RAIRO-Oper. Res. 60 (2026) 2121–2129
https://doi.org/10.1051/ro/2026073

RAIRO Operations Research
www.rairo-ro.org

SATURATION NUMBERS FOR LINEAR FORESTS $P_5 \cup P_4 \cup tP_2$

Renying Chang[1,*] and Xueliang Li[2,3]

Abstract. We say that G is F*-saturated* if G contains no copy of F and for all $e \in E(\bar{G})$ the graph $G + e$ does contain a copy of F. The saturation number, denote by $\mathrm{sat}(n, F)$, is the minimum size of a graph with order n in all F-saturated graphs. In this paper, we determine the saturation numbers for linear forests $P_5 \cup P_4 \cup tP_2$ and characterize the extremal graphs.

Mathematics Subject Classification. 05C35.

Received June 9, 2025. Accepted June 7, 2026.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. Throughout we use the terminology and notation of [3].

Let G and H be graphs. If $V(H) \subset V(G)$ and $E(H) \subset E(G)$, we say that H is a *subgraph* of G. If H is a subgraph of G such that $E(H) = \{uv \mid u, v \in V(H) \text{ and } uv \in E(G)\}$, we say that H is an induced subgraph of G and we may write $H = G[V(H)]$. We say that two graphs G and H are *isomorphic* if there exists an adjacency-preserving bijection between their vertex sets and we write $G \cong H$. If there exists a subset $V' \subset V(G)$ and a subset $E' \subset E(G)$ such that H is isomorphic to the subgraph $H' = (V', E')$, we say that G *contains a copy* of H. Note that by our definition above, when we say that G contains a copy of H, the subgraph H' of G such that $H \cong H'$ need not be an induced subgraph of G. P_n, K_n denote the path on n vertices, the complete graph on n vertices.

We say that G is F*-saturated* if G contains no copy of F and for any $e \in E(\bar{G})$ the graph $G+e$ does contain a copy of F. Denote by $\mathrm{SAT}(n, F)$ the set of all F-saturated graphs of order n. The maximum number of edges in an n-vertex F*-saturated* graph is the well known as the *Turán* number [17] and is usually denoted by $ex(n, F)$. The minimum number of edges in an n-vertex F*-saturated* graph is called *saturation number*, denoted by $\mathrm{sat}(n, F)$ [9]. That is,

$$\mathrm{sat}(n, F) = \min\{|E(G)| : G \in \mathrm{SAT}(n, F)\}.$$

The set of all F-saturated graphs of order n having size $\mathrm{sat}(n, F)$ is denoted $\underline{\mathrm{SAT}}(n, F)$.

Keywords. Saturation number, saturated graph, linear forest.

[1] Business School, Shanghai Dianji University, Shanghai 201306, P.R. China.

[2] Center for Combinatorics, Nankai University, Tianjin 300071, P.R. China.

[3] School of Mathematical Sciences, Xinjiang Normal University, Urumchi, Xinjiang 830017, P.R. China.

[*]Corresponding author: changrysd@163.com

The original paper established $\mathrm{sat}(n, K_k)$ and the uniqueness of the graph in $\underline{\mathrm{SAT}}(n, K_k)$ by *Erdös*, Hajnal and Moon in [9]. In 1986 *Kászonyi* and Tuza [14] found the best known general upper bound for $\mathrm{sat}(n, F)$, where F is a class of forbidden graphs. Since then, $\mathrm{sat}(n, F)$ and $\underline{\mathrm{SAT}}(n, F)$ have been investigated for a range of graphs F, including unions of cliques [2], complete bipartite graphs [13], nearly complete graphs [11], tripartite graphs [15], cycles [6, 16] and paths [5, 10, 12]. For a summary of known results see [8].

Bushaw *et al.* [4] gave the Turán number for equibipartite forest. Corresponding to that, Chen *et al.* focused on the saturation numbers for linear forests in [7]. They obtained the following results about the saturation number for linear forests $P_k \cup tP_2$.

Theorem 1.1 ([7]).

(i) *For n sufficiently large,* $\mathrm{sat}(n, P_3 \cup tP_2) = 3t$ *and* $tK_3 \cup (n-3t)K_1 \in \underline{\mathrm{SAT}}(n, P_3 \cup tP_2)$.

(ii) *For n sufficiently large,* $\mathrm{sat}(n, P_4 \cup tP_2) = 3t + 7$ *and*

$$K_5 \cup (t-1)K_3 \cup (n-3t-2)K_1 \in \underline{\mathrm{SAT}}(n, P_4 \cup tP_2).$$

(iii) *For n sufficiently large,* $\mathrm{sat}(n, P_5 \cup P_2) = 15$ *and* $K_6 \cup (n-6)K_1 \in \underline{\mathrm{SAT}}(n, P_5 \cup P_2)$.

In this paper, we prove the following theorem. In the following, let $V_i(G)$ be the set of vertices of G with degree i.

Theorem 1.2. *Let n and t be positive integers.*

(i) *For $1 \le t \le 2$ and n sufficiently large,* $\mathrm{sat}(n, P_5 \cup P_4 \cup tP_2) = 3t + 42$ *and* $\underline{\mathrm{SAT}}(n, P_5 \cup P_4 \cup tP_2) = K_{10} \cup (t-1)K_3 \cup (n-3t-7)K_1$.

(ii) $\mathrm{sat}(n, P_5 \cup P_4 \cup tP_2) = 3t + 21$ *and* $\underline{\mathrm{SAT}}(n, P_5 \cup P_4 \cup tP_2) = 3K_5 \cup (t-3)K_3 \cup (n-3t-6)K_1$*, where $t \ge 3$ and $n \ge 6t + 44$.*

2. Preliminaries

Lemma 2.1 (The Berge–Tutte Formula, [1]). *For a graph G,*

$$\alpha'(G) = \frac{1}{2}\min\{|V(G)| + |V(X)| - o(G - X) : X \subseteq V(G)\},$$

where $o(G)$ denotes the number of odd components of G and $\alpha'(G)$ denotes the number of edges in a maximum matching of G.

