RAIRO Operations Research
Will be set by the publisher

BOUNDED-DEGREE ROOTED TREE AND TDI-NESS

Herve L.M. Kerivin[1] and Jinhua Zhao[2]

Abstract. This paper contributes to the polyhedral aspect of the Maximum-Weight Bounded-Degree Rooted Tree Problem, where only edge-indexed variables are considered. An initial formulation is given, followed by an analysis of the dimension and a facial study for the polytope. Several families of new valid inequalities are proposed, which enables us to characterize the polytope on trees and cycles with a totally dual integral system.

Keywords: Bounded-Degree Rooted Tree; Polytope; Facets; Total Dual Integrality

Mathematics Subject Classification. 05C85, 52B05, 68R05, 68W01

1. Introduction

Given an undirected graph G with node set V and edge set E, and a specified node r of V, hereafter called the *root*, a *rooted tree* is either the empty graph $(\emptyset, \emptyset)$ or a tree (i.e., a connected and acyclic subgraph) of G containing node r. If a positive integer c_v is associated with each node v of V, then a rooted tree T of G is called *bounded-degree* whenever the degree of each node v in T does not exceed its *degree requirement*, or *capacity*, c_v. This paper deals with the polyhedral structure of the *Bounded-Degree Rooted Tree (BDRT) polytope*, that is, the convex hull of the incidence vectors of edge sets inducing bounded-degree rooted trees of G.

January 28, 2022.

[1] LIMOS CNRS UMR 6158, Universite Clermont Auvergne, Clermont-Ferrand, France; e-mail: `kerivin@isima.fr`

[2] Central China Normal University Wollongong Joint Institute, Faculty of Artificial Intelligence in Education, Central China Normal University, Wuhan 430079, China; e-mail: `jzhao1101@ccnu.edu.cn`

© EDP Sciences 2001

This provisional PDF is the accepted version. The article should be cited as: RAIRO: RO, doi: 10.1051/ro/2022101

To the best of our knowledge, this polytope has not been previously studied in the literature. Actually, the problem of considering bounded-degree rooted trees quite recently arises in the delivery of video streams in under-provisioned peer-to-peer networks, where the scarce resources lie at the peers' level (e.g., average available upload capacity below the stream bit-rate) and not at the links' one [2]. Such peer-to-peer networks usually are represented by non-necessarily complete graphs (due to peering agreements, too-long transmission delays, or too-high jitters) where the video stream's source naturally corresponds to the root and for each peer, its upload-capacity limit can be converted into an upper bound on the number of peers it can send the video stream to [2]. The problem, considered in [2] and called the *Maximum Bounded-Degree Rooted Tree (MBDRT)* problem, then consists of finding a rooted tree which respects the degree constraints and maximizes the number of nodes it contains. In [10], the MBDRT problem was showed to be an NP-hard combinatorial optimization problem by reducing the 3-SAT problem [8] to it, and polynomial-time algorithms were given on certain classes of graphs such as trees, cycles and complete graphs.

If a real-value edge-weight vector is given, the *Maximum-Weight Bounded-Degree Rooted Tree Problem*, hereafter denoted MWBDRTP, consists of finding a maximum-weight subset of edges which induces a bounded-degree rooted tree of G. The NP-hardness of MWBDRTP is twofold since it comes from both the degree restriction and the non-spanning property.

On the one hand, bounded-degree versions of combinatorial optimization problems have received a significant amount of research interest over the last two decades. Several approximation algorithms have been devised for the bounded-degree spanning tree problem (see [14] and the references therein), the bounded-degree Steiner tree problem [7], and bounded-degree matroids and submodular flows [12], but no intensive polyhedral studies seem to exist. Most of these algorithms are based on polyhedral combinatorics [13] in the sense that they use a linear relaxation for the problem to first provide a dual bound and then use the optimal solution to this relaxation to generate a primal solution.

On the other hand, combinatorial optimization problems with the non-spanning property, such as the Steiner tree problem (see [4] and the references therein), the survivable network design problem [11], and the maximum-weight edge-induced connected subgraph problem [5], have been considered in the literature for several decades. The polyhedral structures of these problems have been intensively studied (see [4, 5, 11] and the references therein) and some linear-relaxation based approximation algorithms have been designed (see, e.g., [9]). Besides, the feasibility problem of MWBDRTP is not NP-complete, contrary to the case for other bounded-degree problems such as bounded-degree spanning tree problem and bounded-degree Steiner tree problem.

This paper explores the polyhedral structure of MWBDRTP, and is organized as follows. In Section 2, the Bounded-Degree Rooted Tree (BDRT) polytope, denoted $\mathcal{B}(G, r, \mathbf{c})$, is defined and its dimension is studied. An initial formulation is provided, followed by some necessary and sufficient conditions for the inequalities in the formulation to be facet-defining. Then in Section 3, two families of new valid inequalities are introduced. By using these new valid inequalities, a complete polyhedral characterization of $\mathcal{B}(G, r, \mathbf{c})$ is given in Section 4 on either trees and cycles. In fact, the proposed linear systems not only lead to integral polytopes but also are Totally Dual Integral (TDI) [13].

This introduction is concluded with some definitions and notation, which have been mainly taken from [6] and [13].

Let G be a simple, connected, and undirected graph with node set $V(G)$ and edge set $E(G)$; when there is no confusion on the graph from the context, the graph is labeled as $G = (V, E)$. If $e \in E$ is an edge with extremities u and v, uv is also used to denote e.

Let U be a subset of V. The set of edges having one extremity in U and the other one in $\overline{U} = V \setminus U$ is called a *cut* and is denoted by $\delta(U)$. If $U = \{v\}$ for some $v \in V$, then we write $\delta(v)$ for $\delta(\{v\})$. We denote by $E[U]$ the set of edges having both extremities in U, and $G[U]$ the subgraph induced by U (*i.e.,* $G[U] = (U, E[U])$). Similarly, given $F \subseteq E$, $V[F]$ is used to denote the node set composed of extremities of edges in F, and $G[F]$ the subgraph induced by F (*i.e.,* $G[F] = (V[F], F)$). Given two sets of nodes W and U with $W \subseteq V \setminus U$, the set of edges having one extremity in U and the other one in W is denoted by $[U, W]$. If $\pi = \{V_1, \cdots, V_p\}$, $p \geq 2$, is a partition of V, then we denote by $E(\pi)$ the set of edges having their extremities in different classes of π. For any node $v \in V$, let $N(v) \subseteq V$ denote the set of neighbours of v in G. Besides, in this paper we represent a path by its edge set. For any node $v \in V$, let P_{rv} denote a path between r and v. P_{rv} can also be refereed to as an r-v path. Similarly, for any edge $e \in E$, let P_{re} denote a path between r and e, and P_{re} is also refereed to as an r-e path. Given any edge subset $F \subseteq E$, its *incidence vector* is the vector $\mathbf{x}^F$ in $\{0,1\}^E$ such that $x_e^F = 1$ if and only if $e \in F$. Given any vector $\mathbf{x} \in \mathbb{R}^E$ and any edge set $F \subseteq E$, $\mathbf{x}(F)$ is used for $\sum_{e \in F} x_e$.

Of all the nodes of G, we need to distinguish the non-root nodes having unit degree requirements from others, for any of the former, if present in a bounded-degree rooted tree of G, always appears as a leaf. Thus the set of these unit-capacity nodes is denoted by

$$O = \{v \in V \setminus \{r\} : c_v = 1\}.$$

A node v (edge e, respectively) of G is *unreachable* from root r if there does not exist any bounded-degree r-v path (r-e path, respectively) in G or equivalently, each r-v path (r-e path, respectively) in G contains an inner node in O. Let V_u and E_u be the sets composed of the unreachable nodes and edges of G, respectively. Notice that there might exist edges in E_u whose extremities do not belong to V_u, that is, $\{\delta(u) : u \in V_u\} \subseteq E_u$.

Solving MBDRT problem on G can hence be reduced to solving MBDRT problem on $G' = (V \setminus V_u, E \setminus E_u)$, the graph obtained from G by deleting both unreachable nodes and edges. Notice that getting rid of the unreachable elements in a graph can be performed in linear time by various search algorithms. We therefore make the following assumption for the remainder of the paper.

Assumption 1. *Graph G contains no unreachable nodes or edges.*

2. The BDRT polytope

Let $\mathcal{S} \subseteq \{0,1\}^E$ be the set composed of all the incidence vectors of the edge sets inducing bounded-degree rooted trees of G, that is,

$$\mathcal{S} = \{\mathbf{x}^F \in \{0,1\}^E : G[F] \text{ is a bounded-degree rooted tree of } G\}.$$

The BDRT polytope hence is the convex hull of $\mathcal{S}$ and hereafter is denoted $\mathcal{B}(G, r, \mathbf{c})$. The initially proposed formulation consists of the following inequalities.

$$x_e - \mathbf{x}(\delta(S)) \leq 0 \qquad \text{for all } e \in E[S], S \subseteq V \setminus \{r\}, \tag{1}$$

$$\mathbf{x}(E[S]) \leq |S| - 1 \qquad \text{for all } S \subseteq V, |S| \geq 3, \tag{2}$$

$$\mathbf{x}(\delta(v)) \leq c_v \qquad \text{for all } v \in V, \tag{3}$$

$$x_e \leq 1 \qquad \text{for all } e \in E, \tag{4}$$

$$x_e \geq 0 \qquad \text{for all } e \in E. \tag{5}$$

The *connectivity inequalities* (1) guarantee that each selected edge is connected to root r through a path. The well-known *subtour elimination inequalities* (2) ensure that there is no cycles in the resulting graph [3]. The degree requirement imposed on each node is handled by the *capacity inequalities* (3). The bounds on the variables are guaranteed by the trivial *box inequalities* (4) and (5).

These aforementioned inequalities clearly give a formulation for $\mathcal{B}(G, r, \mathbf{c})$, or equivalently they induce a polytope $P(G, r, \mathbf{c})$ whose integer hull is $\mathcal{B}(G, r, \mathbf{c})$.

