1-k^2)\prod_{i=1}^{n}\left(\frac{(1-b_i)(2-(k+1)(1-b_i))}{1-k}\right)^{\omega_i}}}{k+1},$$

$$\left. \sqrt{2}\cdot\frac{1-\sqrt{1-(1-k^2)\prod_{i=1}^{n}\left(\frac{\left(\frac{\gamma_i}{\sqrt{2}}\right)\left(2-(k+1)\frac{\gamma_i}{\sqrt{2}}\right)}{1-k}\right)^{\omega_i}}}{k+1} \right).$$

$$\mathrm{CIFWGA}^1_{\max} = \left(\frac{1-\sqrt{1-(1-k^2)\prod_{i=1}^{n}\left(\frac{a_i(2-(k+1)a_i)}{1-k}\right)^{\omega_i}}}{k+1}, \right.$$

$$\frac{k+\sqrt{1-(1-k^2)\prod_{i=1}^{n}\left(\frac{(1-b_i)(2-(k+1)(1-b_i))}{1-k}\right)^{\omega_i}}}{k+1},$$

$$\left. \sqrt{2}\cdot\frac{k+\sqrt{1-(1-k^2)\prod_{i=1}^{n}\left(\frac{\left(1-\frac{\gamma_i}{\sqrt{2}}\right)\left(2-(k+1)\left(1-\frac{\gamma_i}{\sqrt{2}}\right)\right)}{1-k}\right)^{\omega_i}}}{k+1} \right).$$

CIFWGA using $T_p^2(x,y)$ and $S_p^2(x,y)$ is defined as follows:

$$\mathrm{CIFWGA}^2_{\min}(\tilde{a}_1,\ldots,\tilde{a}_n) = \left(1-\left[1-\prod_{i=1}^{n}\Big(1-\big(1-a_i\big)^{m+1}\Big)^{w_i}\right]^{\frac{1}{m+1}}, \right.$$

$$\left[1-\prod_{i=1}^{n}\big(1-b_i^{m+1}\big)^{w_i}\right]^{\frac{1}{m+1}},$$

$$\left. \sqrt{2}\left(1-\left[1-\prod_{i=1}^{n}\left(1-\left(1-\frac{\gamma_i}{\sqrt{2}}\right)^{m+1}\right)^{w_i}\right]^{\frac{1}{m+1}}\right)\right).$$

$$\mathrm{CIFWGA}^2_{\max}(\tilde{a}_1,\ldots,\tilde{a}_n) = \left(1-\left[1-\prod_{i=1}^{n}\Big(1-\big(1-a_i\big)^{m+1}\Big)^{w_i}\right]^{\frac{1}{m+1}},\right.$$
$$\left[1-\prod_{i=1}^{n}\big(1-b_i^{m+1}\big)^{w_i}\right]^{\frac{1}{m+1}},$$
$$\left.\sqrt{2}\left[1-\prod_{i=1}^{n}\Big(1-\big(\tfrac{\gamma_i}{\sqrt{2}}\big)^{m+1}\Big)^{w_i}\right]^{\frac{1}{m+1}}\right).$$

7.3. Properties of the proposed CIFWAA and CIFWGA operators

Theorem 7.1 (Idempotency)**.** *Let $\tilde{a} = (a, b, \gamma)$ be a CIFS, where $0 \le a, b \le 1$, $a+b \le 1$, and $0 \le \gamma \le \sqrt{2}$. Then, for any integer $n \ge 1$ and weights $\omega_i = \frac{1}{n}$, the proposed aggregation operator satisfies: $\mathcal{O}(\underbrace{\tilde{a}, \tilde{a}, \ldots, \tilde{a}}_{n \text{ times}}) = \tilde{a}$. Here, $\mathcal{O}$ refers to any of the proposed operators such as* $\mathrm{CIFWAA}_{\min}$*,* $\mathrm{CIFWGA}_{\min}$*,* $\mathrm{CIFWAA}_{\max}$*, or* $\mathrm{CIFWGA}_{\max}$*.*

Proof. When all input CIFSs are identical, the structural operations (addition, multiplication, Archimedean composition) preserves the value. Since all weights $w_i = \frac{1}{n}$ are equal, the operator simplifies to the identity element of the operation, yielding the same fuzzy value. $\mathcal{O}(\underbrace{\tilde{a}, \tilde{a}, \ldots, \tilde{a}}_{n \text{ times}}) = \tilde{a}$. □

Theorem 7.2 (Monotonicity)**.** *Let $\tilde{a}_i = (a_i, b_i, \gamma_i)$ and $\tilde{a}'_i = (a'_i, b'_i, \gamma'_i)$ be two collections of CIFS elements such that for all $i = 1, \ldots, n$, the following condition holds $a_i \le a'_i$, $\quad b_i \ge b'_i$, $\quad \gamma_i \le \gamma'_i$. Then, the aggregation operator $\mathcal{O}$ satisfies $\mathcal{O}(\tilde{a}_1, \ldots, \tilde{a}_n) \le \mathcal{O}(\tilde{a}'_1, \ldots, \tilde{a}'_n)$, which implies, $\alpha_2 \le \alpha'_2$, $\beta_2 \ge \beta'_2$ and $\delta_2 \le \delta'_2$. Where, $\mathcal{O}(\tilde{a}_1, \ldots, \tilde{a}_n) = (\alpha_2, \beta_2, \delta_2)$, $\quad \mathcal{O}(\tilde{a}'_1, \ldots, \tilde{a}'_n) = (\alpha'_2, \beta'_2, \delta'_2)$.*

Proof. α_2, β_2, and δ_2 are strictly increasing with respect to a_i, b_i and γ_i. That is, $a_i \le a'_i \Rightarrow \alpha_2 \le \alpha'_3$, $\quad b_i \ge b'_i \Rightarrow \beta_2 \ge \beta'_3$, $\quad \gamma_i \le \gamma'_i \Rightarrow \delta_2 \le \delta'_3$. Therefore, the operator $\mathcal{O}$ preserves monotonicity under the partial order defined on CIFS tuples. □

Theorem 7.3 (Boundedness)**.** *Let $\tilde{a}_i = (a_i, b_i, \gamma_i)$ be CIFSs for $i = 1, 2, \ldots, n$, where $0 \le a_i, b_i \le 1$, $a_i + b_i \le 1$, and $0 \le \gamma_i \le \sqrt{2}$. Then, the aggregation result: $\mathcal{O}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = (\alpha_2, \beta_2, \delta_2)$ satisfies: $\min_i a_i \le \alpha_2 \le \max_i a_i$, $\min_i b_i \le \beta_2 \le \max_i b_i$, $\min_i \gamma_i \le \delta_2 \le \max_i \gamma_i$.*

Proof. Each component of the operator involves either combinations, monotonic functions, or product of monotonic functions. These preserve bounds of the input variables. Since all operations are confined to the input range, the result must lie within the bounds of the input CIFSs. □

Proposed ranking function for CIFSs

Definition 7.3. Let $\tilde{a} = (a, b, \gamma)$ be a CIFS, where $0 \le a, b \le 1$, $a + b \le 1$, and $0 \le \gamma \le \sqrt{2}$. A ranking function $R : [0, 1]^2 \times [0, \sqrt{2}] \to \mathbb{R}$ is defined as:

$$R(\tilde{a}) = \left(\frac{(a+b)(a+b+1)}{2} + a\right)\left(\frac{\gamma}{\sqrt{2}} + 1\right).$$

This ranking function satisfies the following properties:

(i) If $R(\tilde{a}_i) > R(\tilde{a}_j)$, then $\tilde{a}_i > \tilde{a}_j$
(ii) If $R(\tilde{a}_i) < R(\tilde{a}_j)$, then $\tilde{a}_i < \tilde{a}_j$
(iii) If $R(\tilde{a}_i) = R(\tilde{a}_j)$, then $\tilde{a}_i = \tilde{a}_j$.

8. Proposed operators-based MCDM method with CIFS data

In this section, the proposed CIFWHA, CIFWAA, and CIFWGA aggregation operators are employed to solve MCDM problems under the CIFS environment.

Let $Z = \{z_1, z_2, \ldots, z_m\}$ denote the set of CIF alternatives, and $J = \{J_1, J_2, \ldots, J_n\}$ denote the set of CIF criteria. The weight vector is denoted by $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$, where $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

Each evaluation of an alternative z_i with respect to criterion J_j is represented by CIFSs $\Theta_{ij} = (\mu_{ij}, \nu_{ij}, \gamma_{ij})$. where:

- μ_{ij} is the degree of membership,
- ν_{ij} is the degree of non-membership,
- γ_{ij} is the hesitancy radius.

The CIF decision matrix $\Gamma = [\Theta_{ij}]_{m\times n}$ is:

$$\Gamma = \begin{pmatrix} (\mu_{11}, \nu_{11}, \gamma_{11}) & \cdots & (\mu_{1n}, \nu_{1n}, \gamma_{1n}) \\ \vdots & \ddots & \vdots \\ (\mu_{m1}, \nu_{m1}, \gamma_{m1}) & \cdots & (\mu_{mn}, \nu_{mn}, \gamma_{mn}) \end{pmatrix}.$$

8.1. Algorithm: CIF aggregation-based MCDM procedure

Input: Alternatives $Z = \{z_1, z_2, \ldots, z_m\}$, criteria $J = \{J_1, J_2, \ldots, J_n\}$, weight vector $\mathbf{w} = (w_1, \ldots, w_n)^T$ and decision matrix $\Gamma = [\Theta_{ij}]$.

