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CONFLICT HYPERGRAPHS TO DEFINE NEW FAMILIES OF FACETS FOR THE INDEPENDENCE SYSTEM POLYTOPE

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Abstract. In this paper, we develop a technique for exact simultaneous upliftings of circuit inequalities of an independence system polytope. The resulting inequalities define new families of valid inequalities for this polytope. They are obtained by simultaneously adding the most appropriate set of variables with the highest possible values of the lifting coefficient that maintain the validity. More specifically, in this technique, we introduce a procedure to generate two conflict hypergraph structures types: hypertrees and clutter. In this setting, we use the hyperedges cardinalities of these structures to compute the suitable lifting coefficient values. We then give necessary and sufficient conditions for both the circuit inequalities and the new families of valid inequalities to be facet-defining. We also give a condition of the positivity of the lifted circuit inequality coefficient values.

Keywords: Independence system, conflict hypergraphs, rank inequality, facet, lifting.

Mathematics Subject Classification. 90C10, 90C27, 90C57

1. Introduction

An independence system on a finite set $W = \{1, \ldots, n\}$ is a nonempty family $\mathcal{I}$ of subsets of W closed under inclusion, i.e. $\forall\, I \in \mathcal{I}, \forall\, J \subset W, J \subset I \Rightarrow J \in \mathcal{I}$. The members of $\mathcal{I}$ are called *independent sets* and those of $2^W \setminus \mathcal{I}$ *dependent sets*, where 2^W is the power set of W. Let I be a subset of W. Then I is a *minimal*

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dependent set, if $I \notin \mathcal{I}$ and $I \setminus \{j\} \in \mathcal{I}$ for any $j \in I$. A minimal dependent set is called a *circuit*. Similarly, I is a *maximal independent set* if $I \in \mathcal{I}$ and $I \cup \{i\} \notin \mathcal{I}$ for any $i \in W \setminus I$ (see for example [26]).

In a wide range of applications, a non-negative weight c_i is associated with each element i, $i = 1, \ldots, n$, of W. The Independence System Problem ISP consists of finding an independent set of maximum total weight. Formally, it can be formulated as follows:

$$\text{ISP} : \max\{cx \mid x \in \mathcal{I}\}, \tag{1}$$

where x is a real n-vector, and $x \in \mathcal{I}$ means that there is $I \in \mathcal{I}$ such that $x_j = 1$ if $j \in I$, and $x_j = 0$ otherwise.

Several combinatorial optimization problems, like the spanning tree problem, the stable set problem, the knapsack problem, and the matching problem are special cases of ISP.

The polytope associated with an ISP, denoted by $\mathcal{P}$, is the convex hull of the incidence vectors x of the independent sets in $\mathcal{I}$, i.e.:

$$\mathcal{P} = \text{conv}\{x \in \{0,1\}^n : x \in \mathcal{I}\}. \tag{2}$$

Recall that, given a subset S of W, the rank of S, denoted by $r(S)$, is equal to the maximum cardinality of an independent set contained in S and the corresponding rank inequality, given by $\sum_{i \in S} x_i \leq r(S)$, is valid for $\mathcal{P}$. For instance, the cover inequality for the knapsack polytope is a rank inequality. It is one of the most known valid inequalities for $\mathcal{P}$. In particular, when $r(S) = |S| - 1$, the rank inequality is called a *circuit inequality.*

Theoretically, an ISP can be solved by using linear programming techniques if a complete description of $\mathcal{P}$ by a linear inequality system is known. This is, for instance, the case when the independence system is a matroid. Edmonds [7] showed that the matroid polytope is fully described by the rank inequalities and the non-negativity inequalities $x_j \geq 0$, for all $j \in W$. In practice, a partial description of $\mathcal{P}$ may be sufficient to solve efficiently the underlying ISP. Hence, finding some valid classes of facet-defining inequalities for $\mathcal{P}$ would be of great interest.

An independence system instance can be represented by using a conflict hypergraph $H = (V, E)$ by associating each element j of W to a vertex j of V, and associating a hyperedge for a set of vertices of V for which the corresponding elements in W conflict. Rank inequalities that are based on some structured subhypergraphs represent the most known facet-defining inequalities for $\mathcal{P}$. Conflict hypergraphs are used by numerous authors to generate both valid and facet-defining inequalities for $\mathcal{P}$. For instance, Euler et al. [8] generalized the clique inequalities to $\mathcal{P}$, and Fouilhoux et al. [9] derived a new family of facet-defining inequalities for $\mathcal{P}$. For more examples, the reader can refer to [9]. In our work, we mainly describe two conflict hypergraph structures: hypertree and clutter, that we use to develop new families of valid inequalities for $\mathcal{P}$.

The rank inequality is a facet of a lower-dimensional polytope. Nevertheless, it can be lifted to define a facet for $\mathcal{P}$ by using lifting techniques. Lifting is a technique introduced by Gomory [10] which is used to strengthen valid inequalities. It also increases the coefficients of the variables while maintaining the validity of the inequalities. More specifically, in the lifting process, the variables can be lifted sequentially or simultaneously and the computation of the lifting coefficients can be either exact or approximate. In this framework, if the lifted variables are fixed at their lower bounds then the lifting is called *uplifting*, and if the lifted variables are fixed at their upper bounds, then the lifting is called *down-lifting*. Consequently, lifting can be conducted in at least eight ways depending on the combined properties of lifting: one from {up, down}, one from {exact, approximate}, and one from {sequential, simultaneous}.

In the literature, numerous researchers extensively studied lifting techniques since the 1970s. In this setting, we can list the relevant works on sequential lifting [2, 5, 15, 16, 17, 21, 23, 27, 28, 31, 32], some other works on sequence independent lifting [1, 12, 13, 14, 25, 29, 33] and also on simultaneous lifting [6, 15, 19, 22, 24, 30]. This number of contributions is due to their effectiveness in developing strong valid inequalities that have been successfully used to solve optimization problems.

In this work, we focus on the case of simultaneous lifting which is introduced by Zemel [30] in 1978. This technique is based on the search for the extreme points from the solutions of exponentially many linear integer programs.

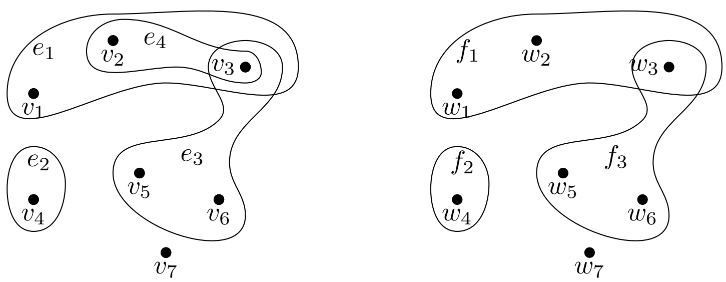
More specifically, in this contribution, we develop a technique for exact simultaneous uplifting of circuit inequalities. Thereby, we aim to increase the dimension of the circuit inequality to construct new families of valid inequalities that we call *CH inequalities*. These inequalities are obtained by simultaneously adding a set of variables with the largest possible coefficient values that maintain the validity in $\mathcal{P}$. To this end, we introduce a procedure that we call *CH procedure* to generate two conflict hypergraph structures kinds: hypertree and clutter. Then, we use the hyperedges cardinalities of these conflict hypergraph structures to compute the lifting coefficient values of the added variables into the CH inequality. We describe a condition of the positivity of the lifted circuit inequality coefficient values. We also describe necessary and sufficient conditions for the circuit and CH inequalities to be facets-defining.

We have organized our paper as follows: Section 2 begins with some preliminaries and definitions of hypergraphs, ISP, and $\mathcal{P}$. In Section 3, we give necessary and sufficient conditions for the circuit inequality to be facet-defining. In Section 4, we state a condition of the positivity of the lifted circuit inequality coefficient values. We also present the steps of our lifting technique, describe the CH inequality and give the formula of its lifting coefficients. In Section 5, we describe the CH procedure and the resulting conflict hypergraph structures. In Section 6, we derive new families of valid inequalities for $\mathcal{P}$ by using the conflict hypergraph structures. In Sections 7 and 8, we give the theoretical results for hypertree and clutter structures, respectively, which provide necessary and sufficient conditions for the CH inequalities to be facets-defining. We end the paper in Section 9 with a conclusion and some directions for future research.

2. Preliminaries and definitions

2.1. Hypergraphs

The concept of a hypergraph is an essential part of this work. It was introduced in 1970, with the work by Claude Berge [3]. Formally, a *hypergraph* H is an ordered pair (V, E) of finite sets, where $V = \{1, \ldots, n\}$ is the set of vertices, and $E = \{E_1, \ldots, E_m\}$ is the set of hyperedges, where $E_i \subseteq V$ for all $i = 1, \ldots, m$. Note that a hypergraph whose all hyperedges have cardinality two is a *graph*. For background information on hypergraphs, the reader can refer to [4]. In this concept, we concentrate on two structures: hypertree and clutter. Before defining these structures, we need to present some essential definitions. A realization of a given hypergraph $H = (V, E)$ is the undirected graph $G = (V, A)$ whose each edge is contained in a hyperedge of H, and the subgraph of G induced by each hyperedge of H is connected. If G is a tree, it is called a *tree-realization* of H. We now turn to define these structures: a *hypertree* is a hypergraph having a tree-realization [18], and a *clutter* is a hypergraph in which no hyperedge contains another one [11].