Lemma 2.2 ([7]). *Let $k_1, \ldots, k_m \ge 2$ be m integers and G be a $(P_{k_1} \cup P_{k_2} \cup \ldots \cup P_{k_m})$-saturated graph. Let x be a degree 2 vertex in G and u, v be the neighbors of x. Then $uv \in E(G)$.*

Lemma 2.3 ([10]). *Let G be a $(P_5 \cup tP_2)$-saturated and H be the graph spanned by all the nontrivial components $H_1, \ldots, H_k$ of G, where k is the number of components of G. If $|V(H)| \ge 2t + 5$, $\delta(H) \ge 2$ and $|V(H_i)| \ge 5$ for $1 \le i \le k$, then G is $(t+2)P_2$-saturated and $|E(G)| > 3t + 12$ if $V_0(G) \ne \emptyset$ where $V_0(G)$ is the set of isolated vertices of G.*

Now we discuss some properties of $(P_5 \cup P_4 \cup tP_2)$-saturated graphs.

Lemma 2.4. *Let G be a $(P_5 \cup P_4 \cup tP_2)$-saturated graph, where t is a positive integer. Then G has the following properties.*

(i) *If $|V_0(G)| > 0$, then $V_1(G) = \emptyset$.*

(ii) *If $|V_0(G)| > 0$, for any $x \in V(G) - V_0(G)$,*

we have $N_G(x) \cup \{x, y\} \subseteq V(F)$ where F is a copy of $P_5 \cup P_4 \cup tP_2$ in $G + xy$ and $y \in V_0(G)$.

Proof. (i) If $V_1(G) \neq \emptyset$, let x_1 be one vertex in $V_1(G)$ and x_1x_2 be the edge incident to x_1 ($d_G(x_2) \geq 1$). Then the graph $G + x_2x_3$ contains a copy F of $P_5 \cup P_4 \cup tP_2$ where $x_3 \in V_0(G)$ (obviously $x_1 \notin V(F)$). By replacing the edge x_2x_3 with x_1x_2, we get a copy of $P_5 \cup P_4 \cup tP_2$ in G, a contradiction. Therefore, $V_1(G) = \emptyset$.

(ii) If there is one vertex $x \in V(G) - V_0(G)$ satisfying $N_G(x) \cup \{x, y\} \nsubseteq V(F)$ where F is any copy of $P_5 \cup P_4 \cup tP_2$ in $G + xy$ and $y \in V_0(G)$, then there exists a vertex $x' \in N_G(x)$ such that $x' \in N_G(x) - V(F)$. Since G is a $P_5 \cup P_4 \cup tP_2$-saturated graph, $G + xy$ contains a copy of $P_5 \cup P_4 \cup tP_2$, say F. Thus $xy \in E(F)$. Then by replacing the edge xy with xx' in F, we obtain a copy of $P_5 \cup P_4 \cup tP_2$ in G, a contradiction. $\square$

Lemma 2.5. *Let G be $(P_5 \cup P_4 \cup tP_2)$-saturated with $|V_0(G)| \geq 2$ and H be the graph spanned by all the nontrivial components $H_1, \ldots, H_k$ of G, where k is the number of components of G. If $|V(H)| \geq 2t + 9$, $\delta(H) \geq 2$ and $|V(H_i)| \geq 5$ for $1 \leq i \leq k$, then G is $(P_5 \cup (t + 2)P_2)$-saturated.*

Proof. Since G is a $(P_5 \cup P_4 \cup tP_2)$-saturated graph, the additional edge $e \notin E(G)$ will result in a $P_5 \cup (t+2)P_2$ in $G + e$. Meanwhile, G contains a copy of $P_5 \cup P_4 \cup (t - 1)P_2$ since $|V_0(G)| \geq 2$.

By the contrary, if G is not $(P_5 \cup (s + 2)P_2)$-saturated, G contains a copy of $P_5 \cup (s + 2)P_2$. Let M be a copy of $P_5 \cup (s + 2)P_2$ in G with $E(M) = \{\omega_1\omega_2, \omega_2\omega_3, \omega_3\omega_4, \omega_4\omega_5\} \cup \{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$ which satisfies the following:

(i) s is as large as possible, and

(ii) Subject to (i), $\sum_{i=1}^{s+2}(d_G(u_i) + d_G(v_i))$ is maximality.

Observe that $s \geq t$. By the choice of M, $V(H) - V(M)$ is an independent set or an empty set. Without loss of generality, we may assume $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} \subseteq V(H_1)$. Since G is $(P_5 \cup P_4 \cup tP_2)$-saturated, G has no copy of $P_5 \cup P_4 \cup tP_2$, we have

$$G[V(M)\backslash\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}] \cong (s + 2)P_2. \tag{1}$$

Since H is the graph spanned by all the nontrivial components $H_1, \ldots, H_k$ of G with $\delta(H) \geq 2$ and $|V(H_i)| \geq 5$, we obtain that $k = 1$, otherwise G has a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction.

Claim 1. For each $1 \leq i \leq s + 2$, we have $N_G(\{u_i, v_i\}) \cap \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} \neq \emptyset$ where $N_G(\{u_i, v_i\})$ is the set of neighbors of $\{u_i, v_i\}$.

Proof. Assume that there exists some $i \in \{1, 2, \ldots, s+2\}$ such that $N_G(\{u_i, v_i\}) \cap \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} = \emptyset$. This together with (1) implies that

$$N_G(u_i) \cap V(M) = \{v_i\} \text{ and } N_G(v_i) \cap V(M) = \{u_i\}. \tag{2}$$

Since $\delta(H) \geq 2$, it follows from (2) that there exist $x, y \in V(H) - V(M)$ such that $xu_i \in E(H)$ and $yv_i \in E(H)$. If $x \neq y$, we replace the edge u_iv_i in $E(M)$ by xu_i, u_iv_i and yv_i, then G has a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. Therefore $x = y$ and $d_G(u_i) = d_G(v_i) = 2$. It is obvious that x is not adjacent with any vertex in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_iv_i\}$ since G has no copy of $P_5 \cup P_4 \cup tP_2$. Then $N_G(x) \cap \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} \neq \emptyset$. That is, $d_G(x) \geq 3$. Then we replace the edge u_iv_i in $E(M)$ by xu_i, $\sum_{j=1, j\neq i}^{s+2}(d_G(u_j) + d_G(v_j)) + d_G(x) + d_G(u_i) > \sum_{j=1}^{s+2}(d_G(u_j) + d_G(v_j))$, which contradicts the choice of M. Claim 1 is true. $\square$

Now we consider the following cases.

