Bounds on the disjunctive domination number of a tree *

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Abstract

A set D of vertices in a graph G is a disjunctive dominating set in G if every vertex not in D is adjacent to a vertex of D or has at least two vertices in D at distance 2 from it in G. The disjunctive domination number, $\gamma_2^d(G)$, of G is the minimum cardinality of a disjunctive dominating set in G. We show that if T is a tree of order n with l leaves and s support vertices, then $\frac{n-l+3}{4} \leq \gamma_2^d(T) \leq \frac{n+l+s}{4}$. Moreover, we characterize the families of trees which attain these bounds.

Keywords: Disjunctive dominating set, disjunctive domination number, tree.

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1 Introduction

Let $G = (V, E)$ be a simple graph, and v be a vertex in G. The *open neighborhood* of v is $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d(v) = |N(v)|$. For two vertices u and v in a connected graph G, the *distance* $d(u, v)$ between u and v is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of a graph G which is denoted by $diam(G)$. A *leaf* of G is a vertex of degree 1 and a *support vertex* of G is a vertex adjacent to a leaf. Denote the sets of leaves and support vertices of G by $L(T)$ and $S(T)$, respectively. Let $l(T) = |L(T)|$ and $s(T) = |S(T)|$. A *double star* is a tree that contains exactly two vertices that are not leaves. A *subdivided star* $K^*_{1,t}$ is a tree obtained from a star $K_{1,t}$ on at least two vertices by subdividing each edge exactly once.

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A *dominating set* in a graph G is a set S of vertices of G such that every vertex in $V(G)\setminus S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [5, 6].

Motivated by the concepts of distance domination and exponential domination (see, [1, 2, 7]), Goddard, Henning and McPillan [4] introduced and studied the concept of disjunctive domination in a graph. A set S of vertices in a graph G is a *disjunctive dominating set*, abbreviated *2DD-set*, in G if every vertex not in S is adjacent to a vertex of S or has at least two vertices in S at distance 2 from it in G. We say a vertex v in G is *disjunctively dominated*, abbreviated *2D-dominated*, by the set S, if $N[v]\cap S\neq\emptyset$ or there exist at least two vertices in S at distance 2 from v in G. The *disjunctive domination number* of G, denoted by $\gamma_2^d(G)$, is the minimum cardinality of a $2DD$-set in G. A disjunctive dominating set of G of cardinality $\gamma_2^d(G)$ is called a $\gamma_2^d(G)$-set. If the graph G is clear from the context, we simply write γ_2^d-set rather than $\gamma_2^d(G)$-set. Every dominating set is a $2DD$-set. The concept of disjunctive domination in graphs has been studied in [4, 8–12, 18] and elsewhere.

An area of research in domination of graphs that has received considerable attention is to bound various domination parameters, some related results can be referred to [3, 13–17, 19–24]. In [4], Goddard et al. proved the following theorem:

Theorem 1.1 ([4]) *If G is a connected graph with $n\geq 5$, then $\gamma_2^d(G)\leq\frac{n-1}{2}$.*

Moreover, they improved this bound when restrict the connected graph G to be a claw-free graph.

Theorem 1.2 ([4]) *If G is a connected claw-free graph of order n, then $\gamma_2^d(G)\leq\frac{2n}{5}$ unless $G\in\{K_1,P_2,P_4,C_4,H_3\}$, where H_3 is the graph obtained from $K_{1,3}^*$ by adding an edge joining two of these support vertices.*

Our aim in this paper is to improve the bound of Theorem 1.1 when we restrict the graph G to be a tree. More precisely, we give a lower bound and an upper bound for the disjunctive domination number of a tree in terms of its order, the number of leaves and support vertices in the tree. Further, we provide the constructive characterization of trees that achieve equality in the two bounds.

2 Main results

We first present the following lemmas, which are helpful for our investgation.

Observation 2.1 *[10] If T is a tree of order at least 3, then we can choose a γ_2^d-set of T contains no leaf.*

Corollary 2.2 *Let T be a tree of order at least 3 and D be a γ_2^d-set of T contains no leaf, if a support vertex has degree two, then it belongs to D.*

As mentioned above, one of our aim is provide the constructive characterization of trees that achieve equality in the upper bound and the lower bound. For our purpose, we define a *labeling* of a tree T as a weak partition $S = (S_A, S_B, S_C, S_D)$ of $V(T)$ (Some of the subsets may be empty). We will refer to the pair (T, S) as a *labeled tree*. The label or *status* of a vertex v, denoted $\mathrm{sta}(v)$, is the letter $x \in \{A, B, C, D\}$ such that $v \in S_x$. Next, we ready to give two families $\mathscr{T}_1$ and $\mathscr{T}_2$, each member of $\mathscr{T}_1$ ($\mathscr{T}_2$, respectively) is obtained from the labeled tree (P_3, S') ((P_4, S''), respectively) by a series of operations (see Fig.1(a), (b)). Before this, we give two definitions. If a labeled tree $(T, S) \in \mathscr{T}_2$, the path P_4 (which comes from the labeled tree (P_4, S'')) is an induced path of T, and we call it the *basic path* of T. For a vertex $v \notin S(T)$, which has status A and does not belong to the basic path, if there exists a vertex u such that vv_1v_2u is an induced path of T and $\mathrm{sta}(v_1) = C$, $\mathrm{sta}(v_2) = D$, $\mathrm{sta}(u) = B$, we call u a *corresponding vertex* of v. In addition, for a vertex u, which has status B, if there exists a vertex v such that vv_1v_2u is an induced path of T and $\mathrm{sta}(v) = A$, $\mathrm{sta}(v_1) = C$, $\mathrm{sta}(v_2) = D$, we also call v a *corresponding vertex* of u.

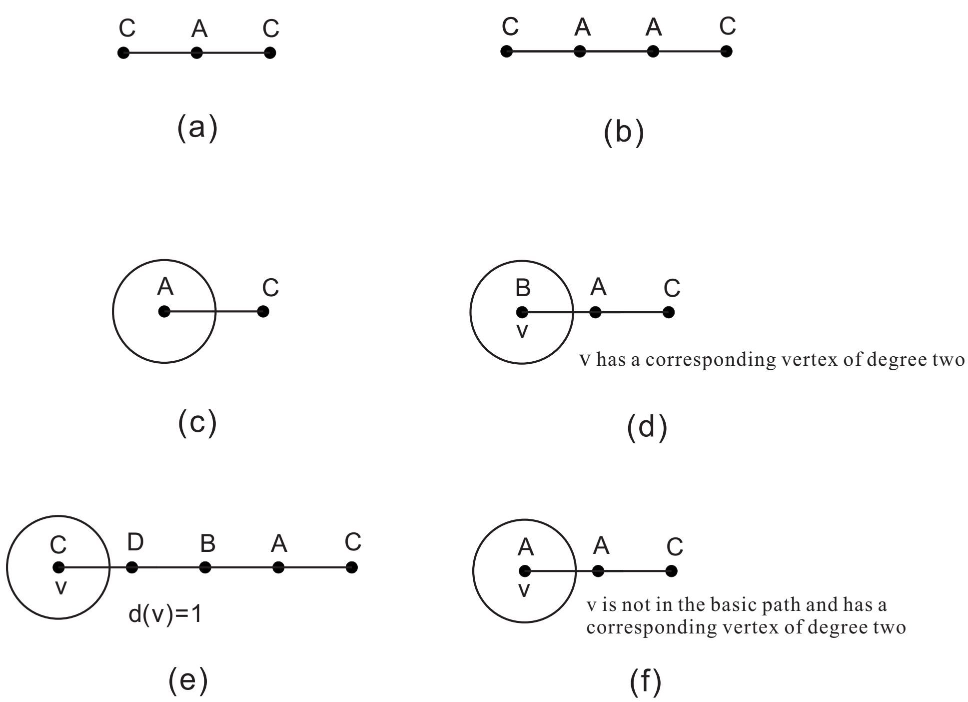

Fig.1

In what follows, we give four operations as follows:

Operation $\mathscr{O}_1$: Let v be a vertex with $\mathrm{sta}(v) = A$. Add a vertex u and the edge uv. Let $\mathrm{sta}(u) = C$.

Operation $\mathscr{O}_2$: Let v be a vertex with $\mathrm{sta}(v) = B$ that has a corresponding vertex of degree two. Add a path u_1u_2 and the edge u_1v. Let $\mathrm{sta}(u_1) = A$, $\mathrm{sta}(u_2) = C$.

Operation $\mathscr{O}_3$: Let v be a vertex with $\mathrm{sta}(v) = C$ that has degree one. Add a path $u_1u_2u_3u_4$ and the edge u_1v. Let $\mathrm{sta}(u_1) = D$, $\mathrm{sta}(u_2) = B$, $\mathrm{sta}(u_3) = A$, $\mathrm{sta}(u_4) = C$.

