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WEAK AND STRONG DOMINATION ON SOME GRAPHS

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Abstract. Let $G = (V(G), E(G))$ be a graph and $uv\varepsilon E$. A subset $D \subseteq V$ of vertices is a dominating set if every vertex in $V - D$ is adjacent to at least one vertex of D. The domination number is the minimum cardinality of a dominating set. Let u and v be elements of V. Then, u strongly dominates u and v weakly dominates u if $(i)uv\varepsilon E$ and $(ii)deg(u) \geq deg(v)$. A set $D \subseteq V$ is a strong (weak) dominating set (sd-set)(wd-set) of G if every vertex in $V - D$ is strongly dominated by at least one vertex in D. The strong (weak) domination number $\gamma_s(\gamma_w)$ of G is the minimum cardinality of a sd-set (wd-set). In this paper, the strong and weak domination numbers of comet, double comet, double star and theta graphs are given. The theta graphs are important geometric graphs that have many applications, including wireless networking, motion planning, MST construction and real-time animation.

Keywords: graph theory, graph operations, domination

Mathematics Subject Classification. 68R10,05C76, 05C70

1. Introduction and Preliminaries

In this paper we consider finite, undirected graphs G without multiple edges and loop. Let $G = (V(G), E(G))$ be a graph. $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G. The order $|(V(G)|$ of G is denoted by n. Let $u, v\varepsilon V(G)$,$uv\varepsilon E(G)$. Two adjacent vertices are referred to as neighbors of each other. The set of neighbors of a vertex v is called the open neighborhood of v (or simply the neighborhood of v) and is denoted by $N(v)$. The set $N[v] = N(v) \cup v$ is called the closed neighborhood of v. The degree of a vertex v in a graph G is

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the number of vertices in G that are adjacent to v. Thus, the degree of v is the number of vertices in its neighborhood $N(v)$. Equivalently, the degree of v is the number of edges incident with v. The degree of a vertex v is denoted by $deg(v)$. Hence $deg(v) = |N(v)|$. The largest degree among the vertices of G is called the maximum degree of G and is denoted by $\triangle(G)$. The minimum degree of G is denoted by $\delta(G)$ [4]. A subset $D \subseteq V(G)$ of vertices is a dominating set if every vertex v in $V(G) - D$ is adjacent to at least one vertex of D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. Let u and v be elements of $V(G)$. Then u, strongly dominates u and v weakly dominates u if $(i)uv\varepsilon E(G)$ and $(ii)deg(u) \geq deg(v)$. The strong (weak) domination number $\gamma_s(\gamma_w)$ of G is the minimum cardinality of a $sd-set(wd-set)$. The domination number $\gamma(G)$ is a well-studied parameter as one can see from the bibliography [14] on domination. In this paper, we use $\gamma_s = \gamma_s(G), \gamma_w = \gamma_w(G), V = V(G)$ and $E = E(G)$ shortly. In general dominating sets in graph theory finds variety of applications of communication networks. The minimum dominating set of sites plays an important role in the network for it dominates the whole network with the minimum cost. The concept of strong and weak domination were introduced by Sampathkumar and Pushpa Latha in [21]. Strong and weak domination come on the scene inartificially practical situations. Consider a network of roads connecting a number of locations. In such a network, the degree of a vertex v is the number of roads meeting at v. Suppose $deg(u) \geq deg(v)$. Inherently, the traffic at u is heavier than that at v. If we consider the traffic between u and v, preference should be given to the vehicles going from u to v. Therefore, in some sense, u strongly dominates v and v weakly dominates u. They give this definition by this motivation. It is shown that the problems of computing γ_s and γ_w are NP-hard [21]. In 1999 and 2000, Rautenbach investigated some bounds on γ_s [18] [19]. Also, some bounds are given by Bhat et al. [2]. Rautenbach and Zverovich have studied results on NP-completeness related to strong and weak dominating set [20]. Desai and Gangadharapppa have investigated upper bounds a strong domination number for trees [7]. The concept of strong and weak domination in fuzzy graphs are introduced by Gani and Ahamed [10]. The strong domination number of some path related graphs given by Vaidya and Karkar [24] and also the strong domination number of some wheel related graphs are studied by Vaidya and Mehta [25]. The relations between strong domination and weak domination number are given by Boutrig and Chellali [3]. Swaminathan and Thangaraju investigated some relations between strong domination and maximum degree of the graph as well as weak domination and minimum degree of the graph [22]. In 2015 Doğan Durgun et al. investigated strong domination number of some graphs [8] and in 2020 they investigated strong and weak domination number in thorn graphs [9]. An algorithm have introduced by Berberler et al. in 2020 [23].

