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GEODETIC COLORING OF GRAPHS

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Abstract. A vertex subset S of a graph $G = (V, E)$ is said to be *geodetic set* if every vertex in $V - S$ lies in some geodesic of any two vertices in S. The minimum cardinality of a geodetic set of G is called as *geodetic number* of G, and is denoted by $g(G)$. A *geodetic coloring* of a graph G is a proper vertex coloring of G in which every vertex in at least one geodetic set of G receives a different color. The minimum number of colors used in a geodetic coloring of G is called the *geodetic chromatic number* of G, and is denoted by $\chi_g(G)$. In this paper, we conduct a detailed study of $\chi_g(G)$, presenting existence results, bounds in relation to other graph parameters, and characterizations of graphs for which $\chi_g(G) = 2$ and $\chi_g(G) = n$, where n is the order of the graph. Furthermore, we prove that for every tree T, $\chi_g(T)$ equals the number of pendant vertices in T, unless T is a star or a path on odd vertices.

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1. Introduction and motivation

Many research articles have been published on the combined study of graph coloring with other graph concepts, such as domination (dominator coloring [7, 8], dominated coloring [11], gamma coloring [9], and others) and irredundance (irredundance coloring [5] and irredundance compelling coloring [6]). Motivated by these works, we initiate a study on parameters combining geodetic and coloring concepts. For this purpose, we introduce the concept of geodetic coloring of graphs in this paper.

All graphs considered in this paper are finite, undirected, simple (without loops and multiple edges). The sets $V(G)$ and $E(G)$ denote the vertex set and edge set of the graph G, respectively, and the cardinality of $V(G)$ is the order of G. The degree of a vertex v of G is the number of edges incident to v in G, and is denoted by $\deg(v)$. A vertex with degree 1 is called a pendant vertex. The open neighborhood $N(u)$ of a vertex u is the set of all vertices of G adjacent to u, and the closed neighborhood is $N[u] = N(u) \cup \{u\}$. A subset of the vertex set is said to be a vertex independent set if no two vertices in it are adjacent. The maximum cardinality among all vertex independent sets of the graph G is called the independence number, and is denoted by $\alpha(G)$. A universal vertex of a graph G is a vertex that is adjacent with all the remaining vertices of G. A graph is said to be bipartite if its vertex set can be partitioned into two independent sets. The diameter of a graph G is the maximum distance between any two vertices, and is denoted by $diam(G)$. A geodesic between any two vertices of the graph G is

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a shortest path between them in G. A graph G is said to be a cyclic graph if it contains at least one cycle. A tree is a connected acyclic (cycle-free) graph. The clique number of a graph G is the maximum order among all complete subgraphs of G, and is denoted by $\omega(G)$. For a set K of vertices of G, $\langle K\rangle$ is the subgraph of G induced by K. The union $G_1 \cup G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G = (V, E)$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. For any positive integer k, kG denoted the graph of k disjoint copies of G. The join of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is a graph formed by connecting each vertex of G_1 to each vertex of G_2. The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2. For a graph $G = (V, E)$, the Mycielskian of G is the graph with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' \mid x \in V\}$ and V' is disjoint from V, and edge set $E' = E \cup \{xy' \mid xy \in E\} \cup \{x'u \mid x' \in V'\}$, and is denoted by $\mu(G)$. A dominating set S of a graph G is the set of vertices of G such that each vertex in $V(G) - S$ has an adjacent vertex in S. The minimum cardinality of a dominating set of G is called the domination number of G, denoted by $\gamma(G)$. For more details on basic definitions and terminology, refer to [2].

A proper vertex coloring of the graph G is the assignment of colors to the vertices in G in which no two adjacent vertices receive same color. The minimum number of colors used in a proper vertex coloring of G is called the chromatic number of G, and is denoted by $\chi(G)$. A χ-coloring of G is the proper vertex coloring of G using $\chi(G)$ colors. A coloring $\mathcal{C} = (V_1, \dots V_k)$ of G, partitions $V(G)$ into independent sets V_i $(1 \le i \le k)$ and each V_i is set of vertices receiving color i, called as color class and any set S is said to be colorful in $\mathcal{C}$ if no two vertices in S receive same color. For more details on coloring of graphs refer to [1, 3].

For two vertices u and v of a graph G, the set $I(u, v)$ consists of all vertices lying on some $u - v$ geodesic (a shortest path between u and v) of G. If S is a set of vertices of G, then we let $I(S) = \bigcup_{u,v\in S} I(u, v)$. A set S of vertices of G is said to be *geodetic set* if $I(S) = V(G)$. The *geodetic number* $g(G)$ is the minimum cardinality among the geodetic sets of G. For more detailed study on geodetic sets, refer to [4, 10].

The following results are utilized in the proofs of some results in this paper.