Output: Optimal alternative.

(1) **Normalization:** Convert all cost-type (minimization-type) criteria into benefit-type (maximization-type) criteria using the complement of CIFS, so that all criteria follow a maximization (benefit) form.
For each criterion J_j, the normalized element $\bar{\Theta}_{ij}$ is defined as:

$$\bar{\Theta}_{ij} = \begin{cases} \Theta_{ij}, & \text{if } J_j \text{ is a benefit-type criterion;} \\ \Theta_{ij}^c, & \text{if } J_j \text{ is a cost-type criterion,} \end{cases}$$

where the complement of $\Theta_{ij} = (\mu_{ij}, \nu_{ij}, \gamma_{ij})$ is given by $\Theta_{ij}^c = (\nu_{ij}, \mu_{ij}, \gamma_{ij})$.

(2) **Weight determination:** If the criteria weights are not given in advance, they are derived from the CIFS decision matrix using the proposed ranking function and a fuzzy AHP based geometric mean approach.
   - First, each normalized CIF evaluation $\bar{\Theta}_{ij} = (\mu_{ij}, \nu_{ij}, \gamma_{ij})$ is converted into a crisp score using the ranking function $s_{ij} = R(\bar{\Theta}_{ij})$.
   - Then, the geometric mean of the scores corresponding to each criterion is computed as:

$$\gamma_j = \left(\prod_{i=1}^{m} s_{ij}\right)^{1/m}, \quad j = 1, 2, \ldots, n.$$

   - Finally, the criteria weights are obtained by normalization:

$$w_j = \frac{\gamma_j}{\sum_{j=1}^{n} \gamma_j}, \quad j = 1, 2, \ldots, n.$$

(3) **Aggregation:** Apply one of the proposed CIF aggregation operators (CIFWHA, CIFWAA, or CIFWGA) to the normalized evaluations for each alternative: $\Theta_i = \mathrm{CIFAgg}(\bar{\Theta}_{i1}, \bar{\Theta}_{i2}, \ldots, \bar{\Theta}_{in})$.

(4) **Scoring:** Compute the score $R(\Theta_i)$ for each aggregated alternative using the proposed ranking function.

(5) **Ranking and decision making:** Rank the alternatives z_i according to $R(\Theta_i)$. The alternative that receives the greatest score is chosen as the most preferred one.

8.1.1. Flowchart of proposed operators based MCDM method with CIFS data

The flowchart illustrating the proposed approach is presented in Figure 3.

9. Real life MCDM example: selection of a dominant farming strategy for small-scale farmers

This section describes a real life inspired MCDM problem to show the practical applicability of the proposed CIFS based aggregation framework. The problem deals with the selection of one suitable farming strategy for small scale farmers working in semi-arid regions. In such regions, farmers usually face several limitations, including limited financial resources, insufficient technical knowledge, poor infrastructure and uncertain external support. Because of these constraints, it is often not possible for farmers to adopt several farming strategies at the same time. Therefore, they are required to choose a single strategy that best matches their local conditions and available resources.

Although different farming strategies may complement each other in long-term agricultural planning, the present decision problem scrutinizes them as separate and competing options.

9.1. Decision alternatives and criteria

Let $Z = \{z_1, z_2, z_3, z_4, z_5\}$ represent the farming strategies considered for small-scale farmers in semi-arid areas. Because of limited income, infrastructure, and technical support, farmers are assumed to follow only one strategy at a time.

- z_1 **Rainfed conservation farming**, where cultivation mainly depends on rainfall and soil moisture is conserved through reduced tillage and crop residue retention.
- z_2 **Crop diversification**, in which farmers grow short-duration and low-water crops to reduce production and market risks.
- z_3 **Improved seed-based farming**, focusing on the use of drought-tolerant and stress-resistant seed varieties to maintain yield under uncertain weather conditions.
- z_4 **Solar-assisted micro-irrigation**, where solar-powered pumps are used along with drip irrigation and organic inputs to improve water use and farm productivity.
- z_5 **Digital and institutional support-based farming**, which includes the use of mobile advisory services, weather forecasts, information on government schemes, crop insurance, and market prices.

Each alternative requires a different type of investment and management effort. Due to practical constraints, farmers are expected to choose only one option. Hence, the alternatives are treated as competing choices in the MCDM analysis.

The alternatives are evaluated using the following criteria:

- J_1 Cost required for adoption and implementation (cost-type),
- J_2 Expected increase in crop yield (benefit-type),
- J_3 Improvement in water-use efficiency (benefit-type),
- J_4 Ease of adoption for farmers with limited technical skills (benefit-type),
- J_5 Long-term sustainability and effects on soil quality (benefit-type).

These criteria represent the main economic, technical and sustainability factors that influence farming decisions in semi-arid regions.

9.2. CIFS based evaluation and decision matrix construction

In agricultural decision problems, expert assessments are often not exact. Criteria such as sustainability, ease of use or long-term impact are usually judged using experience and personal understanding. To depict this practical situation, the decision matrix is constructed using CIFS. Each evaluation of an alternative z_i with respect to a criterion J_j is expressed by a CIFS triple $(\mu_{ij}, \nu_{ij}, \gamma_{ij})$, where:

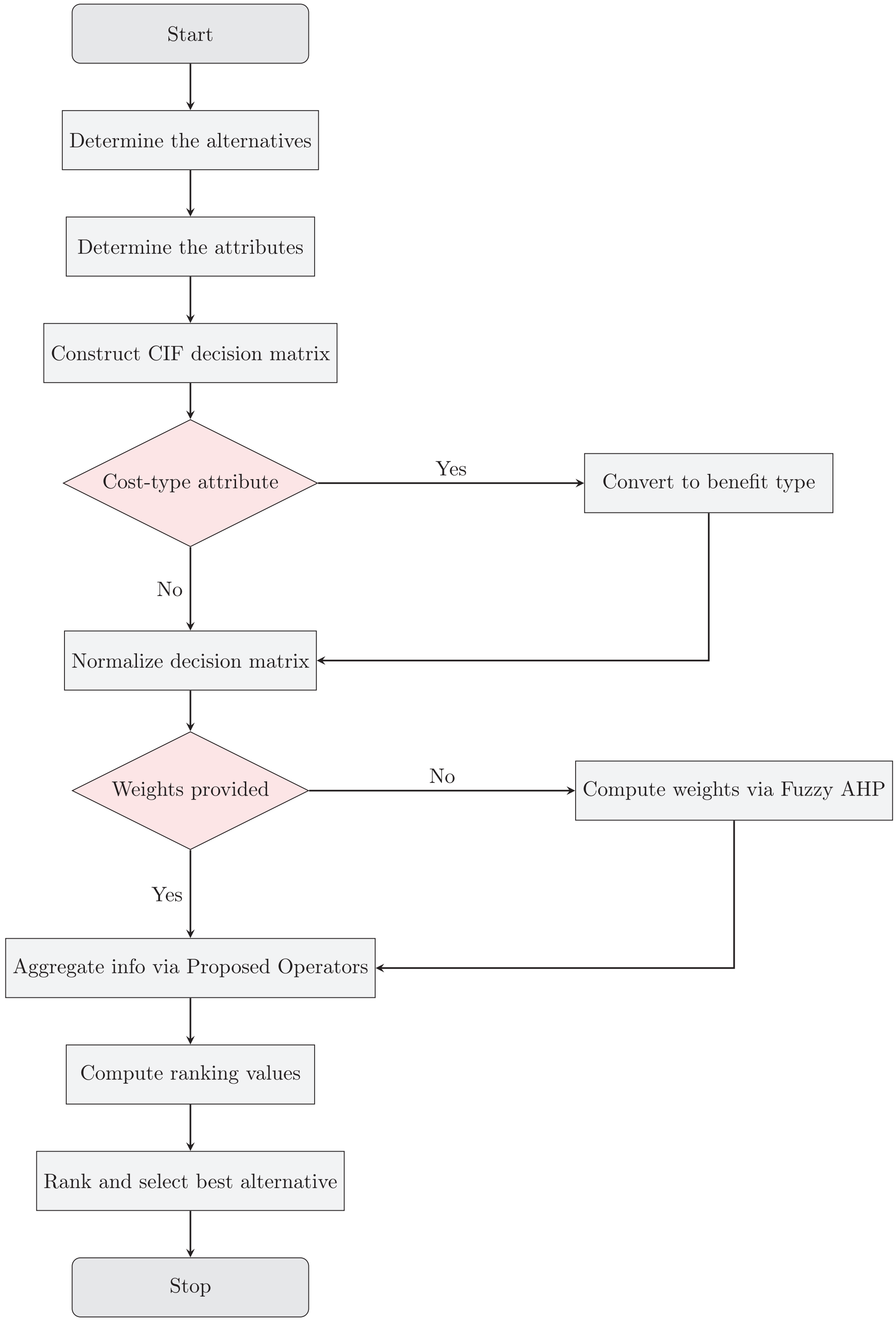


FIGURE 3. Flowchart of proposed operators based MCDM method with CIFS data.

TABLE 6. CIFS decision matrix.