A hypergraph, A clutter.

Figure 1. Examples of a hypergraph and a clutter.

2.2. ISP and $\mathcal{P}$

This article aims to define valid inequalities that could be facets for $\mathcal{P}$. In this context, and more specifically, in certain proofs developed in the following sections, we first need to know the dimension of $\mathcal{P}$ to guide the construction of the facet proof. Before computing the dimension of $\mathcal{P}$, let us first describe how we can write ISP and $\mathcal{P}$ in matrix forms.

$$\text{ISP} : \max\{cx \mid Ax \leq b, x \in \{0,1\}^n\}, \tag{3}$$

$$\mathcal{P} = \text{conv}\{x \in \{0,1\}^n : Ax \leq b\}, \tag{4}$$

where $A \in \mathbb{R}_+^{t\times n}$ is a matrix of resource requirements, and $b \in \mathbb{R}_+^t$ is a vector of resource capacities. Without loss of generality, we assume that $a_{ij} \leq b_i$, for $i \in \{1,\ldots,t\}$ and $j \in \{1,\ldots,n\}$ in every ISP instance. Hence, every dependent set has at least two elements. In this case, $\mathcal{P}$ is of full dimension (i.e. $\dim(\mathcal{P}) = n$), since the $(n+1)$ vectors $\{e_1,\ldots,e_n,0\}$, where e_j denotes the j^{th} unit vector (the vector with 1 in the j^{th} component, and 0's elsewhere), are affinely independent points. The facets associated with the inequalities $x_j \geq 0, j = 1,\ldots,n$ are known as *trivial facets.*

Remark 1. As $\dim(\mathcal{P}) = n$, it is clear that, in the facet proof of the sufficient condition of the CH inequality, we should exhibit n affinely independent points that fulfill this inequality with equality.

Consequently, to avoid repetition, we only have to list the n affinely independent points without repeating Remark 1 in each proof.

Alternatively, as an independence system is fully characterized by its family of circuits, then one can describe $\mathcal{P}$ by (see [20]):

$$\mathcal{P} = \mathrm{conv}\{x \in \{0,1\}^n : \sum_{j\in\mathcal{C}} x_j \leq |\mathcal{C}| - 1 \text{ for all } \mathcal{C} \in \mathfrak{C}\}, \tag{5}$$

where $\mathfrak{C}$ denotes the family of circuits of the ISP associated with the polytope $\mathcal{P}$.

2.3. Examples of ISP

In combinatorial optimization, many problems can be considered in terms of maximizing an objective function over an independence system. We cite below some examples of ISP.

The stable set problem consists of finding in a vertex weighted given graph a stable set (a subset of pairwise non-adjacent vertices) of maximum weight. It is an ISP, where a stable set represents an independent set.

The triangle-free induced subgraph problem (TFISP) consists of finding, in a vertex weighted given graph G, a triangle-free induced subgraph whose weight is maximum [9]. The feasible solutions of TFISP are the independents of the independence system, where each triangle-free induced subgraph of G represents an independent set.

The knapsack problem is also an ISP, for which the associated independence system is given as follows: the set W is the set of items, and the set $\mathcal{I}$ is the family of subsets of W whose incidence vectors satisfy the knapsack constraint $\sum_{j\in W} a_j x_j \leq d$ with $a_j > 0$ for all $j \in W$, where d is the capacity of the knapsack and a_j is the weight of the item j of W. The knapsack problem consists of finding an independent set with maximum profit, a positive profit being assigned to each item.

3. Circuit inequality

The rank inequality is among the most crutial classes of valid inequalities for $\mathcal{P}$. For any independence system $(W, \mathcal{I})$ and any subset S of W, the inequality

$$\sum_{i \in S} x_i \leq r(S), \tag{6}$$

which is called *rank inequality*, is valid for $\mathcal{P}$, where $r(S)$ denotes the rank of S in $(W, \mathcal{I})$. It is determined by using the rank set function r defined as follows:

$$\begin{aligned} r: \ & 2^W \to \mathbb{R} \\ & S \mapsto r(S), \end{aligned}$$

where

$$r(S) = \max\{|I|, I \subseteq S, \ I \in \mathcal{I}\}. \tag{7}$$

The rank $r(S)$, of a subset S represents the maximum cardinality of an independent set in $(W, \mathcal{I})$ that is containd in S.

When the set S is a circuit, we have $r(S) = |S| - 1$, and the rank inequality is called *circuit inequality*. Sufficient conditions for the rank inequality to define a facet for $\mathcal{P}$ are given by Laurent [20]. In this section we present necessary and sufficient conditions for the circuit inequality to define a facet for $\mathcal{P}$.

Theorem 1. *Let us consider an independence system $(W, \mathcal{I})$, and a subset S of W, where S is a circuit. The circuit inequality:*

$$\sum_{i \in S} x_i \leq |S| - 1, \tag{8}$$

induced by S defines a facet for $\mathcal{P}$ if and only if the following condition holds:

(i) $\forall \, j \in W \setminus S$, $\exists \, i \in S$, such that $S_{ij} = (S \setminus \{i\}) \cup \{j\}$ is not a circuit.

Proof. Necessary condition

For the sake of contradiction, assume that the condition (i) of Theorem 1 is not satisfied. If there exists $j \in W \setminus S$ such that $\forall \, i \in S$ the set $S_{ij} = (S \setminus \{i\}) \cup \{j\}$ is a circuit, then any solution of the independent set problem containing j can not include more than $|S| - 2$ elements from S. Hence, its incidence vector cannot satisfy Inequality (8) at equality. It follows that j cannot belong to any solution of the problem whose incidence vector satisfies Inequality (8) at equality. We then deduce that Inequality (8) is equivalent to $x_j \geq 0$. Since Inequality (8) is not a positive multiple of $x_j \geq 0$, this implies that it does not define a facet for $\mathcal{P}$.

Sufficient condition

We know that a circuit inequality is valid. The n following feasible points are affinely independent and satisfy Inequality (8) at equality.

- The set S is a circuit, so the points $x^{\mathcal{S}_h}$, where $\mathcal{S}_h = S \setminus \{h\}$ and $h \in S$, are feasible and satisfy Inequality (8) at equality. We get $|S|$ points.

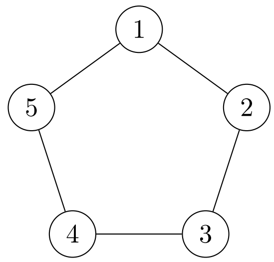


Figure 2. Odd hole C_5.

- From the condition (i) of Theorem 1, for any element $j \in W \setminus S$, we can find an element $i \in S$, such that $S_{ij} = (S \setminus \{i\}) \cup \{j\}$ is not a circuit. We thus obtain the remaining $|W \setminus S| = n - |S|$ feasible points, denoted by $x^{S_{ij}}$, that satisfy Inequality (8) at equality.

Note that these n points are affinely independent. Thus, the circuit inequality define de facet for $\mathcal{P}$. □

Example 1. Vertex packing polytope. To illustrate Theorem 1, we present, in this example, circuit inequalities for the polytope associated with the vertex packing problem. Consider the odd hole $G = (V, A)$, in Figure 2, where $c = (1, 1, 1, 1, 1)$ is a weight vector. The circuits of $G = (V, A)$ are: $S^1 = \{1, 2\}$, $S^2 = \{2, 3\}$, $S^3 = \{3, 4\}$, $S^4 = \{4, 5\}$ and $S^5 = \{5, 1\}$. Each of these circuits induces a circuit inequality, for which the condition (i) of Theorem 1 holds. For example, for the circuit $S^1 = \{1, 2\}$, the sets $S^1_{23} = (S^1 \setminus \{2\}) \cup \{3\} = \{1, 3\}$, $S^1_{14} = (S^1 \setminus \{1\}) \cup \{4\} = \{2, 4\}$ and $S^1_{15} = (S^1 \setminus \{1\}) \cup \{5\} = \{2, 5\}$ are not circuits. Thus, $x_1 + x_2 \leq 1$ defines a facet for $\mathcal{P}$.

In fact, these results hold for any circuit of $G = (V, A)$, since $\forall\, j \in V \setminus S^k$, $\exists i \in S^k$, such that $S^k_{ij} \setminus \{i\}) \cup \{j\}, k = 1, \ldots, 5$, are not circuits. It follows that the circuit inequality induced by each of the circuits of G is a facet-defining inequality for $\mathcal{P}$.

$$\begin{aligned} \mathcal{P} &= \text{conv}\{x \in \{0,1\}^5 : x_1 + x_2 \leq 1, x_2 + x_3 \leq 1, x_3 + x_4 \leq 1, x_4 + x_5 \leq 1, x_5 + x_1 \leq 1\} \\ &= \{x \in [0,1]^5 : x_1 + x_2 \leq 1, x_2 + x_3 \leq 1, x_3 + x_4 \leq 1, x_4 + x_5 \leq 1, x_5 + x_1 \leq 1\}. \end{aligned}$$

4. Lifted circuit inequality

The circuit inequality induced by a subset S of W :

$$\sum_{i \in S} x_i \leq |S| - 1, \tag{9}$$

is a facet of the lower dimensional polytope $\mathcal{P}_S = \text{conv}\{x \in \{0,1\}^{|S|}, A_S x \leq b\}$, where $A_S \in \mathbb{R}_+^{t \times |S|}$. Nevertheless, we can lift it to define a facet for $\mathcal{P}$ using lifting techniques, where the variables x_j, $j \in W \setminus S$, could be introduced into the inequality by finding appropriate lifting coefficients.

