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TOUGHNESS AND DISTANCE SPECTRAL RADIUS IN GRAPHS

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Abstract. Let G be a connected graph with vertex set $V(G)$. A variation $\tau(G)$ of toughness of G, proposed by Enomoto in 1988, is defined as

$$\tau(G)=\min_{S\subset V(G)}\left\{\frac{|S|}{c(G-S)-1}:c(G-S)>1\right\},$$

where $c(G-S)$ is the number of components of $G-S$ for a subset S of $V(G)$. For a positive real number τ, G is called *τ-tough* if $\tau(G)\geq\tau$. Chen *et al.* [Discrete Math. 347 (2024) 114191] gave sufficient spectral radius conditions for a graph G to be τ-tough, where the spectral radius is the maximum eigenvalue of the adjacent matrix of G and τ or $1/\tau$ is a positive integer. Analogously, in this paper, we investigate the case of the distance spectral radius of G and present three tight distance spectral radius conditions, respectively, and analyze the extreme cases, which are similar to Chen *et al.*'s results.

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1. Introduction

Let $G=(V(G),E(G))$ be a graph with edge set $E(G)$ and vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ where $n=|V(G)|$ denotes the order of G. Denote $\delta(G)$ as the minimum degree and $c(G)$ as the number of components of G. For $S\subset V(G)$, the subgraph of G induced by $V(G)\setminus S$ is written as $G-S$. The *disjoint union* of two vertex-disjoint graphs G_1 and G_2 is denoted as $G_1\cup G_2$. Moreover, the graph obtained from $G_1\cup G_2$ by adding all possible edges between $V(G_1)$ and $V(G_2)$ is called the *join* $G_1\vee G_2$. We use $H\subseteq_{sp} G$ to denote that graph H is a spanning subgraph of G.

The *distance matrix* $D(G)$ of G is defined to be an $n\times n$ real symmetric matrix, where for $1\leq i,j\leq n$, the element at the (i,j) position represents the distance $d_{ij}(G)$ between vertices v_i and v_j, which is the length of the shortest path connecting them. We arrange the eigenvalues of $D(G)$ in descending order as $\partial_1(G)\geq\partial_2(G)\geq\cdots\geq\partial_n(G)$. According to the Perron–Frobenius theorem, when G is a non-trivial graph, $\partial_1(G)$ is always positive and $\partial_1(G)\geq|\partial_k(G)|$ for $2\leq k\leq n$. We define $\partial(G)=\partial_1(G)$ as the *distance spectral radius* of G. Moreover, there exists a unique positive unit eigenvector corresponding to $\partial(G)$, which is called the *Perron vector* and denoted as $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T$.

Keywords. Toughness, distance spectral radius, quotient matrix.

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In 1973, to understand the connectivity of a connected graph G, Chvátal [6] first introduced a new quantitative indicator, called *toughness*, into graph theory. Specifically, toughness $t(G)$ of G is defined as the minimum value of the ratio of $|S|$ to $c(G-S)$ under the condition that $S \subset V(G)$ and $c(G-S) > 1$, that is

$$t(G) = \min_{S \subset V(G)} \left\{ \frac{|S|}{c(G-S)} : c(G-S) > 1 \right\}.$$

From the definition, it is easy to see that the smaller the value is, the more likely the graph will split when a small number of vertices are lost, indicating that its connectivity is fragile. For a positive real number t, G is called $\bar{t}$-*tough* if $t(G) \geq t$. Beyond pure graph theory, the concept of toughness has found applications in network security and design. Gao [11] theoretically studied the case where a network meets specific toughness conditions, and he found that such a network can ensure network robustness and data transmission feasibility. Furthermore, toughness can synergize with parameters such as isolated toughness and binding number to assess network vulnerability [11].

The investigation on the connection between toughness and eigenvalues was started in 1995 by Alon [1]. Subsequently, Brouwer [2] proved a better lower bound for a d-regular graph G and conjectured that $t(G) \geq \frac{d}{\lambda} - 1$, where λ is the second largest in absolute value among adjacency matrix eigenvalues of G. Later, Gu [13] first proved part of this conjecture and fully proved it [14] in 2021. Given that eigenvalues of different matrices encode distinct graph structures, recent studies have expanded to determine the conditions for a graph to be $\bar{t}$-tough using eigenvalues of different types of matrices, such as the adjacency matrix [8] and the distance matrix [17].

In 1998, Enomoto ingeniously proposed a variation of toughness of a connected graph G, denoted as $\tau(G)$, in the literature [7], which opened up a new direction for the research on connectivity in graph theory. Specifically, it is to find the minimum value of the ratio of $|S|$ to $c(G-S)-1$ under the condition that $S \subset V(G)$ and $c(G-S) > 1$, that is

$$\tau(G) = \min_{S \subset V(G)} \left\{ \frac{|S|}{c(G-S)-1} : c(G-S) > 1 \right\}.$$

For a positive real number τ, G is called τ-*tough* if $\tau(G) \geq \tau$. The emergence of this concept has raised a crucial question: Is it possible to find exact sufficient conditions to determine whether G is τ-tough or not? Previously, Chen, Fan and Lin [4], as well as Chen, Li and Xu [5] have respectively carried out research. Specifically, the former made use of the spectral radius, while the latter utilized the signless Laplacian spectral radius. Moreover, they each gave sufficient conditions for determining that G is τ-tough, where τ or $\frac{1}{\tau}$ is a positive integer. Very recently, Gao *et al.* [12] established tight sufficient conditions for the existence of fractional k-factors via $\tau(G)$, which further expands the theoretical research and application scenarios of $\tau(G)$.