Operation $\mathscr{O}_4$: Let v be a vertex not in the basic path that has status A and has a corresponding vertex of degree two. Add a path u_1u_2 and the edge u_1v. Let $\mathrm{sta}(u_1) = A$, $\mathrm{sta}(u_2) = C$.

The four operations $\mathscr{O}_1$, $\mathscr{O}_2$, $\mathscr{O}_3$ and $\mathscr{O}_4$ are illustrated in Fig.1(c), (d), (e) and (f).

Let $\mathscr{T}_1$ be the minimum family of labeled trees that: (i) contains (P_3, S') and S' is the labeling that assigns to the two leaves of the path P_3 status C, and the central vertex status A; and (ii) is closed under the two operations $\mathscr{O}_1$ and $\mathscr{O}_3$ that are listed as above, which extend the tree T' to a tree T by attaching a tree to the vertex $v \in V(T')$.

Let $\mathscr{T}_2$ be the minimum family of labeled trees that: (i) contains (P_4, S'') where S'' is the labeling that assigns to the two leaves of the path P_4 status C, and the remaining vertices status A; and (ii) is closed under the three operations $\mathscr{O}_2$, $\mathscr{O}_3$ and $\mathscr{O}_4$ that are listed as above, which extend the tree T' to a tree T by attaching a tree to the vertex $v \in V(T')$.

Let $(T, S) \in \mathscr{T}_1$ ($\mathscr{T}_2$, respectively) be a labeled tree for some labeling S. Then there is a sequence of labeled trees (T_0, S_0), $(T_1, S_1), \cdots, (T_{k-1}, S_{k-1})$, (T_k, S_k) such that $(T_0, S_0) = (P_3, S')$ (or (P_4, S'')), $(T_k, S_k) = (T, S)$. The labeled tree (T_i, S_i) can be obtained from (T_{i-1}, S_{i-1}) by one of the operations $\mathscr{O}_1$ and $\mathscr{O}_3$ ($\mathscr{O}_2$, $\mathscr{O}_3$ and $\mathscr{O}_4$, respectively), where $i \in \{1, 2, \cdots, k\}$. We call the number of terms in such a sequence of labeled trees that is used to construct (T, S), the *length* of the sequence. Clearly, the above sequence has length k. We remark that a sequence of labeled trees used to construct (T, S) is not necessarily unique.

Two main conclusions of our paper are listed as follows.

Theorem 2.3 *If T is a nontrivial tree of order $n(T)$ with $l(T)$ leaves, then $\gamma_2^d(T) \geq \frac{n(T)-l(T)+3}{4}$, with equality if and only if $(T, S) \in \mathscr{T}_1$ for some labeling S.*

Theorem 2.4 *If T is a nontrivial tree of order $n(T)$ with $l(T)$ leaves and $s(T)$ support vertices, then $\gamma_2^d(T) \leq \frac{n(T)+l(T)+s(T)}{4}$, with equality if and only if $(T, S) \in \mathscr{T}_2$ for some labeling S.*

Next, we take some examples to make it easier for reader to understand the families $\mathscr{T}_1$, $\mathscr{T}_2$ and Theorem 2.3, 2.4. In Fig 2(a), by a simple calculation, we have that $\gamma_2^d(T) = 2 = \frac{n(T)-l(T)+3}{4}$. And moreover, it is easy to see that (T_1, S_1') is obtained from (P_3, S') by operation $\mathscr{O}_1$, (T_2, S_2') is obtained from (T_1, S_1') by operation $\mathscr{O}_3$, (T, S_3') is obtained from (T_2, S_2') by operation $\mathscr{O}_1$. It follows from the definition of $\mathscr{T}_1$ that $(T, S_3') \in \mathscr{T}_1$. In Fig 2(b), by a simple calculation, we have that $\gamma_2^d(T') = 6 = \frac{n(T)+l(T)+s(T)}{4}$. And moreover, it is easy to see that (T_1', S_1'') is obtained from (P_4, S'') by operation $\mathscr{O}_3$, (T_2', S_2'') is obtained from (T_1', S_1'') by operation $\mathscr{O}_2$, (T_3', S_3'') is obtained from (T_2', S_2'') by operation $\mathscr{O}_4$ and (T', S_4'') is obtained from (T_3', S_3'') by operation $\mathscr{O}_3$. It follows from the definition of $\mathscr{T}_2$ that $(T', S_4'') \in \mathscr{T}_2$.

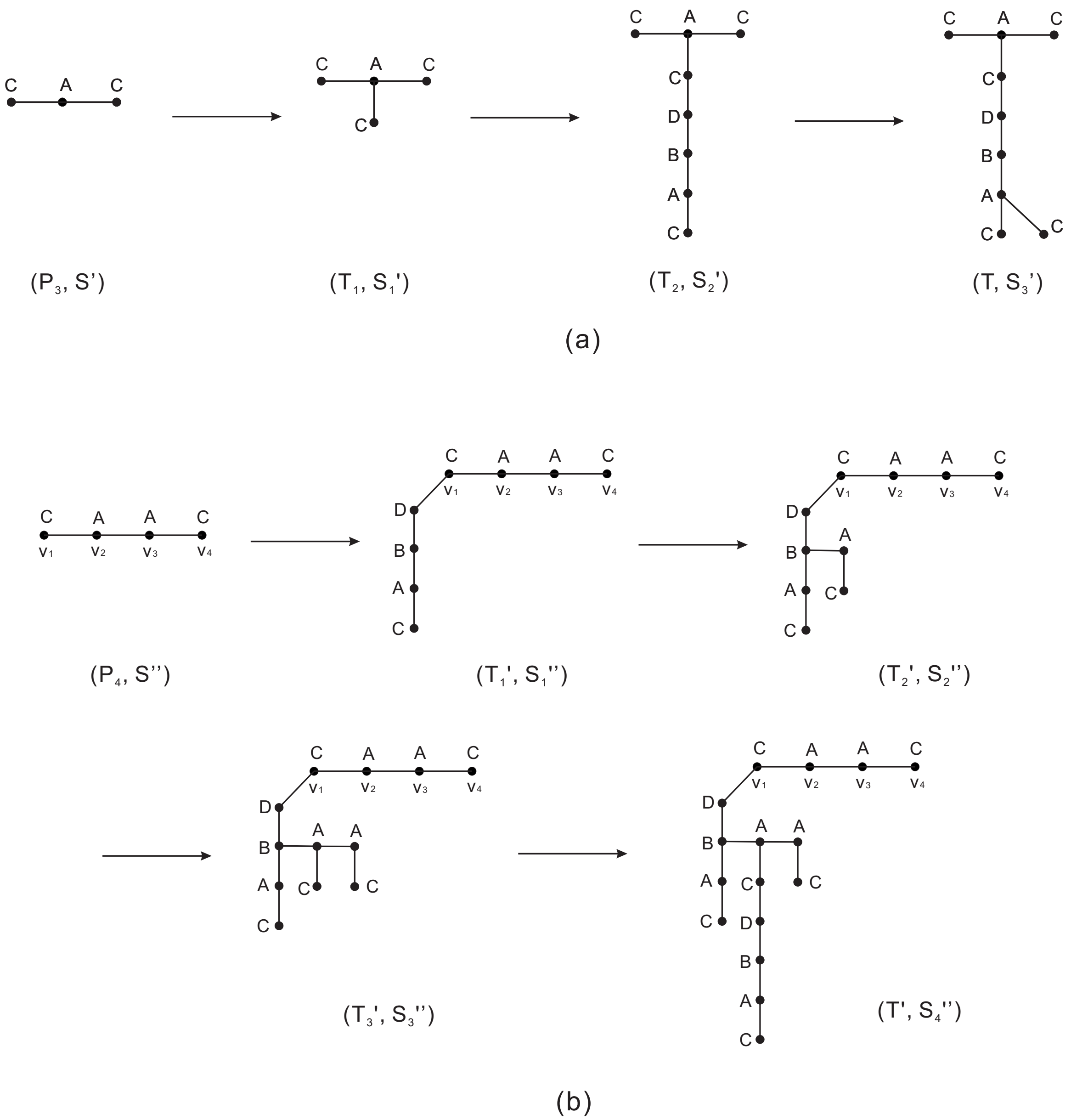

Fig.2

Furthermore, we can slightly improve the upper bound of Theorem 2.4.

Corollary 2.5 *If T is a nontrivial tree of order $n(T)$ with $l(T)$ leaves and $s(T)$ support vertices, then* $\gamma_2^d(T) \leq \frac{n(T)+3s(T)-l(T)}{4}$.