Proposition 2.1. *Polytope $P(G, r, \mathbf{c}) = \{\mathbf{x} \in \mathbb{R}^E : \mathbf{x}$ satisfies $(1) - (5)\}$ is a formulation for $\mathcal{B}(G, r, \mathbf{c})$, that is, $P(G, r, \mathbf{c}) \cap \mathbb{Z}^E = \mathcal{B}(G, r, \mathbf{c}) \cap \mathbb{Z}^E$.* □

2.1. Dimension

We first establish a technical lemma that will come in handy in the forthcoming proofs of the dimension and the facets of $\mathcal{B}(G, r, \mathbf{c})$.

Lemma 2.2. *Given an undirected and connected graph $G = (V, E)$, a node $r \in V$, and a node-capacity vector $\mathbf{c}$, let F be any nonempty subset of edges of G. For any $e \in F$, consider an r-e path P_{re} in G that satisfies the capacity constraints and contains as few edges as possible. The vectors in the set $\{\mathbf{x}^{P_{re}} : e \in F\}$ are affinely independent.*

It is worth noting that here we intentionally avoid the notion of shortest path between r and e (or v) in G, since a shortest path in G does not necessarily satisfy the capacity constraints due to the nodes of $\mathcal{O}$.

Proof of Lemma 2.2. G being connected guarantees that P_{re} exists for any edge $e \in F$. Moreover for any two distinct edges $e_1, e_2 \in F$, if $|P_{re_1}| \geq |P_{re_2}|$ then one trivially has $e_1 \notin P_{re_2}$.

Suppose that there exists a non-zero vector $\boldsymbol{\lambda} \in \mathbb{R}^F$ such that

$$\sum_{e \in F} (\lambda_e \mathbf{x}^{P_{re}}) = \mathbf{0}.$$

Let $F_+ = \{e \in F : \lambda_e \neq 0\}$ and let e_m be an edge in F_+ such that $|P_{re_m}| \geq |P_{re}|$ for any $e \in F_+$. One therefore has $e_m \notin P_{re}$ for any $e \in F_+ \setminus \{e_m\}$. Consequently one deduces $\lambda_{e_m} = 0$, a contradiction with $e_m \in F_+$. The vectors in the set $\{\mathbf{x}^{P_{re}} : e \in F\}$ thus are linearly independent and hence affinely independent. □

The next theorem states that $\mathcal{B}(G, r, \mathbf{c})$ is full-dimensional.

Theorem 2.3. $\dim \mathcal{B}(G, r, \mathbf{c}) = |E|$.

Proof. Let $G_r = (V_r, E_r)$ be the connected component containing r of the subgraph $G[V \setminus O]$. By Assumption 1 O is a stable set of G and therefore, $V_r = V \setminus O$ and $E_r = E \setminus \delta(O)$.

Given any $e \in E_r$, let P_{re} be a shortest r-e path in $G[V_r]$. Since G_r contains no nodes in $\mathcal{O}$, P_{re} is a r-e path that satisfies the capacity constraints and contains as few edges as possible. According to Lemma 2.2, the vectors in the set $S_r = \{\mathbf{x}^{P_{re}} : e \in E_r\}$ are affinely independent. Each of these vectors also satisfies the capacity requirement for no unit-capacity nodes are involved in those paths.

Consider any edge $e_o = vv_o \in \delta_G(O)$ where $v_o \in O$ and then $v \in V_r$. Let P_{rv} denote an r-v path in $G[V_r]$ and $P_{re_o} = P_{rv} \cup \{vv_o\}$. All the inner nodes of P_{re_o} have capacity of value at least 2 for they belong to V_r. P_{re_o} then is a bounded-degree rooted tree of G. Clearly the set $S = S_r \cup \{\mathbf{x}^{P_{re_o}} : e_o \in \delta_G(O)\}$ is composed of $|E_r| + |\delta_G(V_r)| = |E|$ affinely independent vectors for $\mathbf{x}^{P_{re_o}}$ is the only vector satisfying $x_{e_o} = 1$.

Combining all these non-zero affinely independent vectors of S with the zero vector gives a set of $|E| + 1$ affinely independent vectors, each of which inducing a bounded-degree rooted tree of G. One therefore has $\dim \mathcal{B}(G, r, \mathbf{c}) \geq |E|$ and our proof is complete for $|E|$ is a trivial upper bound on $\dim \mathcal{B}(G, r, \mathbf{c})$. □

If Assumption 1 was dropped, we would clearly have $\dim \mathcal{B}(G, r, \mathbf{c}) = |E| - |E_u|$ for any incidence vector of the edge set of a bounded-degree rooted tree of G straightforwardly would satisfy the following equations

$$x_e = 0 \text{ for all } e \in E_u. \tag{6}$$

Moreover there would exist a one-to-one correspondence between the facet-defining inequalities of $\mathcal{B}(G, r, \mathbf{c})$ and those of $\mathcal{B}(G', r, \mathbf{c}')$, where $\mathbf{c}'$ is the restriction of $\mathbf{c}$ to E'. In fact, any facet-defining inequality $\mathbf{a}^T\mathbf{x} \leq b$ of $\mathcal{B}(G, r, \mathbf{c})$ could be written as $\mathbf{a}'^T\mathbf{x}' + \sum_{e \in E_u} \lambda_e x_e \leq b$, where $\mathbf{x}'$ is the restriction of $\mathbf{x}$ to E', $\mathbf{a}'^T\mathbf{x}' \leq b$ is a facet-defining inequality of $\mathcal{B}(G', r, \mathbf{c}')$, and $\boldsymbol{\lambda} \in \mathbb{R}^{E_u}$. Consequently in terms of polyhedral characterizations of BDRT polytope, any complete polyhedral characterization of $\mathcal{B}(G, r, \mathbf{c})$ could easily be deduced from any of $\mathcal{B}(G', r, \mathbf{c}')$, and vice-versa.

2.2. Properties of Facet-defining inequalities

Necessary and sufficient conditions for inequalities (1)-(5) to be facet-defining of $\mathcal{B}(G, r, \mathbf{c})$ have been established. We refer to [17] for detailed statements of these conditions and for their proofs. The latter follow standard techniques and are based on the following general properties facet-defining inequalities of $\mathcal{B}(G, r, \mathbf{c})$ must satisfy. These properties will hereafter be used to devise new facet-defining inequalities and manage redundancy in $\mathcal{B}(G, r, \mathbf{c})$ on trees and cycles.

The following lemma describes a property related to the mandatory nonnegative coefficients of the edges of $\delta(O)$ in any facet-defining inequalities.

Lemma 2.4. *Let $\mathbf{a}^T\mathbf{x} \leq b$ be a valid inequality for $\mathcal{B}(G, r, \mathbf{c})$ different from a negative scalar multiple of any nonnegativity inequality* (5)*. Inequality $\mathbf{a}^T\mathbf{x} \leq b$ is facet-defining of $\mathcal{B}(G, r, \mathbf{c})$ only if $a_e \geq 0$ for any edge $e \in \delta(O)$ or any pendant edge e of G.*

Proof. Suppose that $\mathbf{a}^T\mathbf{x} \leq b$ defines a facet $\mathcal{F}$ of $\mathcal{B}(G, r, \mathbf{c})$ and there exists an edge $e_o \in \delta(O)$ such that $a_{e_o} < 0$. (The proof is similar if e_o is a pendant edge of G.)
Theorem 2.3, combined with the assumption on $\mathbf{a}^T\mathbf{x} \leq b$, implies $\mathcal{F} \neq \{\mathbf{x} \in \mathcal{B}(G, r, \mathbf{c}) : x_e = 0\}$ for all $e \in E$. There then must exist an edge set $F \subseteq E$ such that $e_o \in F$, $G[F]$ is a bounded-degree rooted tree of G, and $\mathbf{a}^T\mathbf{x}^{\mathbf{F}} = b$. Edge e_o is incident to a unit-capacity node, so e_o is a pendant edge of $G[F]$. Therefore $G[F \setminus \{e_o\}]$ also is a bounded-degree rooted tree of G. One thus obtains $b \leq \mathbf{a}^T\mathbf{x}^{\mathbf{F}\setminus\{\mathbf{e_o}\}} = \mathbf{a}^T\mathbf{x}^{\mathbf{F}} - a_{e_o} = b - a_{e_o} < b$, a contradiction. □

A necessary and sufficient condition for the root node's capacity inequality

$$\mathbf{x}(\delta(r)) \leq c_r \tag{7}$$

to be facet-defining was given in [17] as stated in the next proposition.

Proposition 2.5. *Inequality* (7) *defines a facet of $\mathcal{B}(G, r, \mathbf{c})$ if and only if $|\delta(r)| > c_r$.* □

The capacity c_r of the root node impacts the possible values the right-hand sides of facet-defining inequalities take. If the root node has unit capacity, then the right-hand side of any facet-defining inequality of $\mathcal{B}(G, r, \mathbf{c})$ but $\mathbf{x}(\delta(r)) \leq c_r$ must equal 0 as stated in the following lemma.

Lemma 2.6. *Let $\mathbf{a}^T\mathbf{x} \leq b$ be a valid inequality for $\mathcal{B}(G, r, \mathbf{c})$ different from a negative scalar multiple of inequality* (7)*. If $c_r = 1$ inequality $\mathbf{a}^T\mathbf{x} \leq b$ is facet-defining of $\mathcal{B}(G, r, \mathbf{c})$ only if $b = 0$.*

Proof. Suppose that $\mathbf{a}^T\mathbf{x} \leq b$ defines a facet $\mathcal{F}$ of $\mathcal{B}(G,r,\mathbf{c})$ and $b \neq 0$. Let $\mathcal{F}_r = \{\mathbf{x} \in \mathcal{B}(G,r,\mathbf{c}) : \mathbf{x}(\delta(r)) = 1\}$. Theorem 2.3, combined with the assumption on $\mathbf{a}^T\mathbf{x} \leq b$, implies $\mathcal{F} \neq \mathcal{F}_r \neq \mathcal{B}(G,r,\mathbf{c})$.