Theorem 1 ([4])**.** *If v is a vertex of graph G such that $\langle N(v)\rangle$ is complete then v belongs to every geodetic set of G.*

Theorem 2 ([4])**.** *For any tree T, the set of all pendant vertices of T is contained in every geodetic set of T and is the unique minimum geodetic set of G.*

2. Definition and example

Definition 1. A geodetic coloring $\mathcal{C}$ of a graph G is a proper vertex coloring of G in which there exists a geodetic set S of G such that no two vertices in S receive the same color in $\mathcal{C}$ (S is colorful in $\mathcal{C}$). The minimum number of colors used in a geodetic coloring of G is called the geodetic chromatic number of G, and is denoted by $\chi_g(G)$. A geodetic coloring of G with $\chi_g(G)$ colors is said to be the minimum geodetic coloring or χ_g-coloring of the graph G.

For any graph G, assigning of colors to the vertices of G such that no two vertices receive the same color gives rise to a geodetic coloring of G with $V(G)$ as a colorful geodetic set. This shows that every graph admits a geodetic coloring.

Example 1. For the graph G shown in Figure 1, $\chi(G) = 3$ and $g(G) = 3$. By Theorem 1, it follows that $\{v_1, v_3, v_5\}$ is contained in every geodetic set of G. Any proper vertex coloring with v_1, v_3 and v_5 receive different colors requires five colors, implying that $\chi_g(G) \ge 5$. Assign different colors to different vertices in G, this coloring is a geodetic coloring with $\{v_1, v_3, v_5\}$ as a colorful geodetic set. Therefore, $\chi_g(G) = 5$.

Proposition 1. *For the cycle C_n of order $n \ge 3$,* $\chi_g(C_n) = \begin{cases} 2, & \textit{if } gcd(n,\ 4) = 2 \\ 3, & \textit{otherwise} \end{cases}$.

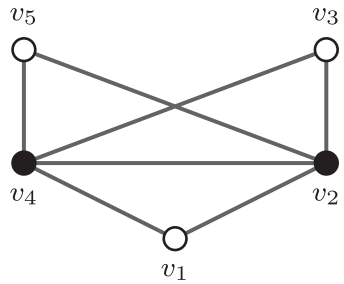

FIGURE 1. Graph G with $\chi(G) = 3$, $g(G) = 3$ and $\chi_g(G) = 5$.

Proof. Let $C_n = v_1 v_2 \dots v_n v_1$ be the cycle of length n.

Case 1. $\gcd(n, 4) = 2$.

In this case, the coloring $(\{v_1, v_3, \dots, v_{n-1}\}, \{v_2, v_4, \dots v_n\})$ is a geodetic coloring of G with colorful geodetic set $\{v_1, v_{\frac{n}{2}+1}\}$. Therefore, $\chi_g(C_n) = 2$.

Case 2. $\gcd(n, 4) \neq 2$.

Case 2.1. $\gcd(n, 4) = 1$.

Clearly, for $\gcd(n, 4) = 1$, $\chi(C_n) = 3$, hence it follows that $\chi_g(C_n) \geq 3$.

If $\lceil \frac{n}{2} \rceil$ is even, then the coloring $(\{v_1, v_3, \dots, v_n\} - \{v_{\lceil \frac{n}{2} \rceil+1}\}, \{v_2, v_4, \dots v_{n-1}\}, \{v_{\lceil \frac{n}{2} \rceil+1}\})$ is a geodetic coloring of G with colorful geodetic set $\{v_1, v_{\lceil \frac{n}{2} \rceil}, v_{\lceil \frac{n}{2} \rceil+1}\}$.

If $\lceil \frac{n}{2} \rceil$ is odd, then the coloring $(\{v_1, v_3, \dots, v_n\} - \{v_{\lceil \frac{n}{2} \rceil}\}, \{v_2, v_4, \dots v_{n-1}\}, \{v_{\lceil \frac{n}{2} \rceil}\})$ is a geodetic coloring of G with colorful geodetic set $\{v_1, v_{\lceil \frac{n}{2} \rceil}, v_{\lceil \frac{n}{2} \rceil+1}\}$.

Therefore, $\chi_g(C_n) = 3$.

Case 2.2. $\gcd(n, 4) = 4$.

Clearly, $\{v_1, v_{\frac{n}{2}+1}\}$ is a minimum geodetic set, and in every χ-coloring of G, the vertices v_1 and $v_{\frac{n}{2}+1}$ receive the same color. It follows that at least three colors are required for any proper vertex coloring of G that makes the geodetic set colorful. The coloring $(\{v_1, v_3, \dots, v_{n-1}\} - \{v_{\frac{n}{2}+1}\}, \{v_2, v_4, \dots v_n\}, \{v_{\frac{n}{2}+1}\})$ is a geodetic coloring of G with colorful geodetic set $\{v_1, v_{\frac{n}{2}+1}\}$. Therefore, $\chi_g(C_n) = 3$.