Alternative	J_1	J_2	J_3	J_4	J_5
z_1	$(0.7, 0.2, 0.1)$	$(0.8, 0.1, 0.1)$	$(0.9, 0.05, 0.05)$	$(0.6, 0.2, 0.2)$	$(0.7, 0.2, 0.1)$
z_2	$(0.6, 0.25, 0.15)$	$(0.85, 0.1, 0.05)$	$(0.5, 0.35, 0.15)$	$(0.7, 0.02, 0.1)$	$(0.6, 0.25, 0.15)$
z_3	$(0.9, 0.05, 0.05)$	$(0.5, 0.3, 0.2)$	$(0.85, 0.1, 0.5)$	$(0.8, 0.1, 0.1)$	$(0.7, 0.2, 0.1)$
z_4	$(0.5, 0.35, 0.15)$	$(0.95, 0.02, 0.03)$	$(0.75, 0.15, 0.1)$	$(0.55, 0.3, 0.15)$	$(0.9, 0.05, 0.05)$
z_5	$(0.65, 0.25, 0.1)$	$(0.7, 0.2, 0.1)$	$(0.6, 0.3, 0.1)$	$(0.85, 0.05, 0.1)$	$(0.7, 0.05, 0.05)$

TABLE 7. Normalized CIFS matrix.

Alternative	J_1	J_2	J_3	J_4	J_5
z_1	(0.2, 0.7, 0.1)	(0.8, 0.1, 0.1)	(0.9, 0.05, 0.05)	(0.6, 0.2, 0.2)	(0.7, 0.2, 0.1)
z_2	(0.25, 0.6, 0.15)	(0.85, 0.1, 0.05)	(0.5, 0.35, 0.15)	(0.7, 0.02, 0.1)	(0.6, 0.25, 0.15)
z_3	(0.05, 0.9, 0.05)	(0.5, 0.3, 0.2)	(0.85, 0.1, 0.5)	(0.8, 0.1, 0.1)	(0.7, 0.2, 0.1)
z_4	(0.35, 0.5, 0.15)	(0.95, 0.02, 0.03)	(0.75, 0.15, 0.1)	(0.55, 0.3, 0.15)	(0.9, 0.05, 0.05)
z_5	(0.25, 0.65, 0.1)	(0.7, 0.2, 0.1)	(0.6, 0.3, 0.1)	(0.85, 0.05, 0.1)	(0.7, 0.05, 0.05)

– μ_{ij} indicates the membership (level of support) for alternative z_i with respect to criterion J_j.
– ν_{ij} indicates the non-membership (degree of rejection or disagreement).
– γ_{ij} represents the uncertainty radius.

The radius γ_{ij} permits variation around the central assessment (μ_{ij}, ν_{ij}) within a circular region satisfying $\mu_{ij} + \nu_{ij} \leq 1$, thereby capturing uncertainty caused by incomplete information and differing expert views.

The CIFS decision matrix is formed using evaluations provided by experts. Experts directly provide numerical assessments in the form of membership and non-membership degrees by considering field experience, available data and practical feasibility of each farming strategy under the given criteria. Since such evaluations are often affected by incomplete information and variability in local conditions, an uncertain radius γ_{ij} is assigned to each assessment to reflect the degree of hesitation.

The resulting decision matrix is presented in Table 6. The proposed CIFS aggregation operators are applied to obtain the overall evaluation and ranking of the farming strategies.

Step 1: Normalized CIFS matrix: The complement of CIFS is used when converting a cost criterion into a benefit-type criterion. After this transformation, the normalized CIFS matrix is given in Table 7.

Step 2: Decision maker weights: The decision maker provided the following weight vector: $\mathbf{w} = \big(0.3\ 0.1\ 0.3\ 0.1\ 0.2\big)^T$.

Step 3: Aggregated decision matrix using proposed operators: The proposed operators are used to aggregate the decision makers' evaluation data.

Step 4: Compute ranking value using the proposed ranking function: Based on Definitions 7.1–7.3, the ranking values are computed from the highest to the lowest. Results are presented in Tables 8, 9, and 10 for the CIFWAA, CIFWGA, and CIFWHA operators, respectively.

Results and discussion

This section discusses the numerical results obtained for the real life farming strategy selection problem using the proposed aggregation operators, namely CIFWAA, CIFWGA, and CIFWHA. The aggregated CIFNs, corresponding score values and final ranking of the farming strategies are summarized in Tables 8, 9, and 10.

TABLE 8. Results of proposed CIFWAA aggregation operators under different parameter.

Operator	Parameter	Scores $R(\Theta_i)$	Ranking (Best $\rightarrow$ Worst)
$\text{CIFWAA}^1_{\min}$	$k=-0.9$	(1.6671, 1.3872, 1.5957, 1.7261, 1.4179)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^1_{\min}$	$k=-0.7$	(1.6729, 1.3902, 1.6026, 1.7309, 1.4204)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^1_{\min}$	$k=-0.5$	(1.6804, 1.3949, 1.6121, 1.7368, 1.4244)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^1_{\min}$	$k=-0.3$	(1.6902, 1.4022, 1.6253, 1.7444, 1.4303)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^1_{\min}$	$k=-0.1$	(1.7044, 1.4141, 1.6453, 1.7553, 1.4396)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^1_{\min}$	$k=0$	(1.7174, 1.4235, 1.6647, 1.7663, 1.4469)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^1_{\max}$	$k=-0.9$	(1.6575, 1.3815, 1.5848, 1.7113, 1.4150)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^1_{\max}$	$k=-0.5$	(1.6702, 1.3888, 1.6005, 1.7213, 1.4213)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^1_{\max}$	$k=-0.1$	(1.6911, 1.4067, 1.6297, 1.7361, 1.4359)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^1_{\max}$	$k=0$	(1.6981, 1.4143, 1.6405, 1.7409, 1.4419)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^2_{\min}$	$m=-0.9$	(1.6087, 1.3541, 1.4879, 1.6833, 1.3894)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^2_{\min}$	$m=0$	(1.6646, 1.3861, 1.5929, 1.7241, 1.4170)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^2_{\min}$	$m=10$	(1.9277, 1.6359, 1.8799, 1.9356, 1.5894)	$z_4 \succ z_1 \succ z_3 \succ z_2 \succ z_5$
$\text{CIFWAA}^2_{\min}$	$m=30$	(1.7905, 1.6008, 1.7374, 1.8342, 1.6073)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^2_{\min}$	$m=100$	(1.8164, 1.6475, 1.7634, 1.8833, 1.6748)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^2_{\max}$	$m=-0.9$	(1.6055, 1.3518, 1.4847, 1.6778, 1.3883)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^2_{\max}$	$m=0$	(1.6551, 1.3805, 1.5822, 1.7093, 1.4141)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWAA}^2_{\max}$	$m=10$	(1.8323, 1.6133, 1.7678, 1.8709, 1.5768)	$z_4 \succ z_1 \succ z_3 \succ z_2 \succ z_5$
$\text{CIFWAA}^2_{\max}$	$m=50$	(1.9004, 1.7460, 1.8270, 1.9486, 1.7378)	$z_4 \succ z_1 \succ z_3 \succ z_2 \succ z_5$
$\text{CIFWAA}^2_{\max}$	$m=100$	(1.9005, 1.7410, 1.8340, 1.9572, 1.7580)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$

TABLE 9. Results of proposed CIFWGA aggregation operators under different parameter.

Operator	Parameter	Scores $R(\Theta_i)$	Ranking (Best $\rightarrow$ Worst)
$\text{CIFWGA}^1_{\min}$	$k=-0.9$	(1.4275, 1.3392, 1.2201, 1.5272, 1.3921)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^1_{\min}$	$k=-0.5$	(1.4120, 1.3346, 1.2073, 1.5127, 1.3886)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^1_{\min}$	$k=-0.1$	(1.3923, 1.3331, 1.1962, 1.4870, 1.3911)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^1_{\min}$	$k=0$	(1.3883, 1.3352, 1.1944, 1.4801, 1.3952)	$z_4 \succ z_5 \succ z_1 \succ z_2 \succ z_3$
$\text{CIFWGA}^1_{\max}$	$k=-0.9$	(1.4358, 1.3447, 1.2284, 1.5404, 1.3950)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^1_{\max}$	$k=-0.5$	(1.4206, 1.3404, 1.2160, 1.5264, 1.3916)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^1_{\max}$	$k=-0.1$	(1.4033, 1.3400, 1.2076, 1.5034, 1.3946)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^1_{\max}$	$k=0$	(1.4040, 1.3439, 1.2120, 1.5017, 1.4000)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^2_{\min}$	$m=-0.9$	(1.4739, 1.3332, 1.2580, 1.5791, 1.3747)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^2_{\min}$	$m=0$	(1.4305, 1.3402, 1.2231, 1.5296, 1.3931)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^2_{\min}$	$m=10$	(1.2166, 1.2318, 1.0947, 1.2983, 1.2679)	$z_4 \succ z_5 \succ z_2 \succ z_1 \succ z_3$
$\text{CIFWGA}^2_{\min}$	$m=30$	(1.1516, 1.1594, 1.0558, 1.2319, 1.2058)	$z_4 \succ z_5 \succ z_2 \succ z_1 \succ z_3$
$\text{CIFWGA}^2_{\min}$	$m=100$	(1.1204, 0.4458, 1.0279, 2.0785, 0.4428)	$z_4 \succ z_1 \succ z_3 \succ z_2 \succ z_5$
$\text{CIFWGA}^2_{\max}$	$m=-0.9$	(1.4769, 1.3355, 1.2608, 1.5843, 1.3758)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^2_{\max}$	$m=0$	(1.4387, 1.3457, 1.2315, 1.5427, 1.3959)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^2_{\max}$	$m=10$	(1.2799, 1.2491, 1.1642, 1.3432, 1.2780)	$z_4 \succ z_1 \succ z_5 \succ z_2 \succ z_3$
$\text{CIFWGA}^2_{\max}$	$m=30$	(1.0908, 1.0725, 1.0101, 1.1687, 1.1389)	$z_4 \succ z_5 \succ z_1 \succ z_2 \succ z_3$
$\text{CIFWGA}^2_{\max}$	$m=100$	(1.0709, 0.4218, 0.9883, 2.0000, 0.4218)	$z_4 \succ z_1 \succ z_3 \succ z_2 \succ z_5$

TABLE 10. Results of proposed CIFWHA operators under different parameter.