Inspired by these achievements, we delved into the relationship between τ-tough graphs and the distance spectral radius. The main results obtained in this exploration process will be presented below.

Theorem 1.1. *Let G be a connected graph of order $n \geq \max\{9\delta, \frac{\delta^2}{2} + 3\delta + 3\}$ with minimum degree $\delta \geq 2$. If $\partial(G) \leq \partial(K_\delta \vee (K_{n-2\delta-1} \cup (\delta+1)K_1))$, then G is 1-tough unless $G \cong K_\delta \vee (K_{n-2\delta-1} \cup (\delta+1)K_1)$.*

Theorem 1.2. *Let G be a connected graph of order $n \geq 4\tau^2 + 5\tau + 1$, where $\tau \geq 2$ be an integer. If $\partial(G) \leq \partial(K_{\tau-1} \vee (K_{n-\tau} \cup K_1))$, then G is τ-tough unless $G \cong K_{\tau-1} \vee (K_{n-\tau} \cup K_1)$.*

Theorem 1.3. *Let G be a connected graph of order $n \geq 4\tau + \frac{1}{\tau} + 5$, where $\frac{1}{\tau} \geq 1$ be an integer. If $\partial(G) \leq \partial(K_1 \vee (K_{n-\frac{1}{\tau}-2} \cup (\frac{1}{\tau}+1)K_1))$, then G is τ-tough unless $G \cong K_1 \vee (K_{n-\frac{1}{\tau}-2} \cup (\frac{1}{\tau}+1)K_1)$.*

2. Preliminary lemmas

In this section, we aim to introduce several essential concepts and present some key lemmas. The following inequality, which relates the distance spectral radius of a graph to that of its spanning subgraph, is fundamental to our analysis.

Lemma 2.1 ([9]). *If e is an edge of a connected graph G such that $G-e$ remains connected, then*

$$\partial(G) < \partial(G-e).$$

In the field of graph theory and matrix analysis, equitable partition and equitable quotient matrix are crucial concepts, which provide a unique perspective to analyze the structures of matrices and the relevant properties of graphs. We begin by analyzing matrices, as they serve as the foundational objects for defining equitable partitions. Suppose there is a real matrix M of order $n \times n$. To analyze its structure more clearly, we perform a block partition on it. The rows and columns of matrix M are partitioned into pairwise disjoint nonempty subsets $X_1, X_2, \ldots, X_m$, which together form a partition of the index set $\{1, 2, \ldots, n\}$. Through this partition, M can be presented as a block matrix in the following form:

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

where $M_{i,j}$ denotes the submatrix formed by rows in X_i and columns in X_j. Based on this block partition, we construct the *quotient matrix* $R(M)$, which is a matrix of dimension $m \times m$. Let the sub-block $M_{i,j}$ be a $p \times q$ matrix. The sum of the elements in its k-th row is

$$s_k = \sum_{t=1}^{q} M_{i,j}(k,t) \quad (k = 1, 2, \ldots, p).$$

The element at the (i,j)-position in $R(M)$ is the average of the row-sums of the sub-block $M_{i,j}$, that is,

$$R(M)_{i,j} = \frac{1}{p} \sum_{k=1}^{p} s_k = \frac{1}{p} \sum_{k=1}^{p} \sum_{t=1}^{q} M_{i,j}(k,t).$$

If all elements of $R(M)$ are constant, the partition and its quotient matrix are said to be *equitable*.

Lemma 2.2 ([3,10,15]). *For a real symmetric matrix M, all the eigenvalues of $R(M)$ are also the eigenvalues of M, where $R(M)$ is the corresponding equitable quotient matrix of M. In addition, the spectral radius of $R(M)$ is the same as that of M if M is nonnegative and irreducible.*

Both of these ideas are crucial to the following arguments. Two inequalities pertaining to the distance spectral radius will then be introduced.

Lemma 2.3 ([16]). *If G is a connected graph with order n, then*

$$\partial(G) = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^T(G)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \geq \frac{\mathbf{1}^T(G)\mathbf{1}}{\mathbf{1}^T\mathbf{1}} = \frac{2W(G)}{n},$$

where $\mathbf{1} = (1, 1, \ldots, 1)^T$ and $W(G) = \sum_{i<j} d_{ij}(G)$.

Lemma 2.4 ([19–21]). *If G is a graph with order $n \geq 2$ and $W^{(2)}(G) = \sum_{1 \leq i < j \leq n} d_{ij}^2(G)$, then*

$$\partial(G) \leq \sqrt{\frac{2(n-1)W^{(2)}(G)}{n}}.$$

The equality holds if and only if G is K_n.