In this work, we focus our interest on the lifted circuit inequality of the form:

$$\sum_{j \in S} x_j + \sum_{j \in N} \alpha_j x_j \leq |S| - 1, \tag{10}$$

where $N = W \setminus S$, obtained from introducing the variables x_j, $j \in W \setminus S$ into (9) with coefficients $\alpha_j \geq 0$. In this context, we will develop new families of valid inequalities using conflict hypergraph structures that we call *CH inequality* (Conflict Hypergraph inequality). Before presenting the CH inequality, let's first answer the following question: *Could the variables x_j, $j \in N \subseteq W \setminus S$, be lifted into the circuit inequality?*

To guarantee the lifting of the variables x_j, $j \in W \setminus S$, the coefficients α_j should be strictly positive. In Corollary 1, we state a condition of the positivity of the coefficients α_j in Inequality (10). This condition is based on the conflict relationship between every two elements i and j in an independence system instance on W, where i belongs to a circuit S and j belongs to $W \setminus S$. To represent this conflict relationship for any circuit $S \subset W$, we associate a critical bipartite graph $G_S = (S, \bar{S}_W, A)$, with $\bar{S}_W \subseteq W \setminus S$, to an independence system instance on W as follows:

- The sets S and $\bar{S}_W$ are the bipartition of the vertex set, and each vertex j in G_S is associated with a binary variable x_j.
- The set A is the edge set. An edge $\{i, j\} \in A$ links vertices i and j, where $i \in S$ and $j \in \bar{S}_W$, if and only if $S_{ij} = (S \setminus \{i\}) \cup \{j\}$ is a circuit.

Corollary 1. *If a critical bipartite graph $G_S = (S, \bar{S}_W, A)$, associated to an independence system on W, is complete bipartite, then we can find $\alpha_j > 0, \forall\, j \in \bar{S}_W$, such that Inequality (10) holds.*

Proof. As the critical bipartite graph $G_S = (S, \bar{S}_W, A)$ is complete bipartite, then the set $S_{ij} = S \setminus \{i\}) \cup \{j\}$ is a circuit, $\forall\, \{i, j\} \in A$ such that $i \in S$ and $j \in \bar{S}_W$. It follows that S_{ij} is not a feasible solution for ISP.

For the sake of clarity, in the rest of this proof, Inequation (10) will be written in the form $\overline{\alpha} x \leq \beta$, where

$$\overline{\alpha}_j = \begin{cases} 1 & \text{if } j \in S, \\ \alpha_j & \text{if } j \in \bar{S}_W. \end{cases}$$

So, we have

$$\begin{aligned} \overline{\alpha} x^{S_{ij}} > \beta \quad & \Leftrightarrow \quad x^{S \setminus \{i\}} + \alpha_j x_j \quad > \quad \beta, \qquad (x_j = 1), \\ & \Leftrightarrow \quad x^{S \setminus \{i\}} + \alpha_j \quad > \quad \beta, \\ & \Leftrightarrow \quad x^{S \setminus \{i\}} \quad > \quad \beta - \alpha_j. \end{aligned} \tag{A}$$

Since S is a circuit, $S \setminus \{i\}$ is a feasible solution for ISP. It follows that;

$$x^{S\setminus\{i\}} \leq \beta. \qquad (B)$$

From (A) and (B), we get:

$$\beta - \alpha_j < x^{S\setminus\{i\}} \leq \beta \qquad \Rightarrow \qquad \beta - \alpha_j < \beta \qquad \Rightarrow \qquad \alpha_j > 0.$$

Hence $\forall\, j \in \bar{S}_W$, $\alpha_j > 0$. □

We now turn to the development of CH inequality. To this end, we will introduce a technique for an exact simultaneous uplifting of the circuit inequality based on conflict hypergraph structures, which are used to compute the lifting coefficient values. This technique consists of the following steps:

(1) **Step 1:** Define the CH inequality.
(2) **Step 2:** Introduce a procedure to generate two conflict hypergraph structures kinds: hypertree and clutter.
(3) **Step 3:** Use the generated conflict hypergraph structures to compute the lifting coefficient of CH inequality.

Steps 2 and 3 will be presented in detail in Sections 5 and 6, respectively. This section is devoted to Step 1, which consists of defining the CH inequality by describing its form and giving the formula of its lifting coefficients.

Let us start with the form of the CH inequality, which we define as follows:

$$\sum_{j\in S} x_j + \alpha \sum_{j\in N} x_j \leq |S| - 1, \qquad (11)$$

where $N \subseteq W \setminus S$.Inequality (11) can be seen as a special case of Inequality (10), where the coefficients α_j are all equal. The difference between them is in the way in which the variables x_j are introduced into the circuit inequality, sequentially for Inequality (10), and simultaneously for Inequality (11).

At the end of this section, we present how the value of the lifting coefficient α could be computed. Before starting, let us first introduce some notations. Given a circuit E of ISP, where $E = I \cup Q$, $I \subset S$, $0 \leq |I| \leq |S| - 1$, $Q \subset N$ and $2 \leq |Q| \leq |N| - 1$, one can observe that $E \setminus \{i\}$, where $i \in Q$, is a feasible solution of ISP. We can compute the lifting coefficient α by using the following formula:

$$\alpha = \frac{|S| - 1 - |I|}{|Q| - 1}, \qquad (12)$$

where $\quad |I| \geq \sum_{j\in I} x_j^{E\setminus\{i\}} \quad$ and $\quad |Q| - 1 \geq \sum_{j\in Q} x_j^{E\setminus\{i\}}$.

To make Formula (12) simpler, let us set $L = S \setminus I$, $|Q| = q + 1$ and let $l = |L|$, where l is the number of elements of S that do not belong to E, and q is the number of elements of N that we can add to I to form a maximal independent. It follows that $|I| = |S| - l$, and thus we can use the following formula to compute the lifting coefficients α:

$$\alpha = \frac{l-1}{q}. \tag{13}$$

5. Conflict hypergraph structures

In this section, we introduce a procedure, which we call *CH procedure* (Conflict Hypergraph procedure), to generate two conflict hypergraph structures kinds: hypertree and clutter. This procedure allows forming a conflict hypergraph, denoted by $H^c = (V^c, E^c)$, for a given independence system instance ISP, on a finite set W of n elements and a given value of the parameter l. The set W defines the set of vertices V^c.

Procedure *Conflict Hypergraph*(Input: W, l, Output: $H^c = (V^c, E^c)$**)**

Begin

Step 0 Initialization: $V^c \leftarrow W$, $r \leftarrow 0$, $E^c \leftarrow \emptyset$, $E_0 \leftarrow \emptyset$, $N^{(r)} \leftarrow V^c$, $L \leftarrow \emptyset$.

Step 1 Select from V^c elements forming a circuit and put them in E_0.

Step 2 If E_0 is empty, then go to Step 9.

Step 3 $p \leftarrow |E_0|$. If E_0 induces a facet for $\mathcal{P}$ (Condition (i) of Theorem 1), then go to Step 1, else select l elements from E_0 and put them in L, and set $Y \leftarrow E_0$, $I \leftarrow E_0 \setminus L$.

Step 4 $r \leftarrow r+1$, set $E^c \leftarrow E^c \cup Y$, $N^{(r)} \leftarrow N^{(r)} \setminus Y$, $Y \leftarrow \emptyset$.

Step 5 If $N^{(r)} = \emptyset$ then go to Step 9.

Step 6 Set $E_r \leftarrow I$ and $Q^{(r)} \leftarrow \emptyset$.

Step 7 Select a subset of elements from $N^{(r)}$, namely $Q^{(r)}$, to add to E_r in order to form a circuit, such that $|Q^{(r)}| \geq 2$.

Step 8 If $E_r \neq \emptyset$ then set $Y \leftarrow E_r$ and $q^{(r)} \leftarrow |Q^{(r)}| - 1$, go to Step 4.

Step 9 Stop and do the following test:
If $E_0 \neq \emptyset$ and $E^c \setminus E_0 \neq \emptyset$, then sort the elements of the set $E^c \setminus E_0$ in increasing order of their cardinal. Then according to this sort, re-index sets E_r, for $r = 1, \ldots, m$, from the largest one to the smallest one with index k from m to 1, i.e. $E^c = \{E_0, E_m, \ldots, E_k, \ldots, E_1\}$. Thereafter, re-index the elements of the set V^c. It follows that:

- $V^c = \{1, \ldots, p, \ldots, n-q^{(k)}-\gamma_k, \ldots, n-\gamma_k, \ldots, n-q^{(1)}, \ldots, n\}$,
- $E_0 = \{1, \ldots, p\}$, $|E_0| \geq 2$,
- $E_1 = \{l+1, \ldots, p, n-q^{(1)}, \ldots, n\}$, where $1 \leq q^{(1)} < n-|E_0|$,
- $E_k = \{l+1, \ldots, p, n-q^{(k)}-\gamma_k, \ldots, n-\gamma_k\}$, for $k = 2, \ldots, m$,

where $\gamma_k = \sum_{i=1}^{k-1}(q^{(i)} + 1)$ and $1 \leq q^{(k)} \leq q^{(k-1)}$.