As $c_r = 1$ any bounded-degree rooted tree of G whose edge set is nonempty satisfies $\mathbf{x}(\delta(r)) = 1$, that is, $\mathcal{B}(G,r,\mathbf{c}) \setminus \{\mathbf{0}\} \subseteq \mathcal{F}_r$. Moreover $\mathbf{0} \notin \mathcal{F}$ for $b \neq 0$ and consequently, $\mathcal{F} \subseteq \mathcal{B}(G,r,\mathbf{c}) \setminus \{\mathbf{0}\} \subseteq \mathcal{F}_r$. Combining these inclusions with $\mathcal{F} \neq \mathcal{F}_r \neq \mathcal{B}(G,r,\mathbf{c})$ gives $\mathcal{F} \subsetneq \mathcal{F}_r \subsetneq \mathcal{B}(G,r,\mathbf{c})$, a contradiction with the maximality of $\mathcal{F}$. □

Besides, for those inequalities having only non-negative coefficients, the following lemma can be developed.

Lemma 2.7. *Given a valid inequality* $\mathbf{ax} \leq b$ *for* $\mathcal{B}(G,r,\mathbf{c})$ *with* $\mathbf{a} \geq 0, \mathbf{a} \neq \mathbf{0}, b > 0$*, let* $E^+ := \{e \in E \mid a_e > 0\}$*. If* $r \notin V[E^+]$*, then* $\mathbf{ax} \leq b$ *defines a facet of* $\mathcal{B}(G,r,\mathbf{c})$ *only if*

(1) *there does not exist an edge* $e_b \in E$ *such that it is in each bounded-degree* $r - e$ *path for any edge* $e \in E^+$*, unless* $E^+ = \{e_b\}$*;*
(2) *there does not exist a node* $v_a \in V \setminus \{r\}$ *with* $c_{v_a} = 2$ *such that it is an inner node in each bounded-degree* $r - e$ *path for any edge* $e \in E^+$*.*

Proof. Suppose that there exists a valid inequality $\mathbf{ax} \leq b$ and an edge $e_b \in E$ such that the first condition is not satisfied. Consider any $F \subseteq E$ with its incidence vector $\mathbf{x}^F$ that satisfies $\mathbf{ax} = b > 0$. One must have $e^+ \in F$ for some $e^+ \in E^+$. Since $G[F]$ is a bounded-degree rooted tree, it must contain a bounded-degree $r - e^+$, which has to include e_b. Hence $e_b \in F$, which implies the inequality $x_{e_b} \leq 1$ induces a larger face than the one defined by $\mathbf{ax} \leq b$.

Now suppose there exists a node $v_a \in V \setminus \{r\}$ that is an inner node in every bounded-degree $r-e$ path for any edge $e \in E^+$. For any $F \subseteq E$ with its incidence vector $\mathbf{x}^F$ satisfying $\mathbf{ax} = b$, $\mathbf{x}(\delta(v_a)) = c_{v_a}$ is also satisfied. As a result, $\mathbf{x}(\delta(v_a)) \leq c_{v_a}$ induces a larger face than the one defined by $\mathbf{ax} \leq b$. □

This lemma expresses that in the circumstances where the coefficients are non-negative and the right-hand side of a facet-defining inequality are positive, the associated graph does not contain certain substructures, specifically the bridges or articulation nodes with properties described above. It is worth noting that here the notion of bridges and articulation nodes needs to respect the capacity factor. For instance, an edge might not be a bridge in the graph, but regarding capacity, removing it might lead to the removal of all bounded-degree paths between r and some other edges. In this case, it can be deemed as a bridge regarding capacity.

To illustrate how these lemmas are reflected on previously introduced valid inequalities, we hereafter give an example. According to Lemma 2.4, any connectivity inequality associated with a set $S \subseteq V \setminus \{r\}$ that satisfies $\delta(S) \cap \delta(O) \neq \emptyset$ is not facet-defining, as stated in the following proposition.

Proposition 2.8. *Given* $e \in E[S]$ *with* $S \subseteq V \setminus \{r\}$*, inequality* $x_e - \mathbf{x}(\delta(S)) \leq 0$ *defines a facet of* $\mathcal{B}(G,r,\mathbf{c})$ *only if* $\delta(S) \cap \delta(O) = \emptyset$*.* □

Furthermore, based on Proposition 2.8, we propose a new version of the connectivity inequalities.

$$x_e - \mathbf{x}(\delta(S) \setminus \delta(O)) \leq 0 \text{ for all } e \in E[S], S \subseteq V \setminus \{r\}. \tag{8}$$

It is worth mentioning that (8) covers certain facets which (1) does not. Hence, in latter discussion (8) is always considered instead of (1).

Necessary and sufficient conditions for each family of the inequalities (2)-(8) and their detailed proof can be found in [17].

3. New valid inequalities

Besides the inequalities introduced previously, there are a few sets of new constraints that have been discovered during our work.

Let $\pi = \{S_0, S_1, \cdots, S_k\}$, $k \geq 1$, be a partition of V with $r \in S_0$ and let $M = \{e_1, \cdots, e_k\}$ be a matching of G with $e_i \in E[S_i]$ for all $i \in \{1, \cdots, k\}$. The pair (M, π) is called a *rooted matching-partition* of G. The concept of matching-partition originates from [5] for the connected subgraph problem.

Let $\mathcal{MP}(G)$ denote the set composed of all the rooted matching-partitions of G, and denote by $E(\pi)$ the set of edges having their extremities in different classes of partition π. With any rooted matching-partition $(M, \pi) \in \mathcal{MP}(G)$, one can associate the following rooted matching-partition inequality

$$\mathbf{x}(M) - \mathbf{x}(E(\pi) \setminus \delta(O)) \leq 0. \tag{9}$$

Theorem 3.1. *For any $(M, \pi) \in \mathcal{MP}(G)$, inequality (9) is valid for $\mathcal{B}(G, r, \mathbf{c})$.*

Proof. Assume there exists an integral vector $\mathbf{x}^* \in \mathcal{B}(G, r, \mathbf{c}) \cap \mathbb{Z}^E$ with $\mathbf{x}^*(M) - \mathbf{x}^*(E(\pi) \setminus \delta(O)) \geq 1$. Let the support graph of $\mathbf{x}^*$ be (U, F). Since M is a matching of G, one needs at least $|M \cap F|$ edges among $E(\pi) \setminus \delta(O)$ to connect all the edge in $M \cap F$ with r. Hence, (U, F) is not connected, which forms a contradiction. □

The rooted matching-partition inequalities introduced here are different from those proposed by [5], mainly due to the existence of the root and the capacity constraints.

It is worth noting that the rooted matching-partition inequalities generalizes the connectivity inequalities. Particularly, the connectivity inequalities can be seen as a special case of the rooted matching-partition inequalities where one always has $|M| = 1$. Thus, in the facial study we focus on the cases with $|M| \geq 2$.

In order to facilitate the forthcoming discussion on the facial study results of rooted matching-partition inequalities, some definition needs to be introduced beforehand. Given $(M, \pi) \in \mathcal{MP}(G)$, let G'_π be the graph obtained from G by first removing $E(\pi) \cap \delta(O)$ and then shrinking each $S_i \in \pi$ into a node, and each non-empty edge set $[S_i, S_j] \setminus \delta(O) \subseteq E(\pi) \setminus \delta(O)$ into an edge, for any distinct $i, j \in \{1, \cdots, k\}$. The following theorem gives the necessary and sufficient facet-defining conditions for the rooted matching-partition inequalities.

Theorem 3.2. *Consider a rooted matching-partition* $(M,\pi) \in \mathcal{MP}(G)$ *with* $|M| = k \geq 2$. *Inequality* $\mathbf{x}(M) - \mathbf{x}(E(\pi) \setminus \delta(O)) \leq 0$ *defines a facet of* $\mathcal{B}(G,r,\mathbf{c})$ *if and only if*

(1) $G[S_i \setminus O]$ *is connected for* $i \in \{0, 1, \cdots, k\}$;
(2) G'_π *is 2-connected;*
(3) $E[S_i] \cap \delta(v_o) \setminus \{e_i\} = \emptyset$ *if* $e_i \in \delta(v_o), v_o \in O$ *for* $i \in \{1, \cdots, k\}$;
(4) *there does not exist* $w \in S_i \cap N(u_i) \cap N(v_i)$ *with* $e_i = u_i v_i$ *such that removing* $\{u_i w, v_i w\} \cup \delta(O)$ *from* G *disconnects* e_i *and* r *for* $i \in \{1, \cdots, k\}$;
(5) *there does not exist any* $e \in E[S_i]$, *such that removing* $\{e\} \cup \delta(O)$ *from* G *disconnects* e_i *and* r;
(6) *there does not exist any* $v \in S_i$ *with* $c_v = 2$, *such that removing* $\delta(v) \cup \delta(O)$ *from* G *disconnects* e_i *and* r. □

The proof of the necessity and sufficiency of the conditions can be found in [17]. For each of these conditions, if it were not satisfies, one could always construct another valid inequality that would induce a larger face of $\mathcal{B}(G,r,\mathbf{c})$. For instance, if condition (1) or (2) were not satisfied, one could find a lower bound inequality (5) or another distinct matching-partition inequality that would induce a larger face of $\mathcal{B}(G,r,\mathbf{c})$. Conversely, for the sufficiency, we prove that if all the conditions are satisfied, the face induced by the matching-partition inequality is not a proper face of any other proper faces of $\mathcal{B}(G,r,\mathbf{c})$.

Besides the rooted matching-partition inequalities, another new family of inequalities, the *upload-capacity inequalities*, are also found to be facet-defining for $\mathcal{B}(G,r,\mathbf{c})$. Given a node set $S \subset V \setminus \{r\}$ and a node v in $S \setminus O$, the upload-capacity inequality is defined as follows.

$$\mathbf{x}(\delta(v)) - c_v \mathbf{x}(\delta(S) \setminus \delta(O)) \leq 0 \tag{10}$$

Theorem 3.3. *For any* $v \in S \setminus O$, $S \subset V \setminus \{r\}$, *inequality* (10) *is valid for* $\mathcal{B}(G,r,\mathbf{c})$.

Proof. Assume $\mathbf{x}^* \in \mathcal{B}(G,r,\mathbf{c}) \cap \mathbb{Z}^E$, and $\mathbf{x}^*(\delta(v)) - c_v \mathbf{x}^*(\delta(S) \setminus \delta(O)) \geq 1$. If $\mathbf{x}^*(\delta(S) \setminus \delta(O)) = 0$, the connectivity inequality associated with some edge in $\delta(v)$ is then violated. If $\mathbf{x}^*(\delta(S) \setminus \delta(O)) \geq 1$, the capacity of v is then exceeded by $\mathbf{x}^*$. □

The necessary and sufficient conditions for the upload-capacity inequalities to be facet-defining is described in the following proposition.