In this paper the strong and weak domination numbers of comet, double comet, double star and theta graphs are given. Theta graphs are important geometric graphs that have many applications, including wireless networking, motion planning, MST construction and also real-time animation [17]. For further information reader should check [17]. The other graphs in this paper are star related graphs.

Several graph types are defined in terms of star, the star network, a computer network modeled after the star graph which is important in distributed computing. In the next section we give some results and proofs of comet graph $C_{t,r}$, double comet $DC(n,a,b)$, double star $S(a,b)$ and theta graph $\theta(s_1, s_2, s_3, ..., s_n)$.

2. Main results

In this section we mention about strong and weak domination number of $C_{t,r}$, $DC(n,a,b)$, $S(a,b)$, $\theta(s_1, s_2, s_3, ..., s_n)$ graphs.

Definition 2.1. Comet $C_{t,r}$ is a graph which is obtained by identifying one end of the path P_t with the center of the star $K_{1,t}$ [6].

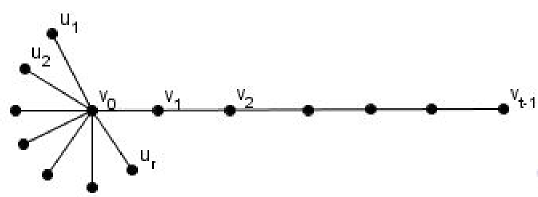

Figure 1. Comet Graph $C_{t,r}$

Theorem 2.2. *Let $G = C_{t,r}$ be a comet graph, where $t \geq 2$ and $r \geq 1$,*

$$\gamma_s(C_{t,r}) = \left\lceil \frac{t-2}{3} \right\rceil + 1, \gamma_w(C_{t,r}) = \left\lceil \frac{t-3}{3} \right\rceil + r + 1.$$

Proof. There is three cases for the proof of strong domination number.Let D be a strong dominating set.

Case1: Let $t \equiv 0(mod3)$. D should contain v_0 to strongly dominate $u_1, u_2, u_3..., u_r, v_1$ and v_{t-2} to strongly dominate v_{t-1} and v_{t-3}. In order to strongly dominate the rest of the vertices $\frac{t}{3} + 1$ number of vertices should be in D. These vertices can be $\{v_0, v_3, v_6, ..., v_{t-2}\}$ or $\{v_0, v_2, v_4, ..., v_{t-2}\}$.Thus,

$$\gamma_s(C_{t,r}) \leq \frac{t}{3} + 1.$$

Let $D = \{v_0, v_3, v_6, ..., v_{t-2}\}$ (or $D = \{v_0, v_2, v_4, ..., v_{t-2}\}$) not be a minimal strong dominating set. Delete v_0 from D. Then $u_1, u_2, u_3..., u_r$ should be in D to strongly dominate themselves.Since

$$deg(u_1) = deg(u_2) = deg(u_3) = ... = deg(u_r) < deg(v_0)$$

then this can not strongly dominate v_0. So v_0 should be in D. If any other vertex deleted from D then to strongly dominate itself or vertices which are its neighbours at least one other vertex must in D. Thus, the cardinality of D invariable. So,

$$\gamma_s(C_{t,r}) \geq \frac{t}{3} + 1.$$

Then ,

$$\gamma_s(C_{t,r}) = \frac{t}{3} + 1$$

where $t \equiv 0(mod3)$.