Case 1. There exist $1 \leq i, j \leq s + 2 (i \neq j)$, $N_G(\{u_i, v_i\}) \cap \{\omega_1, \omega_2\} \neq \emptyset$ and $N_G(\{u_j, v_j\}) \cap \{\omega_3, \omega_4, \omega_5\} \neq \emptyset$.

Since G has no copy of $P_5 \cup P_4 \cup tP_2$, we only consider that $N_G(\{u_i, v_i\}) \cap \{\omega_1, \omega_2\} = \{\omega_2\}$ and $N_G(\{u_j, v_j\}) \cap \{\omega_3, \omega_4, \omega_5\} = \{\omega_4\}$ for $1 \leq i, j \leq s + 2$ (G has a copy of $P_5 \cup P_4 \cup tP_2$ in other situations, a contradiction). Without loss of generality, we may assume that $\omega_2 u_i \in E(H)$ and $\omega_4 u_j \in E(H)$.

Since G has no copy of $P_5 \cup P_4 \cup tP_2$, ω_1 is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$ and ω_1 is not adjacent to ω_3, ω_5 or any vertex in $V(H) - V(M)$. By the symmetry, ω_5 is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$ and ω_5 is not adjacent to ω_1, ω_3 or any vertex in $V(H) - V(M)$. Then $\omega_1\omega_4, \omega_2\omega_5 \in E(H)$ since $\delta(H) \geq 2$. Also ω_3 is not adjacent to any vertex in $V(H) - V(M)$, we obtain that $d_G(\omega_3) = 2$, it follows that $\omega_2\omega_4 \in E(G)$ by Lemma 2.2.

If there exists $x \in V(H) - V(M)$ such that $N_G(x) \cap \{u_i, v_i\} \neq \emptyset$. If $v_ix \in E(H)$, then $P_5 = \omega_1\omega_2u_iv_ix$, $P_4 = \omega_3\omega_4u_jv_j$ and the edges in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_iv_i, u_jv_j\}$ contain a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. Hence $v_ix \notin E(H)$. That is, $u_ix \in E(H)$. Similarly, $\omega_2v_i \notin E(H)$. Then $v_i\omega_4 \in E(H)$ for $\delta(H) \geq 2$. That is, $d_G(v_i) = 2$, it follows that $u_i\omega_4 \in E(G)$ by Lemma 2.2. Since $\delta(H) \geq 2$, x is adjacent to ω_2 or ω_4. If $x\omega_2 \in E(H)$, $P_5 = \omega_1\omega_2xu_iv_i$, $P_4 = \omega_3\omega_4u_jv_j$ and the edges in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_iv_i, u_jv_j\}$ contain a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. So $x\omega_4 \in E(H)$. And if there exists other edge $u_kv_k \in \{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$ such that $N_G(\{u_k, v_k\}) \cap \{\omega_1, \omega_2\} = \{\omega_2\}$ for $k \neq i, k \neq j$, we may assume that $\omega_2u_k \in E(G)$, then $P_5 = \omega_5\omega_4xu_iv_i$, $P_4 = \omega_1\omega_2u_kv_k$ and the edges in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_iv_i, u_kv_k\}$ contain a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. Now we consider $x\omega_4 \in E(H)$ and there only exists one edge $u_iv_i \in \{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$ such that $N_G(\{u_i, v_i\}) \cap \{\omega_1, \omega_2\} = \{\omega_2\}$. Then for $1 \leq j \leq s+2$, $j \neq i$, $N_G(\{u_j, v_j\}) \cap \{\omega_1, \omega_2\} = \emptyset$ and $N_G(\{u_j, v_j\}) \cap \{\omega_3, \omega_4, \omega_5\} = \{\omega_4\}$. We may assume that $\omega_4u_j \in E(H)$. By the symmetry, we obtain that there does not exist $y \in V(H) - V(M)$ such that $N_G(y) \cap \{u_j, v_j\} \neq \emptyset$ for $u_jv_j \in \{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$ $(j \neq i)$. If $\omega_4v_j \notin E(H)$, then$G + \omega_4v_j$ has no copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. Hence $\omega_4v_j \in E(H)$. Choose $z \in V_0(G)$, then $G + w_4z$ has no copy of $P_5 \cup P_4 \cup tP_2$ based on the structure of G, a contradiction. Therefore there does not exist $x \in V(H) - V(M)$ such that $N_G(x) \cap \{u_i, v_i\} \neq \emptyset$ for $1 \leq i \leq s+2$.

Now if $\omega_2v_i \notin E(H)$, $G + \omega_2v_i$ has no copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. Therefore, $\omega_2v_i \in E(H)$. Similarly, $\omega_4u_i \in E(H)$, $\omega_4v_i \in E(H)$. By the symmetry, $\omega_2u_j \in E(H)$, $\omega_2v_j \in E(H)$ and $\omega_4v_j \in E(H)$. That is, for each $1 \leq i \leq s+2$, $N_G(\{u_i, v_i\}) \cap \{\omega_1, \omega_2\} = \{\omega_2\}$ ($\omega_2u_i \in E(H)$ and $\omega_2v_i \in E(H)$) and $N_G(\{u_i, v_i\}) \cap \{\omega_3, \omega_4, \omega_5\} = \{\omega_4\}$ ($\omega_4u_i \in E(H)$ and $\omega_4v_i \in E(H)$). If $V(H) - V(M) \neq \emptyset$, then for each $x \in V(H) - V(M)$, $x\omega_2 \in E(H)$ and $x\omega_4 \in E(H)$ since $\delta(H) \geq 2$. If $G + \omega_4z$ has some copy of $P_5 \cup P_4 \cup tP_2$ for $z \in V_0(G)$, then G contains one copy of $P_5 \cup P_4 \cup tP_2$ by replacing z with ω_5, a contradiction.

Case 2. For each $1 \leq i \leq s+2$, $N_G(\{u_i, v_i\}) \cap \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} = \{\omega_3\}$.