□

Proposition 2. *For complete bi-partite graph $K_{n,m}$ with $n, m \geq 4$, $\chi_g(K_{n,m}) = 4$.*

Proof. Let $K_{n,m}$ is complete bi-partite graph with $n, m \geq 4$ with partite sets $V_1 = \{v_1, v_2, \dots v_n\}$ and $V_2 = \{u_1, u_2, \dots, u_m\}$. Since, $n, m \geq 4$, it is easy to verify that no set with less than four vertices can be a geodetic set of $K_{m,n}$, hence it follows that $\chi_g(G) \geq g(G) \geq 4$. The coloring $(V_1 - \{v_1\}, V_2 - \{u_1\}, \{v_1\}, \{u_1\})$ is a geodetic coloring of $K_{n,m}$ with colorful geodetic set $\{v_1, v_2, u_1, u_2\}$. Therefore $\chi_g(K_{n,m}) = 4$. □

3. Existence results

Question 1. Is the number of colors required to make a minimum geodetic set colorful in a proper vertex coloring of G always equal to the minimum number of colors used in a geodetic coloring of G?

The answer to Question 1 is provided by the following theorem.

Theorem 3. *There is a graph G such that in any χ_g-coloring of G, no minimum geodetic set is colorful.*

Proof. For the graph G shown in Figure 2, the set $\{u_6, u_7\}$ is the only minimum geodetic set of G, and five colors are required for any vertex coloring of G such that u_6 and u_7 receive different colors. However, the coloring $(\{u_1\}, \{u_2, u_5\}, \{u_3, u_4\}, \{u_6, u_7\})$ is a geodetic coloring of G with the colorful geodetic set $\{u_1, u_4, u_5\}$. Therefore, $\chi_g(G) = \chi(G) = 4$. This shows that in any χ_g-coloring of G, $\{u_6, u_7\}$ is not colorful. □

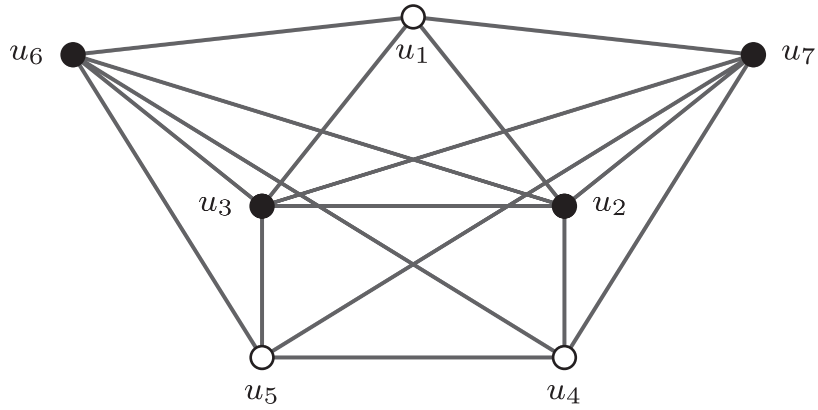

FIGURE 2. A graph G with $\chi_g(G) = 4$.

Theorem 3 proves that making at least one minimum geodetic set colorful is not necessary for minimum geodetic coloring.

Theorem 4. *For any two integers $a \geq b \geq 4$, there exists a connected cyclic graph G of order a with geodetic chromatic number b.*

Proof. For $a = b$, K_a is the required graph. Suppose $a = b + k$ for some positive integer k.

Case 1. b is even.

Clearly, $b = 2m$ for some positive integer $m \geq 2$. For $k = 1$, $(K_m \cup K_m) + K_1$ is the required graph, and for $k = 2$, $K_2 \circ K_m$ is the required graph.

For $k \geq 3$, let us construct a graph G of order a as follows. Take a copy of the complete bi-partite graph $K_{2,a-2m-2}$ with partite sets $V_1 = \{u, v\}$ and $V_2 = \{x_1, \ldots, x_{a-2m-2}\}$, take two copies of complete graph K_m, join u to each vertex of the first copy of K_m (denoted by ${}^1K_m$) and join v to each vertex of the second copy of K_m (denoted by ${}^2K_m$). For illustration, see Figure 3.

By Theorem 1, the set $K = V({}^1K_m) \cup V({}^2K_m) = V(G) - \{u, v, x_1, x_2, \cdots, x_{a-2m-2}\}$ is contained in every geodetic set of G. Therefore, $\chi_g(G) \geq b$. Define the coloring $\mathcal{C}$ where each vertex in K receives a distinct color, v receives a color used for any vertex of ${}^1K_m$, u receives a color used for any vertex of ${}^2K_m$, and all vertices in $\{x_1, x_2, \cdots, x_{a-2m-2}\}$ receive the color of any other vertex (different from the color of v) of ${}^1K_m$. This coloring is a geodetic coloring of G with K as a colorful geodetic set. Hence, $\chi_g(G) = b$.

Case 2. b is odd.

Clearly, $b = 2r + 1$ for some positive integer r. For $k = 1$, the graph $(K_r \cup K_{r+1}) + K_1$ is the required graph, and for $k = 2$, the graph obtained from $K_2 = uv$ by joining u to each vertex of K_r and v to each vertex of K_{r+1} is the required graph. For $k \geq 3$, consider the graph obtained from a copy of the complete bipartite graph $K_{2,a-2r-2}$ with partite sets $V_1 = \{u, v\}$ and V_2, by joining u to each vertex of a copy of K_r and v to each vertex of a copy of K_{r+1}. For this graph, the geodetic chromatic number is equal to b (the proof is similar to that in Case 1). $\square$

Theorem 5. *For any integers a, b and c with $a, b \geq 2$ and $\max\{a, b\} \leq c \leq a + b - 1$, there is a graph G with $\chi(G) = a$, $g(G) = b$ and $\chi_g(G) = c$.*

Proof. Let $S = \{a, b, c\}$. If $|S| = 1$, then K_a is the required graph. If $c = a + b - 1$, then $K_{a-1} + bK_1$ is the required graph.