We finish this section by stating two lemmas that are crucial to the proof.

Lemma 2.5 ([18]). *For $n_1 \geq n_2 \geq \cdots \geq n_c \geq 1$ with $\sum_{i=1}^{c} n_i = n - s$, where n, c, s and n_i $(1 \leq i \leq c)$ be positive integers, then*

$$\partial(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_c})) \geq \partial(K_s \vee (K_{n-s-(c-1)} \cup (c-1)K_1)).$$

The equality holds if and only if $K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_c}) \cong K_s \vee (K_{n-s-(c-1)} \cup (c-1)K_1)$.

Lemma 2.6 ([17]). *For $n_1 \geq 2p$, $n_1 \geq n_2 \geq \cdots \geq n_c \geq p$ and $\sum_{i=1}^{c} n_i = n-s$, where n, c, s, p and n_i $(1 \leq i \leq c)$ be positive integers, then*

$$\partial(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_c})) \geq \partial(K_s \vee (K_{n-s-p(c-1)} \cup (c-1)K_p)).$$

The equality holds if and only if $K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_c}) \cong K_s \vee (K_{n-s-p(c-1)} \cup (c-1)K_p)$.

3. Proof of theorem 1.1

Proof of Theorem 1.1. Suppose to the contrary that G is not 1-tough. Then by the definition of 1-tough, there is a subset S of $V(G)$ such that $|S| < c(G-S) - 1$. Let $|S| = s$ and $c(G-S) = c$. This same notation will be used in the subsequent proof. Since s and c are integers, it follows that $s \leq c-2$. Let $\tilde{G} = K_s \vee \big(K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_{s+2}}\big)$, where $n_1 \geq n_2 \geq \cdots \geq n_{s+2} \geq 1$ are positive integers with $\sum_{i=1}^{s+2} n_i = n-s$. It is straightforward to verify that $G \subseteq_{sp} \tilde{G}$. Thus, by Lemma 2.1, we have

$$\partial(\tilde{G}) \leq \partial(G), \tag{1}$$

with equality if and only if $G \cong \tilde{G}$.

Define $G' = K_s \vee (K_{n-2s-1} \cup (s+1)K_1)$. Then by Lemma 2.5, we get

$$\partial(G') \leq \partial(\tilde{G}), \tag{2}$$

with equality if and only if $\tilde{G} \cong G'$.

Let $\hat{G}$ denote the graph $K_\delta \vee (K_{n-2\delta-1} \cup (\delta+1)K_1)$ mentioned in the Theorem 1.1. To derive a contradiction, we need to show $\partial(\hat{G}) < \partial(G)$. We prove this inequality by introducing intermediate graphs. A classification discussion will be conducted according to the different values of s.

Case 1. $s \geq \delta + 1$.

Consider partitioning the vertices of $\hat{G}$ into $V(\hat{G}) = V((\delta+1)K_1) \cup V(K_{n-2\delta-1}) \cup V(K_\delta)$. It is not difficult to prove that this partition is equitable with respect to the distance matrix $D(\hat{G})$ and corresponding equitable quotient matrix R_δ is given by:

$$R_\delta = \begin{pmatrix} 2\delta & 2(n-2\delta-1) & \delta \\ 2(\delta+1) & n-2\delta-2 & \delta \\ \delta+1 & n-2\delta-1 & \delta-1 \end{pmatrix}.$$

We can calculate that its characteristic polynomial is

$$\begin{aligned} P(R_\delta, x) = x^3 &- (n+\delta-3)x^2 - \big[(2\delta+5)n - 5\delta^2 - 6(\delta+1)\big]x \\ &+ \big(\delta^2 - \delta - 4\big)n - 2\delta^2(\delta-1) + 6\delta + 4. \end{aligned} \tag{3}$$

Note that $V(G')$ has a partition analogous to $V(\hat{G})$, explicitly given by $V(G') = V((s+1)K_1) \cup V(K_{n-2s-1}) \cup V(K_s)$. Then, by substituting δ with s in R_δ (owing to the identical partition structure), we obtain the equitable quotient matrix R_s, whose characteristic polynomial is

$$\begin{aligned} P(R_s, x) = x^3 &- (n+s-3)x^2 - \big[(2s+5)n - 5s^2 - 6(s+1)\big]x \\ &+ \big(s^2 - s - 4\big)n - 2s^2(s-1) + 6s + 4. \end{aligned}$$

Through straightforward calculation,

$$P(R_\delta, x) - P(R_s, x) = (s-\delta)f(x),$$

where $f(x) = x^2 + (2n - 5\delta - 5s - 6)x - (s+\delta-1)n + 2(s^2 + s\delta + \delta^2 - s - \delta) - 6$. By Lemma 2.3 and $n \geq 9\delta$, we note that

$$\begin{aligned}\partial(\hat{G}) &\geq \frac{2W(\hat{G})}{n} = \frac{(\delta+1)(2n-\delta-2) + (n-2\delta-1)(n+\delta) + \delta(n-1)}{n}\\ &= (n+\delta+1) + \delta - \frac{3\delta^2 + 5\delta + 2}{n} > n + \delta + 1.\end{aligned}$$