Proof. Let T' be the tree obtained from T by deleting all but one leaf from each support vertex of T. Then, $n(T') = n(T) - [l(T) - s(T)]$, $s(T') = s(T)$, $l(T') = s(T)$ and $\gamma_2^d(T) = \gamma_2^d(T')$. By Theorem 2.4, we have that $\gamma_2^d(T) = \gamma_2^d(T') \le \frac{n(T')+l(T')+s(T')}{4} = \frac{n(T)-[l(T)-s(T)]+2s(T)]}{4} = \frac{n(T)+3s(T)-l(T)}{4}$. $\square$

We conclude this section by comparing the bound of Theorem 1.1 with the bound of Corollary 2.5. We see that $\frac{n(T)+3s(T)-l(T)}{4} < \frac{n(T)-1}{2}$ when $n(T) > 3s(T) - l(T) + 2$. It implies that for almost all trees, the bound of Corollary 2.5 is better than that of Theorem 1.1.

3 Proof of Theorem 2.3

The following observation establishes properties of trees in the family $\mathscr{T}_1$.

Observation 3.1 *If $(T, S) \in \mathscr{T}_1$, then (T, S) has the following properties.*

(a) *Every support vertex of T has status A and every leaf has status C.*

(b) *Let v be a vertex has status A, then $sta(u) \in \{B, C\}$ for $u \in N(v)$.*

(c) *The set S_A is a $2DD$-set of T.*

(d) *The set S_A, S_B, S_C and S_D are independent sets.*

(e) *If $sta(v) \neq A$, then $d(v) \le 2$.*

Lemma 3.2 *If T is a tree of order $n(T) \ge 3$ with $l(T)$ leaves, and $(T, S) \in \mathscr{T}_1$ for some labeling S, then $\gamma_2^d(T) = |S_A| = \frac{n(T)-l(T)+3}{4}$, and the set S_A is the unique γ_2^d-set of T.*

Proof. We proceed by induction on the length k of a sequence required to construct the labeled tree (T, S). Let D be any γ_2^d-set of T.

When $k = 0$, $(T, S) = (P_3, S')$, $\gamma_2^d(T) = |S_A| = 1$, the set S_A is the unique γ_2^d-set of T. This establishes the base case. Let $k \ge 1$ and assume that if the length of sequence used to construct a labeled tree $(T', S^*) \in \mathscr{T}_1$ is less than k, then $\gamma_2^d(T') = |S_A^*| = \frac{n(T')-l(T')+3}{4}$, S_A^* is the unique γ_2^d-set of T'. Now, $(T, S) \in \mathscr{T}_1$ and let (T_0, S_0), (T_1, S_1), $\cdots$, (T_{k-1}, S_{k-1}), (T_k, S_k) be a sequence of length k used to construct (T, S), where $(T_0, S_0) = (P_3, S')$, $(T_k, S_k) = (T, S)$, (T_i, S_i) can be obtained from (T_{i-1}, S_{i-1}) by one of the operations $\mathscr{O}_1$ and $\mathscr{O}_3$, $i \in \{1, 2, \cdots, k\}$. Let $T' = T_{k-1}$ and $S^* = S_{k-1}$. Note that $(T', S^*) \in \mathscr{T}_1$. By the inductive hypothesis, $\gamma_2^d(T') = |S_A^*| = \frac{n(T')-l(T')+3}{4}$, S_A^* is the unique γ_2^d-set of T'. (T, S) can be obtained from (T', S^*) by operation $\mathscr{O}_1$ or $\mathscr{O}_3$.

In the former case, we have that $n(T) = n(T') + 1$, $l(T) = l(T') + 1$, and $|S_A| = |S_A^*|$. It follows Observation 3.1(c) that $\gamma_2^d(T) \le |S_A| = |S_A^*| = \frac{n(T')-l(T')+3}{4} = \frac{n(T)-1-l(T)+1+3}{4} = \frac{n(T)-l(T)+3}{4}$. On the other hand, assume that $V(T) \setminus V(T') = \{u\}$, and v is the support vertex of u. Take a set $D' = (D \setminus (L(T) \cap N(v))) \cup \{v\}$ when $(L(T) \cap N(v)) \cap D \neq \emptyset$, otherwise, $D' = D$. D' is a $2DD$-set of T'. That is, $\gamma_2^d(T) \ge \gamma_2^d(T') = |S_A^*| = |S_A|$. In summary, $\gamma_2^d(T) = |S_A| = \frac{n(T)-l(T)+3}{4}$. By the inductive hypothesis, S_A^* is the unique γ_2^d-set of T'. Hence, $D' = S_A^*$. In addition, if $u \in D$, then $v \notin D$. It follows from $(T, S) \in \mathscr{T}_1$ and Observation 3.1(a), (b) that v has status A, and the non-leaf neighbor of v, say w, has status B or C. From the choice of D' and $D' = S_A^*$, u is the unique vertex in D which is within distance two from w. It conclude that w is not $2D$-dominated by D, a contradiction. Therefore, $u \notin D$. Similarly, all leaf-neighbors of v do not belong to D, and then $D = D' = S_A^* = S_A$.

In the latter case, the tree T obtained from T' by attaching a path $P_4 = u_1u_2u_3u_4$ to a leaf v of T', where u_4 is a leaf in T. We have that $n(T) = n(T') + 4$, $l(T) = l(T')$ and $|S_A| = |S_A^*| + 1$. It follows Observation 3.1(c) that $\gamma_2^d(T) \le |S_A| = |S_A^*| + 1 = \frac{n(T')-l(T')+3}{4} + 1 = \frac{n(T)-4-l(T)+3}{4} + 1 = \frac{n(T)-l(T)+3}{4}$. Let $D' = (D \setminus \{u_4\}) \cup \{u_3\}$ when $u_4 \in D$ and $D' = D$ when $u_4 \notin D$, $D'' = (D' \setminus \{u_1, u_2\}) \cup \{v\}$ when u_1 or u_2 belong to D', otherwise, $D'' = D'$. Then $u_3 \in D$ and $D'' \setminus \{u_3\}$ is a $2DD$-set of T'. That is, $\gamma_2^d(T) - 1 \ge \gamma_2^d(T') = |S_A^*| = |S_A| - 1$. In summary, $\gamma_2^d(T) = |S_A| = \frac{n(T)-l(T)+3}{4}$. By the inductive hypothesis, S_A^* is the unique γ_2^d-set of T'. Hence, $D'' \setminus \{u_3\} = S_A^*$. If $|\{u_1, u_2, u_3, u_4, v\} \cap D| \ge 2$, the set $(D \setminus \{u_1, u_2, u_3, u_4\}) \cup \{v\}$ is a $2DD$-set of T'. More precisely, $(D \setminus \{u_1, u_2, u_3, u_4\}) \cup \{v\}$ is a γ_2^d-set of T'. By the uniqueness of γ_2^d-set of T', $(D \setminus \{u_1, u_2, u_3, u_4\}) \cup \{v\} = S_A^*$, a contradiction. Hence, $|\{u_1, u_2, u_3, u_4, v\} \cap D| = 1$. It implies that $\{u_1, u_2, u_3, u_4, v\} \cap D = \{u_3\}$. It is easy to see that $D \setminus \{u_3\}$ is a γ_2^d-set of T'. By the uniqueness of γ_2^d-set of T', $D \setminus \{u_3\} = S_A^*$. So, $D = S_A$. □

In what follows, we begin to prove Theorem 2.3.

Proof. The sufficiency follows immediately from Lemma 3.2. So we prove the necessity only. If $diam(T) \le 2$, T is a star, $\gamma_2^d(T) = 1 \ge \frac{n(T)-l(T)+3}{4}$. Suppose that $\gamma_2^d(T) = \frac{n(T)-l(T)+3}{4}$, it is easy to see that there exists a labeling S of the vertices of T such that (T, S) can be obtained from (P_3, S') by repeated applications of operation $\mathscr{O}_1$. Hence, $(T, S) \in \mathscr{T}_1$. If $diam(T) = 3$, T is a double star, and then $\gamma_2^d(T) = 2 > \frac{n(T)-l(T)+3}{4}$. So, we assume that $diam(T) \ge 4$. The proof is by induction on $n(T)$. The result is immediate for $n(T) \le 5$. For the inductive hypothesis, let $n(T) \ge 6$. Assume that for every nontrivial tree T' of order less than $n(T)$, we have that $\gamma_2^d(T') \ge \frac{n(T')-l(T')+3}{4}$, with equality only if $(T', S^*) \in \mathscr{T}_1$ for some labeling S^*.

Let D be a γ_2^d-set of T which contains no leaf and $P = v_1v_2 \cdots v_t$ be a longest path in T such that $d(v_3)$ as large as possible.

We now proceed with a series of claims that we may assume are satisfied by the tree T, for otherwise the desired result holds.

Claim 1. Each support vertex in T has exactly one leaf-neighbor.

If not, assume that there is a support vertex u which is adjacent to at least two leaves. Deleting one of its leaf-neighbors, say u_1, and denote the resulting tree by T'. Observe that $n(T) = n(T') + 1$, $l(T) = l(T') + 1$ and D is still a $2DD$-set of T'. That is, $\gamma_2^d(T) \ge \gamma_2^d(T') \ge \frac{n(T')-l(T')+3}{4} = \frac{n(T)-1-l(T)+1+3}{4} = \frac{n(T)-l(T)+3}{4}$.

