RAIRO Operations Research
Will be set by the publisher

ON DUALITY THEORY FOR MULTIOBJECTIVE SEMI-INFINITE FRACTIONAL OPTIMIZATION MODEL USING HIGHER ORDER CONVEXITY $^{*,\,**,\,***}$

Tamanna Yadav[1] and S. K. Gupta[2]

Abstract. In the article, a semi-infinite fractional optimization model having multiple objectives is first formulated. Due to the presence of support functions in each numerator and denominator with constraints, the model so constructed is also non-smooth. Further, three different types of dual models viz Mond-Weir, Wolfe and Schaible are presented and then usual duality results are proved using higher-order $(K \times Q) - (\mathcal{F}, \alpha, \rho, d)-$type I convexity assumptions. To show the existence of such generalized convex functions, a nontrivial example has also been exemplified. Moreover, numerical examples have been illustrated at suitable places to justify various results presented in the paper. The formulation and duality results discussed also generalize the well known results appeared in the literature.

Mathematics Subject Classification. 90C29; 90C32; 90C46

Introduction

A semi-infinite model (SIM) is an optimization problem having finite number of variables with the infinite number of constraints. Initially, in 1962, SIM is named by Charnes et al. [1], in which a survey of SIM mainly about a linear model and duality results with convex property have been done. In this direction, some important theorems for the linear model have been generalized using the pairing of finite space of sequences and vector space of finite dimension. Later, the application of SIM in Euclidean space has been shown by Charnes et al. [2] and its implication in duality results for a $n-$dimensional convex minimization problem have been demonstrated. For these convex problems, Karney [3] proposed

---

April 5, 2021.

* The first author is grateful to the Council of Scientific and Industrial Research, India, for financial support, to carry out this work.

** *Keywords:* Semi-infinite programming; Fractional optimization model; Support function; Higher-order; Generalized convexity

*** *Corresponding author: skgiitr@gmail.com*

[1], [2] Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, 247 667, India

This provisional PDF is the accepted version. The article should be cited as: RAIRO: RO, doi: 10.1051/ro/2021064

duality results using its Lagrangian dual. Jeyakumar [10] introduced new constraints qualifications for convex SIM and then developed a strong duality relation. SIM is important for both its results and latent applications in different mathematical fields. It is not only used in the practical problems in which constraints have time or space parameters but also in the areas related to statistics, robotics, transportation problems, game theory and engineering. For more details about significance of SIM, we refer to [8, 12-15, 17]. Ito et al. [4] have derived optimality conditions and duality results for the convex SIM using Slater's constraint qualification. After that, considering constraints over arbitrary cones, Shapiro [7] has developed weak and strong duality relations for convex SIM. Next, Gupta and Srivastava [24] have discussed KKT results for the nonsmooth multiobjective programming problem and then developed usual duality relations. An algorithm based on parametric dual for the quadratic semi-infinite problem have been proposed and convergence of the method is shown in Liu et al. [6]. Further, Basu et al. [16] have discussed the duality gap for SIM with the support dual.

Using generalized $(\eta, \rho)-$invexity, Zalmai and Zhang [9] have developed non-parametric duality relations for semi-infinite discrete minimax fractional problem and further, second-order parameter free duality results are established by Zalmai [15]. After that, Antczak and Zalmai [18] have established second-order relations for semi-infinite minimax type fractional optimization using $(\Phi, \rho) - V-$invexity assumptions. These invexity conditions are later on extended to higher-order in Stancu-Minasian et al. [23]. Considering the same optimization model [18], Verma and Zalmai [27] have studied a parameter free dual model and established duality results using $(\phi, \rho, \theta, \tilde{m})-$sonvexity. Later, the approximate duality relations for nonsmooth minimax fractional optimization model using higher order $B-(p, r)$ invexity have been discussed in Sonali et al. [32].

Mishra and Jaiswal [19] have discussed SIM involving equilibrium constraints and derived optimality conditions with duality results using invexity property. For SIM, considering the concept of convexificators, Pandey and Mishra [20, 25] have proposed necessary as well as sufficient optimality conditions. Further, they formulated Mond-Weir and Wolfe type duals, and proved related theorems with the help of ∂^*-convexity\generalized convexity. Slater's constraints qualification is used for a quasiconvex SIM and then optimality theorems with duality relations are established by Kanzi and Soleimani-damaneh [21]. A fractional semi-infinite problem with $(H_p, R)-$ invexity have been studied in Patel and Patel [22].