The axis of symmetry of the function $f(x)$ is given by $x = -n + \frac{5}{2}s + \frac{5}{2}\delta + 3 \leq n + \delta + 1$ (since $n \geq s + c \geq 2s + 2$), which implies that

$$\begin{aligned}f(x) &\geq f(n+\delta+1)\\ &= 2s^2 - (6n + 3\delta + 7)s + 3n^2 - 2\delta n - n - 2\delta^2 - 11\delta - 11\\ &\geq 2\left(\frac{n-2}{2}\right)^2 - (6n + 3\delta + 7)\left(\frac{n-2}{2}\right)\\ &\quad + 3n^2 - 2\delta n - n - 2\delta^2 - 11\delta - 11 \quad (\text{since } n \geq s + c \geq 2s + 2)\\ &= \frac{1}{2}n^2 - \left(\frac{7}{2}\delta + \frac{1}{2}\right)n - 2\delta^2 - 8\delta - 2\\ &\geq 2\delta^2 - 3\delta - 1 \quad (\text{since } \delta \geq 2)\\ &> 0.\end{aligned}$$

Lemma 2.2 shows that $\partial(\hat{G})$ (resp. $\partial(G')$) is the largest root of $P(R_\delta, x)$ (resp. $P(R_s, x)$). Thus, combining with (1) and (2), we obtain

$$\partial(\hat{G}) < \partial(G') \leq \partial(\tilde{G}) \leq \partial(G),$$

a contradiction.

Case 2. $1 \leq s \leq \delta - 1$.

It is easy to see that $\delta(\tilde{G}) \geq \delta(G) = \delta$, so $n_{s+2} - 1 + s \geq \delta$. This implies $n_1 \geq n_2 \geq \cdots \geq n_{s+2} \geq \delta - s + 1$. We now claim that $n_1 \geq 2(\delta - s + 1)$. Suppose that $n_1 \leq 2\delta - 2s + 1$. Recalling that $n_1 \geq n_2 \geq \cdots \geq n_{s+2} \geq 1$ and $1 \leq s \leq \delta - 1$, we have

$$\begin{aligned}n &= s + n_1 + n_2 + \cdots + n_{s+1} + n_{s+2}\\ &\leq s + (s+2)(2\delta - 2s + 1)\\ &= -2s^2 + (2\delta - 2)s + 4\delta + 2 := n(s).\end{aligned}$$

The axis of symmetry of the function $n(s)$ is given by $s_{\text{axis}} = \frac{\delta-1}{2}$. When $\delta \geq 3$, $1 \leq s_{\text{axis}} \leq \delta - 1$. Substituting $s = s_{\text{axis}}$ into the function $n(s)$, we obtain

$$n(s) \leq -2\left(\frac{\delta-1}{2}\right)^2 + (2\delta - 2)\left(\frac{\delta-1}{2}\right) + 4\delta + 2 = \frac{\delta^2}{2} + 3\delta + \frac{5}{2}.$$

When $2 \leq \delta \leq 3$, we have $s_{\text{axis}} \leq 1$. Evaluating the function $n(s)$ at $s = 1$, we get

$$n(s) \leq -2 + 2\delta - 2 + 4\delta + 2 < 6\delta.$$

The results in both cases lead to a contradiction with the condition that $n \geq \max\{9\delta, \frac{\delta^2}{2} + 3\delta + 3\}$. Now we define a new intermediate graph $G^* = K_s \vee \left(K_{n-s-(\delta-s+1)(s+1)} \cup (s+1)K_{\delta-s+1}\right)$. By Lemma 2.6, we have

$$\partial(G^*) \leq \partial(\tilde{G}), \tag{4}$$

with equality if and only if $G^* \cong \tilde{G}$. Next, we will prove $\partial(\hat{G}) < \partial(G^*)$ by dividing it into two cases based on the value of s, thereby deriving a contradiction.

Subcase 2.1. $s = 1$.

Consider the Perron vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ corresponding to $\partial(\hat{G})$. Since the vertex partition implies that vertices in the same partition class have identical components in $\mathbf{x}$, we have

$$\begin{aligned}\partial(\hat{G})x_1 &= 2\delta x_1 + 2(n-2\delta-1)x_2 + \delta x_3,\\ \partial(\hat{G})x_3 &= (\delta+1)x_1 + (n-2\delta-1)x_2 + (\delta-1)x_3.\end{aligned}$$

Since $D(\hat{G})\mathbf{x} = \partial(\hat{G})\mathbf{x}$, we derive

$$\partial(\hat{G})(2x_3 - x_1) = 2x_1 + (\delta-2)x_3.$$

We have $2x_1+(\delta-2)x_3 \geq 2x_1 > 0$ since $\delta \geq 2$. Therefore, $2x_3 \geq x_1 > 0$. In this case, $G^* = K_1 \vee (K_{n-2\delta-1} \cup 2K_\delta)$. By the definition of Rayleigh quotient, it follows that

$$\begin{aligned}\partial(G^*) - \partial(\hat{G}) &\geq \mathbf{x}^T(D(G^*) - D(\hat{G}))\mathbf{x}\\ &= -\delta(\delta-1)x_1^2 + 2\delta(\delta-1)x_1x_3 + (\delta-1)(n-2\delta-1)x_2x_3\\ &\quad + (\delta-1)(n-2\delta-1)x_2^2\\ &= \delta(\delta-1)(2x_3-x_1)x_1 + (\delta-1)(n-2\delta-1)x_2x_3\\ &\quad + (\delta-1)(n-2\delta-1)x_2^2 > 0.\end{aligned}$$

Therefore, $\partial(\hat{G}) < \partial(G^*)$.