Case2: Let $t \equiv 1(mod3)$. D should contain v_0 to strongly dominate $u_1, u_2, u_3..., u_r, v_1$. v_{t-2} to strongly dominate v_{t-1} and v_{t-3} so v_{t-2} should be in D.

$$deg(v_2) = deg(v_3) = ... = deg(v_{t-5}) = deg(v_{t-4}) = 2$$

In order to strongly dominate other vertices which are not dominated $\frac{t-7}{3}$ number of vertices should be in D. There are still two vertices which are not strongly dominated. To dominate them one of the vertices should be in D. Then

$$\gamma_s(C_{t,r}) \leq \frac{t+2}{3}.$$

$D = \{v_0, v_3, v_6, ..., v_{t-3}, v_{t-2}\}$ or $D = \{v_0, v_2, v_4, ..., v_{t-4}, v_{t-2}\}$. Now we consider $D = \{v_0, v_3, v_6, ..., v_{t-3}, v_{t-2}\}$. Let D not be a minimal strong dominating set. Delete one of the vertices of D. Then D is not strong dominated set.Therefore, D is a minimal strong dominating set. So,

$$\gamma_s(C_{t,r}) \geq \frac{t+2}{3}.$$

Then,

$$\gamma_s(C_{t,r}) = \frac{t+2}{3}.$$

where $t \equiv 1(mod3)$.

Case3: $t \equiv 2(mod3)$. By using same idea

$$\gamma_s(C_{t,r}) \geq \frac{t+1}{3}.$$

From these three cases

$$\gamma_s(C_{t,r}) = \left\lceil \frac{t-2}{3} \right\rceil + 1.$$

Now we consider weak domination number of $C_{t,r}$. Since

$$deg(u_1) = deg(u_2) = deg(u_3) = ... = deg(u_r) < deg(v_0)$$

and there is no edges among $u_1, u_2, u_3..., u_r$ these vertices should be in weak dominating set. These vertices weakly dominate themselves and v_0. Since $deg(v_{t-1}) < deg(v_{t-2})$, v_{t-1} weakly dominates v_{t-2}. Remained vertices $v_1v_2v_3...v_{t-4}v_{t-3}$ can consider as a path. There is three cases. Let D be a weak dominating set.

Case4: $t \equiv 0(mod3)$. Let $D = \{u_1, u_2, u_3, ..., u_r, v_2, v_5, ..., v_{t-4}, v_{t-1}\}$ be a weak dominating set. Then

$$\gamma_w(C_{t,r}) \leq \frac{t-3}{3} + r + 1.$$

Let D not be a minimal dominating set. Delete one of the elements of D. Let u_1 be this vertex. Then v_0 can not be weakly dominated. So

$$\gamma_w(C_{t,r}) \geq \frac{t-3}{3} + r + 1.$$

$t \equiv 1(mod3)$ and $t \equiv 2(mod3)$ cases can be obtain by using same process. Therefore,

$$\gamma_w(C_{t,r}) = \frac{t-3}{3} + r + 1.$$

□

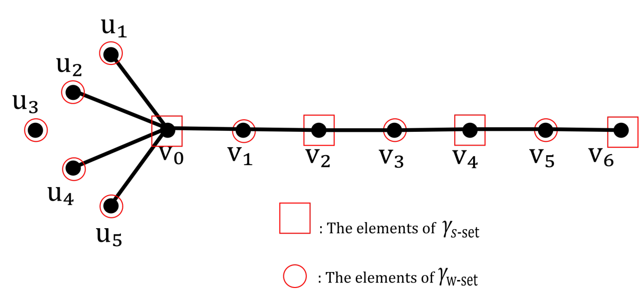

FIGURE 2. Comet Graph $C(7, 5)$

$$\gamma_s(C_{7,5}) = 4$$
$$\gamma_w(C_{7,5}) = 8$$

Definition 2.3. For $a, b \geq 1, n \geq a + b + 2$ by $DC(n, a, b)$ we denote a double comet, which is a tree composed of a path containing $n - a - b$ vertices with a pendent vertices attached to one of the ends of the path and b pendent vertices attached to the other end of the path [14].