Theorem 3.4. *Inequality* (10) *defines a facet of* $\mathcal{B}(G,r,\mathbf{c})$ *if and only if*

(1) $|\delta(v) \setminus (\delta(S) \setminus \delta(O))| \geq c_v, |\delta(v)| \geq c_v + 1$;
(2) $G[S \setminus O]$ *and* $G[\overline{S} \setminus O]$ *are connected respectively;*
(3) *if* $\delta(v) \cap \delta(S) \setminus \delta(O) = \emptyset$ *there does not exist an edge* $e_b \in E[S] \cup \delta(S)$ *such that removing* $\delta(O) \cup \{e_b\}$ *disconnects* v *and* $\overline{S}$;
(4) *if* $\delta(v) \cap \delta(S) \setminus \delta(O) = \emptyset$ *there does not exist a node* $v_a \in S \setminus \{v\}$ *such that* $c_{v_a} = 2$ *and removing* $\delta(O) \cup \delta(v_a)$ *disconnects* v *and* $\overline{S}$. □

Similar to the case of matching-partition inequalities, for each of these conditions, if it is violated, one can always construct another valid inequality that induces a larger face of $\mathcal{B}(G, r, \mathbf{c})$. For instance, if condition (1) or (2) is violated, one can find a connectivity inequality or a lower bound inequality that induces a larger face of $\mathcal{B}(G, r, \mathbf{c})$. Conversely, for the sufficiency, we prove that if all the conditions are satisfied, the face induced by the upload-capacity inequality is not a proper face of any other faces.

For the nodes in O, the presentation of upload-capacity inequalities is slightly different. Given a node set $S \subseteq V \setminus \{r\}$ with $v_o \in S \cap O$, the upload-capacity inequality associated with S and v_o is as follows.

$$\mathbf{x}(\delta(v_o) \setminus \delta(S)) - \mathbf{x}(\delta(S) \setminus \delta(O)) \leq 0 \tag{11}$$

Its validity can also be proved and is stated in the following theorem.

Theorem 3.5. *Inequality* (11) *is valid for* $\mathcal{B}(G, r, \mathbf{c})$.

Proof. Assume $\mathbf{x}^* \in \mathcal{B}(G, r, \mathbf{c}) \cap \mathbb{Z}^E$, and $\mathbf{x}^*(\delta(v_o) \setminus \delta(S)) - \mathbf{x}^*(\delta(S) \setminus \delta(O)) \geq 1$. From the capacity inequality of v_o, one has $\mathbf{x}^*(\delta(v_o) \setminus \delta(S)) \leq \mathbf{x}^*(\delta(v_o)) \leq 1$. Moreover, since $\mathbf{x}^*$ is non-negative, it can be deduced that $\mathbf{x}^*(\delta(v_o) \setminus \delta(S)) = 1$ and $\mathbf{x}^*(\delta(S) \setminus \delta(O)) = 0$. For the edge $e^* \in \delta(v_o) \setminus \delta(S)$ with $x^*_{e^*} = 1$, the connectivity inequality associated with e^* and S is violated. Thus, it forms a contradiction, and hence completes the proof. $\square$

Given $U_S = \{v \in S \setminus O : \delta(v) \cap \delta(S) \neq \emptyset\}$, the necessary and sufficient conditions for inequality (11) to be facet-defining are as follows.

Proposition 3.6. *For any* $v_o \in O$, $S \subseteq V \setminus \{r\}$ *with* $|\delta(v_o) \setminus \delta(S)| \geq 2$, *inequality* (11) *defines a facet of* $\mathcal{B}(G, r, \mathbf{c})$ *if and only if*

(1) $G[S \setminus O]$ *is connected,* $G[\overline{S} \setminus O]$ *is connected;*
(2) *there does not exist an edge* $e \in E[S]$ *such that removing* $e \cup \delta(O \setminus v_o)$ *from* G *disconnects* r *and* v_o*;*
(3) *there does not exist a node* $v \in S \setminus \{v_o\}$ *with* $c_v = 2$ *such that removing* $\delta(v) \cup \delta(O \setminus v_o)$ *from* G *disconnects* r *and* v_o*;*
(4) *if* $E[S \setminus O] \setminus E[U_S] \neq \emptyset$ *there exists* $v \in U_S$ *with* $c_v \geq 3$.

The proof for the necessary and sufficient conditions is similar to the upload-capacity inequalities associated with $v \in S \setminus O$. Detailed proof can be found in [17].

These two aforementioned families of inequalities play an important role in the characterization of $\mathcal{B}(G, c, r)$ on trees and cycles.

4. Characterizations of $\mathcal{B}(G, c, r)$ and TDI-ness

In this section, we show that with the rooted matching-partition inequalities and upload-capacity inequalities being introduced, $\mathcal{B}(G, r, \mathbf{c})$ can be characterized on trees and cycles with TDI systems.

4.1. On trees

4.1.1. *Primal formulation and subproblems*

According to the aforementioned results on valid inequalities and their facet-defining conditions, one can deduce that some of these inequalities is redundant. After getting rid of the redundant inequalities, one can get the following formulation for $\mathcal{B}(G, r, \mathbf{c})$ on trees,

$$
\begin{aligned}
\max\ & \mathbf{wx} & & \\
\text{s.t. } & x_e - x_{f_e} \leq 0 & & \text{for all } e \in E \setminus \delta(r), & (12)\\
& \mathbf{x}(\delta(v)) - c_v x_{f_v} \leq 0 & & \text{for all } v \in V \setminus \{r\}, & (13)\\
& \mathbf{x}(\delta(r)) \leq c_r, & & & (14)\\
& x_e \leq 1 & & \text{for all } e \in \delta(r), & (15)\\
& x_e \geq 0 & & \text{for all } e \text{ is a leaf edge}, & (16)
\end{aligned}
$$

where f_v denotes the edge in $\delta(v)$ and also in the r-v path P_{rv} for $v \in V \setminus \{r\}$ (i.e., $f_v \in \delta(v) \cap P_{rv}$), and f_e denotes the edge that is adjacent to e and also in the r-e path P_{re} for $e \in E \setminus \delta(r)$.

Note that inequalities (12) and inequalities(13) are special cases of connectivity inequalities (8) and upload capacity inequalities (10) respectively. Let the polytope defined by the linear system composed of (12)-(16) be

$$P_T(G, r, \mathbf{c}) = \{\mathbf{x} \in \mathbb{R}^E : \mathbf{x} \text{ satisfies } (12) - (16)\}.$$

We hereafter show that it is a ideal formulation for $\mathcal{B}(G, r, \mathbf{c})$ on trees and that the linear system defining $P_T(G, r, \mathbf{c})$ is TDI. Note that since TDI-ness is a sufficient condition for integrality [18], the integrality of $P_T(G, r, \mathbf{c})$ could be seen as a direct consequence of the next theorem.

Theorem 4.1. *The linear system composed of* (12)-(16) *is TDI.*

Consider the linear program

$$\max\{\mathbf{wx} : \mathbf{x} \in P_T(G, r, \mathbf{c})\}, \tag{17}$$

where $\mathbf{w} \in \mathbb{R}^E$.

Theorem 4.1 is proved by showing that one can always obtain an optimal dual solution to (17) and this solution is integral if $\mathbf{w} \in \mathbb{Z}^E$. We break the proof of Theorem 4.1 into several technical lemmas and propositions, and then provide a proof at the end of this section.

Given any node $v \in V$, let $g(v)$ be the value of a maximum bounded-degree tree rooted at v of the subgraph $G[\lfloor v \rfloor]$ (of G induced by $\lfloor v \rfloor$, the up-closure of v), where the capacity vector $\mathbf{c}^v \in \mathbb{Z}_+^{\lfloor v \rfloor}$ satisfies

$$c_s^v = \begin{cases} c_v - 1 & \text{if } s = v \text{ and } v \neq r, \\ c_v & \text{otherwise.} \end{cases}$$

In other words,

$$g(v) = \max\{\mathbf{x}(F) : G[F] \text{ is a tree of } G[\lfloor v \rfloor] \text{ rooted at } v \text{ and bounded by } \mathbf{c}^v\}.$$

It is straightforward that

$$g(r) = \max\{\mathbf{wx} : \mathbf{x} \in \mathcal{B}(G, r, \mathbf{c})\}. \tag{18}$$

For each node $v \in V$, let $\{v^1, \cdots, v^{q_v}\}$ be the set of nodes in $\lfloor v \rfloor$ adjacent to v, that is,

$$N(v) \cap \lfloor v \rfloor = \{v^1, \cdots, v^{q_v}\}$$

with $q_v \geq 0$. Note that if v is a leaf, then $q = 0$ and $N(v) \cap \lfloor v \rfloor = \emptyset$.

Additionally, for any leaf $v \in V \setminus \{r\}$, it is straightforward to see that $g(v) = 0$.

The following technical lemma is given as a support of our later results.

Lemma 4.2. *If a maximum bounded-degree tree rooted at v contains v^k, $k \in \{1, \cdots, q\}$, it also contains a maximum bounded-degree tree rooted at v^k.*

Proof. Suppose otherwise that a maximum bounded-degree tree T_v rooted at v contains a bounded-degree tree T_{v^k} rooted at v^k which is not maximum. By replacing T_{v^k} in T_v by a maximum bounded-degree tree rooted at v^k, one obviously obtains a bounded-degree tree rooted at v whose weight is larger than T_v. Hence, it contradicts with the assumption. □

Effectively, Lemma 4.2 reduces the MWBDRTP to a series of subproblems, which can be solved with a dynamic programming approach. Details on a dynamic programming algorithm proposed for MWBDRTP on trees can be found in [17]. In this paper, we emphasize on the algorithm that obtains the dual solution.

The following part provides some notation and parameters that will be crucial in the TDI-ness proof.