Subcase 2.2. $2 \leq s \leq \delta - 1$.

In this case, $G^* = K_s \vee \big(K_{n-s-(\delta-s+1)(s+1)} \cup (s+1)K_{\delta-s+1}\big)$. Similar to Case 1, consider partitioning the vertices of G^* into $V(G^*) = V((s+1)K_{\delta-s+1}) \cup V\big(K_{n-s-(\delta-s+1)(s+1)}\big) \cup V(K_s)$. It is easy to show that this partition is equitable with respect to the distance matrix $D(G^*)$ and the corresponding equitable quotient matrix $R_{s,\delta}$ is given by:

$$R_{s,\delta} = \begin{pmatrix} (\delta-s)+2s(\delta-s+1) & 2[n-s-(\delta-s+1)(s+1)] & s\\ 2(s+1)(\delta-s+1) & n-s-(\delta-s+1)(s+1)-1 & s\\ (s+1)(\delta-s+1) & n-s-(\delta-s+1)(s+1) & s-1 \end{pmatrix}.$$

Through direct calculation, we find that its characteristic polynomial is

$$\begin{aligned}P(R_{s,\delta}, x) = x^3 &+ \big[s^2 - (\delta+1)s - n + 3\big]x^2 + \big[2s^4 - (4\delta+2)s^3\\ &+ \big(2\delta^2 - 3\delta + 2n - 3\big)s^2 + \big(5\delta^2 - 2n\delta + n + 5\delta\big)s\\ &+ 3\delta^2 - 3n\delta + 6\delta - 5n + 6\big]x\\ &- s^5 + (2\delta+3)s^4 - \big(\delta^2 + 3\delta + n\big)s^3 + (n\delta - 6\delta + 2n - 5)s^2\\ &+ \big(4\delta^2 - n\delta + 4\delta + 2n\big)s + 3\delta^2 - 3n\delta + 6\delta - 4n.\end{aligned} \tag{5}$$

Since $K_{n-\delta-1}$ is a proper subgraph of $\hat{G}$, $\partial(\hat{G}) > \partial(K_{n-\delta-1}) = n - \delta - 2$. By Lemma 2.2, $\partial(G^*) = \partial(R_{s,\delta})$, which is the largest solution of $P(R_{s,\delta}, x) = 0$. We can then compare the largest solutions of the equations to determine the magnitudes of the matrix eigenvalues. When (3) and (5) are combined, we get

$$\begin{aligned}&P(R_\delta, n-\delta-2) - P(R_{s,\delta}, n-\delta-2)\\ &\quad= (\delta-s)\big[3sn^2 + \big(4\delta + 2s^3 - (2\delta+3)s^2 - (9\delta+9)s + 4\big)n - s^4 - (\delta+1)s^3\\ &\qquad + \big(2\delta^2 + 6\delta + 4\big)s^2 + \big(6\delta^2 + 11\delta + 5\big)s - 5\delta^2 - 9\delta - 4\big]\\ &\quad:= (\delta-s)g(n).\end{aligned}$$

Since $2 \leq s$ and $\delta \geq s+1 \geq 3$, the axis of symmetry of $g(n)$ is given by

$$\begin{aligned}
&\frac{-2s^3+(2\delta+3)s^2+(9\delta+9)s-4\delta-4}{6s}\\
&\leq -\frac{1}{3}s^2+\frac{2\delta+3}{6}s+\frac{3\delta+3}{2}\\
&\leq -\frac{1}{3}\left(\frac{2\delta+3}{4}\right)^2+\frac{2\delta+3}{6}\left(\frac{2\delta+3}{4}\right)+\frac{3\delta+3}{2}\\
&=\frac{1}{12}\delta^2+\frac{7}{4}\delta+\frac{27}{16}\\
&<\frac{1}{2}\delta^2+2\delta+2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
g(n) &\geq g\left(\frac{1}{2}\delta^2+2\delta+2\right)\\
&=\frac{\delta}{4}\big[\delta\big(3s\delta^2-(4s^2-6s-8)\delta+4s^3-14s^2+6s+20\big)\\
&\quad+12s^3-16s^2-4s+28\big]-(s-1)(s^3-2s^2+1)+3\\
&\geq \frac{\delta}{4}\big[\delta\big(3s^3-6s^2+23s+28\big)+12s^3-16s^2-4s+28\big]-(s-1)(s^3-2s^2+1)+3\\
&\geq \frac{\delta}{4}\big(3s^4+6s^3+7s^2+24s+28\big)-(s-1)(s^3-2s^2+1)+3\\
&\geq \frac{1}{4}\big(3s^5+2s^4+19s^3+16s^2+24s+16\big)\\
&>0.
\end{aligned}$$