Operator	k	m	(p,q)	Scores $R(\Theta_i)$	Ranking (Best $\rightarrow$ Worst)
$\text{CIFWHA}^1_{\min}$	-0.1	–	(1,1)	(1.6131, 1.4109, 1.5677, 1.7037, 1.4573)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWHA}^1_{\min}$	-0.1	–	(5,5)	(1.7907, 1.4826, 1.7440, 1.8572, 1.5120)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWHA}^1_{\min}$	-0.9	–	(1,1)	(1.5937, 1.3872, 1.5313, 1.6941, 1.4328)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWHA}^1_{\min}$	-0.5	–	(1,1)	(1.5985, 1.3919, 1.5356, 1.6973, 1.4361)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWHA}^1_{\max}$	-0.1	–	(1,1)	(2.8863, 2.4501, 2.8469, 3.0555, 2.6356)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWHA}^1_{\max}$	-0.1	–	(10,10)	(3.3498, 2.5470, 3.3057, 3.3040, 2.7096)	$z_1 \succ z_3 \succ z_4 \succ z_5 \succ z_2$
$\text{CIFWHA}^1_{\max}$	-0.9	–	(1,1)	(2.8533, 2.3980, 2.7849, 3.0298, 2.5860)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWHA}^1_{\max}$	-0.9	–	(10,10)	(3.5113, 2.5559, 3.4696, 3.3699, 2.7676)	$z_1 \succ z_3 \succ z_4 \succ z_5 \succ z_2$
$\text{CIFWHA}^2_{\min}$	–	0	(1,1)	(1.5931, 1.3868, 1.5314, 1.6935, 1.4328)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWHA}^2_{\min}$	–	0	(5,5)	(1.8575, 1.4890, 1.7944, 1.8916, 1.5228)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWHA}^2_{\min}$	–	1	(1,1)	(3.8297, 3.8185, 3.7839, 3.8360, 3.7974)	$z_4 \succ z_1 \succ z_2 \succ z_5 \succ z_3$
$\text{CIFWHA}^2_{\max}$	–	0	(1,1)	(1.6236, 1.3961, 1.5608, 1.6981, 1.4522)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWHA}^2_{\max}$	–	0	(3,2)	(1.7309, 1.4165, 1.6935, 1.7437, 1.4705)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$
$\text{CIFWHA}^2_{\max}$	–	1	(5,5)	(1.9271, 1.8363, 1.9046, 1.9676, 1.8793)	$z_4 \succ z_1 \succ z_3 \succ z_5 \succ z_2$

Results based on CIFWAA operators. The results obtained using $\text{CIFWAA}^1_{\min}$ and $\text{CIFWAA}^1_{\max}$ exhibits stable decision making behavior with respect to variations in the parameter $k \in [-0.9, 0]$. Although the score values change gradually as k varies, the ranking order of farming strategies remains unchanged across all cases. In particular, strategy z_4 consistently achieves the highest score, followed by z_1 and z_3.

For the type-2 operators $\text{CIFWAA}^2_{\min}$ and $\text{CIFWAA}^2_{\max}$, the parameter m has a stronger influence on the aggregation results. As m increases, more importance is given to dominant membership values, leading to larger score magnitudes and stronger discrimination among strategies. Despite this greater sensitivity, the best farming strategy remains largely unchanged, demonstrating that CIFWAA operators are reliable even under different decision attitudes.

Results based on CIFWGA operators. The CIFWGA operators exhibit a more sensitive aggregation behavior due to their geometric structure. As shown in Table 9, $\text{CIFWGA}^1_{\min}$ and $\text{CIFWGA}^1_{\max}$ are more sensitive to changes in the parameter k than the CIFWAA operators. This results in wider score dispersion among alternatives.

For $\text{CIFWGA}^2_{\min}$ and $\text{CIFWGA}^2_{\max}$, the parameter m influences the scores more clearly. Although the score values vary with different choices of m, the overall ranking pattern remains stable and strategy z_4 is selected as the best option in most cases.

Results based on CIFWHA operators. The CIFWHA operators show a stronger response to parameter changes compared to the averaging based operators, as they consider the combined effect of multiple criteria. Variations in the parameters (p, q), along with (k, m) mainly affect the score values, as reported in Table 10. Although some changes are observed in intermediate rankings, the most preferred farming strategy remains the same in most cases. This shows that the CIFWHA operators are able to reflect interactions among criteria while producing stable final decisions.

Comparative discussion. The three proposed operator families combine information in different ways.

CIFWAA operators combine the criteria by giving similar importance to all input values. When the parameter values change, the scores may increase or decrease slightly, but the overall order of the alternatives usually remains the same. This means that CIFWAA operators focus on the general level of performance rather than reacting strongly to small differences between criteria.

TABLE 11. CIFS decision matrix for the 10×10 MCDM problem.

Alt.	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
z_1	$(0.62, 0.25, 0.10)$	$(0.58, 0.30, 0.12)$	$(0.65, 0.22, 0.08)$	$(0.60, 0.27, 0.11)$	$(0.63, 0.24, 0.09)$	$(0.59, 0.29, 0.10)$	$(0.61, 0.26, 0.09)$	$(0.64, 0.23, 0.08)$	$(0.57, 0.31, 0.12)$	$(0.60, 0.28, 0.10)$
z_2	$(0.54, 0.34, 0.11)$	$(0.57, 0.30, 0.10)$	$(0.60, 0.28, 0.12)$	$(0.55, 0.33, 0.11)$	$(0.58, 0.29, 0.10)$	$(0.56, 0.32, 0.11)$	$(0.59, 0.27, 0.10)$	$(0.61, 0.26, 0.09)$	$(0.55, 0.34, 0.11)$	$(0.57, 0.31, 0.10)$
z_3	$(0.48, 0.40, 0.14)$	$(0.52, 0.36, 0.13)$	$(0.50, 0.38, 0.12)$	$(0.49, 0.39, 0.13)$	$(0.51, 0.37, 0.12)$	$(0.47, 0.41, 0.14)$	$(0.53, 0.35, 0.12)$	$(0.54, 0.34, 0.11)$	$(0.50, 0.38, 0.13)$	$(0.52, 0.36, 0.12)$
z_4	$(0.71, 0.18, 0.09)$	$(0.69, 0.20, 0.10)$	$(0.73, 0.17, 0.08)$	$(0.70, 0.19, 0.09)$	$(0.72, 0.18, 0.08)$	$(0.68, 0.21, 0.10)$	$(0.74, 0.16, 0.08)$	$(0.75, 0.15, 0.07)$	$(0.69, 0.20, 0.10)$	$(0.71, 0.18, 0.09)$
z_5	$(0.60, 0.27, 0.11)$	$(0.58, 0.29, 0.12)$	$(0.62, 0.25, 0.10)$	$(0.61, 0.26, 0.10)$	$(0.59, 0.28; 0.11)$	$(0.57, 0.30, 0.12)$	$(0.63, 0.24, 0.10)$	$(0.64, 0.23, 0.09)$	$(0.58, 0.29, 0.11)$	$(0.60, 0.27, 0.10)$
z_6	$(0.66, 0.23, 0.09)$	$(0.64, 0.25, 0.10)$	$(0.67, 0.22, 0.08)$	$(0.65, 0.24, 0.09)$	$(0.68, 0.21, 0.08)$	$(0.63, 0.26, 0.10)$	$(0.69, 0.20, 0.09)$	$(0.70, 0.19, 0.08)$	$(0.64, 0.25, 0.10)$	$(0.66, 0.23; 0.09)$
z_7	$(0.55, 0.33, 0.12)$	$(0.53, 0.35, 0.13)$	$(0.57, 0.31, 0.11)$	$(0.56, 0.32, 0.12)$	$(0.54, 0.34, 0.13)$	$(0.52, 0.36, 0.14)$	$(0.58, 0.30, 0.11)$	$(0.59, 0.29, 0.10)$	$(0.55, 0.33, 0.12)$	$(0.57, 0.31, 0.11)$
z_8	$(0.63, 0.24, 0.09)$	$(0.61, 0.26, 0.10)$	$(0.65, 0.22, 0.08)$	$(0.64, 0.23, 0.09)$	$(0.62, 0.25, 0.10)$	$(0.60, 0.27, 0.11)$	$(0.66, 0.21, 0.08)$	$(0.67, 0.20, 0.07)$	$(0.61, 0.26, 0.10)$	$(0.63, 0.24, 0.09)$
z_9	$(0.59, 0.28, 0.11)$	$(0.57, 0.30, 0.12)$	$(0.61, 0.26, 0.10)$	$(0.60, 0.27, 0.11)$	$(0.58, 0.29, 0.12)$	$(0.56, 0.31, 0.13)$	$(0.62, 0.25, 0.10)$	$(0.63, 0.24, 0.09)$	$(0.57, 0.30, 0.12)$	$(0.59, 0.28, 0.11)$
z_{10}	$(0.70, 0.19, 0.09)$	$(0.68, 0.21, 0.10)$	$(0.72, 0.18, 0.08)$	$(0.71, 0.19, 0.09)$	$(0.69, 0.20, 0.10)$	$(0.67, 0.22, 0.10)$	$(0.73, 0.17, 0.08)$	$(0.74, 0.16, 0.07)$	$(0.68, 0.21, 0.10)$	$(0.70, 0.19, 0.09)$

TABLE 12. Results of the higher dimensional MCDM problem for CIFWAA and CIFWGA.