In this case, ω_1 is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$. And ω_1 is not adjacent to ω_4, ω_5 to avoid a copy of $P_5 \cup P_4 \cup tP_2$ in G. If there exists $x \in V(H) - V(M)$ such that $x\omega_1 \in E(H)$, then x is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$ and x is not adjacent to ω_4, ω_5. Moreover, x is not adjacent to any vertex in $V(H) - V(M)$ by the choice of M. That is, $x\omega_2 \in E(H)$ or $x\omega_3 \in E(H)$ since $\delta(H) \geq 2$. And there does not exist $y \in V(H) - V(M)$ such that $y\omega_2 \in E(H)$ for $y \neq x$, otherwise G has a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. By the symmetry, this situation holds true for ω_5.

Similarly, ω_2 is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$. And ω_2 is not adjacent to ω_4, ω_5. If there exists $z \in V(H) - V(M)$ such that $z\omega_2 \in E(H)$, then z is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$ and z is not adjacent to ω_4, ω_5. Moreover, z is not adjacent to any vertex in $V(H) - V(M)$. That is, $z\omega_1 \in E(H)$ or $z\omega_3 \in E(H)$ since $\delta(H) \geq 2$. And there does not exist $z' \in V(H) - V(M)$ such that $z'\omega_1 \in E(H)$ for $z' \neq z$, otherwise G has a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. By the symmetry, this situation holds true for ω_4.

By the above arguments,we can not obtain any copy of $P_5 \cup P_4 \cup (t-1)P_2$ in G, a contradiction.

Case 3. For $1 \leq i, j \leq s+2$, $N_G(\{u_i, v_i\}) \cap \{\omega_1, \omega_2\} \neq \emptyset$ and $N_G(\{u_j, v_j\}) \cap \{\omega_1, \omega_2\} \neq \emptyset$.

Subcase 3.1. $N_G(\{u_i, v_i\}) = \{\omega_1, \omega_2\}$, $N_G(\{u_j, v_j\}) \subseteq \{\omega_1, \omega_2\}$.

First we consider $N_G(u_i) \cap \{\omega_1, \omega_2\} \neq \emptyset$ and $N_G(v_i) \cap \{\omega_1, \omega_2\} \neq \emptyset$. Without loss of generality, we may assume $u_i\omega_1 \in E(H)$ and $v_i\omega_2 \in E(H)$.

If $\omega_1 \in N_G(\{u_j, v_j\})$, and we may assume that $\omega_1u_j \in E(H)$, then $P_5 = v_iu_i\omega_1u_jv_j$, $P_4 = \omega_2\omega_3\omega_4\omega_5\}$ and the edges in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_iv_i, u_jv_j\}$ contain a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. Hence, we just consider $N_G(\{u_j, v_j\}) \cap \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} = \{\omega_2\}$ for $j \neq i$. And we may assume that $\omega_2u_j \in E(H)$

for $j \neq i$. Thus ω_1 is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_iv_i\}$. And in order to avoid a copy of $P_5 \cup P_4 \cup tP_2$ in G, ω_1 is not adjacent to ω_3, ω_4 or ω_5. Moreover, ω_1 is not adjacent to any vertex in $V(H) - V(M)$. Similarly, ω_3 (and ω_5) is not adjacent any vertex in $V(H) - V(M)$. Together with Case 1, ω_3 (and ω_5) is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$. Then $\omega_2\omega_5 \in E(H)$ or $\omega_3\omega_5 \in E(H)$ since $\delta(H) \geq 2$. And if there exists $z \in V(H) - V(M)$ such that $N_G(z) \cap \{u_i, v_i\} \neq \emptyset$, then $G[z, u_i, v_i, \omega_1]$ does not contain a copy of P_4.

If ω_4 is adjacent to some vertex $x \in V(H) - V(M)$, then x is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$ to avoid a copy of $P_5 \cup P_4 \cup tP_2$ in G. Thus $\omega_2 x \in E(H)$ since $\delta(H) \geq 2$. If $\omega_3\omega_5 \in E(H)$, then $P_5 = v_ju_j\omega_2u_kv_k$ $(k \neq j, k \neq i)$, $P_4 = x\omega_4\omega_3\omega_5$ and the edges in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_jv_j, u_kv_k\}$ contain a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. Therefore we have $\omega_3\omega_5 \notin E(H)$ in this situation. Now we consider $\omega_2\omega_5 \in E(H)$, which means $d(\omega_3) = 2$. Then $\omega_2\omega_4 \in E(H)$ by Lemma 2.2. However, G has no copy of $P_5 \cup P_4 \cup (t-1)P_2$ based on the discussion of the structure of the graph G, a contradiction. Thus, there does not exist $x \in V(H) - V(M)$ such that $\omega_4 x \in E(H)$, which means $d(\omega_4) = 2$. It follows that $\omega_3\omega_5 \in E(H)$ by Lemma 2.2. Whether $\omega_2\omega_5 \in E(H)$ or not, G has no copy of $P_5 \cup P_4 \cup (t-1)P_2$ based on the discussion of the structure of the graph G, a contradiction.

Therefore now we consider $N_G(u_i) \cap \{\omega_1, \omega_2\} = \emptyset$ or $N_G(v_i) \cap \{\omega_1, \omega_2\} = \emptyset$. Without loss of generality, we may assume that $N_G(v_i) \cap \{\omega_1, \omega_2\} = \emptyset$ and $\omega_1u_i \in E(H)$, $\omega_2u_i \in E(H)$. Together with Case 1, we obtain that v_i is not adjacent to $\omega_3, \omega_4, \omega_5$. Thus v_i is adjacent to some vertex $y \in V(H) - V(M)$ since $\delta(H) \geq 2$.

If $\omega_2 \in N_G(\{u_j, v_j\})$, and we may assume that $\omega_2u_j \in E(H)$, then $P_5 = v_ju_j\omega_2\omega_3\omega_4$, $P_4 = yv_iu_i\omega_1$, and the edges in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_iv_i, u_jv_j\}$ contain a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. Hence, $N_G(\{u_j, v_j\}) \cap \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} = \{\omega_1\}$ for $j \neq i$ and we assume that $\omega_1u_j \in E(H)$. Now $P_5 = v_ju_j\omega_1u_iv_i$, $P_4 = \omega_2\omega_3\omega_4\omega_5$ and the edges in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_iv_i, u_jv_j\}$ contain a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction.

Subcase 3.2. For each $1 \leq i \leq s+2$, $|N_G(\{u_i, v_i\}) \cap \{\omega_1, \omega_2\}| = 1$.