Given a non-leaf node $v \in V$, for any edge $vv^k \in E$ with $v^k \in N(v) \cap \lfloor v \rfloor$, we define a function

$$h(v^k) = w_{vv^k} + g(v^k).$$

According to Lemma 4.2, the problem of calculating $g(v)$ reduces to

$$\max\{\sum_{v^k \in S} h(v^k) : S \subseteq N(v) \cap \lfloor v \rfloor, |S| \leq c_v^v\}.$$

As it is a maximization problem over a uniform matroid if $g(v^k)$ is known for all $k \in \{1, \cdots, q^v\}$, it can be easily solved by a greedy algorithm in linear time, where at each step one selects a node v^k with the maximum non-negative $h(v^k)$ until there is no such nodes or c_v^v nodes have been selected. Without loss of generality, assume that $h(v^1) \geq h(v^2) \geq \cdots \geq h(v^{t_v}) > 0 \geq h(v^{t_v+1}) \geq \cdots \geq h(v^{q_v})$. Let

$$j_v = \min\{t_v, c_v^v\} \text{ for non-leaf node } v \in V.$$

The following equation holds.

$$g(v) = \sum_{k=1}^{j_v} h(v^k) \text{ for non-leaf node } v \in V. \tag{19}$$

4.1.2. *Dual algorithm and TDI-ness*

For any $e \in E$, let α_e be the dual variable corresponding to inequality (12) and (15) associated with e. For any $v \in V$, let β_v be the dual variable corresponding to inequality (13) and (14) associated with v. The dual linear program of (17) is

$$\min\ c_r\beta_r + \sum_{e\in\delta(r)} \alpha_e$$

$$\text{s.t. } \alpha_e + \beta_{u_e} - \sum_{e'\in\delta(v_e)\setminus\{e\}} \alpha_{e'} - (c_{v_e} - 1)\beta_{v_e} = w_e \quad \text{for all } e \text{ is not a leaf edge,} \tag{20}$$

$$\alpha_e + \beta_{u_e} \geq w_e \quad \text{for all } e \text{ is a leaf edge,} \tag{21}$$

$$\alpha_e, \beta_v \geq 0 \quad \text{for all } e \in E, v \in V, \tag{22}$$

where for any edge $e \in E$, one has $e \in \delta(u_e) \cap E[\lfloor u_e \rfloor]$ and $e \in \delta(v_e) \cap E[\lceil v_e \rceil]$, that is, $e = u_e v_e$ and v_e is the extremity of e the further away from r. Note that the reason why we can only use two sets of dual variables (α and β) is that although we need to define a dual variable for each inequality, (12) and (15) have similar forms and are associated with edges, and (13) and (14) have similar forms and are associated with nodes. As a result, we can combine them into only two sets rather than four.

Hereafter we provide a solution to the dual program, and prove its feasibility and optimality. First of all, the value of β can be first decided as follows

$$\beta_v = \begin{cases} h(v^{j_v}) & \text{if } v \in V, v \text{ is not a leaf and } j_v = c_v^v, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

By the definition of j_v, one can deduce that $\beta_v \geq 0$ for any $v \in V$. For any edge $e = u_e v_e$, let

$$\alpha'_{u_e v_e} = \max\{h(v_e) - \beta_{u_e}, 0\}. \tag{24}$$

Note that for any leaf edge $u_e v_e \in E$, since $\beta_{u_e} = 0$ one has

$$\begin{aligned}\alpha'_{u_e v_e} + \beta_{u_e} &= \max\{h(v_e), 0\}\\ &= \max\{w_{u_e v_e}, 0\}.\end{aligned}$$

Consider a non-leaf node $v \in V$. By the definition of j_v, $j_v \leq c_v^v$ holds, whereas from the definition of β_v, one also has $\beta_v = 0$ if $j_v < c_v^v$. Thus

$$j_v \beta_v = c_v^v \beta_v \tag{25}$$

always holds. For any $i > j_v$, we have that $h(v^i) \leq h(v^{j_v}) = \beta_v$ if $j_v = c_v^v$, whereas $h(v^i) \leq 0 = \beta_v$ if $j_v < c_v^v$. Hence $h(v^i) \leq \beta_v$ holds and thus

$$\alpha'_{vv^i} = 0 \text{ for all } i > j_v. \tag{26}$$

Similarly, for any $i \leq j_v$ it can be deduced from $h(v^i) \geq \beta_v$ that

$$\alpha'_{vv^i} = h(v^i) - \beta_v \geq 0 \text{ for all } i \leq j_v. \tag{27}$$

Therefore, for any non-leaf node $v \in V$, we have

$$\begin{aligned}g(v) &= \sum_{i=1}^{j_v} h(v^i)\\ &= \sum_{i=1}^{j_v} (\alpha'_{vv^i} + \beta_v) + \sum_{i=j_v+1}^{q_v} \alpha'_{vv^i}\\ &= j_v \beta_v + \sum_{i=1}^{q_v} \alpha'_{vv^i}\\ &= c_v^v \beta_v + \sum_{i=1}^{q_v} \alpha'_{vv^i},\end{aligned} \tag{28}$$

where the first equality comes from (19), the second equality is from (26) and (27), and the last equality is from (25).

Immediately, for any non-leaf edge $e = u_e v_e \in E$, one has

$$h(v_e) = w_{u_e v_e} + c_{v_e}^{v_e} \beta_{v_e} + \sum_{i=1}^{q_{v_e}} \alpha'_{v_e v_e^i}.$$

Another result one can deduce from (28) is

$$g(r) = c_r \beta_r + \sum_{i=1}^{q_r} \alpha'_{rr^i}. \tag{29}$$

Now we can construct a solution (α, β) based on (α', β), and then prove that it is dual-feasible, optimal, and is integral if $\mathbf{w}$ is integral.

For each non-leaf edge $e = u_e v_e \in E$, the difference between the left-hand side and right-hand side of the dual constraint (20) associated with e and (α', β) is denoted as

$$\begin{aligned} d^{(\alpha',\beta)}(e) &= \alpha'_{u_e v_e} + \beta_{u_e} - (w_{u_e v_e} + c_{v_e}^{v_e} \beta_{v_e} + \sum_{i=1}^{q_{v_e}} \alpha'_{v_e v_e^i}) \\ &= \max\{h(v_e) - \beta_{u_e}, 0\} + \beta_{u_e} - h(v_e) \\ &= \max\{\beta_{u_e} - h(v_e), 0\}. \end{aligned}$$

Let the set of non-leaf edges that satisfy $d^{(\alpha',\beta)}(e) = \beta_{u_e} - h(v_e) > 0$ be

$$F = \{e \in E : d^{(\alpha',\beta)}(e) > 0, e \text{ is a non-leaf edge}\}.$$

Now we show that there exists a vector $\mathbf{\Delta} \in \mathbb{R}_+^E$ such that $\alpha = \alpha' + \mathbf{\Delta}$ and (α, β) is dual-feasible and optimal. Algorithm 1 computes the vector $\mathbf{\Delta}$.

Algorithm 1: Algorithm on trees to obtain $\mathbf{\Delta}$

Input : Tree $G = (V, E)$ and $h(v)$ for all $v \in V$.
Output: $\mathbf{\Delta}$.
begin
  Set $\mathbf{\Delta} = \mathbf{0}$.
  while $F \neq \emptyset$ **do**
    Take an edge $e = u_e v_e \in F$ such that $P_{ru_e} \cap F = \emptyset$.
    Pick one path $P_{v_e v_l}$ between v_e and any leaf $v_l \in \lfloor v_e \rfloor$.
    For each edge in $e' \in P_{v_e v_l}$
      Set $\Delta_{e'} = \Delta_{e'} + d^{(\alpha',\beta)}(e)$.
    Set $F = F \setminus \{e\}$.

The feasibility and optimality of (α, β) can then be proved.

Proposition 4.3. *(α, β) is an optimal dual solution to* (17).

Proof. For each non-leaf edge $e = u_e v_e$, one clearly has

$$d^{(\alpha,\beta)}(e) = d^{(\alpha',\beta)}(e) + \Delta_{u_e v_e} - \sum_{i=1}^{q_{v_e}} \Delta_{v_e v_e^i}.$$

For any non-leaf edge $e = u_e v_e$, Algorithm 1 guarantees that

$$\Delta_e + d^{(\alpha',\beta)}(e) = \sum_{i=1}^{q_{v_e}} \Delta_{v_e v_e^i}.$$

As a result, one has

$$\begin{aligned} d^{(\alpha,\beta)}(e) &= d^{(\alpha',\beta)}(e) + \Delta_{u_e v_e} - \sum_{i=1}^{q_{v_e}} \Delta_{v_e v_e^i} \\ &= 0. \end{aligned}$$

Hence, all the equations in (20) are satisfied by (α, β).

Furthermore, for any leaf edge $e = u_e v_e \in E$, as $\beta_{u_e} = 0$, $\alpha'_e = \max\{w_e, 0\}$ and $\Delta_e \geq 0$, we then have

$$\begin{aligned} \alpha_e + \beta_{u_e} &= \alpha'_e + \Delta_e \\ &\geq w_e, \end{aligned}$$

which indicates that all inequalities (21) are satisfied.

In addition, for any $e \in E$, $\alpha'_e, \Delta_e \geq 0$ leads to $\alpha_e \geq 0$. Therefore, (α, β) is dual-feasible.

Notice that for any edge $rr^i \in \delta(r)$, Algorithm 1 also guarantees $\Delta_{rr^i} = 0$. Combining with (28) and (29) gives us the following equation

$$\begin{aligned} c_r\beta_r + \sum_{i=1}^{q_r} \alpha_{rr^i} &= c_r\beta_r + \sum_{i=1}^{q_r} \alpha'_{rr^i} \\ &= g(r). \end{aligned}$$

This implies that (α, β) is dual-optimal according to (18). □

Theorem 4.1 can then be proved based on Proposition 4.3.

*Proof of **Theorem** 4.1.* According to Proposition 4.3, (α, β) is an optimal dual solution to (17). Moreover, α and β are obtained by additions and subtractions involving only the components of $\mathbf{w}$. So (α, β) is integral if $\mathbf{w}$ is integral, which completes our proof. □

To summarize, MWBDRTP on trees can be reduced to a series of subproblems, and it thus can be solved using dynamic programming. An enhanced formulation incorporating some of the proposed new constraints has been proved to be TDI by showing that a integral dual optimal solution can always be obtained using a dual algorithm whenever the weights are integral.