In particular, if $\gamma_2^d(T) = \frac{n(T)-l(T)+3}{4}$, then $\gamma_2^d(T') = \frac{n(T')-l(T')+3}{4}$. It means that $(T', S^*) \in \mathscr{T}_1$ for some labeling S^*. By Observation 3.1(a), u has status A. Let S be obtained from S^* by labeling u_1 with label C. Then (T, S) can be obtained from (T', S^*) by operation $\mathscr{O}_1$. Thus, $(T, S) \in \mathscr{T}_1$. $\square$

By Claim 1, we can assume that $d(v_2) = 2$. And by Corollary 2.2, $v_2 \in D$. Now, we consider the vertex v_3.

Claim 2. $d(v_3) = 2$.

Suppose that $d(v_3) \geq 3$. If $v_3 \in D$, let $T' = T - \{v_1, v_2\}$. Clearly, $D \setminus \{v_2\}$ is a $2DD$-set of T'. Note that $n(T) = n(T') + 2$, $l(T) = l(T') + 1$, then $\gamma_2^d(T) \geq \gamma_2^d(T') + 1 \geq \frac{n(T')-l(T')+3}{4} + 1 = \frac{n(T)-2-l(T)+1+3}{4} + 1 > \frac{n(T)-l(T)+3}{4}$. So we assume that $v_3 \notin D$. If v_3 is adjacent to a support vertex outside P, say v_2'. It follows from Claim 1 and Corollary 2.2 that $v_2' \in D$. Moreover, $(D \setminus \{v_2, v_2'\}) \cup \{v_3\}$ is a $2DD$-set of the tree T' obtained from T by removing all leaf-neighbors of v_2 and v_2'. Hence, $\gamma_2^d(T) \geq \gamma_2^d(T') + 1 \geq \frac{n(T')-l(T')+3}{4} + 1 = \frac{n(T)-2-l(T)+3}{4} + 1 > \frac{n(T)-l(T)+3}{4}$. Combining the assumption that $d(v_3) \geq 3$, v_3 is a support vertex of degree three of T. We remove its leaf-neighbor, say u, and D is still a $2DD$-set of the resulting tree T' from $u \notin D$. Hence, $\gamma_2^d(T) \geq \gamma_2^d(T') \geq \frac{n(T')-l(T')+3}{4} = \frac{n(T)-l(T)+3}{4}$. We show that in fact $\gamma_2^d(T) > \frac{n(T)-l(T)+3}{4}$. Suppose to the contrary that $\gamma_2^d(T) = \frac{n(T)-l(T)+3}{4}$. Then we have equality throughout the above inequality chain. In particular, $\gamma_2^d(T) = \gamma_2^d(T') = \frac{n(T')-l(T')+3}{4}$. By the inductive hypothesis, $(T', S^*) \in \mathscr{T}_1$ for some labeling S^*. By Observation 3.1(a) and (b), the vertex v_3 has status B or C in S^*. Since D contains no leaf, D is also a γ_2^d-set of T'. On the other hand, by Lemma 3.2, S_A^* is the unique γ_2^d-set of T'. So, $D = S_A^*$. It implies that u can not be $2D$-dominated by D, a contradiction. $\square$

Claim 3. $d(v_4) = 2$.

Assume that $d(v_4) \geq 3$ and v_3' is a neighbor of v_4 outside P. From Claim 1 and the choice of P, one of the three cases as following holds:

(1) v_3' is adjacent to a support vertex, say v_2', where v_2' and v_3' have degree two;

(2) v_3' is a support vertex of degree two in T;

(3) v_3' is a leaf.

In the first case, let T' be a tree obtained from T by removing v_1, v_2, v_3 and the leaf-neighbor of v_2'. We have that $n(T) = n(T') + 4$, $l(T) = l(T') + 1$ and $\gamma_2^d(T') \leq \gamma_2^d(T) - 1$. In the latter two cases, let $T' = T - \{v_1, v_2, v_3\}$. We have that $n(T) = n(T') + 3$, $l(T) = l(T') + 1$ and $\gamma_2^d(T') \leq \gamma_2^d(T) - 1$. In either case, we always have $\gamma_2^d(T) > \frac{n(T)-l(T)+3}{4}$ by an argument similar to the proof of Claim 2. $\square$

Let $T' = T - \{v_1, v_2, v_3, v_4\}$. Note that $n(T) = n(T') + 4$, $\gamma_2^d(T') \leq \gamma_2^d(T) - 1$. In addition, $l(T) = l(T') + 1$ when $d(v_5) \geq 3$, and $l(T) = l(T')$ when $d(v_5) = 2$. Hence, we always have that $\gamma_2^d(T) \geq \gamma_2^d(T') + 1 \geq \frac{n(T')-l(T')+3}{4} + 1 \geq \frac{n(T)-4-l(T)+3}{4} + 1 = \frac{n(T)-l(T)+3}{4}$. Suppose that $\gamma_2^d(T) = \frac{n(T)-l(T)+3}{4}$, then we have equality throughout the above inequality chain. In particular, $d(v_5) = 2$ and $\gamma_2^d(T) - 1 = \gamma_2^d(T') = \frac{n(T')-l(T')+3}{4}$. By the inductive hypothesis, $(T', S^*) \in \mathscr{T}_1$ for some labeling S^*. Since v_5 is a leaf in T', by Observation 3.1(a), it has status C. Let S be obtained from the labeling S^* by labeling the vertices v_1, v_2, v_3, v_4 with label C, A, B, D, respectively. Then, (T, S) can be obtained from (T', S^*) by operation $\mathscr{O}_3$. Thus, $(T, S) \in \mathscr{T}_1$. □

4 Proof of Theorem 2.4

The following observation establishes properties of trees in the family $\mathscr{T}_2$.

Observation 4.1 *If $(T, S) \in \mathscr{T}_2$, then (T, S) has the following properties.*

(*a*) *Every support vertex of T has status A and every leaf has status C.*

(*b*) *The set S_A is a $2DD$-set of T.*

(*c*) *Let v be a vertex which has status A or B, v has at most one corresponding vertex. In particular, if there is no corresponding vertex of degree two of v in T, then $d(v) = 2$.*

(*d*) *If v is a support vertex, then v has degree two.*

(*e*) *Let v be a vertex of degree two which has status C, then it is adjacent to two vertices, say u and w, which are labeled A and D, respectively. In particular, if $d(u) = 2$, the component of $T - vw$ containing v, say T', containing the basic path of T, and $(T', S^*) \in \mathscr{T}_2$ for some labeling S^*.*

Lemma 4.2 *Let T be a tree and S be a labeling of T such that $(T, S) \in \mathscr{T}_2$. Then, $\gamma_2^d(T) = \frac{n(T)+s(T)+l(T)}{4}$.*

Proof. By Observation 4.1(b), S_A is a $2DD$-set of T and $S_A = \frac{n(T)+s(T)+l(T)}{4}$ (We can obtain this conclusion by induction on $n(T)$, it is similar to the proof of Lemma 3.2, so we omit it). So, $\gamma_2^d(T) \leq \frac{n(T)+s(T)+l(T)}{4}$. Since $(T, S) \in \mathscr{T}_2$, $T = P_4$ when $n \leq 4$, and $\gamma_2^d(T) = 2 = \frac{n(T)+s(T)+l(T)}{4}$. So, we assume that $n(T) \geq 5$. Combining the definition of $\mathscr{T}_2$, we have that $diam(T) \geq 7$. Suppose that T is a tree with minimum order which satisfy the two properties:

(1) $(T, S) \in \mathscr{T}_2$;

(2) $\gamma_2^d(T) < \frac{n(T)+s(T)+l(T)}{4}$.

Let D be a γ_2^d-set of T which contains no leaf, $u_1u_2u_3u_4$ be the basic path of T, and v_1 be a leaf of T that at maximum distance from u_2, let $P = v_1v_2v_3 \cdots v_tu_2$ be the path between v_1 and u_2. Note that $v_t = u_1$ or u_3. It follows from $(T, S) \in \mathscr{T}_2$ and Observation 4.1(d) that $d(v_2) = 2$ and v_1, v_2 have status C, A, respectively. And moreover, by the definition of $\mathscr{T}_2$, v_3 has status A or B.

In the form case, if $d(v_3) = 2$, then $v_1v_2v_3v_4$ is the basic path of T, a contradiction. So, $d(v_3) \geq 3$. It implies that there exists a sequence of length k used to construct (T, S):

(P_4, S''), $(T_1, S_1), \cdots, (T_{k-1}, S_{k-1})$, (T, S), such that (T, S) is obtained from (T_{k-1}, S_{k-1}) by operation $\mathscr{O}_4$. That is, T is obtained from T_{k-1} by adding the path v_1v_2 and joining v_2 to v_3. But in this case, by the definition of $\mathscr{O}_4$, we can always obtain a leaf which is farther away from u_2 than v_1, contradicting the choice of v_1. So we assume that v_3 has status B.