Then

$$P(R_\delta, n-\delta-2) > P(R_{s,\delta}, n-\delta-2). \tag{6}$$

Now, we consider the derivatives of (3) and (5). After subtracting one from the other, we obtain

$$\begin{aligned}
&P'(R_\delta,x)-P'(R_{s,\delta},x)\\
&=(\delta-s)\big[(2s-2)x+2s^3-(2\delta+2)s^2-(5\delta-2n+3)s+2\delta+n\big]\\
&\geq(\delta-s)\big[(2s-2)(n-\delta-2)+2s^3-(2\delta+2)s^2-(5\delta-2n+3)s+2\delta+n\big]\\
&=(\delta-s)\big[2s^3-(2\delta+2)s^2+(4n-7\delta-7)s+4\delta-n+4\big]\\
&:=(\delta-s)h(s).
\end{aligned}$$

We divide the value of δ into two cases for consideration: $3 \leq \delta \leq 5$ and $\delta > 5$. The derivative of $h(s)$ is greater than 0 in both cases. Thus, we have that $h(s)$ is monotonically increasing within the interval $(2, \delta-1)$, and therefore

$$h(s) \geq h(2) = 7n-18\delta-2 \geq 45\delta-2 > 0.$$

Therefore, we can conclude that

$$P'(R_\delta, x) > P'(R_{s,\delta}, x). \tag{7}$$

Next, we will determine the monotonicity of $P(R_\delta, x)$ according to the sign (positive or negative) of $P'(R_\delta, x)$. The axis of symmetry of $P'(R_\delta, x)$ is

$$\frac{\delta+n-3}{3} \leq (n-\delta-2)-\frac{14}{3}\delta+1 < n-\delta-2,$$

and then

$$\begin{aligned}P'(R_\delta, x) &\geq P'(R_\delta, n-\delta-2)\\ &= n^2-(8\delta+7)n+10\delta^2+16\delta+6\\ &\geq 19\delta^2-47\delta+6\\ &> 0.\end{aligned} \tag{8}$$

By comprehensively considering the conditions given by (6), (7), and (8), we can draw the conclusion that $\partial(\hat{G}) < \partial(G^*)$.

Furthermore, by combining the results of (1), (4), Subcases 2.1 and 2.2, we can obtain the following relationship:

$$\partial(\hat{G}) < \partial(G^*) \leq \partial(\tilde{G}) \leq \partial(G),$$

which is a contradiction.

Lastly, there is a particular case where $s=\delta$. In this case, $G' = K_\delta \vee (K_{n-2\delta-1} \cup (\delta+1)K_1) = \hat{G}$. Combining (1), (2) with the condition in the theorem, we have $G \cong \hat{G}$. Let $S = V(K_\delta)$, and we can get

$$\frac{|S|}{c(G-S)-1} = \frac{\delta}{\delta+1} < 1.$$

Thus, G is not 1-tough. □

4. Proof of theorem 1.2

Proof of Theorem 1.2. Suppose G does not satisfy the τ-tough condition. Then $|S| < \tau(c(G-S)-1)$ for $\emptyset \neq S \subset V(G)$. Since $\tau \geq 2$, $s \leq \tau(c-1)-1$. For positive integers $n_1 \geq n_2 \geq \cdots \geq n_c \geq 1$ with $\sum_{i=1}^{c} n_i = n-\tau(c-1)+1$, $G \subseteq_{sp} \tilde{G} = K_{\tau(c-1)-1} \vee (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_c})$. Lemma 2.1 yields

$$\partial(\tilde{G}) \leq \partial(G), \tag{9}$$

with equality if and only if $G \cong \tilde{G}$.

Let $G' = K_{\tau(c-1)-1} \vee (K_{n-(\tau+1)(c-1)+1} \cup (c-1)K_1)$. By Lemma 2.5, we have

$$\partial(G') \leq \partial(\tilde{G}), \tag{10}$$

with equality if and only if $\tilde{G} \cong G'$. According to Lemma 2.3, it follows that

$$\partial(G') \geq \frac{2W(G')}{n} = \frac{-(2\tau+1)(c-1)^2+(2n+1)(c-1)+n^2-n}{n} := \frac{\Phi(c)}{n}.$$

From $\sum_{i=1}^{c} n_i = n-\tau(c-1)+1$, we have $n-\tau(c-1)+1 \geq c$. If $c \geq 3$, then $2 \leq c-1 := x \leq \frac{n}{\tau+1}$ and

$$\Phi(x) = -(2\tau+1)x^2+(2n+1)x+n^2-n.$$

Therefore,

$$\Phi\left(\frac{n}{\tau+1}\right) - \Phi(2) = \frac{(n-2\tau-2)\big(n-4\tau^2-5\tau-1\big)}{(\tau+1)^2} \geq 0,$$

and $\min \Phi(x) = \Phi(2)$. We obtain

$$\partial(G') \geq \frac{\Phi(2)}{n} = \frac{n^2+3n-8\tau-2}{n} \geq n+3. \tag{11}$$