If there exits $1 \leq i \leq s+2$ such that $N_G(\{u_i, v_i\}) \cap \{\omega_1, \omega_2\} = \{\omega_1\}$, and we may assume that $\omega_1u_i \in E(H)$, then any edge $u_jv_j \in \{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_iv_i\}$ is only incident to ω_2 in order to avoid a copy of $P_5 \cup P_4 \cup tP_2$ in G. And we assume that $\omega_2u_j \in E(H)$ for $j \neq i$. Combined with Case1, now we consider v_i is not adjacent to $\omega_2, \omega_3, \omega_4, \omega_5$. If $v_i\omega_1 \notin E(H)$, then v_i is adjacent to some vertex $x \in V(H) - V(M)$ since $\delta(H) \geq 2$. Now $P_5 = v_ju_j\omega_2\omega_3\omega_4$, $P_4 = xv_iu_i\omega_1$ and the edges in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_iv_i, u_jv_j\}$ contain a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. So we consider $v_i\omega_1 \in E(H)$. It follows that u_i(and v_i) is not adjacent to any vertex $x \in V(H) - V(M)$. In order to avoid a $P_5 \cup P_4 \cup tP_2$ in G, ω_1 is not adjacent to ω_3, ω_5 and ω_1 is not adjacent to any vertex in $V(H) - V(M)$. Similarly, ω_3 (and ω_5) is not adjacent to any vertex in $V(H) - V(M)$. And in this case, ω_3 (ω_5) is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$. Therefore $\omega_3\omega_5 \in E(H)$ or $\omega_2\omega_5 \in E(H)$ since $\delta(H) \geq 2$. If $\omega_3\omega_5 \in E(H)$, then ω_4 is not adjacent to any vertex in $V(H) - V(M)$ in order to avoid a copy of $P_5 \cup P_4 \cup tP_2$ in G. And in this case ω_4 is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$. If $\omega_1\omega_4 \in E(H)$, then $P_5 = v_ju_j\omega_2u_kv_k$, $P_4 = \omega_1\omega_4\omega_3\omega_5$ and the edges in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_jv_j, u_kv_k\}$ for $k, j \neq i$ contain a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. Thus we consider $\omega_1\omega_4 \notin E(H)$ in this situation. However, we will not obtain a copy of $P_5 \cup P_4 \cup (t-1)P_2$ in G, a contradiction. If $\omega_2\omega_5 \in E(H)$, then ω_4 is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$. If $\omega_1\omega_4 \in E(H)$, then $P_5 = v_iu_i\omega_1\omega_4\omega_5$, $P_4 = v_ju_j\omega_2\omega_3$ and the edges in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\} - \{u_iv_i, u_jv_j\}$ contain a copy of $P_5 \cup P_4 \cup tP_2$, a contradiction. Thus we obtain $\omega_1\omega_4 \notin E(H)$ in this situation. And we will not obtain a copy of $P_5 \cup P_4 \cup (t-1)P_2$ in G, a contradiction.

Now we consider $N_G(\{u_i, v_i\}) \cap \{\omega_1, \omega_2\} = \{\omega_2\}$ for each $1 \leq i \leq s+2$. Without loss of generality, we may assume $u_i\omega_2 \in E(H)$.

If ω_1 is adjacent to some vertex $x \in V(H) - V(M)$, then x is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$ and x is not adjacent to ω_3, ω_4 or ω_5 in order to avoid a copy of $P_5 \cup P_4 \cup tP_2$ in G. Thus $\omega_2x \in E(H)$. Similarly, ω_1 is not adjacent to ω_3, ω_4 or ω_5. And in this case ω_3 (and ω_5) is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$. Meanwhile, ω_3 (and ω_5) is not adjacent to any vertex in $V(H)-V(M)$. Therefore $\omega_3\omega_5 \in E(H)$ or $\omega_2\omega_5 \in E(H)$ since $\delta(H) \geq 2$. If $\omega_3\omega_5 \in E(H)$, then ω_4 is not adjacent to any vertex in $V(H)-V(M)$ and ω_4 is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$. However, we will not obtain a copy of $P_5 \cup P_4 \cup (t-1)P_2$ in G, a contradiction. It follows that $\omega_3\omega_5 \notin E(H)$. We consider $\omega_2\omega_5 \in E(H)$ in this situation. Meanwhile, G does not contain a copy of $P_5 \cup P_4 \cup (t-1)P_2$, a contradiction.

Therefore ω_1 is not adjacent to any vertex $x \in V(H)-V(M)$. Thus ω_1 is only adjacent to ω_4. And ω_3 (and ω_5) is not adjacent to any vertex in $V(H)-V(M)$ in order to avoid a copy of $P_5 \cup P_4 \cup tP_2$ in G. In this case, ω_3 (and ω_5) is not incident with any edge in $\{u_1v_1, u_2v_2, \ldots, u_{s+2}v_{s+2}\}$. Furthermore, $\omega_3\omega_5 \notin E(H)$. Thus $\omega_2\omega_5 \in E(H)$ since $\delta(H) \geq 2$. And $\omega_2\omega_4 \in E(H)$ since $d(\omega_3) = 2$ by Lemma 2.2. However, we will not obtain a copy of $P_5 \cup P_4 \cup (t-1)P_2$ in G, a contradiction.

Hence, Lemma 2.5 is true. □

Lemma 2.6. *Let G be $(P_5 \cup P_4 \cup tP_2)$-saturated with $|V_0(G)| \geq 2$ $(t \geq 1)$ and H be the graph spanned by all the nontrivial components $H_1, \ldots, H_k$ of G, where k is the number of components of G. If $|V(H)| \geq 2t+9$, $\delta(H) \geq 2$ and $|V(H_i)| \geq 5$ for $1 \leq i \leq k$, then $|E(G)| > 3t+21$ for $t > 3$, $|E(G)| \geq 3t+21$ for $t = 3$, and $|E(G)| > 3t+42$ for $1 \leq t \leq 2$.*

Proof. Assume that $|E(G)| \leq 3t+21$ for $t \geq 4$ and $|E(G)| \leq 3t+20$ for $t = 3$. It follows from Lemma 2.5 that G is $(P_5 \cup (t+2)P_2)$-saturated. And G contains a copy of $P_5 \cup P_4 \cup (t-1)P_2$ since $|V_0(G)| \geq 2$. Hence $\alpha'(G) = \alpha'(H) \geq t+3$. If $\alpha'(H) \geq t+4$, since $\delta(H) \geq 2$ and $|V(H_i)| \geq 5$ for $1 \leq i \leq k$, then H contains a copy of $(P_5 \cup (t+2)P_2)$, which contradicts G is $(P_5 \cup (t+2)P_2)$-saturated. Therefore, we have $\alpha'(G) = \alpha'(H) = t+3$. By the Berge-Tutte Formula, we have

$$t+3 = \frac{1}{2}\min\{|V(H)| + |V(X)| - o(H-X) : \ X \subseteq V(H)\}. \tag{3}$$

Choose a maximal subset S of $V(H)$ such that

$$t+3 = \frac{1}{2}(|V(H)| + |V(S)| - o(H-S)). \tag{4}$$

Let $H'_1, \ldots, H'_p$ be the components of $H-S$. We have the following claims.