If $d(v_3) \geq 3$, by Observation 4.1(d), v_3 is not a support vertex. From the choice of v_1 and the fact that $diam(T) \geq 7$, v_3 is adjacent to s support vertices of degree two other than v_2, where $s \geq 1$. These support vertices are labeled A, and the leaf-neighbor of each of them is labeled C. From the choice of D and Corollary 2.2, $S(T) \cap N(v_3) \subseteq D$. v_4, v_5, v_6 has status D, C, A, respectively, and $d(v_4) = d(v_5) = 2$. Moreover, there exists no a corresponding vertex of degree two of v_6 in T, so $d(v_6) = 2$. Note that $\{v_3, v_4, v_5, v_6\} \cap D \neq \emptyset$, then $(D \setminus \{v_3, v_4, v_5\}) \cup \{v_6\}$ is also a γ_2^d-set of T. Hence, $D' = D \setminus \{v_2\}$ is a $2DD$-set of T' with order at most $\gamma_2^d(T) - 1$, where $T' = T - \{v_1, v_2\}$. On the other hand, note that $(T', S^*) \in \mathscr{T}_2$ for some labeling S^*, from the choice of T, $\gamma_2^d(T') = \frac{n(T')+s(T')+l(T')}{4} = \frac{n(T)+s(T)+l(T)}{4} - 1 > \gamma_2^d(T) - 1$. A contradiction.

If $d(v_3) = 2$, from the definition of $\mathscr{T}_2$, v_4 has status D, and furthermore, v_5, v_6 have status C, A, respectively. In particular, $d(v_4) = d(v_5) = 2$. Note that $v_2 \in D$, and $\{v_3, v_4, v_5, v_6\} \cap D \neq \emptyset$, so the set $D' = (D \setminus \{v_3, v_4, v_5\}) \cup \{v_6\}$ is also a γ_2^d-set of T. Now, we distinguish two cases as follows.

Case 1. $d(v_6) = 2$.

The set $D'' = D' \setminus \{v_2\}$ is a $2DD$-set of T' with order at most $\gamma_2^d(T) - 1$, where $T' = T - \{v_1, v_2, v_3, v_4\}$. On the other hand, from the choice of T and the fact that $(T', S^*) \in \mathscr{T}_2$ for some labeling S^*, $\gamma_2^d(T') = \frac{n(T')+s(T')+l(T')}{4} = \frac{n(T)+s(T)+l(T)}{4} - 1 > \gamma_2^d(T) - 1$. A contradiction.

Case 2. $d(v_6) \geq 3$.

We have that $\text{sta}(v_7) = A$ or B. If $\text{sta}(v_7) = B$, then all neighbors of v_6 outside P have status A, and note that these neighbors are support vertices of degree two (From the choice of v_1 and the definition of $\mathscr{T}_2$). We remove one of these support vertices, say u_1, and its leaf-neighbor, say u_2, denote the resulting tree by T'. Clearly, $(T', S^*) \in \mathscr{T}_2$ for some labeling S^*. We know that $v_2, v_6 \in D'$, and $\{u_1, u_2\} \cap D' \neq \emptyset$, so $D'' = D' \setminus \{u_1, u_2\}$ is a $2DD$-set of T' with order at most $\gamma_2^d(T) - 1$, where $T' = T - \{u_1, u_2\}$. On the other hand, from the choice of T, $\gamma_2^d(T') = \frac{n(T')+s(T')+l(T')}{4} = \frac{n(T)+s(T)+l(T)}{4} - 1 > \gamma_2^d(T) - 1$. A contradiction.

If $\text{sta}(v_7) = A$, then one of the two cases as following holds:

(1) There exists a neighbor of v_6 outside P, say u_1, has status B.

(2) All neighbors of v_6 outside P have status A.

In the former case, there exists a neighbor u_2 of u_1 which has status D. Similarly, there exists a neighbor u_3 of u_2 which has status C, and there exists a neighbor u_4 of u_3 which has status A. Moreover, let u_5 be a neighbor of u_4 other than u_3, then u_5 has status A or B. In either case, u_5 has degree at least two, which contradicts the choice of v_1.

In the latter case, we take any neighbor of v_6 outside P, say u_1, and we have that u_1 has a neighbor which has status C, say u_2. From the choice of v_1, u_2 is a leaf. By Observation 4.1(d), $d(u_1) = 2$. And we can obtain a contradiction by an argument similar to the case that $\mathrm{sta}(v_7) = B$ as above.

In summary, if $(T, S) \in \mathscr{T}_2$. Then, $\gamma_2^d(T) = \frac{n(T)+s(T)+l(T)}{4}$. □

Lemma 4.3 *Let T be a tree and S be a labeling of T such that $(T, S) \in \mathscr{T}_2$. Then for any leaf v, there exists a set D with order $\frac{n(T)+s(T)+l(T)}{4} - 1$ such that each vertex of T is $2D$-dominated by D except for v, and the non-leaf neighbor of the support vertex of v belongs to D.*

Proof. Take any leaf v_1 of T. We proceed by induction on the length k of a sequence required to construct the labeled tree (T, S). When $k = 0$, $(T, S) = (P_4, S'')$, the result is immediate. Let $k \geq 1$ and assume that if the length of sequence used to construct a labeled tree $(T', S^*) \in \mathscr{T}_2$ is less than k, the result holds. Since $(T, S) \in \mathscr{T}_2$, there exists always a sequence of length k used to construct (T, S): (P_4, S''), $(T_1, S_1), \cdots, (T_{k-1}, S_{k-1})$, (T, S).

First, we assume that v_1 is in the basic path of T. Since $(T_{k-1}, S_{k-1}) \in \mathscr{T}_2$, v_1 is still a leaf of T_{k-1}. By the inductive hypothesis, there exists a set D' with order $\frac{n(T_{k-1})+s(T_{k-1})+l(T_{k-1})}{4} - 1$ such that each vertex of T_{k-1} is $2D$-dominated by D' except for v_1, and v_3 belongs to D', where v_3 is the neighbor of the support vertex of v_1. We know that (T, S) is obtained from (T_{k-1}, S_{k-1}) by one of the operations $\mathscr{O}_2$, $\mathscr{O}_3$ and $\mathscr{O}_4$. In the first or third case, let D be the set consisting of D' and the support vertex which belongs to $V(T) \setminus V(T_{k-1})$, and D is the desired set. In the second case, the tree T is obtained from T_{k-1} by adding a path $u_1u_2u_3u_4$ and joining u_1 to a leaf u of T_{k-1}. Note that u has status C, and by Observation 4.1(d), the neighbor of u in T_{k-1}, say u', has degree two. By the inductive hypothesis, there exists a set D' with order $\frac{n(T_{k-1})+s(T_{k-1})+l(T_{k-1})}{4} - 1$ such that each vertex of T_{k-1} is $2D$-dominated by D' except for v_1, and v_3 belongs to D'. Moreover, one of u and u' belongs to D'. Let D be the set consisting of D' and the vertex u_3, and D is the desired set.

Next, we consider the case that v_1 is not in the basic path. Since $(T, S) \in \mathscr{T}_2$, this leaf has status C and its support vertex v_2 is labeled A. By Observation 4.1(d), v_2 has degree two. Let $P = v_1v_2 \cdots v_tv$ be the path between v_1 and v, where v is the vertex of basic path which has minimum distance from v_1. Note that the neighbor of v_2, say v_3, has status A or B.

Next, we distinguish two cases as follows.

Case 1. $\mathrm{sta}(v_3) = A$.

If $d(v_3) = 2$, then it is easy to see that $v_1v_2v_3v_4$ is the basic path of T, a contradiction.

If $d(v_3) \geq 3$, from the definition of $\mathscr{T}_2$ and the fact that a sequence of labeled trees used to construct (T, S) is not necessarily unique, we have that there exists a sequence of length k used to construct (T, S): (P_4, S''), $(T_1', S_1'), \cdots, (T_{k-1}', S_{k-1}')$, (T, S), such that (T, S) is obtained from (T_{k-1}', S_{k-1}') by $\mathscr{O}_4$. That is, the tree T is obtained from T_{k-1}' by adding the path v_1v_2 and joining v_2 to a vertex v_3. Note that v_3 has a neighbor of degree two, say u, which is labeled C (Otherwise, no vertex of T is the corresponding vertex of v_3). By Observation 4.1(e), the component of $T_{k-1}' - uu'$ containing u, say T', containing the basic path, and $(T', S^*) \in \mathscr{T}_2$ for some S^*, where u' is the neighbor of u other than v_3. It implies that there always exists a sequence of length k used to construct (T, S): (P_4, S''), $(T_1'', S_1''), \cdots, (T_{k-1}'', S_{k-1}'')$, (T, S), satisfying the two conditions as follows:

(1) $(T_{k-1}'', S_{k-1}'') = (T_{k-1}', S_{k-1}')$;

(2) There is a $i \in \{1, 2, \cdots, k-2\}$ in this sequence such that $(T_i'', S_i'') = (T', S^*)$.