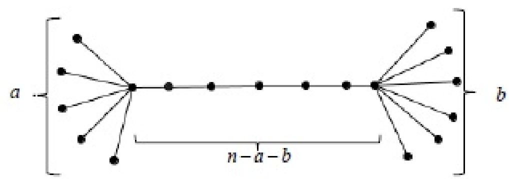

FIGURE 3. Double Comet Graph $DC(18, 5, 6), n = 18, a = 5, b = 6$

Theorem 2.4. *Let $DC(n, a, b)$ be a double comet where $a, b \geq 1$ and $n \geq a+b+2$. Then,*

$$\gamma_s(DC(n, a, b)) = 2 + \left\lceil \frac{n-a-b-4}{3} \right\rceil$$

$$\gamma_w(DC(n, a, b)) = a + b + \left\lceil \frac{n-a-b-2}{3} \right\rceil .$$

Proof. Let $DC(n, a, b)$ be a double comet where $a, b \geq 1$, $n \geq a + b + 2$.
proof of this theorem can be shown easily by using same process with the previous proof. □

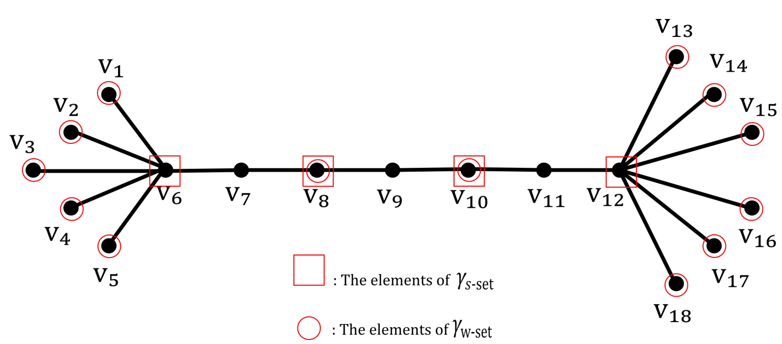

FIGURE 4. Double Comet Graph
$DC(18, 5, 6), (n = 18, a = 5, b = 6)$

$$\gamma_s(DC(18, 5, 6)) = 4$$
$$\gamma_w(DC(18, 5, 6)) = 13$$

Definition 2.5. Double star $S(a,b)$ graph where $a, b \geq 0$ is the graph consisting of the union of two starts $K_{1,n}$ and $K_{1,m}$ together with a line joining their centers [11].

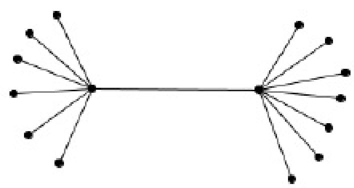

FIGURE 5. Double Star Graph $S(6,7)$

Proposition 2.6. *Let $S(a,b)$ be a double star graph where $a, b \geq 0$. Then $\gamma_s(S(a,b)) = 2$ and $\gamma_w(S(a,b)) = a + b$.*

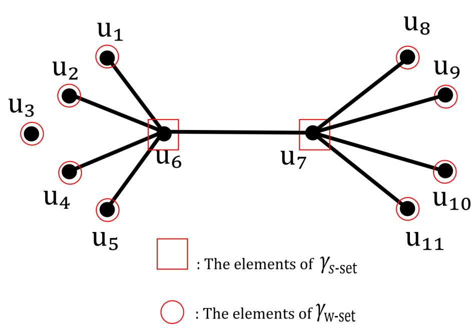

FIGURE 6. Double Star Graph $S(5,4), (a = 5, b = 4)$

$$\gamma_s(S(5,4)) = 2 \qquad \gamma_s(S(a,b)) = 2$$
$$\gamma_w(S(5,4)) = 9 \qquad \gamma_w(S(a,b)) = a + b$$