Else write "$H^c = (V^c, E^c)$ could not be constructed using these data".

End

From this procedure, we can derive a conflict hypergraph $H^c = (V^c, E^c)$ with a set of vertices $V^c = W$, and a set of hyperedges $E^c = \{E_0, E_m, \dots, E_1\}$ which are circuits. According to the value of parameter l, we can distinguish two structures of $H^c = (V^c, E^c)$:

- **Hypertree structure:** If $l = 1, \dots, |E_0| - 1$.
- **Clutter structure:** If $l = |E_0|$.

We suppose that there exist two oracles, the first one checks in unitary time if a given set is a circuit, and the second one checks in unitary time the condition (i) of Theorem 1. Thus, under this hypothesis, we can deduce that the running time of the CH procedure in the worst case is quadratic $O(n^2)$. So, the conflict hypergraphs $H^c = (V^c, E^c)$ can be generated in polynomial time.

In the following sections, the conflict hypergraph $H^c = (V^c, E^c)$ will be referred to as a conflict hypergraph associated with an independence system instance.

Remark 2. Let $c = (c_1, \dots, c_n)$ be the weight vector associated with the elements of W. For problems with capacity constraints, like the knapsack problem, it is recommended to sort the elements of W in decreasing order according to their weight. This allows us to easily have the hypergraph structures.

Let $I = \{l{+}1, \dots, p\}$, where $l \in \{1, \dots, p\}$. If $l = p$, then $I = \emptyset$. The hyperedges of $H^c = (V^c, E^c)$ could be written as follows :

- $E_0 = \{1, \dots, l\} \cup I$,
- $E_1 = I \cup \{n - q^{(1)}, \dots, n\}$,
- $E_k = I \cup \{n - q^{(k)} - \gamma_k, \dots, n - \gamma_k\}$, for $k = 2, \dots, m$, with $\gamma_k = \sum_{i=1}^{k-1}(q^{(i)} + 1)$.

The hyperedges cardinalities $|E_k|$, where $k = 0, \dots, m$, are

- $|E_0| = l + |I|$,
- $|E_k| = |I| + q^{(k)} + 1$, where $k = 1, \dots, m$,
- $|E_m| \leq \dots \leq |E_k| \leq \dots \leq |E_1|$.

In the next section, we will use the hyperedge cardinalities of the conflict hypergraph structures to compute the lifting coefficient α.

6. New families of valid inequalities for $\mathcal{P}$

In this section, we present the use of the CH procedure to simultaneously exact uplifting the circuit inequalities into CH inequalities. Indeed, to compute the lifting coefficient α, we use hyperedge cardinalities of the conflict hypergraph $H^c =$

(V^c, E^c) to define the values of the parameters $|I|$ and $|Q^{(k)}|$, where $Q^{(k)} = E_k \setminus I$. The CH inequalities induced by a circuit E_0 of $H^c = (V^c, E^c)$ are given by:

$$\sum_{j \in E_0} x_j + \alpha^{(k)} \sum_{j \in N^{(k)}} x_j \leq |E_0| - 1, \tag{14}$$

where $N^{(k)} = V^c \setminus \bigcup_{j=0}^{k-1} E_j$, $\alpha^{(k)} > 0$.

As the hyperedges E_k, $k = 1, \ldots, m$, are circuits, it follows that $E_k \setminus \{i\}$, where $i \in Q^{(k)}$, is a feasible solution of ISP. Thus, by Formula (12), when $S = E_0$, the lifting coefficients $\alpha^{(k)}$ are given by $\alpha^{(k)} = \frac{l-1}{q^{(k)}}$.

Remark 3. The values of the coefficients $\alpha^{(k)}$ vary according to the cardinalities of the hyperedges E_0 and E_k, $k = 1, \ldots, m$. We can distinguish two cases:

- **Case 1:** $|E_k| \geq |E_0|$, we have

$$|E_0| \leq |E_k| \quad \Leftrightarrow \quad |I| + l \leq |I| + q^{(k)} + 1 \tag{15}$$

$$\Leftrightarrow \quad \alpha^{(k)} = \frac{l-1}{q^{(k)}} \leq 1. \tag{16}$$

- **Case 2:** $|E_k| < |E_0|$, we have

$$|E_0| > |E_k| \quad \Leftrightarrow \quad |I| + l > |I| + q^{(k)} + 1 \tag{17}$$

$$\Leftrightarrow \quad \alpha^{(k)} = \frac{l-1}{q^{(k)}} > 1. \tag{18}$$

The CH inequalities represent new families of valid inequalities for $\mathcal{P}$. We state the validity of the CH inequalities in Theorem 2. Throughout this paper, we use the following notations to ease the proofs of the theorems.

$$\tilde{z}_1 = \sum_{j \in I} x_j^{E_k \setminus \{i\}} \leq |I| \quad \text{and} \quad \tilde{z}_2 = \sum_{j \in Q^{(k)}} x_j^{E_k \setminus \{i\}} \leq |Q^{(k)}| - 1 = q^{(k)}.$$

Theorem 2. *Let us consider an independence system instance with the corresponding conflict hypergraph $H^c = (V^c, E^c)$. Then Inequality (14) is valid for $\mathcal{P}$.*

Proof. Conversely, suppose that Inequality (14) is not valid. Thus, there exists $x^* \in \mathcal{P}$ such that

$$\sum_{j \in E_0} x_j^* + \frac{l-1}{q^{(k)}} \sum_{j \in N^{(k)}} x_j^* > |E_0| - 1 \tag{19}$$

Since E_0 is a circuit, we necessarily have $\tilde{z}_1 \leq |E_0| - 1$. In this setting, we can distinguish the following cases:

Case 1, $\tilde{z}_1 = 0$ and $\tilde{z}_2 = 0$: From Inequality (19) we have $0 > |E_0| - 1$, which is absurd.

Case 2, $\tilde{z}_1 = 0$ and $\tilde{z}_2 \geq 1$: From Inequality (19) we have

$$\frac{l-1}{q^{(k)}}\tilde{z}_2 > |E_0| - 1 \tag{20}$$

By assumption, we have $1 \leq l \leq |E_0|$, $\tilde{z}_2 \leq q^{(k)}$ and $1 \leq q^{(k)}$, for $k \geq 1$. It follows that:

$$\frac{l-1}{q^{(k)}} \leq \frac{|E_0|-1}{q^{(k)}} \quad \Leftrightarrow \quad \frac{l-1}{q^{(k)}}\tilde{z}_2 \leq \frac{|E_0|-1}{q^{(k)}}\tilde{z}_2 \tag{21}$$

$$\Leftrightarrow \quad \frac{l-1}{q^{(k)}}\tilde{z}_2 \leq \frac{|E_0|-1}{q^{(k)}}q^{(k)} \tag{22}$$

$$\Leftrightarrow \quad \frac{l-1}{q^{(k)}}\tilde{z}_2 \leq |E_0| - 1 \tag{23}$$

The last inequality contradict Inequality (20).

Case 3, $\tilde{z}_1 \geq 1$ and $\tilde{z}_2 = 0$: Inequality (19) reduces to the inequality

$$\sum_{j \in E_0} x_j^* > |E_0| - 1. \tag{24}$$

Inequality (24) means that the circuit inequality induced by the circuit E_0

$$\sum_{j \in E_0} x_j \leq |E_0| - 1, \tag{25}$$

is not valid for $\mathcal{P}$, which is absurd.

Case 4, $\tilde{z}_1 \geq 1$ and $\tilde{z}_2 \geq 1$: In this case, and by the construction of the hyperedges E_k, $k \geq 1$, we have $\tilde{z}_1 \leq |E_0| - l$ and $\tilde{z}_2 \leq q^{(k)}$. Consequently, Inequality (19) reduces to the following contradiction:

$$\tilde{z}_1 + \frac{l-1}{q^{(k)}}\tilde{z}_2 > |E_0| - 1 \quad \Leftrightarrow \quad \frac{l-1}{q^{(k)}} > \frac{|E_0| - 1 - \tilde{z}_1}{\tilde{z}_2}. \tag{26}$$

$$\Leftrightarrow \quad \frac{l-1}{q^{(k)}} > \frac{l-1}{q^{(k)}}. \tag{27}$$

So Inequality (14) is valid for $\mathcal{P}$. □

Remark 4.
For $\alpha = 1$, when the elements of the vertex set V^c are arranged in increasing order, according to their weight, Inequalities (14) reduce to the extended cover inequality.