By the inductive hypothesis, there exists a set D' with order $\frac{n(T_i'')+s(T_i'')+l(T_i'')}{4} - 1$ such that each vertex of T_i'' is $2D$-dominated by D' except for u, and u'' belongs to D', where u'' is a neighbor of v_3 in T_i'' other than u. Then $D_i = D' \cup \{v_3\}$ is a γ_2^d-set of T_i''. For each $j \in \{i, i+1, \cdots, k-2\}$, we know that (T_{j+1}'', S_{j+1}'') is obtained from (T_j'', S_j'') by one of the operations $\mathscr{O}_2$, $\mathscr{O}_3$ and $\mathscr{O}_4$. Let $D_{j+1} = D_j \cup \{w\}$, where $w \in V(T_{j+1}'') \setminus V(T_j)''$ and has status A. It is easy to see that D_{j+1} is a γ_2^d-set of T_{j+1}'', and moreover, D_{k-1} is the desired set.

Case 2. sta$(v_3) = B$.

In this case, if $d(v_3) \geq 3$, there must be a neighbor of v_3, say u, which has status A. From the definition of $\mathscr{T}_2$, the component of $T - v_3u$ containing v_3, say T', containing the basic path, and $(T', S^*) \in \mathscr{T}_2$ for some S^*. We can obtain the desired set by an argument similar to the case of sta$(v_3) = A$ and $d(v_3) \geq 3$.

If $d(v_3) = 2$, then v_4, v_5, v_6 have status D, C, A, respectively, and $d(v_4) = d(v_5) = 2$. If $d(v_6) = 2$, let $T' = T - \{v_1, v_2, v_3, v_4\}$. Note that $(T', S^*) \in \mathscr{T}_2$ for some S^*. By the inductive hypothesis, there exists a set D' with order $\frac{n(T')+s(T')+l(T')}{4} - 1$ such that each vertex of T' is $2D$-dominated by D' except for v_5, and v_7 belongs to D', then the set $D' \cup \{v_3\}$ is the desired set. So we consider the case of $d(v_6) \geq 3$. From the definition of $\mathscr{T}_2$, there must exist a neighbor of v_6, say u, such that sta$(u) = A$ and the component of $T - v_6u$ containing v_6, say T', containing the basic path, and $(T', S^*) \in \mathscr{T}_2$ for some S^*. By the inductive hypothesis, there exists a set D' with order $\frac{n(T')+s(T')+l(T')}{4} - 1$ such that each vertex of T' is $2D$-dominated by D' except for v_1, and v_3 belongs to D', We can obtain the desired set by an argument similar to the case of sta$(v_3) = A$ and $d(v_3) \geq 3$. □

In what follows, we begin to prove Theorem 2.3.

Proof. The sufficiency follows immediately from Lemma 4.2. So we prove the necessity only. If $diam(T) \leq 2$, T is a star, and $\gamma_2^d(T) = 1 < \frac{n(T)+s(T)+l(T)}{4}$. If $diam(T) = 3$, T is a double star, and then $\gamma_2^d(T) = 2 \le \frac{n(T)+s(T)+l(T)}{4}$. Support that $\gamma_2^d(T) = \frac{n(T)+s(T)+l(T)}{4}$, it is easy to see that $T = P_4$, let S be the labeling that assigns to the two leaves of the path P_4 status C, and the remaining vertices status A, then the label tree $(P_4, S) \in \mathscr{T}_2$. So we assume that $diam(T) \ge 4$. The proof is by induction on $n(T)$. The result is immediate for $n(T) \le 4$. For the inductive hypothesis, let $n(T) \ge 5$. Assume that for every nontrivial tree T' of order less than $n(T)$, we have that $\gamma_2^d(T') \le \frac{n(T')+s(T')+l(T')}{4}$, with equality only if $(T', S^*) \in \mathscr{T}_2$ for some labeling S^*.

Let D be a γ_2^d-set of T which contains no leaf and $P = v_1v_2\cdots v_t$ be a longest path in T such that

(i) $d(v_5)$ as large as possible, and subject to this condition

(ii) $d(v_4)$ as large as possible, and subject to this condition

(iii) $d(v_3)$ as large as possible.

We now proceed with a series of claims that we may assume are satisfied by the tree T, for otherwise the desired result holds.

Claim 1. Each support vertex in T has exactly one leaf-neighbor.

If not, assume that there is a support vertex u which is adjacent to at least two leaves, say u_1, u_2. Deleting u_1, and denote the resulting tree by T'. Take a γ_2^d-set of T' contains no leaf, say D'. It follows that u is either contained in D' or has at least two non-leaf neighbors in D', and then D' is also a $2DD$-set of T. That is, $\gamma_2^d(T) \le \gamma_2^d(T')$. Observe that $n(T) = n(T') + 1$, $l(T) = l(T') + 1$ and $s(T) = s(T')$. We have that $\gamma_2^d(T) \le \gamma_2^d(T') \le \frac{n(T')+s(T')+l(T')}{4} = \frac{n(T)-1+s(T)+l(T)-1}{4} < \frac{n(T)+s(T)+l(T)}{4}$. □

By Claim 1, we can assume that $d(v_2) = 2$. And by Corollary 2.2, $v_2 \in D$. Now, we consider the vertex v_3.

Claim 2. v_3 is not a support vertex.

In other words, all neighbors of v_3 are support vertices of degree two, except possibly the vertex v_4. If not, support that v_3 is a support vertex and u is the leaf-neighbor. Let $T' = T - \{v_1, v_2\}$. Note that $n(T) = n(T') + 2$, $l(T) = l(T') + 1$ and $s(T) = s(T') + 1$, then $\gamma_2^d(T) \le \gamma_2^d(T') + 1 \le \frac{n(T')+s(T')+l(T')}{4} + 1 = \frac{n(T)-2+s(T)-1+l(T)-1}{4} + 1 = \frac{n(T)+s(T)+l(T)}{4}$. In particular, if $\gamma_2^d(T) = \frac{n(T)+s(T)+l(T)}{4}$, then $\gamma_2^d(T') = \frac{n(T')+s(T')+l(T')}{4}$. It means that $(T', S^*) \in \mathscr{T}_2$ for some labeling S^*. By Lemma 4.3, there exists a $2DD$-set S of $T' - \{u\}$ with cardinality $\gamma_2^d(T') - 1$, and the non-leaf neighbor of v_3 in T' belongs to S. It is easy to see that $S \cup \{v_2\}$ is a $2DD$-set of T with cardinality $\gamma_2^d(T')$. That is, $\gamma_2^d(T) \le \gamma_2^d(T')$, Contradicting the fact that $\gamma_2^d(T) = \gamma_2^d(T') + 1$. Hence, we have that $\gamma_2^d(T) < \frac{n(T)+s(T)+l(T)}{4}$. □

Let $(S(T) \cap N(v_3)) \setminus \{v_4\} = \{w_1, w_2, \cdots, w_t\}$, where $w_1 = v_2$, $t \ge 1$.

Claim 3. $d(v_4) = 2$.

Assume that $d(v_4) \geq 3$, let T' be the component of $T - v_3v_4$ containing v_4. It follows from $n(T) = n(T')+1+2t$, $l(T) = l(T')+t$ and $s(T) = s(T')+t$ that $\gamma_2^d(T) \leq \gamma_2^d(T')+t \leq \frac{n(T')+s(T')+l(T')}{4} + t = \frac{n(T)-1-2t+s(T)-t+l(T)-t}{4} + t < \frac{n(T)+s(T)+l(T)}{4}$. □

Claim 4. $d(v_5) = 2$.

Assume that $d(v_5) \geq 3$ and v_4' be a neighbor of v_5 outside P. If $t = 2$, from the choice of P and Claim 1, we only need to consider the two case as follows (In other cases, let $T' = T - \{v_1, v_2, v_3, v_4\}$. We can always obtain a γ_2^d-set of T' which contains a vertex $u \in N[v_5] \cap V(T')$. It means that $\gamma_2^d(T) \leq \gamma_2^d(T') + 1$. Observe that $n(T) = n(T') + 4$, $l(T) = l(T') + 1$ and $s(T) = s(T') + 1$. We always have that $\gamma_2^d(T) < \frac{n(T)+s(T)+l(T)}{4}$):

(1) v_5 is not a support vertex, v_4' is adjacent to a support vertex v_3', where v_3' and v_4' have degree two.

(2) v_5 is not a support vertex and v_4' is adjacent to h support vertices of degree two, where $h \geq 2$.