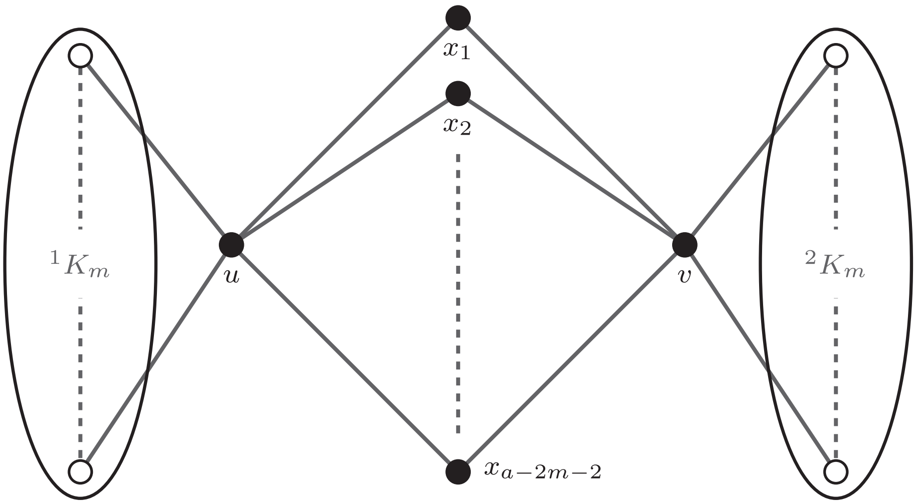

FIGURE 3. A graph G of order a and $\chi_g(G) = b$.

Suppose $|S| \geq 2$ and $c \leq a + b - 2$. We construct the graph G as follows: take a copy of $K_{a-1} + \langle S_1 \rangle$ and a copy of $K_{c-b+1} + \langle S_2 \rangle$ where $S_1 = \{v_a, u_a\}$ and $S_2 = \{u_1, \ldots, u_{b-1}\}$ are independent sets, and join v_a to exactly one vertex of K_{c-b+1}, let it be labeled as z_1. For illustration of the resulting graph G, see the Figure 4.

Claim 1. $\chi(G) = a$.

Proof of the Claim 1. Since $\langle \{v_1, v_2, \ldots, v_a\} \rangle$ is a complete graph, it follows that $\chi(G) \geq a$. Consider the coloring $\mathcal{C}$ where vertices $v_1, v_2, \ldots, v_a$ receive colors $c_1, c_2, \ldots, c_a$, respectively, all vertices in $\{u_1, u_2, \ldots, u_b\}$ receive color c_a, and vertices $z_1, z_2, \ldots, z_{c-b+1}$ receive colors $c_1, c_2, \ldots, c_{c-b+1}$, respectively. The coloring $\mathcal{C}$ is a proper vertex coloring of G with a colors. Therefore, $\chi(G) = a$.

Claim 2. $g(G) = b$.

Proof of the Claim 2. By Theorem 1, the set $\{u_1, u_2, \ldots, u_b\}$ is contained in every geodetic set of G. Since $\{u_1, u_2, \ldots, u_b\}$ is a geodetic set of G, it follows that $g(G) = b$.

Claim 3. $\chi_g(G) = c$.

Proof of the Claim 3. The set $\{u_1, u_2, \ldots, u_{b-1}\}$ is contained in every geodetic set of G, it implies that in any geodetic coloring of G, at least c colors are required to color the vertices in the set $\{u_1, u_2, \ldots, u_{b-1}, z_1, \ldots, z_{c-b+1}\}$. Therefore, $\chi_g(G) \geq c$. To prove the claim, it enough to obtain a geodetic coloring of G with c colors. Assign colors $r_1, r_2, \ldots, r_c$ to the vertices $u_1, u_2, \ldots, u_{b-1}, z_1, \ldots, z_{c-b+1}$ respectively.

Case 1. $c = b$.

Clearly, $a < b$ (since $a \neq c$). Assign colors $r_1, r_2, \ldots, r_a$ to the vertices $v_1, v_2, \ldots, v_a$ respectively and assign color r_b to the vertex u_b . This coloring is a geodetic coloring of G with a colorful geodetic set $\{u_1, u_2, \ldots, u_b\}$.

Case 2. $c > b$.

Clearly, $c - b + 1 \geq 2$. Assign colors $r_1, r_2, \ldots, r_{a-1}$ to the vertices $v_1, v_2, \ldots, v_{a-1}$ respectively and assign color r_c to both the vertices v_a and u_b . This coloring is a geodetic coloring of G with a colorful geodetic set $\{u_1, u_2, \ldots, u_b\}$.