Definition 2.7. A complete $k-ary$ tree with depth n is all leaves have the same depth and all internal vertices have degree k. A complete $k-ary$ tree has $\frac{k^{n+1}-1}{k-1}$ vertices and $\frac{k^{n+1}-1}{k-1} - 1$ edges [11].

Proposition 2.8. *Let G be a complete $k-ary$ tree where $m, k, n \in Z^+$ with depth n. Then*

$$i)\gamma_s(G) = \begin{cases} \sum_{k=1}^{m} \frac{k^{n+2}+k^3-k^2+1}{k^3-1} & n \equiv 0(mod3) \\ \sum_{k=1}^{m} \frac{k^{n+2}+k^4-k^3-k}{k^3-1} & n \equiv 1(mod3) \\ \sum_{k=1}^{m} \frac{k^{n+2}-k}{k^3-1} & n \equiv 2(mod3) \end{cases}$$

$$ii)\gamma_w(G) = \begin{cases} \sum_{k=1}^{m} \frac{k^{n+3}-1}{k^3-1} & n \equiv 0(mod3) \\ \sum_{k=1}^{m} \frac{k^{n+3}+k^3-k-1}{k^3-1} & n \equiv 1(mod3) \\ \sum_{k=1}^{m} \frac{k^{n+3}+k^3-k^2-1}{k^3-1} & n \equiv 2(mod3) \end{cases}$$

Definition 2.9. The generalized theta graph $\theta(l_1, l_2, l_3, ..., l_p)$ consists in two end-vertices joined by $p \geq 2$ internally vertex disjoint paths with respective lengths $1 \leq l_1 \leq ... \leq l_p$. [16]

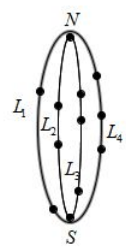

FIGURE 7. Theta Graph $\theta(2, 2, 3, 3)$

Theorem 2.10. *Let $\theta(s_1, s_2, s_3, ..., s_n)$ be a theta graph where $n \geq 3$,($v_{11}, v_{12}, v_{13}, ..., v_{1s_1}$ be vertices of longitude L_1, $v_{21}, v_{22}, v_{23}, ..., v_{2s_2}$ be vertices of longitude L_2 and $v_{n1}, v_{n2}, v_{n3}, ..., v_{ns_n}$ be vertices of longitude L_n). Then,*

$$i)\gamma_s(\theta(s_1, s_2, s_3, ..., s_n)) = \begin{cases} 2 & 1 \leq s_1 = s_2 = s_3 = ... = s_n \leq 2 \\ \left\lceil \frac{s_1-2}{3} \right\rceil + \left\lceil \frac{s_2-2}{3} \right\rceil + ... + \left\lceil \frac{s_n-2}{3} \right\rceil + 2 & 3 \leq s_1 \leq s_2 \leq s_3 \leq ... \leq s_n \end{cases}$$

$$ii)\gamma_w(\theta(s_1, s_2, s_3, ..., s_n)) = \begin{cases} 1 + \left\lceil \frac{s_2}{3} \right\rceil + ... + \left\lceil \frac{s_n}{3} \right\rceil, & s_1 = 1, \quad 1 \leq s_2 = s_3 = ... = s_n \leq 2 \\ 2 + \left\lceil \frac{s_2}{3} \right\rceil + ... + \left\lceil \frac{s_n}{3} \right\rceil, & s_1 = 2, \quad 2 < s_2 = s_3 = ... = s_n \\ 2 + \left\lceil \frac{s_1-2}{3} \right\rceil + \left\lceil \frac{s_2}{3} \right\rceil + ... + \left\lceil \frac{s_n}{3} \right\rceil, & s_1 > 2, \quad 2 < s_2 \leq s_3 \leq ... \leq s_n. \end{cases}$$