Furthermore, in the characterization of $\mathcal{B}(G, r, \mathbf{c})$ on trees, the upload capacity inequalities (10) play a important role, which is present in the form of (13). On the other hand, rooted matching-partition inequalities contribute little to the characterization, as the only facet-defining ones are a subset of connectivity inequalities in the form of (12).

4.2. On cycles

On cycles, the characterization of $\mathcal{B}(G, r, \mathbf{c})$ needs to be considered in the following four different cases depending on the capacity of r and and whether the node set O is empty:

(1) $c_r = 1$ and $O = \{v_o\}$;
(2) $c_r \geq 2$ and $O = \{v_o\}$;
(3) $c_r \geq 2$ and $O = \emptyset$;
(4) $c_r = 1$ and $O = \emptyset$.

Note that according to Assumption 1, $O \leq 1$ holds for cycles.

First of all, some notation needs to be introduced as preparation for the proof. For any edge $e \in E$, let

$$\begin{aligned} \mathcal{M}_e &= \{(M, \pi) \in \mathcal{MP}(G) : e \in M\}, \\ \mathcal{P}_e &= \{(M, \pi) \in \mathcal{MP}(G) : e \in E(\pi) \setminus \delta(O)\}. \end{aligned}$$

Intuitively, $\mathcal{M}_e$ consists of all rooted matching-partitions in which e is present as a matching edge, whereas $\mathcal{P}_e$ consists of all rooted matching-partitions in which e is present in the cut between partitions and is not in $\delta(O)$. Since G is a cycle, we can assume without loss of generality that

$$\begin{aligned} &V = \{r, v_1, \cdots, v_{n-1}\}, \\ &E = \{e_1 = rv_1, e_n = rv_{n-1}\} \cup \{e_i = v_{i-1}v_i : i \in \{2, \cdots, n-1\}\}. \end{aligned}$$

The methodology of the proof is similar to the one for the case on trees. Particularly, for the first case, the same TDI-system composed of (12)-(16) can be used to characterized $\mathcal{B}(G, r, \mathbf{c})$. For the other three cases, an algorithm is devised for each of them to obtain a dual solution that is proved to be optimal and integral if the edge weights are integral.

The proof of the third case is demonstrated in the next part as an example. Proofs for the other cases are constituted of similar process, and can be found in [17].

4.2.1. *Primal and dual formulations*

In the case with $c_r \geq 2$ and $O = \emptyset$, all the capacity constraints (3) are redundant because $|\delta(v)| \leq c_v$ for all $v \in V$. The only pertinent subtour elimination inequality in (2) is $x(E) \leq |E| - 1$, while the others are redundant. Thus, the primal linear program for MWBDRTP which include inequalities (1)-(5) and the rooted matching-partition inequalities (9) can be rewritten as follows.

$$
\begin{aligned}
\max\ & \mathbf{wx} \\
\text{s.t. } & \mathbf{x}(E) \le |E| - 1, && && (30)\\
& \mathbf{x}(M) - \mathbf{x}(E(\pi)) \le 0 && \text{for all } (M,\pi) \in \mathcal{MP}(G), && (31)\\
& x_e \le 1 && \text{for all } e \in E, && (32)\\
& x_e \ge 0 && \text{for all } e \in E. && (33)
\end{aligned}
$$

Therefore the polytope

$$P_C(G, r, \mathbf{c}) = \{\mathbf{x} \in \mathbb{R}^E : \mathbf{x} \text{ satisfies } (30) - (33)\}$$

is a formulation for $\mathcal{B}(G, r, \mathbf{c})$ if G is a cycle, $c_r \ge 2$ and $O = \emptyset$. Hereafter we show that the system composed of (30)-(33) is TDI.

Consider the linear program

$$\max\{\mathbf{wx} : \mathbf{x} \in P_C(G, r, \mathbf{c})\}, \tag{34}$$

where $\mathbf{w} \in \mathbb{R}^E$. Let α be the dual variable corresponding to constraint (30). For any rooted matching-partition $(M, \pi) \in \mathcal{MP}(G)$, let $\beta_{(M,\pi)}$ be the dual variable corresponding to constraint (31) associated with (M, π). For any $e \in E$, let γ_e be the dual variable corresponding to constraint (32) associated with e. The dual linear program of (34) is as follows.

$$
\begin{aligned}
\min\ & (|E| - 1)\alpha + \sum_{e \in E} \gamma_e \\
\text{s.t. } & \alpha + \sum_{(M,\pi) \in \mathcal{M}_e} \beta_{(M,\pi)} - \sum_{(M,\pi) \in \mathcal{P}_e} \beta_{(M,\pi)} + \gamma_e \ge w_e \qquad \text{for all } e \in E, && (35)\\
& \alpha, \beta, \gamma \ge 0. && (36)
\end{aligned}
$$

Given an edge-weight vector $\mathbf{w} \in \mathbb{R}^E$, let the set of edges with positive weights be

$$E_+(\mathbf{w}) = \{e \in E : w_e > 0\},$$

and let the set of edges with negative weights be

$$E_-(\mathbf{w}) = \{e \in E : w_e < 0\},$$

and

$$E_0(\mathbf{w}) = \{e \in E : w_e = 0\}$$

4.2.2. *Dual algorithm*

In order to present the dual algorithm, we introduce a notion called *alternating edge set*. An alternating edge set $F(\mathbf{w})$ regarding the weight vector $\mathbf{w} \in \mathbb{R}^E$ is defined as

$$F(\mathbf{w}) = F_+(\mathbf{w}) \cup F_-(\mathbf{w}),$$

with

$$\begin{aligned} F_+(\mathbf{w}) &= \{e_{j_1}, \cdots, e_{j_q}\}, \\ F_-(\mathbf{w}) &= \{e_{k_1}, \cdots, e_{k_{q+1}}\}, \end{aligned}$$

such that it satisfies the following conditions

$$\begin{aligned} &q \geq 1, \\ &F_+(\mathbf{w}) \subseteq E_+(\mathbf{w}), \\ &F_-(\mathbf{w}) \subseteq E_-(\mathbf{w}), \\ &k_i < j_i < k_{i+1} \text{ for } i \in \{1, \cdots, q\}. \end{aligned}$$

The alternating edge set $F(\mathbf{w})$ can then be written as

$$F(\mathbf{w}) = \{e_{k_1}, e_{j_1}, \cdots, e_{k_q}, e_{j_q}, e_{k_{q+1}}\}.$$

Since in the remaining part of this paper, the weight vector $\mathbf{w}$ is always clear from the context, therefore F (or F_+, F_-, respectively) is used instead of $F(\mathbf{w})$ (or $F_+(\mathbf{w}), F_-(\mathbf{w})$, respectively) for the sake of simplicity.

It is trivial to see that there exists a rooted matching-partition $(M, \pi) \in \mathcal{MP}(G)$ such that $M = F_+$ and $E(\pi) = F_-$. Particularly, one can obtain such a rooted matching-partition by setting $M = F_+$ and $\pi = \{S_0, S_1, \cdots, S_q\}$ with $S_i = \{v_{k_i}, \cdots, v_{k_{i+1}-1}\}$ for $i \in \{1, \cdots, q\}$, and $S_0 = V \setminus (S_1 \cup S_2 \cup \cdots \cup S_q)$. Herein, such rooted matching-partition is referred to as the rooted matching-partition associated with the alternating edge set F.

Algorithm 2 computes an alternating edge set with maximal cardinality. Note that if $F = \emptyset$ is obtained from Algorithm 2, it implies that either $E_-(\mathbf{w}) = \emptyset$ or the connected component containing r in $G[E \setminus E_-(\mathbf{w})]$ also contains all edges in $E_+(\mathbf{w})$.

Based on Algorithm 2, Algorithm 3 is proposed to obtain a dual-feasible solution to (34).

Proposition 4.4. *Algorithm 3 computes a dual-feasible solution to* (34).

Proof. Algorithm 3 guarantees that for any edge $e \in E$, one has

$$w_e = \alpha + \gamma_e - \delta_e + \sum_{(M,\pi) \in \mathcal{M}_e} \beta_{(M,\pi)} - \sum_{(M,\pi) \in \mathcal{P}_e} \beta_{(M,\pi)},$$

Algorithm 2: Algorithm to obtain an alternating edge set on a cycle

Input : Cycle $G = (V, E)$ and $\mathbf{w} \in \mathbb{R}^E$.
Output: Alternating edge set F.
begin
  Set $i = 1, sign = -1$ and $F_+ = F_- = \emptyset$.
  while $i \leq n$ **do**
    if $sign * w_{e_i} > 0$ **then**
      if $w_{e_i} > 0$ **then**
        Add e_i to F_+
      else
        Add e_i to F_-
      Set $sign = sign * -1$
    Set $i = i + 1$
  if $|F_-| \leq 1$ **then**
    Set $F = F_+ = F_- = \emptyset$.
  else
    if $|F_+| = |F_-|$ **then**
      Remove the last edge added to F_+.
    $F = F_+ \cup F_-$.

where $\delta_e = \max\{-w_e^k, 0\}$ for any $e \in E$. Moreover, as all variables are non-negative according to Algorithm 3, it gives us

$$w_e \leq \alpha + \gamma_e + \sum_{(M,\pi)\in\mathcal{M}_e} \beta_{(M,\pi)} - \sum_{(M,\pi)\in\mathcal{P}_e} \beta_{(M,\pi)}.$$

Therefore, (α, β, γ) is a dual-feasible solution to (34). □

Additionally, for any edge $e \in E_0(\mathbf{w})$, one has

$$\sum_{(M,\pi)\in\mathcal{M}_e} \beta_{(M,\pi)} = \sum_{(M,\pi)\in\mathcal{P}_e} \beta_{(M,\pi)} = \gamma_e = \delta_e = 0.$$

Note that $E_0(\mathbf{w}) \neq \emptyset$ implies $\alpha = 0$. Correspondingly, for any edge $e \in E_+(\mathbf{w})$, one has

$$\sum_{(M,\pi)\in\mathcal{P}_e} \beta_{(M,\pi)} = \delta_e = 0,$$

whereas for any edge $e \in E_-(\mathbf{w})$, one has

$$\sum_{(M,\pi)\in\mathcal{M}_e} \beta_{(M,\pi)} = \gamma_e = 0. \quad (37)$$

Algorithm 3: Dual algorithm on cycles with $c_r \geq 2$ and $O = \emptyset$

Input : Cycle $G = (V, E)$ and $\mathbf{w} \in \mathbb{R}^E$.
Output: Dual-feasible solution (α, β, γ).
begin
Set $\alpha = \max\{0, \min\{w_e, e \in E\}\}$.
Initialize $\beta = \mathbf{0}$
Set $w_e^1 = w_e - \alpha$ for all $e \in E$.
Compute an alternating edge set F^1 according to $\mathbf{w}^1$ using Algorithm 2.
Set $k = 1$.
while $F^k \neq \emptyset$ **do**
Set $\beta_{(M^k,\pi^k)} = \min\{|w_e^k| : e \in F^k\}$, where (M^k, π^k) is the rooted matching-partition (M^k, π^k) associated with the alternating edge set F^k.
Set $\mathbf{w}^{k+1}$ as

$$w_e^{k+1} = \begin{cases} w_e^k - \beta_{(M^k,\pi^k)}, & \text{for } e \in M^k, \\ w_e^k + \beta_{(M^k,\pi^k)}, & \text{for } e \in E(\pi^k), \\ w_e^k, & \text{otherwise.} \end{cases}$$

Compute an alternating edge set F^{k+1} according to $\mathbf{w}^{k+1}$ using Algorithm 2.
Set $k = k + 1$.
Set $\gamma_e = \max\{w_e^k, 0\}$ and $\delta_e = \max\{-w_e^k, 0\}$ for all $e \in E$.