Operator	Parameter	Scores $R(\Theta_i)$	Best	Ranking order
$\text{CIFWAA}^1_{\min}$	$k = -0.9$	$(1.5317, 1.5160, 1.4465, \mathbf{1.6564}, 1.5514, 1.5043, 1.5820, 1.4815, 1.5524, 1.6458)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
	$k = -0.5$	$(1.5319, 1.5161, 1.4466, \mathbf{1.6565}, 1.5515, 1.5044, 1.5821, 1.4815, 1.5525, 1.6459)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
	$k = 0$	$(1.5337, 1.5169, 1.4472, \mathbf{1.6570}, 1.5520, 1.5049, 1.5826, 1.4820, 1.5530, 1.6464)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWAA}^1_{\max}$	$k = -0.9$	$(1.5300, 1.5152, 1.4461, \mathbf{1.6560}, 1.5509, 1.5040, 1.5816, 1.4811, 1.5520, 1.6454)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
	$k = 0$	$(1.5305, 1.5155, 1.4465, \mathbf{1.6563}, 1.5511, 1.5042, 1.5818, 1.4813, 1.5522, 1.6456)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWGA}^1_{\min}$	$k = -0.9$	$(1.5292, 1.5146, 1.4453, \mathbf{1.6549}, 1.5502, 1.5034, 1.5811, 1.4806, 1.5515, 1.6442)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
	$k = 0$	$(1.5293, 1.5147, 1.4452, \mathbf{1.6544}, 1.5501, 1.5034, 1.5811, 1.4805, 1.5514, 1.6438)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWGA}^1_{\max}$	$k = -0.9$	$(1.5309, 1.5154, 1.4457, \mathbf{1.6552}, 1.5507, 1.5038, 1.5815, 1.4809, 1.5520, 1.6446)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
	$k = 0$	$(1.5325, 1.5161, 1.4459, \mathbf{1.6551}, 1.5511, 1.5041, 1.5818, 1.4811, 1.5522, 1.6445)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWAA}^2_{\min}$	$m = -0.9$	$(1.5302, 1.5152, 1.4460, \mathbf{1.6559}, 1.5509, 1.5039, 1.5816, 1.4811, 1.5520, 1.6453)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
	$m = 10$	$(1.5517, 1.5264, 1.4548, \mathbf{1.6633}, 1.5588, 1.5105, 1.5883, 1.4876, 1.5586, 1.6527)$	z_4	$z_4 > z_{10} > z_7 > z_5 > z_9 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWAA}^2_{\max}$	$m = -0.9$	$(1.5296, 1.5150, 1.4458, \mathbf{1.6558}, 1.5507, 1.5038, 1.5815, 1.4809, 1.5519, 1.6452)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
	$m = 10$	$(1.5366, 1.5194, 1.4508, \mathbf{1.6595}, 1.5542, 1.5069, 1.5846, 1.4843, 1.5546, 1.6489)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWGA}^2_{\min}$	$m = -0.9$	$(1.5291, 1.5147, 1.4454, \mathbf{1.6552}, 1.5504, 1.5035, 1.5812, 1.4807, 1.5516, 1.6446)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
	$m = 0$	$(1.5292, 1.5146, 1.4453, \mathbf{1.6549}, 1.5502, 1.5035, 1.5811, 1.4806, 1.5515, 1.6442)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWGA}^2_{\max}$	$m = -0.9$	$(1.5291, 1.5147, 1.4454, \mathbf{1.6552}, 1.5504, 1.5035, 1.5812, 1.4807, 1.5516, 1.6446)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
	$m = 0$	$(1.5292, 1.5146, 1.4453, \mathbf{1.6549}, 1.5502, 1.5035, 1.5811, 1.4806, 1.5515, 1.6442)$	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$

CIFWGA operators contributes to criteria with relatively higher or lower values. When alternatives have close evaluations, CIFWGA operators increase the difference between their final scores.

CIFWHA operators consider the interaction among criteria rather than treating them independently. When the effect of one criterion depends on another, CIFWHA operators reflect this combined influence during aggregation. This behavior is useful in problems where criteria are related to each other. Even with these differences, all three operators select the same farming strategy as the best option across a wide range of parameter values. This shows that the proposed framework gives consistent results for real life MCDM problems.

9.3. CIFS based MCDM for higher dimensions

To further illustrate the use of the proposed CIFS based aggregation framework, an MCDM problem involving 10 alternatives and 10 criterion is considered. Each evaluation is expressed in the form $(\mu_{ij}, \nu_{ij}, \gamma_{ij})$. To avoid introducing additional subjectivity, all criteria are assumed to be equally important in this illustration. Hence, the weight vector is defined as

$$\mathbf{w} = (0.1,\, 0.1,\, \ldots,\, 0.1), \qquad \sum_{j=1}^{10} w_j = 1.$$

The resulting 10×10 CIFS decision matrix is presented in Table 11. This example is included to verify the numerical stability and consistent behavior of the proposed aggregation operators with the increase of the the number of alternatives and criteria. The sensitivity analysis for all twelve proposed aggregation operators including score variations and ranking stability, are summarized in Tables 12 and 13.

Table 13. Results of the higher dimensional MCDM problem for CIFWHA operators.

Operator	Parameters	Aggregated scores $R(\Theta_i)$	Best alt.	Ranking order
$\text{CIFWHA}^1_{\text{min}}$	$k=-0.9,\ p=q=1$	(1.5316, 1.5159, 1.4463, **1.6560**, 1.5512, 1.5042, 1.5819, 1.4813, 1.5523, 1.6454)	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWHA}^1_{\text{min}}$	$k=-0.5,\ p=q=5$	(1.5405, 1.5202, 1.4497, **1.6586**, 1.5542, 1.5067, 1.5843, 1.4838, 1.5548, 1.6480)	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWHA}^1_{\text{max}}$	$k=-0.9,\ p=q=10$	(2.7520, 2.7068, 2.5235, **3.0220**, 2.7791, 2.6695, 2.8861, 2.6027, 2.7825, 3.0028)	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWHA}^1_{\text{max}}$	$k=-0.5,\ p=q=10$	(2.7503, 2.7057, 2.5226, **3.0203**, 2.7781, 2.6687, 2.8849, 2.6020, 2.7817, 3.0011)	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWHA}^2_{\text{min}}$	$m=0,\ p=q=1$	(1.5317, 1.5159, 1.4464, **1.6560**, 1.5512, 1.5042, 1.5819, 1.4814, 1.5523, 1.6454)	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWHA}^2_{\text{min}}$	$m=0,\ p=q=5$	(1.5416, 1.5208, 1.4502, **1.6592**, 1.5547, 1.5070, 1.5848, 1.4842, 1.5552, 1.6486)	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWHA}^2_{\text{max}}$	$m=0,\ p=q=1$	(1.5307, 1.5154, 1.4461, **1.6558**, 1.5509, 1.5040, 1.5817, 1.4812, 1.5521, 1.6452)	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$
$\text{CIFWHA}^2_{\text{max}}$	$m=0,\ p=q=10$	(1.5375, 1.5197, 1.4509, **1.6597**, 1.5544, 1.5070, 1.5849, 1.4844, 1.5548, 1.6491)	z_4	$z_4 > z_{10} > z_7 > z_9 > z_5 > z_1 > z_2 > z_6 > z_8 > z_3$

9.4. Sensitivity analysis

A sensitivity analysis is performed to analyze the stability and consistency of the proposed circular intuitionistic fuzzy aggregation operators with respect to changes in their associated parameters. In particular, the influence of parameters k, m and (p, q) on the aggregated CIFNs, score values, and final rankings is examined for the CIFWAA, CIFWGA and CIFWHA operators. The corresponding numerical results are reported in Tables 8–10.

Influence of Parameter k. For type-1 operators, namely $\text{CIFWAA}^1_{\text{min}}$, $\text{CIFWAA}^1_{\text{max}}$, $\text{CIFWGA}^1_{\text{min}}$ and $\text{CIFWGA}^1_{\text{max}}$, the parameter $k \in (-1, 0]$ controls the aggregation behavior of the operators. The numerical results show that varying k leads to gradual changes in the computed score values of the alternatives. However, the overall ranking order remains same across the entire range of k. This observation indicates that the proposed operators do not react sharply to changes in k and are able to produce stable ranking results, which is desirable for practical decision making problems.