Proof. Let $\theta(s_1, s_2, s_3, ..., s_n)$ be a theta graph where $n \geq 3$,($v_{11}, v_{12}, v_{13}, ..., v_{1s_1}$ be vertices of longitude L_1, $v_{21}, v_{22}, v_{23}, ..., v_{2s_2}$ be vertices of longitude L_2 and $v_{n1}, v_{n2}, v_{n3}, ..., v_{ns_n}$ be vertices of longitude L_n). Now we consider a strong domination number of $\theta(s_1, s_2, s_3, ..., s_n)$. There is two cases. Let D be a strong dominating set.

Case1: To find strong domination number where $1 \leq s_1 = s_2 = s_3 = ... = s_n \leq 2$ and $s_1 = s_2 = s_3 = ... = s_n = 1$, since

$$\begin{aligned} \deg(N) &= \deg(S) > \deg(v_{11}) = \deg(v_{12}) = \deg(v_{13}) = ... = \deg(v_{1s_1}) \\ \deg(N) &= \deg(S) > \deg(v_{21}) = \deg(v_{22}) = \deg(v_{23}) = ... = \deg(v_{2s_2}) \\ &\vdots \\ \deg(N) &= \deg(S) > \deg(v_{n1}) = \deg(v_{n2}) = \deg(v_{n3}) = ... = \deg(v_{ns_n}) \end{aligned}$$

then vertex N should be in D. So, vertices $v_{11}, v_{12}, v_{13}, ..., v_{1s_1}$ can be strongly dominated. To strongly dominate vertex S itself should be in D. Therefore $D =$

$\{N, S\}$ and $\gamma_s(\theta(s_1, s_2, s_3, ..., s_n)) \leq 2$.
Assume that $D = \{N, S\}$ not be a minimal strong dominating set. Remove any vertex from D, say S. This means S can not be strongly dominated. Hence our assumption is false.
Then $\gamma_s(\theta(s_1, s_2, s_3, ..., s_n)) \geq 2$. If $s_1 = s_2 = s_3 = ... = s_n = 2$, then

$$\begin{aligned}
\deg(N) &= \deg(S) > \deg(v_{11}) = \deg(v_{12}) = \deg(v_{13}) = ... = \deg(v_{1s_1}) \\
\deg(N) &= \deg(S) > \deg(v_{21}) = \deg(v_{22}) = \deg(v_{23}) = ... = \deg(v_{2s_2}) \\
&\vdots \\
\deg(N) &= \deg(S) > \deg(v_{n1}) = \deg(v_{n2}) = \deg(v_{n3}) = ... = \deg(v_{ns_n}).
\end{aligned}$$

Therefore N and S should be in D to strongly dominate vertices $v_{11}, v_{12}, v_{13}, ..., v_{1s_1}$ and $v_{n1}, v_{n2}, v_{n3}, ..., v_{ns_n}$ respectively. So, $D = \{N, S\}$ and $\gamma_s(\theta(s_1, s_2, s_3, ..., s_n)) \leq 2$.
Assume that $D = \{N, S\}$ not be a minimal strong dominating set. By using some process one can obtain $\gamma_s(\theta(s_1, s_2, s_3, ..., s_n)) \geq 2$.
Thus $\gamma_s(\theta(s_1, s_2, s_3, ..., s_n)) = 2$.