Example 2. Let us consider the following subset sum instance:

$$\begin{cases} \max Z = & 30x_1 + 30x_2 + 25x_3 + 25x_4 + 16x_5 + 16x_6 + 15x_7 + 15x_8 + 15x_9 \\ & +14x_{10} + 14x_{11} + 13x_{12} + 13x_{13}, \\ \text{subject to} & \\ & 30x_1 + 30x_2 + 25x_3 + 25x_4 + 16x_5 + 16x_6 + 15x_7 + 15x_8 + 15x_9 \\ & +14x_{10} + 14x_{11} + 13x_{12} + 13x_{13} \leq 92, \\ & \\ & x_j \in \{0,1\} \quad j = 1,\dots,13. \end{cases}$$

The associated conflict hypergraph is $H^c = (V^c, A^c)$, where $I = \{3,4\}$.
$V^c = \{1,2,3,4,5,6,7,8,9,10,11,12,13\}$.
$A^c = \{E_0 = \{1,2,3,4\},\ E_1 = \{3,4,10,11,12,13\}, E_2 = \{3,4,7,8,9\}\}$.

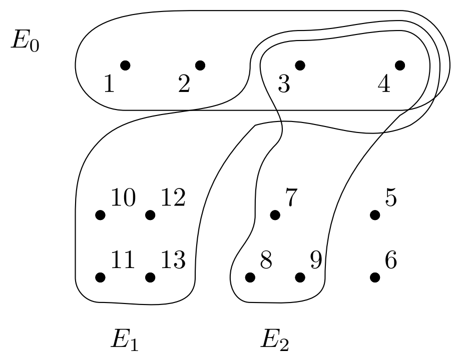


FIGURE 3. The Hypertree structure of $H^c = (V^c, A^c)$

(1) For $k = 1$, $q^{(1)} = 3$, $\alpha^{(1)} = \frac{1}{3}$ and $N^{(1)} = V^c \backslash E_0 = \{5,6,7,8,9,10,11,12,13\}$, we have

$$\sum_{j \in E_0} x_j + \alpha^{(1)} \sum_{j \in N^{(1)}} x_j \leq |E_0| - 1 \quad \Leftrightarrow$$
$$x_1 + x_2 + x_3 + x_4 + \frac{1}{3}(x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13}) \leq 3 \tag{28}$$

(2) For $k = 2$, $q^{(2)} = 2$, $\alpha^{(2)} = \frac{1}{2}$ and $N^{(2)} = V^c \setminus (E_0 \cup E_1) = \{5,6,7,8,9\}$, we have

$$\begin{aligned} &\sum_{j \in E_0} x_j + \alpha^{(2)} \sum_{j \in N^{(2)}} x_j \leq |E_0| - 1 \Leftrightarrow \\ &x_1 + x_2 + x_3 + x_4 + \frac{1}{2}(x_5 + x_6 + x_7 + x_8 + x_9) \leq 3 \end{aligned} \tag{29}$$

By Theorem 2, Inequalities (28) and (29) are valid for $\mathcal{P}$.

The CH procedure allows for forming a conflict hypergraph $H^c = (V^c, E^c)$. The set of hyperedges E^c could contain only one hyperedge, namely E_0. Nevertheless, this does not avoid generating valid inequalities for $\mathcal{P}$. Indeed, from this kind of conflict hypergraphs, we can generate CH inequalities of the form

$$\sum_{j\in E_0\cup\{j^1\}} x_j \leq |E_0|-1, \quad j^1 \in N^{(1)}. \tag{30}$$

In the following theorem, we state the validity of Inequality (30).

Theorem 3. *Let us consider an independence system instance with the corresponding conflict hypergraph $H^c = (V^c, E^c)$. Inequality (30) is valid for $\mathcal{P}$, where $j^1 \in N^{(1)}$.*

Proof. Assume for contradiction that Inequality (30) is not valid. Thus, there exists $x^* \in \mathcal{P}$ such that

$$\sum_{j\in E_0\cup\{j^1\}} x^*_j > |E_0|-1, \tag{31}$$

where $j^1 \in N^{(1)}$. Since E_0 is a circuit, we necessarily have $\tilde{z}_1 \leq |E_0|-1$. Thus, we can distinguish the following cases:

- **Case 1.** If $x_{j^1} = 0$, then Inequality (31) reduces to Inequality (32)

$$\sum_{j\in E_0} x^*_j > |E_0|-1. \tag{32}$$

  which means that the circuit inequality induced by the circuit E_0

$$\sum_{j\in E_0} x_j \leq |E_0|-1, \tag{33}$$

  is not valid for $\mathcal{P}$, which is absurd.
- **Case 2.** If $x_{j^1} = 1$, then Inequality (31) reduces to

$$\tilde{z}_1 + 1 \geq |E_0| \tag{34}$$

  In this case we can distinguish two subcases:
  - **Subcase 2.1.** $\tilde{z}_1 = 0$. Inequality (31) reduces to $1 > |E_0| - 1 \geq 1$, which is absurd.
  - **Subcase 2.2.** $\tilde{z}_1 \geq 1$. As E_0 is a circuit and $x_{j^1} = 1$, it follows that $\tilde{z}_1 < |E_0| - 1 \Leftrightarrow \tilde{z}_1 + 1 < |E_0|$, which contradict Inequality (34).

□

In the next two sections, we will focus on obtaining necessary and sufficient conditions for CH inequalities to be facets defining for $\mathcal{P}$. The first condition consist of the positivity of the lifting coefficient α. Indeed, the critical bipartite graph, associated to an independence system instance on W, for which the sets E_0 and $N^{(k)}$ are the bipartition of the vertex set, should be complete bipartite.

Remark 5. This condition arise in each of the Theorems 4,5,6,7 and 8, and it proof could be done in the same way as in Corollary 1.

In the remaining of this paper, and for the sake of ease and clarity of the theorem proofs, let us note by $\mathcal{F}^{(k)}$, where

$$\mathcal{F}^{(k)} = \{x \in \mathcal{P} : \sum_{j\in E_0} x_j + \alpha^{(k)} \sum_{j\in N^{(k)}} x_j = |E_0| - 1, N^{(k)} = V^c \setminus \bigcup_{j=0}^{k-1} E_j, \alpha^{(k)} > 0\},$$

a face of $\mathcal{P}$ defined by Inequality (14). And let us also note the sets $N^{(k)}$ and $\mathcal{F}^{(k)}$ by $N^{(1)}$ and $\mathcal{F}^{(1)}$, respectively, when $k = 1$, otherwise, when $k > 1$, we let them unchanged. Moreover, to prove that Inequality (14) is facet defining for $\mathcal{P}$, it suffices to show that $\mathcal{F}^{(k)}$ contains n affinely independent feasible points.

7. Theoretical results for the hypertree structure

The lifting technique presented in the previous sections allows extending a circuit inequality into a CH inequality. However, it does not guarantee that the generated CH inequalities define facets for $\mathcal{P}$. In this section, we present, in Theorems 4, 5 and 6, respectively, necessary and sufficient conditions for CH inequalities to be facet-defining for $\mathcal{P}$, for the hypertree structure.

Theorem 4. *Let us consider an independence system instance with the corresponding conflict hypergraph $H^c = (V^c, E^c)$. For $k = 1$, Inequality (14) is a facet-defining for $\mathcal{P}$ if and only if the following conditions are satisfied:*

- (i) *The critical bipartite graph $G_{E_0} = (E_0, N^{(1)}, A)$ is complete bipartite.*
- (ii) *$S_2 = \{3, \dots, |E_0|, n - q^{(1)} + 1, \dots, n\}$, is not a circuit.*
- (iii) *There exist $i_1, i_2 \in E_1 \setminus I$ and $h_3 \in N^{(2)}$ such that the set $S_3 = (E_1 \setminus \{i_1, i_2\}) \cup \{h_3\}$ is not a circuit.*

Proof. Necessary condition.

Let us consider the following inequality:

$$\sum_{j\in E_0} x_j + \alpha_1^{(1)} \sum_{j\in N^{(1)}} x_j \leq |E_0| - 1, \tag{35}$$

where $N^{(1)} = V^c \setminus E_0$, $\alpha_1^{(1)} > 0$. For $k = 1$, Inequality (35) has the same form as Inequality (14) with $\alpha_1^{(1)} \neq \alpha^{(1)}$.

Assume, for the sake of contradiction, that Inequality (14) for $k = 1$ does not satisfy the conditions of Theorem 4 but defines a facet for $\mathcal{P}$.

Condition (i): See Remark 5.

Condition (ii): For the converse, if the set S_2 is a circuit, it follows that $\tilde{z}_1 = |E_0| - 2$ and $\tilde{z}_2 = q^{(1)} - 1$. Thus $\alpha_1^{(1)} = \frac{1}{q^{(1)}-1} > \alpha^{(1)} = \frac{l-1}{q^{(1)}}$, which means that Inequality (35) dominates Inequality (14) for $k = 1$, a contradiction.

Condition (iii) : Conversely, if the set S_3 is a circuit, it follows that $\tilde{z}_1 = |E_0| - l$ and $\tilde{z}_2 = q^{(1)} - 1$.

So, $\alpha_1^{(1)} = \frac{l-1}{q^{(1)}-1} > \alpha^{(1)} = \frac{l-1}{q^{(1)}}$, which means that Inequality (35) dominates Inequality (14) for $k = 1$, a contradiction.