Influence of Parameter m. For type-2 operators, namely $\text{CIFWAA}^2_{\text{min}}$, $\text{CIFWAA}^2_{\text{max}}$, $\text{CIFWGA}^2_{\text{min}}$ and $\text{CIFWGA}^2_{\text{max}}$, the parameter m controls the aggregation behavior of the operators. When the value of m is increased, the resulting score values also change and the differences between alternatives become more visible. In small number of cases, this change slightly alters the positions of middle-ranked alternatives. However, the most preferred alternative remains unchanged in almost all situations. This indicates that the proposed operators remain reliable even when the value of m is varied.

Influence of Parameters (p, q) in CIFWHA. The CIFWHA operators, including $\text{CIFWHA}^1_{\text{min}}$, $\text{CIFWHA}^1_{\text{max}}$, $\text{CIFWHA}^2_{\text{min}}$ and $\text{CIFWHA}^2_{\text{max}}$, use the parameters (p, q) to controls the aggregation behavior of the operators. Changing the values of (p, q) leads to visible differences in the aggregated values and score levels, as shown in the sensitivity results. Although these changes affect the numerical scores, the overall ranking pattern remains mostly the same. In particular, the top-ranked alternative is preserved in most cases. This shows that CIFWHA operators can represent criterion interaction while keeping the final decision stable.

Thus, it can be concluded from the sensitivity analysis that the proposed CIFWAA, CIFWGA and CIFWHA operators provide stable and dependable ranking results under different parameter settings. While the parameters allow flexibility in aggregation behavior, the final decision outcomes remain same, demonstrating the robustness of the proposed CIFS based aggregation framework. Similar results are obtained for higher dimensional CIFS based decision problem presented in Subsection 9.3. The final selection of the optimal alternative remains unchanged under different parameter settings. This confirms the stability of the proposed approach and its applicability to higher dimensional MCDM problems.

10. Comparison with existing operators

This section compares the proposed CIFS based aggregation framework with two well-established approaches from the literature. The first comparison is made with Harish Garg's aggregation method [23], while the second comparison focuses on power aggregation operators [46].

TABLE 14. Comparative study of Harish Garg's method and proposed method.

Method	Ranking values	Order
Harish Garg method [23]	–	$z_2 \succ z_1 \succ z_3$
Proposed CIFS based operators		
$\text{CIFWGA}^1_{\min}$ $(k=0)$	(1.4743, 1.5553, 1.4093)	$z_2 \succ z_1 \succ z_3$
$\text{CIFWGA}^1_{\max}$ $(k=0)$	(1.5259, 1.5750, 1.4187)	$z_2 \succ z_1 \succ z_3$
$\text{CIFWAA}^1_{\min}$ $(k=0)$	(1.5456, 1.6037, 1.4595)	$z_2 \succ z_1 \succ z_3$
$\text{CIFWAA}^1_{\max}$ $(k=0)$	(1.4933, 1.5836, 1.4497)	$z_2 \succ z_1 \succ z_3$
$\text{CIFWGA}^2_{\min}$ $(m=1)$	(1.4743, 1.5553, 1.4093)	$z_2 \succ z_1 \succ z_3$
$\text{CIFWGA}^2_{\max}$ $(m=1)$	(1.5259, 1.5750, 1.4187)	$z_2 \succ z_1 \succ z_3$
$\text{CIFWAA}^2_{\min}$ $(m=1)$	(1.5456, 1.6037, 1.4595)	$z_2 \succ z_1 \succ z_3$
$\text{CIFWAA}^2_{\max}$ $(m=1)$	(1.4933, 1.5836, 1.4497)	$z_2 \succ z_1 \succ z_3$
$\text{CIFWHA}^1_{\min}$ $(k=0,\ p=q=1)$	(1.5497, 1.6022, 1.4563)	$z_2 \succ z_1 \succ z_3$
$\text{CIFWHA}^1_{\max}$ $(k=0,\ p=q=1)$	(2.5574, 2.6337, 2.3654)	$z_2 \succ z_1 \succ z_3$
$\text{CIFWHA}^2_{\min}$ $(m=0,\ p=q=1)$	(1.5193, 1.5864, 1.4428)	$z_2 \succ z_1 \succ z_3$
$\text{CIFWHA}^2_{\max}$ $(m=0,\ p=q=1)$	(1.4957, 1.5778, 1.4387)	$z_2 \succ z_1 \succ z_3$

10.1. Comparison with Garg's aggregation method

As shown in Table 14, Garg's aggregation method produces the ranking $z_2 \succ z_1 \succ z_3$, which is the ranking obtained by the proposed CIFWAA, CIFWGA, and CIFWHA operators. This result indicates that for the given decision matrix where the alternatives are clearly distinguishable, both methods are capable of identifying the same most preferred alternative.

It is also important to note that Garg's approach employs algebraic t-norm along with arithmetic and geometric mean operators to aggregate membership and non-membership degrees. These operators and t-norm are free from parameters. Thus, once the decision matrix is specified, the aggregation behavior becomes fixed and cannot be adjusted to reflect the decision maker's preferences. Furthermore, Garg's method aggregates the radius of uncertainty using min/max-type operators. This aggregation strategy may lead to degenerate results, in which essential uncertain information is lost. For instance, consider the alternatives $(\mu_a, \nu_a, 0.9), (\mu_b, \nu_b, \sqrt{2}), (\mu_c, \nu_c, 0)$. Under min-based aggregation, the resulting aggregated hesitation radius becomes 0, despite the presence of nonzero uncertainty (0.9 and $\sqrt{2}$). Conversely, under max-based aggregation, the aggregated hesitation radius becomes $\sqrt{2}$, ignoring the existence of both uncertainty radius (0 and 0.9) . Such outcomes are fail to provide a balanced and faithful representation of the CIFS.

As a result, the radius component is not combined within the same Archimedean aggregation framework as other CIFS components.

In contrast, the proposed CIFWAA, CIFWGA and CIFWHA operators aggregate the membership, non-membership and hesitation radius simultaneously within a single t-norm. This leads to a uniform treatment of all CIFS components. Although the ranking result coincides with Garg's method for the present example, the proposed framework offers a more structurally consistent aggregation mechanism, which is particularly important when alternatives are close to each other or when expert hesitation is relatively high.

10.2. Comparison with power aggregation methods

To further evaluate the behavior of the proposed aggregation operators, a comparative analysis is conducted with the power aggregation methods [46]. Power aggregation operators extend classical weighted aggregation by introducing power parameters that emphasize dominant criteria or evaluations.

TABLE 15. Comparison between power aggregation and proposed operators

Method	Scores	Ranking order
	Power aggregation methods [46]	
C-IFPWA$_t$	–	$z_2 > z_1 > z_3 > z_4$
C-IFPWA$_{tc}$	–	$z_1 > z_2 > z_3 > z_4$
C-IFPWG$_t$	–	$z_2 > z_1 > z_3 > z_4$
C-IFPWG$_{tc}$	–	$z_1 > z_4 > z_2 > z_3$
	Proposed operators	
	z_1, z_2, z_3, z_4, z_5	
CIFWAA$^1_{\min}$ (at $k=0$)	2.4132, 1.995, 2.4694, 2.3645	$z_3 > z_1 > z_4 > z_2$
CIFWAA$^1_{\max}$ (at $k=0$)	2.1159, 2.9117, 2.2631, 2.1442	$z_3 > z_4 > z_1 > z_2$
CIFWGA$^1_{\min}$ (at $k=0$)	1.8163, 1.6774, 2.0886, 2.0377	$z_3 > z_4 > z_1 > z_2$
CIFWGA$^1_{\max}$ (at $k=0$)	2.0715, 1.7505, 2.2790, 2.2492	$z_3 > z_4 > z_1 > z_2$
CIFWAA$^2_{\min}$ (at $m=1$)	2.4132, 1.995, 2.4694, 2.3645	$z_3 > z_1 > z_4 > z_2$
CIFWAA$^2_{\max}$ (at $m=1$)	2.1159, 2.9117, 2.2631, 2.1442	$z_3 > z_4 > z_1 > z_2$
CIFWGA$^2_{\min}$ (at $m=1$)	1.8163, 1.6774, 2.0886, 2.0377	$z_3 > z_4 > z_1 > z_2$
CIFWGA$^2_{\max}$ (at $m=1$)	2.0715, 1.7505, 2.2790, 2.2492	$z_3 > z_4 > z_1 > z_2$

The criteria weights $\mathbf{w} = (0.20763,\ 0.20739,\ 0.19688,\ 0.15902,\ 0.22957)$ are determined using Step 2 of the proposed MADM framework.

The ranking results obtained using the power aggregation methods and the proposed CIFS based operators are presented in Table 15. These results show that the ranking orders generated by the power aggregation methods change noticeably with the choice of operator (such as t or t_c). This indicates a strong dependence on operator selection, which may complicate decision interpretation, especially in the presence of significant expert hesitation.

The power aggregation methods are not developed specifically for CIFSs. In particular, the aggregation of the hesitancy radius γ may exceed the admissible CIFS domain $[0, \sqrt{2}]$. Consequently, for a higher radius of uncertainty, the aggregated results may fail to preserve a valid CIFS representation.