Case2: Let $3 \leq s_1 \leq s_2 \leq s_3 \leq ... \leq s_n$.

$$\begin{aligned}
\deg(N) &= \deg(S) > \deg(v_{11}) = \deg(v_{12}) = \deg(v_{13}) = ... = \deg(v_{1s_1}) \\
\deg(N) &= \deg(S) > \deg(v_{21}) = \deg(v_{22}) = \deg(v_{23}) = ... = \deg(v_{2s_2}) \\
&\vdots \\
\deg(N) &= \deg(S) > \deg(v_{n1}) = \deg(v_{n2}) = \deg(v_{n3}) = ... = \deg(v_{ns_n}).
\end{aligned}$$

In order to strongly dominate vertices $v_{11}, v_{12}, v_{13}, ..., v_{1s_1}$ and $v_{n1}, v_{n2}, v_{n3}, ..., v_{ns_n}$ vertices N and S should be in D, respectively. There is $2n + 2$ number of vertices which are strongly dominated in each longitude. To strongly dominated rest of the vertices in each longitude $\lceil \frac{s_1-2}{3} \rceil, \lceil \frac{s_2-2}{3} \rceil, \lceil \frac{s_3-2}{3} \rceil, ..., \lceil \frac{s_n-2}{3} \rceil$ number of vertices should be in D, respectively. So,

$$\gamma_s(\theta(s_1, s_2, s_3, ..., s_n)) \leq 2 + \left\lceil \frac{s_1 - 2}{3} \right\rceil + \left\lceil \frac{s_2 - 2}{3} \right\rceil + ... + \left\lceil \frac{s_n - 2}{3} \right\rceil.$$

If we remove any vertex in strong dominate set which is mentioned above then this increases strong domination number, then

$$\gamma_s(\theta(s_1, s_2, s_3, ..., s_n)) \geq \left\lceil \frac{s_1 - 2}{3} \right\rceil + \left\lceil \frac{s_2 - 2}{3} \right\rceil + ... + \left\lceil \frac{s_n - 2}{3} \right\rceil + 2.$$

Hence

$$\gamma_s(\theta(s_1, s_2, s_3, ..., s_n)) = \left\lceil \frac{s_1 - 2}{3} \right\rceil + \left\lceil \frac{s_2 - 2}{3} \right\rceil + ... + \left\lceil \frac{s_n - 2}{3} \right\rceil + 2.$$

From these two cases,

$$\gamma_s(\theta(s_1,s_2,s_3,...,s_n)) = \begin{cases} 2, & 1\le s_1 = s_2 = s_3 = ... = s_n \le 2 \\ 2+\left\lceil \frac{s_1-2}{3}\right\rceil + \left\lceil \frac{s_2-2}{3}\right\rceil + ... + \left\lceil \frac{s_n-2}{3}\right\rceil & 3\le s_1 \le s_2 \le s_3 \le ... \le s_n. \end{cases}$$

Now we consider weak domination number of $\theta(s_1,s_2,s_3,...,s_n)$. There is three cases. Let D be a weak dominating set.

Case3: Let $s_1 = s_2 = s_3 = ... = s_n = 1$. In order to weakly dominate to N , S and other vertices of the graph one vertex from each longitude can be chosen. Thus, $\gamma_w(\theta(s_1,s_2,s_3,...,s_n)) \le s_1+s_2+s_3+...+s_n$. This means that $D=\{v_{11},v_{12},v_{13},...,v_{1s_1}\}$. Let $D=\{v_{11},v_{12},v_{13},...,v_{1s_1}\}$ not be minimal weak dominating set. Then removing one vertex from D can be a weak dominating set. Remove v_{11} from D. Then v_{11} can not be weakly dominated by any other vertex in D. So, $\gamma_w(\theta(s_1,s_2,s_3,...,s_n)) \ge s_1+s_2+s_3+...+s_n$. Let $s_1=1$ and $s_2=s_3=...=s_n=2$. In order to weakly dominate vertices N and S vertex in longitude should be chosen. To weakly dominate other vertices in other longitude $\left\lceil \frac{s_2}{3}\right\rceil, \left\lceil \frac{s_3}{3}\right\rceil, \left\lceil \frac{s_4}{3}\right\rceil, ..., \left\lceil \frac{s_n}{3}\right\rceil$ number of vertices should be in D. Then $\gamma_w(\theta(s_1,s_2,s_3,...,s_n)) \le 1+\left\lceil \frac{s_2}{3}\right\rceil+\left\lceil \frac{s_3}{3}\right\rceil+...+\left\lceil \frac{s_n}{3}\right\rceil$. Let D not be a minimal dominating set. By using same idea with previous proofs, $\gamma_w(\theta(s_1,s_2,s_3,...,s_n)) \ge 1+\left\lceil \frac{s_2}{3}\right\rceil+\left\lceil \frac{s_3}{3}\right\rceil+...+\left\lceil \frac{s_n}{3}\right\rceil$. Therefore $\gamma_w(\theta(s_1,s_2,s_3,...,s_n)) = 1+\left\lceil \frac{s_2}{3}\right\rceil+\left\lceil \frac{s_3}{3}\right\rceil+...+\left\lceil \frac{s_n}{3}\right\rceil$.