Let $\hat{G}$ denote $K_{\tau-1} \vee (K_{n-\tau} \cup K_1)$ as mentioned in the theorem. By calculation,

$$\begin{aligned} W^{(2)}(\hat{G}) &= \sum_{1 \le i < j \le n} d_{ij}^2(\hat{G}) \\ &= \frac{(\tau-1)(n-1) + (n-\tau)(n+2) + (\tau-1) + 4(n-\tau)}{2} \\ &= \frac{1}{2}n^2 + \frac{5}{2}n - 3\tau. \end{aligned}$$

With the help of Lemma 2.4 and taking into account the prerequisite condition that $\tau \ge 2$, we can draw the following inequality

$$\begin{aligned} \partial(\hat{G}) &\le \sqrt{\frac{2(n-1)W^{(2)}(\hat{G})}{n}} = \sqrt{\frac{(n-1)(n^2+5n-6\tau)}{n}} \\ &< \sqrt{\frac{n^3+4n^2-5n}{n}} = \sqrt{(n+2)^2-9} \\ &\le n+2. \end{aligned}$$

Thus, by virtue of (11), we infer that $\partial(\hat{G}) \le n+2 < \partial(G')$. When we combine (9) and (10), we obtain

$$\partial(\hat{G}) < \partial(G') \le \partial(\tilde{G}) \le \partial(G),$$

However, this relationship leads to a contradiction.

If $c = 2$, $G' = K_{\tau-1} \vee (K_{n-\tau} \cup K_1) = \hat{G}$. From (9) and (10), and the theorem condition $\partial(G) \le \partial(\hat{G})$, we have $\partial(\hat{G}) = \partial(G)$. Thus, it follows that $G \cong \hat{G} = K_{\tau-1} \vee (K_{n-\tau} \cup K_1)$. Let $S = V(K_{\tau-1})$, then we can calculate

$$\frac{|S|}{c(G-S)-1} = \tau - 1 < \tau.$$

This means that $\tau(G) < \tau$, and G is not τ-tough. □

5. Proof of theorem 1.3

Proof of Theorem 1.3. By contradiction, consider a non-empty vertex subset $S \subset V(G)$ with $|S| < \tau(c(G-S)-1)$, which implies $c \ge \frac{s}{\tau}+2$. For positive integers $n_1 \ge n_2 \ge \cdots \ge n_{\frac{s}{\tau}+2} \ge 1$ with $n_1+n_2+\cdots+n_{\frac{s}{\tau}+2} = n-s$, it is straightforward to see that $G \subseteq_{sp} \tilde{G} = K_s \vee \left(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{\frac{s}{\tau}+2}}\right)$. Lemma 2.1 implies

$$\partial(\tilde{G}) \le \partial(G), \tag{12}$$

with equality if and only if $G \cong \tilde{G}$.

For simplicity in subsequent notation, let $\hat{G} = K_1 \vee \left(K_{n-1-(\frac{1}{\tau}+1)} \cup (\frac{1}{\tau}+1)K_1\right)$. By Lemma 2.3 and $n \ge 4\tau + \frac{1}{\tau} + 5$, we have

$$\begin{aligned} \partial(\hat{G}) &\ge \frac{2W(\hat{G})}{n} = \frac{(1+\frac{1}{\tau})(2n-3) + (n-\frac{1}{\tau}-2)(n+\frac{1}{\tau}) + n - 1}{n} \\ &= n + \frac{1}{\tau} + 1 + \frac{1}{\tau} - \frac{4}{n} - \frac{5}{n\tau} - \frac{1}{n\tau^2} \\ &> n + \frac{1}{\tau} + 1. \end{aligned}$$

Let $G' = K_s \vee \Big(K_{n-s-(\frac{s}{\tau}+1)} \cup \big(\frac{s}{\tau}+1\big)K_1\Big)$. By Lemma 2.5, we get

$$\partial(G') \le \partial(\tilde{G}), \tag{13}$$

with equality if and only if $\tilde{G} \cong G'$. Consider partitioning $V(G')$ as $V(G') = V((\frac{s}{\tau}+1)K_1) \cup V(K_{n-s-(\frac{s}{\tau}+1)}) \cup V(K_s)$. Then $D(G')$ has the following equitable quotient matrix:

$$R_{\tau,s} = \begin{pmatrix} \frac{2s}{\tau} & 2\big(n-s-(\frac{s}{\tau}+1)\big) & s \\ 2\big(\frac{s}{\tau}+1\big) & n-s-\big(\frac{s}{\tau}+1\big)-1 & s \\ \frac{s}{\tau}+1 & n-s-\big(\frac{s}{\tau}+1\big) & s-1 \end{pmatrix}.$$