In both the cases, we proved there is a geodetic coloring using c colors. Therefore, $\chi_g(G) = c$. $\square$

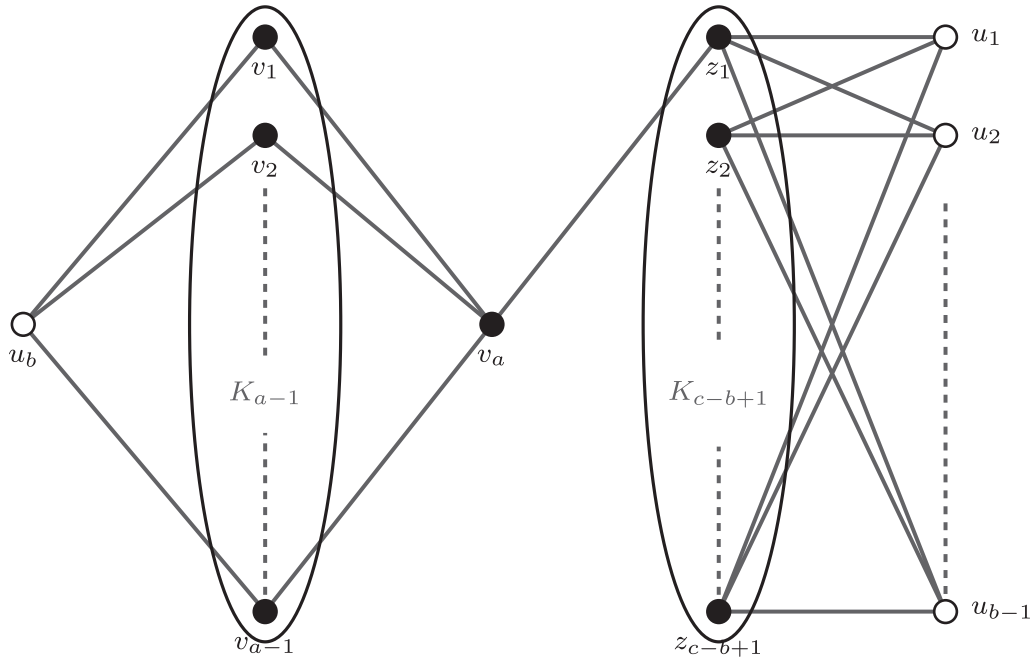

FIGURE 4. A graph G with $\chi(G) = a$, $g(G) = b$ and $\chi_g(G) = c$.

4. Bounds

In this section, we construct bounds for the geodetic chromatic number in terms of various graph parameters, including chromatic number, geodetic number, diameter, and vertex independence number.

Theorem 6. *For any connected graph G,*

$$\max\{\chi(G),\ g(G)\} \leq \chi_g(G) \leq \chi(G) + g(G) - 1.$$

Proof. Every geodetic coloring is a proper vertex coloring, and in any geodetic coloring of G, at least one geodetic set of G is colorful, proves the lower bound.

Let $\mathcal{C}$ be any χ-coloring of G and S be any minimum geodetic set of G. For an arbitrary $a \in S$, assign colors to the vertices in $V(G) - (S - \{a\})$ as in $\mathcal{C}$ and assign new different colors to the vertices in $S - \{a\}$. This coloring is a geodetic coloring of G with the colorful geodetic set S. Therefore, $\chi_g(G) \leq \chi(G) + g(G) - 1$. This bound is sharp for all the graphs $K_a + bK_1$ with $a, b \in \mathbb{N} - \{1\}$. □

Theorem 7. *For any connected graph G with diam$(G) = d$ of order n,*

$$\chi_g(G) \leq \begin{cases} n - d + 1, & d \text{ is odd} \\ n - d + 2, & d \text{ is even} \end{cases}.$$

Proof. Let x and y be vertices in G such that $d(x, y) = \text{diam}(G) = d$, and let $v_0 v_1 \cdots v_{d-1} v_d$ be any x-y geodesic with $v_0 = x$ and $v_d = y$. Clearly, the set $V(G) - \{v_1, v_2, \ldots, v_{d-1}\}$ is a geodetic set of G.

Case 1. d is odd.

Let us define a coloring $\mathcal{C}_1$ as follows. Assign different colors to the vertices in $V(G) - \{v_0, v_1, \ldots, v_d\}$, assign a new colors to all the vertices in $\{v_0, v_2, \ldots, v_{d-1}\}$ and another new color to all the vertices in $\{v_1, v_3, \ldots, v_d\}$.

The coloring $\mathcal{C}_1$ is a geodetic coloring of G with the colorful geodetic set $V(G)-\{v_1, v_2, \ldots, v_{d-1}\}$. Therefore, $\chi_g(G) \leq n-d+1$.

This bound is sharp for all cycles of even order.

Case 2. d is even.