Case4: Let $s_1=2$ and $2<s_2=s_3=...=s_n$. In order to weakly dominate N and S two vertices of L_1 should be in D. For other vertices $\left\lceil \frac{s_2}{3}\right\rceil, \left\lceil \frac{s_3}{3}\right\rceil, \left\lceil \frac{s_4}{3}\right\rceil, ..., \left\lceil \frac{s_n}{3}\right\rceil$ number of vertices should be in D. Then
$\gamma_s(\theta(s_1,s_2,s_3,...,s_n)) = 2+\left\lceil \frac{s_2}{3}\right\rceil+\left\lceil \frac{s_3}{3}\right\rceil+...+\left\lceil \frac{s_n}{3}\right\rceil$.

Case5: Let $s_1>2$ and $2<s_2\le s_3\le ...\le s_n$. By using same process we obtain $\gamma_w(\theta(s_1,s_2,s_3,...,s_n)) = 2+\left\lceil \frac{s_1-2}{3}\right\rceil+\left\lceil \frac{s_2}{3}\right\rceil+...+\left\lceil \frac{s_n}{3}\right\rceil$
From cases above,

$$\gamma_w(\theta(s_1,s_2,s_3,...,s_n)) = \begin{cases} 1+\left\lceil \frac{s_2}{3}\right\rceil+...+\left\lceil \frac{s_n}{3}\right\rceil, & s_1=1, \quad 1\le s_2=s_3=...=s_n\le 2 \\ 2+\left\lceil \frac{s_2}{3}\right\rceil+...+\left\lceil \frac{s_n}{3}\right\rceil, & s_1=2, \quad 2<s_2=s_3=...=s_n \\ 2+\left\lceil \frac{s_1-2}{3}\right\rceil+\left\lceil \frac{s_2}{3}\right\rceil+...+\left\lceil \frac{s_n}{3}\right\rceil, & s_1>2, \quad 2<s_2\le s_3\le ...\le s_n \end{cases}$$

□

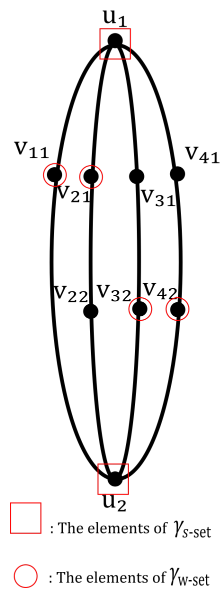

FIGURE 8. Theta Graph $\theta(1,2,2,2)$

$$\gamma_s(\theta(1,2,2,2)) = 2$$
$$\gamma_w(\theta(1,2,2,2)) = 4$$

3. Conclusions

The concept of weak and strong domination in graphs relates dominating sets and degree of vertices. The weak and strong domination number of some graphs are already available in the literature where we have investigated the strong and weak domination number for the comet, double comet, double star and theta graphs. These graphs are important for networks. To obtain similar results for the other graph classes are open areas of research.

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