It is straightforward to verify that this partition is equitable and the characteristic polynomial of $R_{\tau,s}$ is

$$P(R_{\tau,s},x) = x^3 + \frac{-n\tau^2 - s\tau + 3\tau^2}{\tau^2}x^2 + \frac{\begin{matrix}-2ns\tau - 5n\tau^2 + 3s^2\tau + 2s^2 \\ +3s\tau + 3s\tau^2 + 6\tau^2\end{matrix}}{\tau^2}x + \frac{\begin{matrix}ns^2\tau + ns\tau^2 - 2ns\tau - 4n\tau^2 - s^3\tau - s^3 - \tau^2 s^2 + s^2\tau + 2s^2 \\ +2s\tau^2 + 4s\tau + 4\tau^2\end{matrix}}{\tau^2}.$$

For $D(\hat{G})$, the equitable quotient matrix $R_\tau = R_{\tau,1}$ is obtained by substituting $s = 1$ into $R_{\tau,s}$. According to Lemma 2.2, the largest root of $P(R_{\tau,s},x) = 0$ is $\partial(R_{\tau,s}) = \partial(G')$. Similarly, the largest root of $P(R_\tau,x) = 0$ equals $\partial(\hat{G})$. By direct calculation,

$$P(R_\tau,x) - P(R_{\tau,s},x) = (s-1)\alpha(x),$$

where $\alpha(x) = \tau x^2 + (2n\tau - 3s\tau - 2s - 3\tau^2 - 6\tau - 2)x - n(s\tau + \tau^2 - \tau) + s^2\tau + s^2 + s\tau^2 - s - \tau^2 - 4\tau - 1$. Then the axis of symmetry of $\alpha(x)$ is $x = -n + \frac{3}{2}s + \frac{s}{\tau} + \frac{3\tau}{2} + 3 + \frac{1}{\tau}$. Since each $n_i \ge 1$ and $\sum_{i=1}^{\frac{s}{\tau}+2} n_i = n - s$, we have $n - s \ge \frac{s}{\tau} + 2$, so $n \ge s + \frac{s}{\tau} + 2$. If $2 \le s$, we have

$$\begin{aligned} &-n + \frac{3}{2}s + \frac{s}{\tau} + \frac{3\tau}{2} + 3 + \frac{1}{\tau} \le \Big(n + \frac{1}{\tau} + 1\Big) \\ &-2\Big(s + \frac{s}{\tau} + 2\Big) + \frac{3}{2}s + \frac{s}{\tau} + \frac{3}{2}\tau + 2 \le n + \frac{1}{\tau} + 1, \end{aligned}$$

which implies that

$$\begin{aligned} \alpha(x) \ge \alpha\Big(n + \frac{1}{\tau} + 1\Big) &= (\tau+1)x^2 + \Big(-4n\tau - 2n + \tau^2 - 3\tau - 6 - \frac{2}{\tau}\Big)x \\ &\quad - 4\tau^2 - 12\tau - 7 - \frac{1}{\tau} + 3n^2\tau + n\big(-4\tau^2 - \tau + 2\big) \\ &\ge (\tau+1)\Big(\frac{n-2}{1+\frac{1}{\tau}}\Big)^2 + \frac{n-2}{1+\frac{1}{\tau}}\Big(-4n\tau - 2n + \tau^2 - 3\tau - 6 - \frac{2}{\tau}\Big) \\ &\quad - 4\tau^2 - 12\tau - 7 - \frac{1}{\tau} + 3n^2\tau + n\big(-4\tau^2 - \tau + 2\big) \\ &\ge \frac{1}{\tau^2+\tau}\big[n^2\tau^2 - \big(3\tau^4 + 4\tau^3 + \tau^2\big)n - 6\tau^4 - 6\tau^3 - 7\tau^2 - 4\tau - 1\big] \\ &\ge \frac{1}{(\tau^2+\tau)(\tau+1)}\big(-12\tau^6 - 33\tau^5 - 14\tau^4 + 24\tau^3 + 22\tau^2 + 5\tau\big) \\ &> 0. \end{aligned}$$

Thus, $P(R_\tau, x) > P(R_{\tau,s}, x)$ for all $x \geq n + \frac{1}{\tau} + 1$. Hence, $\partial(\hat{G}) < \partial(G')$. By combining (12) and (13), we get

$$\partial(\hat{G}) < \partial(G') \leq \partial(\tilde{G}) \leq \partial(G),$$

which is a contradiction.

If $s = 1$, then $G' = K_1 \vee \Big(K_{n-1-(\frac{1}{\tau}+1)} \cup \big(\frac{1}{\tau}+1\big)K_1\Big) = \hat{G}$. We have $\partial(\hat{G}) \leq \partial(G)$ from (12) and (13). By the theorem's condition that $\partial(G) \leq \partial(\hat{G})$, we get $\partial(\hat{G}) = \partial(G)$. Thus, $G \cong \hat{G} = K_1 \vee \Big(K_{n-1-(\frac{1}{\tau}+1)} \cup \big(\frac{1}{\tau}+1\big)K_1\Big)$. Let $S = V(K_1)$, then we can calculate

$$\frac{|S|}{c(G-S)-1} = \frac{1}{1+\frac{1}{\tau}+1-1} < \tau.$$

This shows that G is not τ-tough. □

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