Unlike power aggregation methods, the proposed CIFWAA and CIFWGA operators preserve the CIFS structure by jointly aggregating all CIFS components. As observed in Table 15, the resulting rankings remain stable(z_3) under parameter variation while still providing clear separation among alternatives.

11. Conclusion

This paper introduces a new family of Archimedean t-norms and t-conorms constructed using the Riemann Liouville fractional integral. Unlike classical Archimedean operators that rely on point-wise additive generators, the proposed construction employs an integral-based generator, allowing aggregation to account for information over an interval $[0, x]$ rather than a single point x. The fractional order α regulates the use of interval-based information, with smaller values emphasizing pointwise information and larger values giving greater importance to information from the interval $[0, x]$. In addition, the use of the Beta-type kernel enables the proposed framework to adapt its aggregation behavior. This allows an appropriate balance between membership and non-membership information, enhancing applicability across diverse decision making scenarios.

From a theoretical viewpoint, the proposed t-norms and t-conorms satisfy all essential Archimedean properties, including commutativity, associativity, monotonicity and boundary conditions. More importantly, the proposed operators guarantee closure of the hesitation radius γ within the CIFS domain $[0, \sqrt{2}]$.

Based on the proposed t-norm, several advanced aggregation operators are constructed, such as Heronian mean-based, weighted average and weighted geometric operators.

These operators not only preserve the structural properties of CIFSs but also effectively capture inter-attribute dependencies, meaning they account for the mutual influence or correlation among multiple criteria during aggregation, thereby improving decision accuracy in complex real world scenarios. In particular:

- The Heronian mean incorporates both additive and multiplicative terms, which naturally encode the degree of interaction between attribute pairs. This enables the aggregation to reflect not only the individual importance of attributes but also how strongly or weakly they support each other.
- The geometric mean-based operators capture nonlinear interactions, where even a single low attribute value can significantly reduce the overall aggregation outcome. This behavior introduces sensitivity to joint weaknesses or strengths across attributes, enhancing the realism of the decision model.

To support the decision analysis, a new ranking function is introduced to improve the separation among CIFS alternatives. In addition, criteria weights are obtained using AHP approach in CIFS environment, which allows uncertainty and hesitation to be naturally incorporated into the weighting process. The combination of CIFS based weighting and the proposed ranking function leads to clearer and more consistent decision outcomes.

The practical applicability of the proposed framework is demonstrated through a real life multi-criteria decision making problem involving the selection of a dominant farming strategy for small scale farmers under uncertain and hesitant expert evaluations. In addition, a higher-order MCDM problem is also analyzed to verify the numerical stability and consistent behavior of the proposed operators.

The numerical results depict that the proposed framework produces ranking outcomes consistent with those obtained using Garg's CIFS aggregation method. This consistency of results confirms the correctness of the proposed aggregation structure. At the same time, further a comparison with power aggregation operators indicates that the proposed method handles CIFS components in a more consistent manner. Although values of the parameters changes, the ranking order remains stable and the differences among alternatives remain clear ($z_3 > z_1 > z_4 > z_2$).

The sensitivity study shows that the proposed operators behave in a stable manner when the parameters k, m, p and q are varied. While the score values change gradually, the overall ranking of alternatives does not show sudden or unexpected changes. This indicates that the proposed framework is sensitive to parameter variations within the considered range.

11.1. Advantages of the proposed framework

The proposed framework offers several practical and methodological benefits, which are summarized below.

- **Generalized fractional Archimedean aggregation framework:** A new family of Archimedean t-norms and t-conorms is developed using the Riemann Liouville fractional integral. The resulting framework provides a general aggregation structure that is not limited to CIFSs and can be adapted to other fuzzy and uncertain models.
- **Aggregation preserving the closure property of CIFSs:** Unlike many existing CIFS approaches that rely on simple selection rules for the hesitation radius, the proposed framework aggregates the radius within the same Archimedean structure as other CIFS components. This ensures the validity of aggregated results in CIFSs. In particular, when the uncertain radius is set to zero, the proposed approach directly reduces to an intuitionistic fuzzy setting, allowing MCDM problems to be solved within the IFS framework without requiring a separate formulation.
- **Stable ranking behavior:** Numerical examples and sensitivity analysis shows that the proposed operators produce consistent ranking order under different choices of the parameter, even though there is change in score values.
- **Consideration of criterion interdependence:** The proposed geometric and Heronian based operators allow the combined influence of multiple criteria to be reflected during aggregation. This feature is particularly useful in decision problems where criteria are not completely independent.

– **Flexible parameterization of aggregation behavior:** The parameters involved in the proposed operators allow decision makers to adjust the aggregation behavior according to their preference, without changing the basic structure of the method.

11.2. Future research directions

The present study focuses on the construction and application of fractional Archimedean aggregation operators for CIFSs. While the results are encouraging, there are several directions in which this work can be further developed.

– **Application to other fuzzy settings:** The proposed fractional t-norms and t-conorms may also be applied to other fuzzy frameworks, such as Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, and bipolar fuzzy models.
– **Extensions to more complex circular models:** Future studies may consider extensions to interval-valued CIFSs or higher-order circular uncertain models, which can represent more complex forms of hesitation.
– **Use with additional decision making methods:** The proposed aggregation operators can be incorporated into other multi-criteria decision making techniques, including EDAS, MARCOS, and group decision making problems involving multiple experts.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

Data availability statement

The data used in the numerical illustrations are hypothetical.

References

[1] F. Ahemad, A.Z. Khan, M.K. Mehlawat, P. Gupta and S.K. Roy, Multi-attribute group decision making for solid waste management using interval-valued q-rung orthopair fuzzy COPRAS. *RAIRO Oper. Res.* **57** (2023) 1239–1265.

[2] D. Alghazzawi, H. Alolaiyan, L. Latif, U. Shuaib, H.A.E.-W. Khalifa, H. Alqahtani and Q. Xin, A cutting-edge TOPSIS approach for navigating MCDM challenges under t-intuitionistic fuzzy environments. *IEEE Access* **12** (2024) 44873–44887.

[3] N. Alkan and C. Kahraman, Circular intuitionistic fuzzy TOPSIS method: pandemic hospital location selection. *J. Intell. Fuzzy Syst.* **42** (2021) 295–316.

[4] K. Atanassov, Intuitionistic fuzzy sets. Intuitionistic Fuzzy Sets: Theory and Applications. Physica-Verlag HD, Heidelberg (1999) 1–137.

[5] K. Atanassov, Circular intuitionistic fuzzy sets. *J. Intell. Fuzzy Syst.* **39** (2020) 5981–5986.

[6] K. Atanassov and E. Marinov, Four distances for circular intuitionistic fuzzy sets. *Mathematics* **9** (2021) 1121.

[7] A. Asif, T. Abdullah, F. Amin, S. Siddique and R. Ali, Hamacher aggregation operators for Pythagorean fuzzy sets and their application in multi-attribute decision making problem. *Spec. Oper. Res.* **2** (2021) 27–40.

[8] E. Ayyildiz, Fermatean fuzzy step-wise weight assessment ratio analysis and its application to prioritizing indicators to achieve sustainable development goal-7. *Renew. Energy* **193** (2022) 136–148.

[9] J. Bajaj and S. Kumar, Pythagorean fuzzy similarity measure with significance in MCDM and transportation challenges. *Evol. Intell.* **18** (2025) 93.

[10] G. Beliakov, A. Pradera and T. Calvo, Aggregation Functions: A Guide for Practitioners, Vol. 221. Springer, Berlin (2007).

[11] A. Biswas, K.H. Gazi, P.S. Mondal and A. Ghosh, A decision making framework for sustainable highway restaurant site selection: AHP-TOPSIS approach based on the fuzzy numbers. *Spec. Oper. Res.* **2** (2025) 1–26.

[12] I. Canco, D. Kruja and T. Iancu, AHP, a reliable method for quality decision making: a case study in business. *Sustainability* **13** (2021) 13932.

[13] E. Çakır and M.A. Taş, Circular intuitionistic fuzzy decision making and its application. *Expert Syst. Appl.* **225** (2023) 120076.

[14] E. Çaloğlu Büyükselçuk and Y.C. Sarı, The best whey protein powder selection via VIKOR based on circular intuitionistic fuzzy sets. *Symmetry* **15** (2023) 1313.

[15] S. Chakraborty, R.D. Raut, T.M. Rofin and S. Chakraborty, On solving a healthcare supplier selection problem using MCDM methods in intuitionistic fuzzy environment. *Opsearch* **61** (2024) 680–708.

[16] H.A. Dağıstanlı, An interval-valued intuitionistic fuzzy VIKOR approach for R&D project selection in defense industry investment decisions. *J. Soft. Comput. Decis. Anal.* **2** (2024) 1–13.

[17] K. Debnath, S.K. Roy, M. Deveci and H. Tomášková, Integrated MADM approach based on extended MABAC method with Aczel–Alsina generalized weighted Bonferroni mean operator. *Artif. Intell. Rev.* **58** (2024) 27.

[18] K. Debnath, E.B. Tirkolaee and S.K. Roy, A decision making model with new Dice–Jaccard similarity measure: green hydrogen electrolyzer selection. *Eng. Appl. Artif. Intell.* **160** (2025) 111817.