Sufficient condition.

Observe that $\mathcal{F}^{(1)}$ contains

- $|E_0|$ feasible points $x^{\mathcal{S}_{h_1}}$, where $\mathcal{S}_{h_1} = E_0 \setminus \{h_1\}$ and $h_1 \in E_0$, since E_0 is a circuit.
- $|E_1 \setminus I|$ feasible points $x^{\mathcal{S}^{(1)}_{h_2}}$, where $\mathcal{S}^{(1)}_{h_2} = E_1 \setminus \{h_2\}$ and $h_2 \in E_1 \setminus I$, by considering the conditions (i) and (ii) and as the set E_1 is a circuit.
- $|N^{(2)}|$ feasible points x^{S_3}, by considering the condition (iii) and as the set E_1 is a circuit.

Then clearly, these n feasible points are affinely independent and satisfy Inequality (14) for $k = 1$ at equality. So, this inequality is a facet defining for $\mathcal{P}$. □

Example 3. Let us consider the subset sum instance of Example 2. As the conditions of Theorem 4 hold, it follows that Inequality (28) defines a facet for $\mathcal{P}$. Indeed

- Condition (i): The critical bipartite graph $G_{E_0} = (E_0, N^{(1)}, A)$ is complete bipartite, and $\forall i \in E_0$ and $\forall j \in N^{(1)}$ the sets $E_0 \setminus \{i\} \cup \{j\}$ are circuits.
- Condition (ii) and (iii): The sets $S_2 = \{3, 4, 11, 12, 13\}$ and $S_3 = \{3, 4, 7, 12, 13\}$ are not circuits.

Theorem 5. *Let us consider an independence system instance with the corresponding conflict hypergraph $H^c = (V^c, E^c)$. For $k > 1$, Inequality (14) is a facet-defining for $\mathcal{P}$ if and only if the following conditions are satisfied:*

(i) *The critical bipartite graph $G_{E_0} = (E_0, N^{(k)}, A)$ is complete bipartite.*

(ii) $\tilde{S}_2 = \{3, \ldots, |E_0|, n - q^{(k)} + 1 - \gamma_k, \ldots, n - \gamma_k\}$, *where* $\gamma_k = \sum_{i=1}^{k-1}(q^{(i)} + 1)$, *is not a circuit.*

(iii) *There exist $i_1, i_2 \in E_k \setminus I$ and $h_3 \in N^{(k+1)}$ such that the set $\tilde{S}_3 = (E_k \setminus \{i_1, i_2\}) \cup \{h_3\}$ is not a circuit, where* $\gamma_k = \sum_{i=1}^{k-1}(q^{(i)} + 1)$.

Proof. Necessary condition.

Let us consider the inequality:

$$\sum_{j \in E_0} x_j + \alpha_1^{(k)} \sum_{j \in N^{(k)}} x_j \leq |E_0| - 1, \tag{36}$$

where $N^{(k)} = V^c \setminus \bigcup_{j=0}^{k-1} E_j$, $\alpha_1^{(k)} > 0$. Inequality (36) has the same form as Inequality (14), with $\alpha_1^{(k)} \neq \alpha^{(k)}$. Assume for contradiction that Inequality (14) for $k > 1$ does not satisfy the conditions of Theorem 5 and defines a facet for $\mathcal{P}$.

Condition (i). See Remark 5.

Condition (ii). In converse of this condition, if the set $\tilde{S}_2$ is a circuit, it follows that $\tilde{z}_1 = |E_0| - 2$ and $\tilde{z}_2 = q^{(k)} - 1$. Thus, $\alpha_1^{(k)} = \frac{1}{q^{(k)}-1} > \alpha^{(k)} = \frac{l-1}{q^{(k)}}$, which means that Inequality (36) dominates Inequality (14), a contradiction.

Condition (iii). Conversely, if the set $\tilde{S}_3$ is a circuit, it follows that $\tilde{z}_1 = |E_0| - l$ and $\tilde{z}_2 = q^{(k)} - 1$. Thus, $\alpha_1^{(k)} = \frac{l-1}{q^{(k)}-1} > \alpha^{(k)} = \frac{l-1}{q^{(k)}}$, which means that Inequality (36) dominates Inequality (14), a contradiction.

Sufficient condition

Note that $\mathcal{F}^{(k)}$ contains

- $|E_0|$ feasible points $x^{\mathcal{S}_{h_1}}$, where $\mathcal{S}_{h_1} = E_0 \setminus \{h_1\}$ and $h_1 \in E_0$, since the set E_0 is a circuit,
- $|E_k \setminus I|$ feasible points $x^{\mathcal{S}_{h_2}^{(k)}}$, where $\mathcal{S}_{h_2}^{(k)} = E_k \setminus \{h_2\}$ and $h_2 \in E_k \setminus I$, by considering the conditions (i) and (ii) and as the set E_k is a circuit.
- $|N^{(k+1)}|$ feasible points $x^{\tilde{S}_3}$, by considering the condition (iii) and as the set E_k is a circuit.
- $|(\bigcup_{j=1}^{k-1} E_j) \setminus I|$ feasible points $x^{\mathcal{S}_h^{(k)}}$, where $\mathcal{S}_h^{(k)} = E_k \setminus \{h_2\} \cup \{h\} = \mathcal{S}_{h_2}^{(k)} \cup \{h\}$, $h_2 \in E_k \setminus I$ and $h \in (\bigcup_{j=1}^{k-1} E_j) \setminus I$, as the set E_k is a circuit.

It is clear that these n feasible points are affinely independent and satisfy Inequality (14) for $k > 1$ at equality. Consequently, this inequality define a facet for $\mathcal{P}$. □

Example 4. Let us consider the subset sum instance of Example 2. The following conditions hold:

- Condition (i): The critical bipartite graph $G_{E_0} = (E_0, N^{(2)}, A)$ is complete bipartite, $\forall i \in E_0$ and $\forall j \in N^{(2)}$ the sets $E_0 \setminus \{i\} \cup \{j\}$ are circuits.
- Condition (ii) and (iii): The sets $\tilde{S}_2 = \{3, 4, 8, 9\}$ and $\tilde{S}_3 = \{3, 4, 5, 9\}$ are not circuits.

By Theorem 5, it follows that Inequality (29) defines a facet for $\mathcal{P}$.

Theorem 6. *Let us consider an independence system instance with the corresponding conflict hypergraph $H^c = (V^c, E^c)$. Inequality (30) defines a facet for $\mathcal{P}$ if and only the following conditions are satisfied:*

(i) *The critical bipartite graph $G_{E_0} = (E_0, \{j^1\}, A)$ is complete bipartite.*
(ii) *$S_1 = I \cup \{j^1, j^2\}$ is not a circuit, where $j^2 \in V^c \setminus (E_0 \cup \{j^1\})$.*

Proof. Necessary condition.

Let us consider Inequality (36). Conversely, assume that Inequality (30) does not satisfy the conditions of Theorem 6, but defines a facet for $\mathcal{P}$.

Condition (i): See Remark 5.

Condition (ii): In converse of this condition, if the set S_1 is a circuit, it follows that $\tilde{z}_1 = |E_0| - l$ and $\tilde{z}_2 = 1$. Thus, $\alpha_1^{(k)} = l - 1$. So Inequality (36) can be written as

$$\sum_{j \in E_0} x_j + (l-1)(x_{j^1} + x_{j^2}) \le |E_0| - 1 \tag{37}$$

Inequality (37) dominates Inequality (30), a contradiction.

Sufficient condition.

We shall exhibit n affinely independent feasible points that satisfy Inequality (30) at equality. These points are determined as follows:

- The set E_0 is a circuit, so the points $x^{\mathcal{S}_{h_1}}$, where $\mathcal{S}_{h_1} = E_0 \setminus \{h_1\}$ and $h_1 \in E_0$, are feasible and satisfy Inequality (30) at equality. We get $|E_0|$ points.
- From the conditions (i) and (ii), and as S_1 is not a circuit, we can get the remaining feasible points x^{S_1} and $x^{\mathcal{S}_{j^1}}$, where $\mathcal{S}_{j^1} = I \cup \{j^1\}$, that satisfy Inequality (30) at equality. Thus, we get $|N^{(1)}|$ points.

These feasible points are clearly affinely independent. Thus, Inequality (30) defines a facet for $\mathcal{P}$. □

Example 5. Let us focus on the triangle-free induced subgraph polytope and consider the complete graph on five vertices $K_5 = (V, A)$ (Figure 4). Let $c = (1,1,1,1,1)$ be a weight vector. The associated conflict hypergraph is $H^c = (V^c, E^c)$, where $V^c = V = \{1,2,3,4,5\}$ and $E^c = \{E_0 = \{1,2,3\}\}$.