4.2.3. *Primal solution and TDI-ness*

Now we are ready to to prove the TDI-ness of the system composed of (30)-(33) by showing that (α, β, γ) is dual-optimal, and is integral if the weights are integral.

Theorem 4.5. *The linear system composed of* (30)-(33) *is TDI.*

Proof. Based on the dual solution, a primal feasible solution can be calculated through Algorithm 4. First, it is straightforward to see E^* induces a bounded-degree rooted tree of G in all the cases.

If $\alpha > 0$, according to Algorithm 3 and Algorithm 4, $E^* = E \setminus \{e\}$ with $\gamma_e = 0$ for some $e \in E$, $\delta_e = 0$ for any $e \in E$, and $\beta_{(M,\pi)} = 0$ for any $(M, \pi) \in \mathcal{MP}(G)$. Hence $\mathbf{x}^{E^*}$ satisfies

$$\begin{aligned} \mathbf{w}\mathbf{x}^{E^*} &= \sum_{e \in E^*} w_e \\ &= \sum_{e \in E^*} (\alpha + \gamma_e) \\ &= (|E| - 1)\alpha + \sum_{e \in E} \gamma_e. \end{aligned}$$

Algorithm 4: Dual algorithm on cycles with $c_r \geq 2$ and $O = \emptyset$

Input : Cycle $G = (V, E)$, weight vector $\mathbf{w} \in \mathbb{R}^E$ and dual-feasible solution (α, β, γ)
Output: Primal-feasible solution $\mathbf{x}^{E^*}$.
begin
  if $\alpha > 0$ **then**
    $E^* = E \setminus \{e\}$, where e is an arbitrary edge with $\gamma_e = 0$.
  else if $\max\{\gamma_e : e \in E\} = 0$ **then**
    Set $E^* = \emptyset$.
  else
    W.l.o.g., let $e = e_a$, $a \in \{1, \cdots, n-1\}$ be the edge with $\gamma_{e_a} > 0$ such that $\gamma_{e_i} \geq 0$ for $i \in \{1, \cdots, a\}$ and a is maximized.
    Then let e_b, $b \in \{a, \cdots, n-1\}$ be the edge in $E_+(\mathbf{w})$ such that $w_{e_i} \geq 0$ for $i \in \{a, a+1, \cdots, b\}$ and b is maximized.
    Let $P_{re_b} = \{e_1, \cdots, e_b\}$.
    Due to the symmetry of cycle, one can reverse the indices of the edges and apply the previous process to obtain $P_{re_{b'}} = \{e_n, e_{n-1}, \cdots, e_{b'}\}$. For the sake of homogeneity, we set $P_{re_{b'}} = \emptyset$ if such path does not exist.
    if $P_{re_b} \cap P_{re_{b'}} = \emptyset$ **then**
      $E^* = P_{re_b} \cup P_{re_{b'}}$.
    else
      $E^* = P_{re_b}$.

Thus, $\mathbf{x}^{E^*}$ and (α, β, γ) are optimal.

If $\alpha = 0$ and $\max\{\gamma_e : e \in E\} = 0$, since $E^* = \emptyset$ according to Algorithm 4, one immediately gets

$$\mathbf{w}\mathbf{x}^{E^*} = (|E| - 1)\alpha + \sum_{e \in E} \gamma_e = 0.$$

Thus, $\mathbf{x}^{E^*}$ and (α, β, γ) are also optimal in this case.

If $\alpha = 0$ and $\max\{\gamma_e : e \in E\} > 0$,

Algorithm 4 guarantees that for all $(M, \pi) \in \mathcal{MP}$ with $\beta_{(M,\pi)} > 0$, the following equation holds,

$$|P_{re_b} \cap E(\pi)| = |P_{re_b} \cap M|.$$

Detailed proof of this equation can be found in [17]. Consequently, one also has

$$|E^* \cap E(\pi)| = |E^* \cap M|,$$

which leads to

$$\sum_{e\in E^*}\Big(\sum_{(M,\pi)\in\mathcal{M}_e}\beta_{(M,\pi)}-\sum_{(M,\pi)\in\mathcal{P}_e}\beta_{(M,\pi)}\Big)=\sum_{(M,\pi)\in\mathcal{MP}(G)}(|E^*\cap M|-|E^*\cap E(\pi)|)\beta_{(M,\pi)}$$
$$=0.$$

Hence it can be deduced that

$$\begin{aligned}(|E|-1)\alpha+\sum_{e\in E}\gamma_e &=\sum_{e\in E}\gamma_e\\ &=\sum_{e\in E^*}\gamma_e\\ &=\sum_{e\in E^*}\Big(w_e-\sum_{(M,\pi)\in\mathcal{M}_e}\beta_{(M,\pi)}+\sum_{(M,\pi)\in\mathcal{P}_e}\beta_{(M,\pi)}\Big)\\ &=\sum_{e\in E^*}w_e\end{aligned}$$

Therefore, $\mathbf{x}^{E^*}$ and (α,β,γ) are always feasible and optimal. Finally, vectors α, β and γ are obtained by additions and subtractions involving only the components of $\mathbf{w}$. So (α,β,γ) is integral if $\mathbf{w}$ is integral, which completes our proof. □

Besides, in [17] it is shown that there exist upload-capacity inequalities and rooted matching-partition inequalities on trees and cycles with Chvátal-Gomory rank at least 2. This indicates that the characterization of $\mathcal{B}(G,r,\mathbf{c})$ on trees and cycles cannot be trivially obtained as the first Chvátal closure of the polytope defined by (1) - (5).

5. Decomposition at the root

Consider a connected graph $G=(V,E)$ where r is an articulation node, such that G is a 1-sum of $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ at r. Given a vector $\mathbf{x}$ in $\mathbb{R}^E$, let $\mathbf{x}^i$ be the restriction of $\mathbf{x}$ to G_i, $i=1,2$. Additionally, let the capacity vector on graph G_i be $\mathbf{c}^i\in\mathbb{Z}^{V_i}$, such that $c^i_v=c_v$ for any $v\in V_i, i=1,2$. The following polytope

$$\begin{aligned}P_R(G,r,\mathbf{c})=&\{\mathbf{x}\in\mathbb{R}^E:\mathbf{x}^1\in\mathcal{B}(G_1,r,\mathbf{c}^1),\mathbf{x}^2\in\mathcal{B}(G_2,r,\mathbf{c}^2),\\ &\mathbf{x}(\delta(r))-c_r\le 0\}\end{aligned}$$

and $\mathcal{B}(G,r,\mathbf{c})$ can be proved to be identical in this case.

Theorem 5.1. $P_R(G,r,\mathbf{c})=\mathcal{B}(G,r,\mathbf{c})$.

Proof. It is straightforward to see that $P_R(G,r,\mathbf{c})\cap\mathbb{Z}^E=\mathcal{B}(G,r,\mathbf{c})\cap\mathbb{Z}^E$, or in other words, a bounded-degree rooted tree of G is composed of two bounded-degree rooted trees of G_1 and G_2 respectively.

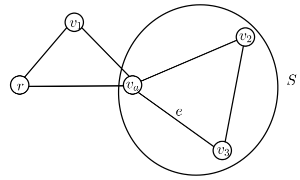

Figure 1. Counter example of decomposition involving two 2-connected components

Assume that there exists a fractional extreme point $\overline{\mathbf{x}}$ in $P_R(G, r, \mathbf{c})$. Let $S(\overline{\mathbf{x}})$ be the linear system of equations that defines $\overline{\mathbf{x}}$. Without loss of generality, assume that $S(\overline{\mathbf{x}})$ contains $|E|$ equations (whose coefficient matrix has full rank).

It can be deduced that $\mathbf{x}(\delta(r)) - c_r = 0$ is in $S(\overline{\mathbf{x}})$. Otherwise, $S(\overline{\mathbf{x}})$ must contain $|E_1|$ and $|E_2|$ equations with respect to G_1 and G_2 respectively. From the integrality of $\mathcal{B}(G_1, r, \mathbf{c}^1)$ and $\mathcal{B}(G_2, r, \mathbf{c}^2)$, one has $\overline{\mathbf{x}}^1$ and $\overline{\mathbf{x}}^2$ are integral and thus $\overline{\mathbf{x}}$ is integral which forms a contradiction.