Let us define a coloring $\mathcal{C}_2$ as follows. Assign different colors to the different vertices in $V(G) - \{v_0, v_1, \ldots, v_{d-1}\}$, assign a new color to all the vertices in $\{v_0, v_2, \ldots, v_{d-2}\}$ and another new color to all the vertices in $\{v_1, v_3, \ldots, v_{d-3}, v_{d-1}\}$. The coloring $\mathcal{C}_2$ is a geodetic coloring of G with the colorful geodetic set $V(G) - \{v_1, v_2, \ldots, v_{d-1}\}$. Therefore, $\chi_g(G) \leq n-d+2$.

This bound is sharp for all paths of odd length. □

Theorem 8. *For any triangle-free graph G of order $n \geq 4$ with $\delta(G) \geq 2$, $\chi_g(G) \leq n-\alpha(G)+1$, where $\alpha(G)$ is vertex independence number of G and the bound is sharp.*

Proof. In order to prove the inequality, it enough to obtain a geodetic coloring with $n-\alpha(G)+1$ colors. Let S be any maximum independent set of G and $v \in S$ be arbitrary. Since $\deg(v) \geq 2$, there exist the vertices u_1 and u_2 such that $u_1, u_2 \in N(v)$. Clearly, u_1 and u_2 are non-adjacent (otherwise, $\langle v, u_1, u_2 \rangle$ forms a C_3), and $u_1, u_2 \in V(G)-S$ since $v \in S$. This shows that v lies in the u_1-u_2 geodesic ($u_1 v u_2$). This concludes that $V(G)-S$ is a geodetic set of G. Now, assign different colors to the vertices in $V(G)-S$ and assign a new common color to all the vertices in S (this is possible since S is independent). This coloring is a geodetic coloring of G with the colorful geodetic set $V(G)-S$. Therefore, $\chi_g(G) \leq n-\alpha(G)+1$.

This bound is sharp for $K_{r,2}$ with $r \geq 2$. □

5. Characterizations

In this section, we characterized trees that have a geodetic chromatic number equal to the number of pendant vertices and connected graphs that have a geodetic chromatic number equal to 2 and equal to its order.

For any tree T of order $n \geq 3$ with k pendant vertices, we have $\chi(T) = 2$ and $g(T) = k$. By Theorem 6, it follows that $\chi_g(T) = k$ or $k+1$.

Theorem 9. *For any tree T of order $n \geq 3$ with k pendant vertices, $\chi_g(T) = k$ if and only if T is neither a star nor a path on odd vertices.*

Proof. Let $\chi_g(T) = k$. It is easy to verify that if T is a star or a path on odd vertices then $\chi_g(T) = k+1$. Therefore, $T \notin \{K_{1,n-1}, P_{2(\frac{n-1}{2})+1}\}$.

Conversely, let T is neither a star nor a path on odd vertices with k pendant vertices. Since $\chi_g(T) \geq k$, to the prove equality, it enough to obtain a geodetic coloring of T with k colors.

Case 1. $k = 2$.

Clearly, T is a path, which implies that $T = v_1 v_2 \cdots v_n$ is a path on even vertices. In any χ-coloring of T, v_1 and v_n receive different colors. It implies that every χ-coloring of T is a geodetic coloring of T with the colorful geodetic set $\{v_1, v_n\}$. Therefore, $\chi_g(T) = 2$.

Case 2. $k \geq 3$.

Let x and y be pendant vertices with $d(x,y) = diam(G) = d$. Clearly, the x-y geodesic contains all non-pendant vertices of T. Since T is not a star, it follows that $d \geq 3$. Let $v_0 v_1 \cdots v_{d-1}$ with $v_0 = x$, $v_d = y$ be the x-y geodesic.

Suppose d is odd. Assign k different colors to the pendant vertices, assign the color of x to the vertices $v_2, v_4, \ldots, v_{d-1}$, and assign the color of y to the vertices $v_1, v_3, \ldots, v_{d-2}$. This coloring is a proper vertex coloring of T where each pendant vertex receives a different color (geodetic coloring).

Suppose d is even. Let z be a pendant vertex different from x and y. Assign k different colors to the pendant vertices, assign the color of x to the vertices $v_2, v_4, \ldots, v_{d-4}$, and assign the color of y to the vertices

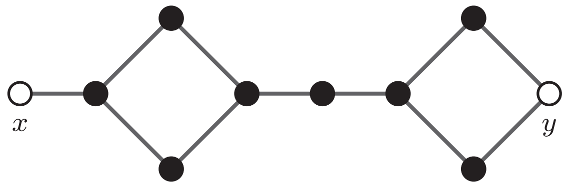

FIGURE 5. A graph in class $\mathcal{F}_1$.

$v_1, v_3, \ldots, v_{d-3}$. If $N(z) = v_{d-1}$, then assign the color of x to the vertex v_{d-1} and the color of z to the vertex v_{d-2}; otherwise, assign the color of x to the vertex v_{d-2} and the color of z to the vertex v_{d-1}. This coloring is a proper vertex coloring of T where each pendant vertex receives a different color (geodetic coloring).

In both the cases, there is geodetic coloring of T with k colors. Therefore, $\chi_g(T) = k$. $\square$

To characterize graphs with $\chi_g = 2$, let us define a class of graphs $\mathcal{F}_1$, as follows.