Claim 1. For $1 \leq i \leq p$, $H[S \cup V(H'_i)]$ is a complete graph.

Proof. On the contrary, assume that there exist two distinct vertices $x,\ y \in S \cup V(H'_i)$ such that $xy \notin E(H)$. Set $H' = H + xy$. Then $|H'| = |H|$ and $o(H'-S) = o(H-S)$. By (4), we have

$$t+3 = \frac{1}{2}(|V(H')| + |V(S)| - o(H'-S)). \tag{5}$$

By the Berge-Tutte Formula, we have $\alpha'(H') \leq t+3$, which implies that $G+xy$ has no copy of $(t+4)P_2$, contrary to G is $(P_5 \cup P_4 \cup tP_2)$-saturated. Claim 1 is true. □

Claim 2. $S \neq \emptyset$.

Proof. If $S = \emptyset$, then $H'_1, \ldots, H'_p$ are all components $H_1, \ldots, H_k$ of H with $p = k$. By Claim 1, H_i is a complete graph of order at least 5 for $1 \leq i \leq k$. It follows that $\delta(H) \geq 4$ and

$$2|E(H)| = \sum_{x \in V(H)} d_H(x) \geq 4|V(H)|. \tag{6}$$

If $t > 3$, then $|E(H)| = |E(G)| \leq 3t+21$. This together with $|V(H)| \geq 2t+9$ implies that $t \leq 3$, a contradiction.

If $t = 3$, then $|E(H)| = |E(G)| \leq 3t + 20$. This together with $|V(H)| \geq 2t + 9$ implies that $t \leq 2$, a contradiction.

If $t = 1$, then $|V(H)| \geq 11$. And H_i is a complete graph of order at least 5, hence $\alpha'(H)| \geq 5 > t + 3$, a contradiction. If $t = 2$, then $|V(H)| \geq 13$. Similarly, we obtain that $\alpha'(H)| \geq 6 > t + 3$ in this situation, a contradiction. This completes the proof of Claim 2. □

Combining Claim 1 with Claim 2, $N_H(x) = V(H) - \{x\}$ for $x \in S$. For $y \in V_0(G)$, we have $\{x, y\} \cup N_H(x) \subseteq V(F)$, where F is a copy of $P_5 \cup P_4 \cup tP_2$ in $G + xy$ by Lemma 2.4. And $2t + 10 \leq |V(H)| + 1 = |\{x, y\} \cup N_H(x)| \leq |V(F)| = 2t + 9$, a contradiction. This completes the proof of Lemma 2.6. □

3. Proof of Theorem 1.2

Proof. Let G be a $(P_5 \cup P_4 \cup tP_2)$-saturated graph. Consider the graph G' obtained from G by deleting all the components of order 3 and the components of order 4, we have $G \cong G' \cup aK_3 \cup bK_4$ where a is the number of components of order 3 and b is the number of components of order 4 in G. Clearly, all components of order 3 or 4 are complete.

Since $|V_0(G')| = |V_0(G)| \geq 2$, it follows that $V_1(G) = \emptyset$ and G contains a copy of $P_5 \cup P_4 \cup (t-1)P_2$. If $b = 1$, then $G - K_4 - aK_3$ contains a copy of $P_5 \cup (t - 1 - a)P_2$ and contains no copy of $P_5 \cup (t-a)P_2$. Thus, $G - K_4 - aK_3$ is $(P_5 \cup (t-a)P_2)$-saturated. It follows that $G - K_4 - aK_3$ is $(t - a + 2)P_2$-saturated by Lemma 2.3. However, $G + xy$ contains a copy of $P_5 \cup P_4 \cup tP_2$ for $x \in V(K_4)$ and $y \in V_0(G)$. It implies that $G - K_4 - aK_3$ contains a copy of $P_4 \cup (t-a)P_2$, then $G - K_4 - aK_3$ contains a copy of $(t - a + 2)P_2$, a contradiction. If $b \geq 2$, let H_1 and H_2 be two components of order 4, then $G + xy$ contains no copy of $P_5 \cup P_4 \cup tP_2$ for $x \in V(H_1)$ and $y \in V(H_2)$, a contradiction. Therefore $b = 0$.

(i) For $1 \leq t \leq 2$, $K_{10} \cup (t-1)K_3 \cup (n - 3t - 7)K_1 \in \mathrm{SAT}(n, P_5 \cup P_4 \cup tP_2)$. Let $G \in \underline{\mathrm{SAT}}(n, P_5 \cup P_4 \cup tP_2)$, then $|E(G)| \leq 3t + 42$. Since n is sufficiently large ($n \geq 6t + 86$), we obtain that $|V_0(G)| \geq 2$. And G contains a copy of $P_5 \cup P_4 \cup (t-1)P_2$. Then $a \leq t - 1$.

Set $G' = G - aK_3$. As $G \in \underline{\mathrm{SAT}}(n, P_5 \cup P_4 \cup tP_2)$, $G' \in \underline{\mathrm{SAT}}(n', P_5 \cup P_4 \cup t'P_2)$ where $n' = n - 3a$ and $t' = t - a$. Then $t' \geq 1$. Suppose H' is a graph spanned by all nontrivial components of G'. Clearly, any component in H' has order at least 5. Since $|E(G)| \leq 3t + 42$, $|E(G')| = |E(G)| - 3a \leq 3t' + 42$. And $|V_0(G')| \geq 2$, we have $|V(H')| \leq 2t' + 8$ by Lemma 2.6. It follows that $H' = K_{2t'+8}$ and $|E(G')| = |E(H')| = \frac{(2t'+8)(2t'+7)}{2} \leq 3t' + 42$, which implies that $t' = 1$ and $H' \cong K_{10}$. Thus $G' \cong K_{10} \cup (n' - 10)K_1$. It follows that $a = t - 1$ and $G \cong K_{10} \cup (t-1)K_3 \cup (n - 3t - 7)K_1$.