[19] M. Deveci, E.A. Varouchakis, P.R. Brito-Parada, A.R. Mishra, P. Rani, M. Bolgkoranou and M. Galetakis, Evaluation of risks impeding sustainable mining using Fermatean fuzzy score function based SWARA method. *Appl. Soft Comput.* **139** (2023) 110220.

[20] D. Diakoulaki, G. Mavrotas and L. Papayannakis, Determining objective weights in multiple criteria problems: the CRITIC method. *Comput. Oper. Res.* **22** (1995) 763–770.

[21] J. Dombi, A general class of fuzzy operators, the DeMorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. *Fuzzy Sets Syst.* **8** (1982) 149–163.

[22] A. Fahmi, A. Khan, Z. Maqbool and T. Abdeljawad, Circular intuitionistic fuzzy Hamacher aggregation operators for multi-attribute decision making. *Sci. Rep.* **15** (2025) 5618.

[23] H. Garg, M. Ünver, M. Olgun and E. Türkaslan, An extended EDAS method with circular intuitionistic fuzzy value features and its application to multi-criteria decision making process. *Artif. Intell. Rev.* **56** (2023) 3173–3204.

[24] C.P. Garg, Ö.F. Görçün, P. Kundu and H. Küçükönder, An integrated fuzzy MCDM approach based on Bonferroni functions for selection and evaluation of industrial robots for the automobile manufacturing industry. *Expert Syst. Appl.* **213** (2023) 118863.

[25] S. Giri, S.K. Roy and M. Deveci, An MADM model using Frank operations based power aggregation operator under quasi-rung orthopair fuzzy sets for highway selection in war-plane landing. *Appl. Soft Comput.* **185** (2025) 113918.

[26] F. Guo, R. Imran, S. Yin, K. Ullah, M. Akram, D. Pamucar and M. Elashiry, Assessment of air purifiers for improving the air quality index using circular intuitionistic fuzzy Heronian means. *Complex Intell. Syst.* **11** (2025) 258.

[27] F. Herrera, E. Herrera-Viedma and F. Chiclana, A study of the origin and uses of the ordered weighted geometric operator in multicriteria decision making. *Int. J. Intell. Syst.* **18** (2003) 689–707.

[28] J.-Y. Huang, Intuitionistic fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. *J. Intell. Fuzzy Syst.* **27** (2014) 505–513.

[29] A. Hussain, T. Abdullah, F. Amin, S. Siddique and R. Ali, Advanced decision making with complex q-rung orthopair fuzzy Muirhead mean models. *Spec. Decis. Mak. Appl.* **3** (2021) 316–338.

[30] M. Ijaz, T. Abdullah, F. Amin, S. Siddique and R. Ali, Dynamic aggregation operators for selection of optimal communication system in the rescue department under complex q-rung orthopair fuzzy environment. *Spec. Decis. Mak. Appl.* **3** (2021) 339–363.

[31] D. Joshi and S. Kumar, Intuitionistic fuzzy entropy and distance measure based TOPSIS method for multi-criteria decision making. *Egypt. Inform. J.* **15** (2014) 97–104.

[32] C. Kahraman and I. Otay, Extension of VIKOR method using circular intuitionistic fuzzy sets, in International Conference on Intelligent and Fuzzy Systems (2021) 48–57.

[33] E. Kawecka, A. Perec and A. Radomska-Zalas, Use of the simple multicriteria decision making (MCDM) method for optimization of the high-alloy steel cutting process by the abrasive water jet. *Spec. Mech. Eng. Oper. Res.* **1** (2024) 111–120.

[34] V. Keršuliene, E.K. Zavadskas and Z. Turskis, Selection of rational dispute resolution method by applying new step-wise weight assessment ratio analysis (SWARA). *J. Bus. Econ. Manag.* **11** (2010) 243–258.

[35] M.J. Khan, W. Kumam and N.A. Alreshidi, Divergence measures for circular intuitionistic fuzzy sets and their applications. *Eng. Appl. Artif. Intell.* **116** (2022) 105455.

[36] M.R. Khan, K. Ullah, Z. Ali, E. Rak, D. Pamucar and Y. Shang, The MCDM approach using preference ranking organization method for enrichment evaluation for intuitionistic fuzzy sets with AHP based weight information. *Int. J. Fuzzy Syst.* (2025) 1–16.

[37] E. Klement, R. Mesiar and E. Pap, Triangular norms. Position paper II: general constructions and parameterized families. *Fuzzy Sets Syst.* **145** (2004) 411–438.

[38] D. Li, W. Zeng and J. Li, Geometric bonferroni mean operators. *Int. J. Intell. Syst.* **31** (2016) 1181–1197.

[39] R. Liang, S. He, J. Wang, K. Chen and L. Li, An extended MABAC method for multi-criteria group decision making problems based on correlative inputs of intuitionistic fuzzy information. *Comput. Appl. Math.* **38** (2019) 112.

[40] P. Liu and S.-M. Chen, Group decision making based on Heronian aggregation operators of intuitionistic fuzzy numbers. *IEEE Trans. Cybern.* **47** (2016) 2514–2530.

[41] I. Otay and C. Kahraman, A novel circular intuitionistic fuzzy AHP&VIKOR methodology: an application to a multi-expert supplier evaluation problem. *Pamukkale Univ. Muh. Bilim. Derg.* **28** (2022) 194–207.

[42] S. Opricovic and G.-H. Tzeng, Extended VIKOR method in comparison with outranking methods. *Eur. J. Oper. Res.* **178** (2007) 514–529.

[43] S. Petchimuthu and A.K. Singh, Sustainable urban development: q-Rung orthopair fuzzy MCDM with generalized power prioritized Yager aggregation. *Spectr. Oper. Res.* **3** (2021) 275–309.

[44] E. Rak, J. Sarzyński and R. Rak, Effectiveness of an ensemble technique based on the distributivity equation in detecting suspicious network activity. *Fuzzy Sets Syst.* **488** (2024) 109015.

[45] S. Rani and S. Dhanasekar, Type-2 trapezoidal Pythagorean fuzzy number with novel entropy measure and aggregation operators extended to MCDM. *Complex Intell. Syst.* **11** (2025) 1–45.

[46] M. Rukhsar, K. Ullah, Z. Ali and A. Hussain, Analysis of power aggregation operators through circular intuitionistic fuzzy information and their applications in machine learning analysis. *Eng. Rev.* **44** (2024) 141–159.

[47] B. Sahoo, B.K. Debnath and A. Saha, An integrated Pythagorean fuzzy multi-criteria decision making model for best supplier selection in sustainable agile manufacturing. *RAIRO Oper. Res.* **59** (2025) 2577–2606.

[48] B. Schweizer and A. Sklar, Statistical metric spaces. *Pac. J. Math.* **10** (1960) 313–334.

[49] T. Senapati, G. Chen, R. Mesiar and R.R. Yager, Intuitionistic fuzzy geometric aggregation operators in the framework of Aczel–Alsina triangular norms and their application to multiple attribute decision making. *Expert Syst. Appl.* **212** (2023) 118832.

[50] M. Tavana, M. Soltanifar and F.J. Santos-Arteaga, Analytical hierarchy process: revolution and evolution. *Ann. Oper. Res.* **326** (2023) 879–907.

[51] Y.-J. Wang and H.-S. Lee, Generalizing TOPSIS for fuzzy multiple-criteria group decision making. *Comput. Math. Appl.* **53** (2007) 1762–1772.

[52] T.-C. Wang and Y.-H. Chen, Applying fuzzy linguistic preference relations to the improvement of consistency of fuzzy AHP. *Inf. Sci.* **178** (2008) 3755–3765.

[53] W.W. Wang and X.W. Liu, Intuitionistic fuzzy geometric aggregation operators based on Einstein operations. *Int. J. Intell. Syst.* **26** (2011) 1049–1075.

[54] W.-D. Wang, J. Guo, L.-G. Fang and X.-S. Chang, A subjective and objective integrated weighting method for landslides susceptibility mapping based on GIS. *Environ. Earth Sci.* **65** (2012) 1705–1714.

[55] Z. Xu and Q.-L. Da, An overview of operators for aggregating information. *Int. J. Intell. Syst.* **18** (2003) 953–969.

[56] Z. Xu and R.R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int. J. Gen. Syst.* **35** (2006) 417–433.

[57] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Trans. Syst. Man Cybern. Syst.* **18** (2002) 183–190.

[58] R.R. Yager, Pythagorean fuzzy subsets, in Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS). IEEE (2013) 57–61.

[59] R.R. Yager, Generalized orthopair fuzzy sets. *IEEE Trans. Fuzzy Syst.* **25** (2016) 1222–1230.

[60] M. Yazdani, A.E. Torkayesh, E.D.R. Santibanez-Gonzalez and S.K. Otaghsara, Evaluation of renewable energy resources using integrated Shannon entropy–EDAS model. *Sustain. Oper. Comput.* **1** (2020) 35–42.

[61] L. Zadeh, Fuzzy sets. *Inf. Control* **8** (1965) 338–353.

[62] H. Zhao, Z. Xu, M. Ni and S. Liu, Generalized aggregation operators for intuitionistic fuzzy sets. *Int. J. Intell. Syst.* **25** (2010) 1–30.

[63] Z. Zhang, Interval-valued intuitionistic fuzzy Frank aggregation operators and their applications to multiple attribute group decision making. *Neural Comput. Appl.* **28** (2017) 1471–1501.

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