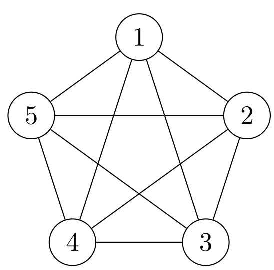


FIGURE 4. The complete graph K_5

Let $j^1 = 4$. Since $\forall i \in E_0$ the sets $S_{ij} = E_0 \setminus \{i\} \cup \{j^1\}$, i.e., $S_{14} = \{2,3,4\}$, $S_{24} = \{1,3,4\}$ and $S_{34} = \{1,2,4\}$, are circuits, it follows that the critical graph $G_{E_0} = (E_0, \{4\}, A)$ is complete bipartite. Furthermore, the condition (ii) of Theorem 6 holds. Then we have

For $l = 1$, $I = \{2,3\}$, the set $S_1 = \{2,3,4,5\}$ is not a circuit,
For $l = 2$, $I = \{3\}$ the set $S_1 = \{3,4,5\}$ is not a circuit,
For $l = 3$, $I = \{\}$, the set $S_1 = \{4,5\}$ is not a circuit.

Thus, the inequality: $x_1 + x_2 + x_3 + x_4 \le 2$ defines a facet for $\mathcal{P}$.

8. Theoretical results for the clutter structure

In this section we present necessary and sufficient conditions for the CH inequalities to be facet-defining for $\mathcal{P}$ for the clutter structure.

Theorem 7. *Let us consider an independence system instance with the corresponding conflict hypergraph $H^c = (V^c, E^c)$. For $k = 1$, Inequality (14) defines a facet for $\mathcal{P}$ if and only if the following conditions hold:*

(*i*) *The critical bipartite graph $G_{E_0} = (E_0, N^{(1)}, A)$ is complete bipartite.*
(*ii*) *$\forall i \in E_0$ and $\forall j \in N^{(1)}$, the sets $\{i, j\}$ are circuits.*
(*iii*) *$\forall h_3 \in N^{(2)}$, $\exists i_1, i_2 \in E_1$ such that the set $S_3 = (E_1 \setminus \{i_1, i_2\}) \cup \{h_3\}$ is not a circuit.*

Proof. Necessary condition

Let us consider Inequality (35). Assuming conversely that Inequality (14) for $k = 1$ does not satisfy the conditions of Theorem 7 and defines a facet for $\mathcal{P}$.

- *Condition* (*i*): See Remark 5.
- *Condition* (*ii*): Assume for contradiction that $\exists\ i \in E_0$, $\exists\ j \in N^{(1)}$, such that the set $\{i, j\}$ is not a circuit. Thus, there exists a set $E_1^{'} \subset V^c$ that is a circuit, such that $E_1^{'} = I_1 \cup I_2$, with $i \in I_1$ and $j \in I_2$, where $I_1 \subset E_0$, $1 \leq |I_1| \leq |E_0| - 2$, $I_2 \subset E_1$ and $1 \leq |I_2| < |E_1|$. It follows that Inequality (35) has a coefficient $\alpha_1^{(1)} = \frac{l-1}{q^{(1)}-\theta} > \alpha^{(1)} = \frac{l-1}{q^{(1)}}$, where $\theta = |E_1 \setminus I_2|$ and $1 \leq \theta \leq |E_1| - 1$, which means that Inequality (35) dominates Inequality (14) for $k = 1$, a contradiction.
- *Condition* (*iii*): In converse of this condition $\exists\ h_3 \in N^{(2)}$ such that $\forall\ i_1, i_2 \in E_1$ the set $S_3 = (E_1 \setminus \{i_1, i_2\}) \cup \{h_3\}$ is a circuit. It follows that $\tilde{z}_1 = 0$ and $\tilde{z}_2 = q^{(1)} - 1$. Thus, $\alpha_1^{(1)} = \frac{l-1}{q^{(1)}-1} > \alpha^{(1)} = \frac{l-1}{q^{(1)}}$, which means that Inequality (35) dominates Inequality (14) for $k = 1$, a contradiction.

Sufficient condition.

The n affinely independent feasible points that are contained in $\mathcal{F}^{(1)}$ are the following:

- The $|E_0|$ feasible points $x^{\mathcal{S}_{h_1}}$, where $\mathcal{S}_{h_1} = E_0 \setminus \{h_1\}$ and $h_1 \in E_0$, since the set E_0 is a circuit.
- From the conditions (*i*) and (*ii*) and as the set E_1 is a circuit, we get $|E_1|$ feasible points $x^{\mathcal{S}_{h_2}}$, where $\mathcal{S}_{h_2} = E_1 \setminus \{h_2\}$ and $h_2 \in E_1$.
- The $|N^{(2)}|$ feasible points x^{S_3}, where $S_3 = (E_1 \setminus \{i_1, i_2\}) \cup \{h_3\}$, $i_1, i_2 \in E_1$ and $h_3 \in N^{(2)}$, by considering the condition (*iii*) and as the set S_3 is not a circuit.

These feasible points are clearly affinely independent. It follows that Inequality (14) for $k = 1$ defines a facet for $\mathcal{P}$. □

Theorem 8. *Let us consider an independence system instance with the corresponding conflict hypergraph $H^c = (V^c, E^c)$. For $k > 1$, Inequality (14) defines a facet for $\mathcal{P}$ if and only if the following conditions hold:*

(i) *The critical bipartite graph* $G_{E_0} = (E_0, N^{(k)}, A)$ *is complete bipartite.*
(ii) $\forall\, i \in E_0$ *and* $\forall j \in N^{(k)}$*, the sets* $\{i, j\}$ *are circuits.*
(iii) $\forall\, h_4 \in N^{(k+1)}$*,* $\exists\, i_1, i_2 \in E_k$ *such that the set* $S_4 = (E_k \setminus \{i_1, i_2\}) \cup \{h_4\}$ *is not a circuit.*

Proof. Necessary condition
Let us consider Inequality (36). Assuming for contradiction that Inequality (14) for $k > 1$ does not satisfy the conditions of Theorem 8 and defines a facet for $\mathcal{P}$.

- *Condition (i)*: See Remark 5.
- *Condition (ii)*: Conversely, assume that $\exists\, i \in E_0$, $\exists\, j \in N^{(k)}$, such that the set $\{i, j\}$ is not a circuits. Thus, there exists a set $E_k^{'} \subset V^c$ that is a circuit, such that $E_k^{'} = I_1 \cup I_3$, with $i \in I_1$ and $j \in I_3$, where $I_1 \subset E_0$, $1 \le |I_1| \le |E_0| - 2$, $I_3 \subset E_k$ and $1 \le |I_3| < |E_k|$. It follows that Inequality (36) has a coefficient $\alpha_1^{(k)} = \frac{l-1}{q^{(k)}-\theta} > \alpha^{(k)} = \frac{l-1}{q^{(k)}}$, where $\theta = |E_k \setminus I_3|$ and $1 \le \theta \le |E_k| - 1$, which means that Inequality (36) dominates Inequality (14) for $k > 1$, a contradiction.
- *Condition (iii)*: Assume for contradiction that $\exists\, h_4 \in N^{(k+1)}$ such that $\forall\, i_1, i_2 \in E_k$ the set $S_4 = (E_k \setminus \{i_1, i_2\}) \cup \{h_4\}$ is a circuit. It follows that $\tilde{z}_1 = 0$ and $\tilde{z}_2 = q^{(k)} - 1$. Thus, $\alpha_1^{(k)} = \frac{l-1}{q^{(k)}-1} > \alpha^{(k)} = \frac{l-1}{q^{(k)}}$, which means that Inequality (36) dominates Inequality (14) for $k > 1$, a contradiction.

Sufficient condition.
The n affinely independent feasible points that $\mathcal{F}^{(k)}$ contains are the following:

- The set E_0 is a circuit, so we get $|E_0|$ feasible points $x^{\mathcal{S}_{h_1}}$, where $\mathcal{S}_{h_1} = E_0 \setminus \{h_1\}$ and $h_1 \in E_0$.
- From the conditions (i) and (ii) and as the set E_k is a circuit, we obtain $|E_k|$ feasible points $x^{\mathcal{S}_{h_2}}$, where $\mathcal{S}_{h_2} = E_1 \setminus \{h_2\}$ and $h_2 \in E_k$.
- From the condition (iii) and as the set S_4 is not a circuit, it results $|N^{(k+1)}|$ feasible points x^{S_4}, where $S_4 = (E_k \setminus \{i_1, i_2\}) \cup \{h_4\}$, $i_1, i_2 \in E_k$ and $h_4 \in N^{(k+1)}$.
- As the set E_k is a circuit, so the remaining feasible points are $x^{\mathcal{S}_h}$, where $\mathcal{S}_h = (E_k \setminus \{h_2\}) \cup \{h\}$, $h_2 \in E_k$ and $h \in (\bigcup_{j=1}^{k-1} E_j)$, which are in number of $|(\bigcup_{j=1}^{k-1} E_j)|$.

Clearly, these feasible points are affinely independent. This establishes that Inequality (14) for $k > 1$ defines a facet for $\mathcal{P}$. □

Next, we give an illustration of Theorem 7 for subset sum and vertex packing polytopes.