Hence, Theorem 1.2(i) is true.

(ii) For $t \geq 3$, $3K_5 \cup (t-3)K_3 \cup (n - 3t - 6)K_1 \in \mathrm{SAT}(n, P_5 \cup P_4 \cup tP_2)$. Let $G \in \underline{\mathrm{SAT}}(n, P_5 \cup P_4 \cup tP_2)$, then $|E(G)| \leq 3t + 21$. Since n is sufficiently large ($n \geq 6t + 44$), we obtain hat $|V_0(G)| \geq 2$. By Lemma 2.2, we have $V_1(G) = \emptyset$. And G contains a copy of $P_5 \cup P_4 \cup (t-1)P_2$. That is, $a \leq t - 1$.

Set $G \cong G' \cup aK_3$. As $G \in \underline{\mathrm{SAT}}(n, P_5 \cup P_4 \cup tP_2)$, $G' \in \underline{\mathrm{SAT}}(n', P_5 \cup P_4 \cup t'P_2)$ where $n' = n - 3a$ and $t' = t - a$. Then $t' \geq 1$. Suppose H' is a graph spanned by all nontrivial components of G'. Clearly, any component in H' has order at least 5. If $|E(G)| < 3t + 21$, then $|E(G')| = |E(G)| - 3a < 3t' + 21$. Since $|V_0(G')| \geq 2$, we have $|V(H')| \leq 2t' + 8$. It follows that $H' = K_{2t'+8}$ and $|E(G')| = |E(H')| = \frac{(2t'+8)(2t'+7)}{2} > 3t' + 21$, a contradiction. Therefore $|E(G')| = |E(H')| = 3t' + 21$ and $|E(G)| = 3t' + 21 + 3a = 3t + 21$. That is, $\mathrm{sat}(n, P_5 \cup P_4 \cup tP_2) = 3t + 21$.

Let $G \in \underline{\mathrm{SAT}}(n, P_5 \cup P_4 \cup tP_2)$. Then $|E(G)| = 3t + 21$. Set $G \cong G' \cup aK_3$ for $a \leq t - 1$.

If $a = t - 1$, then G' is $(P_5 \cup P_4 \cup P_2)$-saturated, and $|E(G')| > 3 \times 1 + 42 = 45$ by Lemma 2.6 and $|E(G)| = |E(G')| + 3(t-1) > 3t + 42$, a contradiction.

If $a = t - 2$, then G' is $(P_5 \cup P_4 \cup 2P_2)$-saturated, then $|E(G')| > 3 \times 2 + 42 = 48$ by Lemma 2.6 and $|E(G)| = |E(G')| + 3(t-2) > 3t + 42$, a contradiction.

If $a < t - 3$, then G' is $(P_5 \cup P_4 \cup (t-a)P_2)$-saturated. Since $t - a > 3$, then $|E(G')| > 3(t-a) + 21$ by Lemma 2.6 and $|E(G)| = |E(G')| + 3a > 3t + 21$, a contradiction.

If $a = t-3$, then G' is $(P_5 \cup P_4 \cup 3P_2)$-saturated and $|E(G')| = 30$. Since $\alpha'(G') = 3+3 = 6$ by Lemma 2.6. And each nontrivial component H_i' of G' has order at least 5 with $\delta(H_i') \geq 2$, we have $\alpha'(H_i') \geq 2$ and each H_i' contains a copy of P_5. Then the number of nontrivial components of G' is at most 3.

If the number of nontrivial components of G' is 3, we may assume that the three nontrivial components are H_1', H_2' and H_3'. Then $\alpha'(H_i') = 2$ for $1 \leq i \leq 3$. If there exists H_i' such that $|V(H_1')| \geq 6$, we may assume that $|V(H_1')| \geq 6$. Then $G' + xy$ contains a copy of $P_5 \cup P_4 \cup 3P_2$ for $xy \notin E(H_1')$, $x, y \in V(H_1')$. Since each component contains a copy of P_5, $H_1' + xy$ is selected as $P_5 \cup P_2$, $P_4 \cup P_2$ or $3P_2$ in some copy of $P_5 \cup P_4 \cup 3P_2$. If $H_1' + xy$ is selected as $P_5 \cup P_2$, then $G' - H_1'$ contains a copy of $P_4 \cup 2P_2$. It implies that $G' - H_1'$ is $(P_4 \cup 3P_2)$-saturated and $|E(G' - H_1')| \geq 3 \times 3 + 7 = 16$ by Theorem 1.1. Meanwhile, $G - H_2' - H_3'$ is $(P_5 \cup P_2)$-saturated, $|E(H_1')| \geq 3 \times 1 + 12 = 15$ by Lemma 2.3. Then $|E(G')| \geq 16 + 15 > 30$, a contradiction. If $H_1' + xy$ is selected as $P_4 \cup P_2$, then $G' - H_1'$ contains a copy of $P_5 \cup 2P_2$. It implies that $G' - H_1'$ is $(P_5 \cup 3P_2)$-saturated. We obtain that $|E(G' - H_1')| \geq 3 \times 3 + 12 = 21$ by Lemma 2.3. Meanwhile, we obtain that $G' - H_2' - H_3'$ is $(P_4 \cup P_2)$-saturated and the number of edges is at least $3 \times 1 + 7 = 10$ by Theorem 1.1. Hence $|E(G')| \geq 21 + 10 > 30$, a contradiction. If $H_1' + xy$ is selected as $3P_2$, then $G' - H_1'$ contains a copy of $P_5 \cup P_4$. Similarly, $G' - H_1'$ is $(P_5 \cup P_4 \cup P_2)$-saturated. Then $|E(G' - H_1')| \geq 3 \times 1 + 42 > 30$ by Theorem 1.2(i). Therefore, $|E(G')| > 30$, a contradiction. So the order of each H_i' is 5 and $H_i' \cong K_5$. Therefore $G \cong 3K_5 \cup (t-3)K_3 \cup (n-3t-6)K_1$.