A graph G is in class $\mathcal{F}_1$ if G is bipartite and $\{x, y\}$ is a geodetic set of G for $x, y \in V(G)$ with $d(x, y)$ being odd. The graph in Figure 5 is an example class-$\mathcal{F}_1$ graph.

Theorem 10. *For a connected graph G of order n, $\chi_g(G) = 2$ if and only if $G \in \mathcal{F}_1$.*

Proof. Let $\chi_g(G) = 2$, and let (V_1, V_2) be any χ_g-coloring of G with a colorful geodetic set $\{x, y\}$ and $d(x, y) = d$. Clearly, G is bipartite. If $d = 1$, then $G = K_2$.

Suppose d is even. Let $v_0 v_1 \ldots v_d$ with $v_0 = x$, $v_d = y$ be any x-y geodesic. It is clear that at least three colors are required to color the vertices in the above x-y geodesic such that color$(x) \neq$ color(y). This shows that $\chi_g(G) \geq 3$, a contradiction that proves d is odd.

Conversely, let $G \in \mathcal{F}_1$. Since G is bipartite, (V_1, V_2) is a χ-coloring of G. If $x, y \in V_1$(*or* V_2), then $d(x, y)$ is even. Therefore, $x \in V_1$ and $y \in V_2$. This shows that (V_1, V_2) is a χ_g-coloring of G with the colorful geodetic set $\{x, y\}$. Therefore, $\chi_g(G) = 2$. $\square$

To characterize graphs with χ_g equal to its order, let us define a class of graphs, denoted as $\mathcal{F}_2$, as follows.

A graph G is in class $\mathcal{F}_2$ if every vertex in the set $\{v \in V(G) \ / \ \langle N(v) \rangle \textit{ is not complete}\}$ is a universal vertex of G.

Lemma 1. *A graph $G \in \mathcal{F}_2$ if and only if $G = K_m + (K_{n_1} \cup \ldots \cup K_{n_r})$ with $n_1 + \cdots + n_r = n - m$.*

Proof. Let $G \in \mathcal{F}_2$ and $S = \{u \in V(G) \mid \langle N(u) \rangle \text{ is complete}\}$ with $|S| = m$. Clearly, $N[v] = V(G)$ for all $v \in V(G) - S$. This proves that $\langle V(G) - S \rangle = K_m$, and hence $G = K_m + \langle S \rangle$. Also, $\langle N(v) \rangle = K_{\deg(v)}$ for all $v \in S$. This proves that any two vertices in S are adjacent if and only if they are in the same component of $\langle S \rangle$. Therefore, $\langle S \rangle = K_{n_1} \cup \ldots \cup K_{n_r}$, where r is the number of components of $\langle S \rangle$. This proves that $G = K_m + (K_{n_1} \cup \ldots \cup K_{n_r})$. The converse is obvious.

$\square$

Theorem 11. *For a connected graph G of order $n \geq 4$, $\chi_g(G) = n$ if and only if $G = K_m + (K_{n_1} \cup \ldots \cup K_{n_r})$ with $n_1 + \cdots + n_r = n - m$.*

Proof. Let G be a connected graph of order n with $\chi_g(G) = n$. Suppose $G \notin \mathcal{F}_2$ and let $S = \{v \in V(G) \ / \ \langle N(v) \rangle \textit{ is not complete}\}$. Clearly, there is a vertex $x \in S$ such that $N[x] \neq V(G)$. Since $x \in S$, it follows that there are non-adjacent vertices v_1 and v_2 such that $\{v_1, v_2\} \subseteq N(x)$. Since $N[x] \neq V(G)$, it follows that there is a vertex $v_3 \in V(G)$ such that $v_3 \notin N(x)$. Clearly, x lies in a $v_1 - v_2$ geodesic, implying that $V(G) - \{x\}$ is a geodetic set of G. Consider the coloring $\mathcal{C}$ as follows. Assign $n - 2$ different colors to the vertices in $V(G) - \{x, v_3\}$ and assign a new color to both x and v_3. This coloring is a geodetic coloring of G with the colorful geodetic set $V(G) - \{x\}$ using $n-1$ colors, a contradiction. Therefore, $G \in \mathcal{F}_2$.

Conversely, let $G \in \mathcal{F}_2$ and $K = \{x \in V(G) \mid \langle N(x) \rangle$ is complete$\}$. By Theorem 1, K is contained in all geodetic sets of G. In any geodetic coloring of G, each vertex in K receives a different color. Since $G \in \mathcal{F}_2$, each vertex in $V(G) - K$ is adjacent to all vertices of G. In any geodetic coloring of G, each vertex in $V(G) - K$ receives a different color. This proves that no two vertices receive the same color in any geodetic coloring of G. Therefore, $\chi_g(G) = n$.