Let T' is the component of $T - v_5v_4'$ containing v_5. In the former case, $n(T) = n(T')+3$, $l(T) = l(T') + 1$, $s(T) = s(T') + 1$ and $\gamma_2^d(T) \leq \gamma_2^d(T') + 1$. In the latter case, note that it is possible that v_4' is a support vertex, then $n(T') + 2h + 1 \leq n(T) \leq n(T') + 2h + 2$, $l(T') + h \leq l(T) \leq l(T') + h + 1$, $s(T') + h \leq s(T) \leq s(T') + h + 1$ and $\gamma_2^d(T) \leq \gamma_2^d(T') + h$. In either case, we conclude that $\gamma_2^d(T) < \frac{n(T)+s(T)+l(T)}{4}$.

If $t \geq 3$, let T' be the component of $T - v_4v_5$ containing v_5. Observe that $n(T) = n(T') + 2 + 2t$, $l(T) = l(T') + t$ and $s(T) = s(T') + t$ and $\gamma_2^d(T) \leq \gamma_2^d(T') + t$. Analogous to the proof of Case 3, we have that $\gamma_2^d(T) < \frac{n(T)+s(T)+l(T)}{4}$. □

Claim 5. $d(v_6) = 2$ or all neighbors of v_6 outside P are support vertices of degree two.

First, we show that v_6 is not a support vertex. If not, it follows from Claim 1 that v_6 has one leaf-neighbor, and construct a tree T' which is obtained from T by removing the leaf-neighbor of v_6 and joining a new vertex to v_2. Let D' be a γ_2^d-set of T' which contains no leaf, then $N(v_3) \cap S(T) \subseteq D'$. We take a set $D'' = (D' \setminus \{v_3, v_4, v_5\}) \cup \{v_6\}$ when $D' \cap \{v_3, v_4, v_5\} \neq \emptyset$, and otherwise, $D'' = D'$. Note that D'' is also a $2DD$-set of T, and moreover, $n(T) = n(T')$, $l(T) = l(T')$, $s(T) = s(T') + 1$. Hence, $\gamma_2^d(T) \leq \gamma_2^d(T') \leq \frac{n(T')+s(T')+l(T')}{4} = \frac{n(T)+s(T)-1+l(T)}{4} < \frac{n(T)+s(T)+l(T)}{4}$.

Let u_1 be a leaf outside P that at maximum distance from v_6, and $P_1 = u_1u_2 \cdots u_{s-1}u_s$ be the path between u_1 and v_6, where $u_s = v_6$. Clearly, $s \leq 6$.

If $s = 4$, then we have that u_3 is adjacent to a support vertices of degree two, where $a \geq 1$. Suppose that u_3 is not a support vertex, let T' be the component of $T - u_3v_6$ containing v_6. It follows from $n(T) = n(T')+2a+1$, $l(T) = l(T')+a$ and $s(T) = s(T')+a$ that $\gamma_2^d(T) \leq \gamma_2^d(T')+a \leq \frac{n(T')+s(T')+l(T')}{4}+a = \frac{n(T)-2a-1+s(T)-a+l(T)-a}{4}+a < \frac{n(T)+s(T)+l(T)}{4}$. So, we assume that u_3 has a leaf-neighbor, say u, and in this case, let $T' = T - \{u_1, u_2\}$. Note that $n(T) = n(T') + 2$, $l(T) = l(T') + 1$ and $s(T) = s(T') + 1$, then $\gamma_2^d(T) \leq$

$\gamma_2^d(T')+1 \le \frac{n(T')+s(T')+l(T')}{4}+1 = \frac{n(T)-2+s(T)-1+l(T)-1}{4}+1 \le \frac{n(T)+s(T)+l(T)}{4}$. In particular, if $\gamma_2^d(T) = \frac{n(T)+s(T)+l(T)}{4}$, then $\gamma_2^d(T') = \frac{n(T')+s(T')+l(T')}{4}$. It means that $(T', S^*) \in \mathscr{T}_2$ for some labeling S^*. By Lemma 4.3, there exists a $2DD$-set S of $T' - \{u\}$ with cardinality $\gamma_2^d(T')-1$, and a non-leaf neighbor of u_3 in T' belongs to S. It is easy to see that $S \cup \{u_2\}$ is a $2DD$-set of T with cardinality $\gamma_2^d(T')$. That is, $\gamma_2^d(T) \le \gamma_2^d(T')$, Contradicting the fact that $\gamma_2^d(T) = \gamma_2^d(T')+1$.

If $s = 5$, by an argument similar to that of Claim 1, Claim 2 and Claim 3, we have that $d(u_2) = d(u_4) = 2$, u_3 is not a support vertex and adjacent to a support vertices of degree two, where $a \ge 1$. Let T' be the component of $T - u_4v_6$ containing v_6 and D' be a γ_2^d-set of T' contains no leaf. If $a \ge 2$, Observe that $D' \cup (S(T) \cap N(u_3))$ is a $2DD$-set of T. Combining the fact that $n(T) = n(T')+2a+2$, $l(T) = l(T')+a$, $s(T) = s(T')+a$. We have that $\gamma_2^d(T) \le \gamma_2^d(T')+a \le \frac{n(T')+s(T')+l(T')}{4}+a = \frac{n(T)-2a-2+s(T)-a+l(T)-a}{4}+a < \frac{n(T)+s(T)+l(T)}{4}$.

So we consider the case of $a = 1$. If there is a vertex belonging to $N[v_6] \cap D'$, then $D' \cup \{u_2\}$ is a $2DD$-set of T, and so $\gamma_2^d(T) \le \gamma_2^d(T')+1 \le \frac{n(T')+s(T')+l(T')}{4}+1 = \frac{n(T)-4+s(T)-1+l(T)-1}{4}+1 < \frac{n(T)+s(T)+l(T)}{4}$. So we can assume that $N[v_6] \cap D' = \emptyset$. If $\{v_3, v_4\} \cap D' \ne \emptyset$, then $(D' \setminus \{v_3, v_4\}) \cup \{v_5\}$ is also a γ_2^d-set of T', and we are done. If $v_3, v_4 \notin D'$, it follows from $d(v_5) = 2$ and $N[v_6] \cap D' = \emptyset$ that v_5 is not $2D$-dominated by D', a contradiction.

If $s = 6$, from Claim 1, Claim 2 and the choice of T, we have that $d(u_2) = d(u_4) = d(u_5) = 2$, and u_3 is not a support vertex and adjacent to a support vertices of degree two, where $a \le t$. Let T' be the component of $T - v_5v_6$ containing v_6 and D_1 be a γ_2^d-set of T' contains no leaf. Note that $S(T) \cap N(u_3) \subseteq D_1$. Take a set $D' = (D_1 \setminus \{u_3, u_4, u_5\}) \cup \{v_6\}$ when $\{u_3, u_4, u_5\} \cap D_1 \ne \emptyset$, and otherwise, $D' = D_1$. Observe that $D' \cup \{w_1, w_2, \cdots, w_t\}$ is a $2DD$-set of T. Combining the fact that $n(T) = n(T')+2t+3$, $l(T) = l(T')+t$, $s(T) = s(T')+t$. We have that $\gamma_2^d(T) \le \gamma_2^d(T')+t \le \frac{n(T')+s(T')+l(T')}{4}+t = \frac{n(T)-2t-3+s(T)-t+l(T)-t}{4}+t < \frac{n(T)+s(T)+l(T)}{4}$. □

We assume that $|N(v_6) \setminus \{v_5, v_7\}| = a$, then $a \ge 0$. In addition, by the claims as above, we have that $d(v_2) = d(v_4) = d(v_5) = 2$, v_3 is not a support vertex and adjacent to t support vertices of degree two, where $t \ge 1$.

If $a = 0$, then $d(v_6) = 2$. Let T' be the component of $T - v_4v_5$ containing v_5 and D' be a γ_2^d-set of T' contains no leaf. Observe that $v_6 \in D'$ and $D' \cup \{w_1, w_2, \cdots, w_t\}$ is a $2DD$-set of T. It follows from $n(T) = n(T')+2t+2$, $l(T) = l(T')+t-1$ and $s(T) = s(T')+t-1$ that $\gamma_2^d(T) \le \gamma_2^d(T')+t \le \frac{n(T')+s(T')+l(T')}{4}+t = \frac{n(T)-2t-2+s(T)-t+1+l(T)-t+1}{4}+t = \frac{n(T)+s(T)+l(T)}{4}$. Suppose that $\gamma_2^d(T) = \frac{n(T)+s(T)+l(T)}{4}$, then we have equality throughout the above inequality chain. In particular, $\gamma_2^d(T') = \frac{n(T')+s(T')+l(T')}{4}$. By the inductive hypothesis, $(T', S^*) \in \mathscr{T}_2$ for some labeling S^*. Since v_5 is a leaf in T', by Observation 4.1(a), it has status C, and then v_6 has status A. Let S be obtained from the labeling S^* by labeling the vertices v_3, v_4 with label B, D, respectively. And moreover, labeling $w_1, w_2, \cdots, w_t$ with label A, and label their leaf-neighbors with label C. Then, (T, S) can be obtained from (T', S^*) by doing the operation $\mathscr{O}_3$ for one time and the operation $\mathscr{O}_2$ for $t-1$ times. Thus, $(T, S) \in \mathscr{T}_2$.