If the number of nontrivial components of G' is 2, we may assume that the two nontrivial components are H_1', H_2', respectively. Then $\alpha'(H_1') = 2$, $\alpha'(H_2') = 4$ or $\alpha'(H_1') = 3$, $\alpha'(H_2') = 3$. If $\alpha'(H_1') = 2$, $\alpha'(H_2') = 4$. We also obtain contradictions through discussions similar to the above. Now we consider $\alpha'(H_1') = \alpha'(H_2') = 3$. If there exists some component H_i' such that $|V(H_i')| = 6$, we may assume that $|V(H_1')| = 6$. Then $H_1' \cong K_6$ and $|V(H_2')| \geq 9$. It implies that H_2' is $(P_5 \cup 2P_2)$-saturated and $|E(H_2')| \geq 3 \times 2 + 12 = 18$ by Lemma 2.3. Then $|E(G')| \geq 18 + 15 > 30$, a contradiction. If there exists some component H_i' such that $|V(H_i')| = 7$, we may assume that $|V(H_1')| = 7$. Then $H_1' \cong K_7$ and $|V(H_2')| \geq 8$. It implies that H_2' is $(P_4 \cup 2P_2)$-saturated and $|E(H_2')| \geq 3 \times 2 + 7 = 13$ by Theorem 1.1. Then $|E(G')| \geq 13 + 21 > 30$, a contradiction. If there exists some component H_i' such that $|V(H_i')| = 8$, we may assume that $|V(H_1')| = 8$. Then $|V(H_2')| \geq 8$. If $|V(H_2')| \geq 9$, then H_2' is $(P_5 \cup 2P_2)$-saturated and $|E(H_2')| \geq 3 \times 2 + 12 = 18$ by Lemma 2.3, H_2' is $(P_4 \cup 2P_2)$-saturated and $|E(H_1')| \geq 3 \times 2 + 7 = 13$ by Theorem 1.1, Then $|E(G')| \geq 18 + 13 > 30$, a contradiction. If $|V(H_2')| = 8$, then $G' + xy$ contains no copy of $P_5 \cup P_4 \cup 3P_2$ for $xy \notin E(H_2')$, a contradiction.

If the number of nontrivial components of G' is 1, we may assume that the nontrivial component is H_1'. Then $\alpha'(H_1') = 6$. Let $M = \{u_1v_1, u_2v_2, u_3v_3, u_4v_4, u_5v_5, u_6v_6\}$ be a maximum matching of H_1'. Since G' has copy of $P_5 \cup P_4 \cup 2P_2$, we may assume that $P_5 = xu_1v_1u_2v_2$ for some $x \in V(H_1') - V(M)$, $P_4 = u_3v_3u_4v_4$ and $2P_2 = \{u_5v_5, u_6v_6\}$. The subgraph induced by $V(G') - \{u_3, v_3, u_4, v_4, u_5, v_5, u_6, v_6\}$ is $P_4 \cup P_2$-saturated and the number of edges is at least $3 \times 1 + 7 = 10$ by Theorem 1.1. Similarly, the subgraph induced by $V(G') - \{u_1, v_1, u_2, v_2, u_5, v_5, u_6, v_6\}$ is $P_4 \cup P_2$-saturated and the number of edges is at least 10. Meanwhile, the subgraph induced by $V(G') - \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}$ is $(P_4 \cup P_2)$-saturated and the number of edges is at least 10. Since he number of nontrivial components of G' is 1 and $|V(H_1') - V(M)| \geq 3$, then $|E(G')| \geq 30 + 1 > 30$, a contradiction.

Hence, Theorem 1.2(ii) is true. □

References

[1] C. Berge, Sur Le couplage maximum d'un graphe. *C. R. Acad. Sci. Paris* **247** (1958) 258–259.

[2] T. Bohman, M. Fonoberova and O. Pikhurko, The saturation function of complete partite graphs. *J. Comb.* **1** (2010)149–170.

[3] J. Bondy and U. Murty, Graph Theory With Applicatins. American Elsevier, New York (1976).

[4] N. Bushaw and N. Kettle, Turán numbers of multiple paths and equibipartite forests. *Comb. Probab. Comput.* **20** (2011) 837–853.

[5] S. Cao, H. Lei, X. Lian, S. Yao and J. Zhang, Saturation numbers for tP_k with k less than 6. *Discrete Appl. Math.* **325** (2023) 108–119.
[6] Y. Chen, All minimum C_5-saturated graphs. *J. Graph Theory* **67** (2011) 9–26.
[7] G. Chen, J.R. Faudree, R.J. Faudree, R.J. Gould, M.S. Jacobson and C. Magnant, Results and problems on saturation numbers for linear forests. *Bull. Inst. Comb. Appl.* **75** (2015) 29–46.
[8] B. Currie, J.R. Faudree, R.J. Faudree and J.R. Schmitt, A survey of minimum saturation graphs. *Electron. J. Comb.* **18** (2021) DS19.
[9] P. Erdös, A. Hajnal and J.W. Moon, A problem in graph theory. *Amer. Math. Monthly* **71** (1964) 1107–1110.
[10] Q. Fan and C. Wang, Saturation numbers for linear forests $P_5 \cup tP_2$. *Graph. Combin.* **31** (2015) 2193–2200.
[11] R. Faudree and R. Gould, Saturation numbers for nearly complete graphs. *Graph. Combin.* **29** (2013) 429–448.
[12] Z. He, M. Lu and Z. Lv, Minimum tP_3-saturation graphs. *Discrete Appl. Math.* **327** (2023) 148–156.
[13] S. Huang, H. Lei, Y. Shi and J. Zhang, The saturation number of $K_{3,3}$. *Discrete Math.* **347** (2024) 13.
[14] L. Kászonyi and Z. Tuza, Saturated graphs with minimal number of edges. *J. Graph Theory* **10** (1986) 203–210.
[15] E. Sullivan and P. Wenger, Saturation numbers in tripartite graphs. *J. Graph Theory* **84** (2017) 428–442.
[16] C. Timmons, C_{2k}-Saturated graphs with no short odd cycles. *Graphs Comb.* **35** (2019) 1023–1034.
[17] P. Turán, Eine extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok* **48** (1941) 436–452.