The above lemma proves that $\chi_g(G) = n$ if and only if $G = K_m + (K_{n_1} \cup \ldots \cup K_{n_r})$ with $n_1 + \cdots + n_r = n - m$. □

6. χ_g of Mycielskian of a graph

Characterizing the geodetic chromatic number χ_g, of the Mycielskian of a graph can be challenging, as there exist graphs satisfying $\chi_g(\mu(G_1)) > \chi_g(G_1)$, $\chi_g(\mu(G_2)) < \chi_g(G_2)$, and $\chi_g(\mu(G_3)) = \chi_g(G_3)$. Examples include:

- $\chi_g(\mu(K_{1,3})) = 3$ and $\chi_g(K_{1,3}) = 4$
- $\chi_g(\mu(K_{1,2})) = 3$ and $\chi_g(K_{1,2}) = 3$
- $\chi_g(\mu(P_2)) = 3$ and $\chi_g(P_2) = 2$.

However, we characterize it for graphs with domination number equal to 1.

Theorem 12. *For a connected graph G with $\gamma(G) = 1$, $\chi_g(\mu(G)) = \chi(G) + 1$.*

Proof. Let G be a connected graph with $\gamma(G) = 1$ and $V(G) = \{u_1, u_2, \ldots, u_n\}$. Let $\{u_1\}$ be a γ-set of G and $\mu(G)$ be the Mycielskian of G with vertex set $V(G) \cup V'(G) \cup \{u\}$. Clearly, $\chi_g(\mu(G)) \geq \chi(\mu(G)) = \chi(G) + 1$.

Claim. $\{u_1, u_1', u\}$ is a geodetic set of $\mu(G)$.

Proof of the Claim. Any vertex u_i in $V(G) - \{u_1\}$ lies in the u_1-u_1' geodesic $(u_1 u_i u_1')$, and any vertex u_j' in $V'(G) - \{u_1'\}$ lies in the u_1-u geodesic $(u_1 u_j' u)$. Therefore, $\{u_1, u_1', u\}$ is a geodetic set of $\mu(G)$.

Let $\mathcal{C}$ be any χ-coloring of G. Consider the new coloring $\mathcal{C}'$ such that the vertices of $V(G)$ are assigned colors as in $\mathcal{C}$, the vertex u is assigned any color used in $\mathcal{C}$ different from color of u_1 and all the vertices in $V'(G)$ are assigned a new different common color. The coloring $\mathcal{C}'$ is a geodetic coloring of $\mu(G)$ with the colorful geodetic set $\{u_1, u_1', u\}$. Therefore $\chi_g(\mu(G)) = \chi(G) + 1$. □

7. Complexity result

This section explores the complexity of finding geodetic chromatic number of a graph. The decision problem of geodetic chromatic number is defined as follows.

Instance: A graph G of order n and a positive integer $k \leq n$

Question: Is there a geodetic coloring of G with k colors?

Theorem 13. *Geodetic chromatic number is NP-complete.*

Proof. The decision problem chromatic number is NP-complete for arbitrary graphs. To prove the result, it is enough to construct a graph $G^{'}$ such that G has a proper vertex coloring with k colors if and only if G' has a geodetic coloring with $k+1$ colors.

For a given graph G, we construct a graph G' by adding three new vertices x, y and z to G, joining x and y with each vertex of G (x and y are not adjacent) and joining z with y. Clearly, $\chi(G') \geq \chi(G) + 1$.

To prove the necessity, let $\mathcal{C} = (V_1, V_2, \ldots, V_k)$ be any χ-coloring of G with k colors. Then the coloring $\mathcal{C}' = (V_1, V_2, \ldots, V_k \cup \{z\}, V_{k+1} = \{x, y\})$ is a geodetic coloring of G' with colorful geodetic set $\{x, z\}$ using $k+1$ colors.

To prove sufficiency, let $\mathcal{D} = (U_1, U_2, \ldots, U_{k+1})$ be a geodetic coloring of G' with colorful geodetic set S. By Theorem 1, $\{x, z\} \subseteq S$, implying x and z receive different colors. If x and y receive different colors, then $\{x\}$ and $\{y\}$ are color classes in $\mathcal{D}$, and the coloring $\mathcal{D}$ restricted to G would be a proper vertex coloring of G with less than k colors, a contradiction. Similarly, if $\{z\}$ is a color class in $\mathcal{D}$, a contradiction arises. Therefore, $\{x, y\}$ must be a color class, and $\{z\}$ is not a color class in $\mathcal{D}$. Hence, the coloring $\mathcal{D}$ restricted to G is a proper vertex coloring of G with k colors.

Thus, for any graph G, G has a proper vertex coloring with k colors if and only if G' has a geodetic coloring with $k+1$ colors. Hence the decision problem geodetic chromatic number is NP-complete. □

8. Open problems

We conclude this paper by a list some open problems.

1. Characterize the graphs G with $\chi_g(G) = n - 1$.
2. Give a structural characterization of graphs G with $\chi_g(G) = 2$.
3. Characterize the graphs G with
   i. $\chi_g(\mu(G)) > \chi_g(G)$
   ii. $\chi_g(\mu(G)) < \chi_g(G)$
   iii. $\chi_g(\mu(G)) = \chi_g(G)$.
4. Characterize the triangle-free graphs without pendant vertices satisfies $\chi_g(G) = n - \alpha(G) + 1$.

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