Therefore, $S(\overline{\mathbf{x}})$ contains $\mathbf{x}(\delta(r)) - c_r = 0$ and other $|E| - 1$ equations with respect to only G_1 or G_2. Without loss of generality, assume that $S(\overline{\mathbf{x}})$ contains $|E_1|$ equations with respect to G_1 and $|E_2| - 1$ equations with respect to G_2, which is denoted by $S_1(\overline{\mathbf{x}})$ and $S_2(\overline{\mathbf{x}})$. Since $\mathcal{B}(G_1, r, \mathbf{c}^1)$ is integral, $S_1(\overline{\mathbf{x}})$ must define an integral point, that is, $\overline{\mathbf{x}}^1$ is integral, and hence $\overline{\mathbf{x}}^1(\delta(r))$ is also integral. Furthermore, since $S_2(\overline{\mathbf{x}})$ and $\mathbf{x}(\delta_{G_2}(r)) - (c_r - \overline{\mathbf{x}}^1(\delta(r))) = 0$ admits a feasible solution $\overline{\mathbf{x}}^2$, there must also exist an integral solution $\mathbf{x}^{2,*}$ which satisfies the same equations. Combining $\overline{\mathbf{x}}^1$ and $\mathbf{x}^{2,*}$ gives us an integral point that satisfies $S(\overline{\mathbf{x}})$ which forms a contradiction.

Thus $P_R(G, r, \mathbf{c})$ is integral and therefore $P_R(G, r, \mathbf{c}) = \mathcal{B}(G, r, \mathbf{c})$. $\square$

On the other hand, if the articulation node is not r, this decomposition will not work as straightforwardly.

Take the graph in Figure 1 as an example. The following inequality

$$x_e - \mathbf{x}(\delta(S)) \leq 0 \tag{38}$$

defines a facet of $\mathcal{B}(G, r, \mathbf{c})$. Inequality (38) has variables associated with edges in both G_1 and edges in G_2. Hence if one wants to decompose G into G_1 and G_2, inequalities such as (38) should be included in addition to the simple combination of polytopes respecting G_1 and G_2.

Similarly, it can also be deduced that the inequality

$$\mathbf{x}(E[S]) - (|S| - 1)\mathbf{x}(\delta(S)) \leq 0$$

defines another facet of $\mathcal{B}(G, r, \mathbf{c})$ and it involves edges in both G_1 and G_2 as well.

Furthermore, consider the graph in Figure 2. Denote an edge set M as $M = \{e_1, e_2, e_3\}$, a partition of V as $\pi = \{S_0, S_1, S_2, S_3\}$. Let the set of edges between

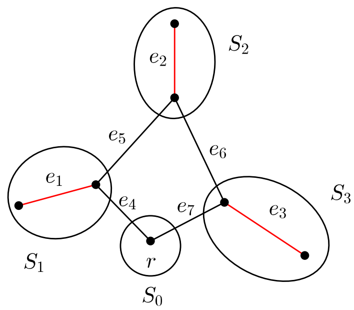

FIGURE 2. Counter example of decomposition involving rooted matching-partition inequalities

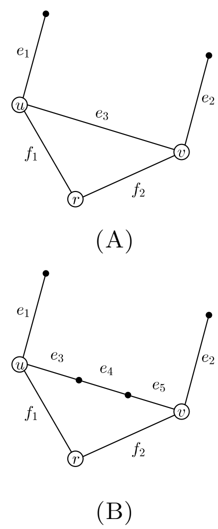

FIGURE 3. Counter example of decomposition involving other inequalities

two different partition classes be $E(\pi) = \{e_4, e_5, e_6, e_7\}$. Even when disregarding the capacity factor, it can be proved that the following rooted matching-partition inequality

$$\mathbf{x}(M) - \mathbf{x}(E(\pi)) \leq 0 \tag{39}$$

defines a facet of $\mathcal{B}(G, r, \mathbf{c})$.

Beside the inequalities introduced previously, there are more inequalities can be found to be facet-defining, which involves the factor of capacity. Given a graph as demonstrated in Figure 3A with $c_u = c_v = 2$ and capacity of any other nodes being sufficiently large, one has an inequality

$$x_{e_1} + x_{e_2} - x_{f_1} - x_{f_2} \leq 0,$$

which is facet-defining.

Additionally, if the edge e_3 is expanded into a path as in Figure 3B, one can also get a facet-defining inequality as the following one.

$$x_{e_1} + x_{e_2} + x_{e_5} - x_{e_3} - x_{f_1} - x_{f_2} \leq 0.$$

Nonetheless, it is found out that the following series of inequalities can be obtained in this case and are all facet-defining.

$$\begin{aligned} x_{e_1} + x_{e_2} + x_{e_3} - x_{e_5} - x_{f_1} - x_{f_2} &\leq 0, \\ x_{e_1} + x_{e_2} + 2x_{e_4} - x_{e_3} - x_{e_5} - x_{f_1} - x_{f_2} &\leq 0, \\ x_{e_1} + x_{e_2} + x_{e_3} + x_{e_5} - 2x_{e_4} - x_{f_1} - x_{f_2} &\leq 0. \end{aligned}$$

It can be noticed that the four aforementioned inequalities only differ in the coefficients of e_3, e_4, and e_5. Furthermore, they do not belong to any set of inequalities that have been introduced previously, and their graphical interpretation is yet to be revealed. Thus, the decomposition over an arbitrary articulation node is hitherto unlikely to work to the best of our knowledge.

6. Concluding remarks

In this paper, the polytope associated with MWBDRTP is studied. The dimension of the polytope is examined first. Several sets of valid inequalities and their facet-defining conditions are discussed. With two families of newly proposed facet-defining inequalities, the polytope is proved to be characterizable with a TDI system in each case on trees and cycles. Additionally, the decomposition of the polytope with respect to the articulation nodes is proved to be feasible if the articulation node is the root.

Besides the aspects examined in this paper, there are a few directions can be further explored for MWBDRTP. On the one hand, the application of MWBDRTP in the telecommunication field considers a packing of potentially more than one rooted trees. This problem is called the Maximum-Weight Bounded-Degree Rooted Tree Packing Problem (MWBDRTPP). Preliminarily, we have looked into the case of 2 rooted trees as the first step. The polyhedral structure turns out to be much more complicated. With a formulation also considering only edge-indexed variables, we have characterized some fractional extreme points in the case where 2 rooted trees are considered and the graph G is a star, which can be cut by the following constraint

$$\mathbf{x}^1(\delta(v)) + \mathbf{x}^2(\delta(v)) - \mathbf{x}^1(\delta(S)) - \mathbf{x}^2(\delta(S)) \leq c_v \text{ for all } v \in S \subseteq V \setminus \{r\},$$

where the superscripts correspond to the index of the rooted trees. Nonetheless, considering a packing of 2 rooted trees, a polynomial-time combinatorial algorithm for MWBDRTPP on trees is proposed in the work with [19].

On the other hand, we have also done some computational testing on different formulations for $\mathcal{B}(G, r, \mathbf{c})$, in order to see how the new inequalities, presented in this paper and some others introduced in [20], affect the performance of a branch-and-cut algorithm on graphs with different properties. Generally, these new inequalities are able to improve the performance of the branch-and-cut algorithm in terms of gap, number of solved instances and running time, although the improvements vary as the graph property changes (e.g., parse graphs vs dense graphs). Detailed results and discussions can be found in [20] (and in [17] as well).

Acknowledgments

We would like to thank the anonymous referees for their helpful comments and suggestions, which help to improve the paper's flow.

References

[1] M. Conforti, G. Cornuéjols, and G. Zambelli, Extended formulations in combinatorial optimization. 4OR-Q J. Oper Res (2010) 1-48.

[2] J. Chakareski, P. Frossard, H. Kerivin, J. Leblet, and G. Simon, A note on the data-driven capacity of p2p networks. Technical Report EPFL-LTS-2009-008, École Polytechnique Fédérale de Lausanne, Switzerland, 2009.

[3] G.B. Dantzig, D.R. Fulkerson, and S.M. Johnson, Solution of a large-scale traveling salesman problem, Operations Research 2 (1954) 393-410.

[4] M. Didi Biha, H. Kerivin, and A.R. Mahjoub, Steiner trees and polyhedra, Discrete Applied Mathematics 112 (2001) 101-120.

[5] M. Didi Biha, H.L.M. Kerivin, and P.H. Ng., Polyhedral study of the connected sugbraph problem, Discrete Mathematics 338 (2015) 80-92.

[6] R. Diestel, Graph Theory, Third ed., Springer-Verlag, Berlin Heidelberg, 2006.

[7] M. Fürer and B. Raghavachari, Approximating the minimum-degree Steiner tree problem to within one of optimal, Journal of Algorithms 17 (1994) 409-423.

[8] M.R. Garey and D.S. Johnson, Computers and intractability: A guide to the theory of NP-completeness. W.H. Freeman and Company, New York, 1979.

[9] K. Jain, A factor 2 approxiamtion algorithm for the generalized Steiner network problem, Combinatorica 21 (2001) 39-60.

[10] H. Kerivin, J. Leblet, G. Simon, and F. Zhou, Maximum bounded rooted-tree packing problem. E-print arXiv:1111.0706, `http://arxiv.org/abs/1111.0706`, 2011.

[11] H. Kerivin and A.R. Mahjoub, Design of survivable networks: A survey, Networks 46 (2005) 1-21.

[12] T. Király, L.C. Lau, and M. Singh, Degree bounded matroids and submodular flows, Combinatorica 32 (2012) 703-720.

[13] A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency, Springer-Verlag, Berlin Heidelberg, 2003.

[14] M. Singh and L. C. Lau, Approximating minimum bounded degree spanning trees to within one of optimal. In Proceedings of the 39th Annual ACM Symposium on Theory of Computing (STOC) (2007) 661-670.

[15] M. X. Goemans, The Steiner tree polytope and related polyhedra, Mathematical Programming 63 (1994) 157-182
[16] G.L. Nemhauser and L.A. Wolsey, Integer and Combinatorial Optimization, Wiley-Interscience, New York, 1988.
[17] J. Zhao, Maximum Bounded Rooted-Tree Problem: Algorithms and Polyhedra, Ph.D. Thesis, LIMOS, Université Clermont Auvergne, 2017.
[18] J. Edmonds and R. Giles, A min-max relation for submodular functions on graphs, Annals of Discrete Mathematics 1 (1977) 185-204.
[19] C. Vinhas de Lima, Le Problème du Packing Borné Maximal de r-Arbre: Etude polyédrale pour le cas du graphe étoile avec 2 arbres, Master's Thesis, Institut Supérieur d'Informatique, de Modélisation et de leurs Applications (ISIMA), 2016,
[20] H. Kerivin and J. Zhao, A computational study on the Maximum-Weight Bounded-Degree Rooted Tree Problem, Applied Mathematics and Computation, Volume 413, 126623, 2022