Example 6. Consider the following subset sum instance:

$$\begin{cases} \max Z = & 6x_1 + 6x_2 + 5x_3 + 5x_4 + 3x_5 + 3x_6 + 3x_7, \\ \textit{subject to} & \\ & 6x_1 + 6x_2 + 5x_3 + 5x_4 + 3x_5 + 3x_6 + 3x_7 \leq 8, \\ & \\ & x_j \in \{0,1\} \quad j = 1,\ldots,7. \end{cases}$$

The associated conflict hypergraph is $H^c = (V^c, A^c)$, with $V^c = \{1,2,3,4,5,6,7\}$, $A^c = \{E_0, E_1\}$, $E_0 = \{1,2\}$, $E_1 = \{5,6,7\}$, $N^{(1)} = \{3,4,5,6,7\}$ and $N^{(2)} = \{3,4\}$. The conditions of Theorem 7 are satisfied. We have

- Condition (i): The critical bipartite graph $G_{E_0} = (E_0, N^{(1)}, A)$ is complete bipartite, $\forall i \in E_0$ and $\forall j \in N^{(1)}$ the sets $E_0 \setminus \{i\} \cup \{j\}$ are circuits.
- Condition (ii): $\forall i \in E_0, \forall j \in N^{(1)}$, the sets $\{1,3\}$,$\{1,4\}$,$\{1,5\}$,$\{1,6\}$,$\{1,7\}$, $\{2,3\}$,$\{2,4\}$,$\{2,5\}$,$\{2,6\}$,$\{2,7\}$ are circuits.
- Condition (iii): $S_3 = \{4,7\}$ is not a circuit.

It follows that the Inequality (38)

$$x_1 + x_2 + \frac{1}{2}(x_3 + x_4 + x_5 + x_6 + x_7) \leq 1 \tag{38}$$

defines a facet for $\mathcal{P}$.

Example 7. Consider the Vertex packing polytope. Let us consider the complete graph with five vertices $K_5 = (V, A)$ given in Figure 4. Let $c = (1,1,1,1,1)$ be a weight vector. The associated conflict hypergraph is $H^c = (V^c, E^c)$, with $V^c = V = \{1,2,3,4,5\}$, $E^c = \{E_0 = \{1,2\}, E_1 = \{4,5\}\}$, $N^{(1)} = \{3,4,5\}$ and $N^{(2)} = \{3\}$. The following conditions are satisfied:

- Condition (i): The critical bipartite graph $G_{E_0} = (E_0, N^{(1)}, A)$ is complete bipartite, $\forall i \in E_0$ and $\forall j \in N^{(1)}$ the sets $E_0 \setminus \{i\} \cup \{j\}$ are circuits,
- Condition (ii): As K_5 is a complete graph, $\forall i \in E_0$, $\forall j \in N^{(1)}$, the sets $\{i, j\}$ are circuits,
- Condition (iii): $S_3 = \{3\}$ is not a circuit.

And hence by Theorem 7, it follows that the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 1 \tag{39}$$

defines a facet for $\mathcal{P}$.

9. Conclusion

In this paper, we have presented a technique for an exact simultaneous uplifting of circuit inequalities. In this technique, we used conflict hypergraph structures to define new families of valid inequalities for the independence system polytope. These inequalities are obtained by simultaneously adding the most appropriate set of variables with the largest coefficient values that maintain their validity. Furthermore, we have introduced a procedure to generate two conflict hypergraph structures kinds: hypertrees and clutters. Moreover, we used the hyperedges cardinalities of these structures to compute the lifting coefficient values of CH inequalities.

On the other hand, we have provided some theoretical results that consist of necessary and sufficient conditions for the circuits and the CH inequalities to be facet-defining for $\mathcal{P}$. We also give a condition of the positivity of the lifted circuit inequality coefficient values.

As perspectives, it would be interesting to test our inequalities in the separation process of branch-and-cut algorithms to investigate their computational potential in solving ISP.

Beside, investigating other structures of conflict hypergraphs and selecting elements of the independence system would be of great interest. This could serve us to generate more new families of facets defining inequalities for a known ISP.

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References

[1] A. Atamtürk. On the facets of the mixed-integer knapsack polyhedron. *Mathematical Programming*, 98(1-3):145–175, 2003.

[2] E. Balas. Facets of the knapsack polytope. *Mathematical Programming*, 8:146–164, 1975.

[3] C. Berge. *Graphes et hypergraphes*. Dunod: Paris, 1970.

[4] C. Berge. *Graphs and Hypergraphs*. North-Holland Publishing Company, 1973.

[5] I. R. De Farias Jr, I.E. Johnson, and G. Nemhauser. Facets of the complimentarity knapsack polytope. *Mathematics of Operations Research*, 27(1):210–227, 2002.

[6] T. Easton and K. Hooker. Simultaneously lifting sets of binary variables into cover inequalities for knapsack polytopes. *Discrete Optimization*, Special Issue: In Memory of George B. Dantzig, 5(2):254–261, 2008.

[7] J. Edmonds. Matroids and the greedy algorithm. *Mathematical Programming*, 1:127–136, 1971.

[8] R. Euler, M. Junger, and G. Reinelt. Generalizations of cliques, odd cycles and anticycles and their relation to independence system polyhedra. *Mathematics of Operations Research*, 12:451–462, 1987.

[9] P. Fouilhoux, M. Labbé, and A. R. Mahjoub. Generating facets for the independence system polytope. *SIAM Journal of Discrete Mathematics*, 23(3):1484–1506, 2009.

[10] R. E. Gomory. Some polyhedra related to combinatorial problems. *Linear Algebra and its Applications*, 2(451–588), 1969.
[11] M. Grötschel, L. Lovasz, and A. Schrijver. *Geometric algorithms and combinatorial optimization*. Springer, 1988.
[12] Z. Gu, G.L. Nemhauser, and M.W.P. Savelsbergh. Lifted cover inequalities for 0-1 integer programs : computation. *INFORMS Journal on Computing*, 10(4):427–437, 1998.
[13] Z. Gu, G.L. Nemhauser, and M.W.P. Savelsbergh. Lifted cover inequalities for 0-1 integer programs : complexity. *INFORMS Journal on Computing*, 11(1):117–123, 1999.
[14] Z. Gu, G.L. Nemhauser, and M.W.P. Savelsbergh. Sequence independent lifting in mixed integer programming. *Journal of Combinatorial Optimization*, 4(1):109–129, 2000.
[15] T. Gutierrez. Lifting general integer variables. Master of science thesis, Department of Industrial and Manufacturing Systems Engineering, Kansas State University, 2007.
[16] P. L. Hammer, E. L. Johnson, and U. N. Peled. Facets of regular 0-1 polytopes. *Mathematical Programming*, 8(1):179–206, 1975.
[17] B. Hunsaker and C. Tovey. Simple lifted cover inequalities and hard knapsack problems. *Discrete Optimization*, 2(3):219–228, 2005.
[18] E. Korach and M. Razgon. Optimal hypergraph tree realization. In *Graph-Theoretic Concepts in Computer Science*, volume 31st International Workshop. Springer, 2005.
[19] L. A. Kubik. Simultaneously lifting multiple sets in binary knapsack integer programs. Master of science thesis, Department of Industrial and Manufacturing Systems Engineering, Kansas State University, 2009.
[20] M. Laurent. A generalization of antiwebs to independence systems and their canonical facets. *Mathematical Programming*, 45:97–108, 1989.
[21] G.L. Nemhauser and P.H. Vance. Lifted cover facets of the 0-1 knapsack polytope with gub constraints. *Operations Research Letters*, 16(5):255–263, 1994.
[22] S. Pahwa. The theory of simultaneously lifting: constellations in conflict hypergraphs. Master of science thesis, Department of Industrial and Manufacturing Systems Engineering, Kansas State University, 2009.
[23] K. Park. Lifting cover inequalities for the precedence-constrained knapsack problem. *Discrete Applied Mathematics*, 72(3):219–241, 1997.
[24] K. Sharma. Simultaneously lifting sets of variables in binary knapsack problems. Master of science thesis, Department of Industrial and Manufacturing Systems Engineering, Kansas State University, 2007.
[25] S. Shebalov and D. Klabjan. Sequence independent lifting for mixed integer programs with variable upper bounds. *Mathematical Programming*, 105:523–561, 2006.
[26] D. Welsh. *Matroid theory*. Academic Press, 1976.
[27] L. A. Wolsey. Faces for a linear inequality in 01 variables. *Mathematical Programming*, 8(1):165–178, 1975.

[28] L.A. Wolsey. Facets and strong valid inequalities for integer programs. *Operations Research*, 24:367–372, 1976.

[29] L.A. Wolsey. Valid inequalities and superadditivity for $0-1$ integer programs. *Mathematic of Operations Research*, 1(1):66–77, 1977.

[30] E. Zemel. Lifting the facets of 0-1 polytopes. *Mathematical Programming*, 15:268–277, 1978.

[31] E. Zemel. Easily computable facets of the knapsack polytope. *Mathematics of Operations Research*, 14(4):760–766, November 1989.

[32] B. Zenga and J. P. P. Richard. A polyhedral study on 01 knapsack problems with disjoint cardinality constraints: Facet-defining inequalities by sequential lifting. *Discrete Optimization*, 8:277–301, 2011.

[33] B. Zenga and J. P. P. Richard. A polyhedral study on 01 knapsack problems with disjoint cardinality constraints: Strong valid inequalities by sequence-independent lifting. *Discrete Optimization*, 8:259–276, 2011.