Next we consider the case of $a \geq 1$. Let $u_1, u_2, \cdots, u_a$ be all neighbors of v_6 outside P and u_i' be the leaf-neighbor of u_i ($i = 1, 2, \cdots, a$). Let $T' = T - \{u_1, u_2, \cdots, u_a, u_1', u_2', \cdots, u_a'\}$ and D' be a γ_2^d-set of T' contains no leaf. Note that v_6 has degree two in T', and $D' \cup \{u_1, u_2, \cdots, u_a\}$ is a $2DD$-set of T. It follows from $n(T) = n(T') + 2a$, $l(T) = l(T') + a$ and $s(T) = s(T') + a$ that $\gamma_2^d(T) \leq \gamma_2^d(T') + a \leq \frac{n(T')+s(T')+l(T')}{4} + a = \frac{n(T)-2a+s(T)-a+l(T)-a}{4} + a = \frac{n(T)+s(T)+l(T)}{4}$. Suppose that $\gamma_2^d(T) = \frac{n(T)+s(T)+l(T)}{4}$, then we have equality throughout the above inequality chain. In particular, $\gamma_2^d(T') = \frac{n(T')+s(T')+l(T')}{4}$. By the inductive hypothesis, $(T', S^*) \in \mathscr{T}_2$ for some labeling S^*.

If $t \geq 2$, by Lemma 4.3, there exists a set D_1 with order $\frac{n(T')+s(T')+l(T')}{4} - 1$ such that each vertex of T' is $2D$-dominated by D_1 except for v_1, and v_3 belongs to D_1. Since leaf-neighbor of each w_i ($i = 2, 3, \cdots, t$) is $2D$-dominated by D_1, without loss of generality, we can assume that each w_i ($i = 2, 3, \cdots, t$) belongs to D_1. Note that $d(v_4) = d(v_5) = d(v_6) = 2$ in T' and $\{v_4, v_5, v_6, v_7\} \cap D_1 \neq \emptyset$, we construct a set $D_2 = (D_1 \setminus \{v_4, v_5, v_6\}) \cup \{v_7\}$, each vertex of T' is $2D$-dominated by D_2 except for v_1 and $|D_2| \leq |D_1|$. Let D_3 be a set which is obtained from D_2 by deleting v_3, and adding all neighbors of v_6 outside P and v_2. It is easy to see that D_3 is a $2DD$-set of T, and $|D_3| \leq \frac{n(T)+s(T)+l(T)}{4} - 1$, it is impossible.

If $t = 1$, the vertices v_1 and v_2 have status C and A, respectively, in S^*. And so, v_3 has status A or B.

In the former case, it follows from $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2$ and the definition of $\mathscr{T}_2$ that $v_1v_2v_3v_4$ is the basic path of T', and then v_4 has status C. Moreover, v_5, v_6 have status D, B, respectively. Let S be obtained from the labeling S^* by labeling each u_i with label A, and each u_i' with label C. Then, (T, S) can be obtained from (T', S^*) by doing the operation $\mathscr{O}_2$ for a times. Thus, $(T, S) \in \mathscr{T}_2$.

In the latter case, from the definition of $\mathscr{T}_2$, v_4, v_5, v_6 have status D, C, A, respectively. And v_7 has status A or B. Assume that $\text{sta}(v_7) = A$. If $d(v_7) = 2$, we have that $v_5v_6v_7v_8$ is the basic path of T'. Let S_1^* be obtained from S^* by changing the status v_3, v_4, v_5, v_6 to A, C, D, B, respectively, and clearly, $(T', S_1^*) \in \mathscr{T}_2$. Let S be obtained from the labeling S_1^* by labeling each u_i with label A, and each u_i' with label C. Then, (T, S) can be obtained from (T', S_1^*) by doing the operation $\mathscr{O}_2$ for a times. Thus, $(T, S) \in \mathscr{T}_2$. If $\text{sta}(v_7) = A$ and $d(v_7) \geq 3$, or $\text{sta}(v_7) = B$, let S be obtained from the labeling S^* by labeling each u_i with label A, and each u_i' with label C. Then, (T, S) can be obtained from (T', S^*) by doing the operation $\mathscr{O}_4$ for a times. Thus, $(T, S) \in \mathscr{T}_2$. □

5 Summary

A network can be modeled by a graph $G = (V, E)$ with the vertices representing nodes such as processors or stations, and the edges representing links between the nodes. We often need to select some special nodes in the computer network to monitor the communication of the entire network. These special nodes correspond to the dominating set we mentioned above. It is well known that in the process of designing and managing computer networks, cost control is a key issue. Hence, we hope that the number of these special nodes to be as small as possible. This requires us to calculate the relevant domination parameters on the computer network, or give the upper and lower bounds of the relevant domination parameters.

Over the last few decades, many new domination parameters were proposed to meet various network design requirements, and disjunctive domination is one of them.

As the tree structure is one of the most important network topology structures, we take it as the research object of this article. We believe that our work will promote the development of computer networks.

References

[1] M. Anderson, R. C. Brigham, J. R. Carrington, R. P. Vitray, J. Yellen, *On exponential domination of $C_m \times C_n$*, AKCE Int. J. Graphs Comb., **6** (2009) 341-351.

[2] P. Dankelmann, D. Day, D. Erwin, S. Mukwembi, H. Swart, *Domination with exponential decay*, Discrete Math., **309** (2009) 5877-5883.

[3] X. Chen, M. Y. Sohn, *Bounds on the locating-total domination number of a tree*, Discrete Appl. Math., **159** (2011) 769-773.

[4] W. Goddard, M. A. Henning, C. A. McPillan, *The disjunctive domination number of a graph*, Quaest. Math., **37** (2014) 547-561.

[5] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker Inc., New York, 1998.

[6] T. W. Haynes, S. T. Hedetniemi, P. J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker Inc., New York, 1998.

[7] M. A. Henning, Distance domination in graphs, in: T. W. Haynes, S. T. Hedetniemi, P. J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker Inc., New York, 1998, pp. 335-365.

[8] M. A. Henning, S. A. Marcon, *A constructive characterization of trees with equal total domination and disjunctive domination numbers*, Quaest. Math., **39** (2016) 531-543.

[9] M. A. Henning, S. A. Marcon, *Domination versus disjunctive domination in graphs*, Quaest. Math., **39** (2016) 261-273.

[10] M. A. Henning, S. A. Marcon, *Domination versus disjunctive domination in trees*, Discrete Appl. Math., **184** (2015) 171-177.

[11] M. A. Henning, S. A. Marcon, *Vertices contained in all or in no minimum disjunctive dominating set of a tree*, Util. Math., **105** (2017) 95-123.

[12] F. P. Jamil, R. P. Malalay, *On disjunctive domination in graphs*, Quaest. Math., **43** (2020) 149-168.

[13] M. Krzywkowski, *An upper bound for the double outer-independent domination number of a tree*, Georgian Math. J., **22** (2015) 105-109.

[14] M. Krzywkowski, *An upper bound on the 2-outer independent domination number of a tree*, Cr. Math., **349** (2011) 1123-1125.

[15] Z. Li, J. Xu, *On the trees with same signed edge and signed star domination number*, Int. J. Comput. Math., **95** (2018) 2388-2395.

[16] W. Ning, M. Lu, J. Guo, *Bounds on the differentating-total domination number of a tree*, Discrete Appl. Math., **200** (2016) 153-160.

[17] W. Ning, M. Lu, K. Wang, *Bounding the locating-total domination number of a tree in terms of its annihilation number*, Discuss. Math. Graph T., **39** (2019) 31-40.

[18] B. S. Panda, A. Pandey, S. Paul, *Algorithmic aspects of b-disjunctive domination in graphs*, J. Comb. Optim., **36** (2018) 572-590.

[19] N. J. Rad, H. Rahbani, *Bounds on the locating roman domination number in trees*, Discuss. Math. Graph T., **38** (2018) 49-62.

[20] N. J. Rad, H. Rahbani, *Bounds on the locating-domination number and differentating-total domination number in trees*, Discuss. Math. Graph T., **38** (2018) 455-462.

[21] Y. B. Venkatakrishnan, B. Krishnakumari, *An improved upper bound of edge-vertex domination number of a tree*, Inform. process. Lett., **134** (2018) 14-17.

[22] Y. B. Venkatakrishnan, H. N. Kumar, B. Krishnakumari, *Bounds on the double edge-vertex domination number of a tree*, Ars Combinatoria, **146** (2019) 29-36.

[23] K. Wang, W. Ning, M. Lu, *Bounds on the locating-total domination number in trees*, Discuss. Math. Graph T., **40** (2020) 25-34.

[24] H. Yang, P. Wu, S. Nazari-Moghaddam, *Bounds for signed double roman k-domination in trees*, Rairo-oper. Res., **53** (2019) 627-643.