Recently, a robust approximation approach is applied in fractional semi-infinite programming and some interesting results for optimality solution with approximation have been established in Zeng et al. [29]. After that, mixed type dual models are formulated and approximate dual relations are discussed for nonlinear SIM in Sun et al. [30, 33]. For a robust vector optimization problem, inspiring from the concept of Quasi $\epsilon-$ solution for SIM in Jiao et al. [31], necessary and sufficient optimality relations between feasible solution and $\epsilon-$ solution are developed in Antczak et al. [34]. Using convex decomposition, optimality conditions and extended duality results are developed for generalized SIM by Aboussoror et al. [36]. In Tung [26, 35], subdifferential in terms of tangential convexity is used for developing KKT and strong KKT optimality results for multiobjective SIM. In terms of invexity and equilibrium constraints, sufficient optimality conditions and duality results for two dual models have been derived in Joshi [37]. Recently, Emam [38] has studied a nonsmooth SIM involving $E-$convexity and support functions, and further established duality results by constructing Mond-Weir type dual model.

Liang et al. [5] have introduced the concept of generalized $(\mathcal{F}, \alpha, \rho, d)$ convexity and further for fractional optimization model, they have derived optimality relations and usual duality theorems. Using the same type of convexity, higher order dual models are formulated and optimality relations are derived for minimax type problems in Ahmad et al. [11]. Motivated by the work in [5, 11, 35], in this paper, we have studied a new class of semi-infinite fractional programming over arbitrary cones. The main outcomes of the paper are briefly explained below:

- *Problem formulation*: A new class of semi-infinite fractional multiple objective problem over arbitrary cones has been formulated. Due to the presence of support functions in each numerator and denominator of the objective function and in each constraint, the problem becomes non-smooth. This not only generalizes all the existing semi-infinite models but also gives infinitely many optimization problems since it involves arbitrary cones.
- *Assumptions*: The concept of higher order $(K \times Q) - (\mathcal{F}, \alpha, \rho, d)-$type I convexity is introduced whose existence is further illustrated by citing a non-trivial example.
- *Dual problems*: Three dual models (Wolfe/Mond-Weir/Schaible) have been constructed and appropriate duality relations have been established under the said assumption.
- *Numerical illustrations*: Various non-trivial examples have been exemplified at suitable places to justify the results obtained in the article. Further, it has been shown by giving examples that without satisfying the assumption of higher order $(K \times Q) - (\mathcal{F}, \alpha, \rho, d)-$type I convexity, the result obtained may not hold.

This paper is organized as : In Section 1, some notations and preliminary results are recalled. Also, the concept of higher order $(K \times Q) - (\mathcal{F}, \alpha, \rho, d)-$type I convexity is introduced and further, a non-trivial example has been demonstrated. In Section 2, 3 and 4, for a class of a non-smooth multiple objective semi-infinite fractional programming problem, higher order Mond-Weir, Wolfe and Schaible type dual models are constructed, and usual duality results are proved under aforesaid assumption. To validate and clarify the duality results, different numerical examples are also shown at suitable places. In the last Section, the conclusion with future scope is given.

1. Preliminaries

Consider the following cone optimization model:

$$\begin{aligned} \textbf{(MP)} \qquad & K - \min\ \psi(x) \\ \text{subject to}\ \ & -\phi(x) \in Q, \\ & x \in B \end{aligned}$$

where $B \subset \mathbb{R}^n$ is open and $\psi : B \to \mathbb{R}^k,\ \phi : B \to \mathbb{R}^m$ are differentiable vector functions. The set $K \subseteq \mathbb{R}^k,\ Q \subseteq \mathbb{R}^m$ are closed convex cones with non-empty interiors and $K \cap -K = \{0\}$. Let $B_0 = \{x \in B : -\phi(x) \in Q\}$ denotes the feasible region of the problem (MP).

Definition 1.1. [5] A point $\tilde{x} \in B_0$ is said to be an efficient (weakly efficient) solution if there exists no $x \in B_0$ such that $\psi(\tilde{x}) - \psi(x) \in K \setminus \{0\}(\text{int } K)$.

Definition 1.2. [11] A functional $\mathcal{F} : B \times B \times \mathbb{R}^n \to \mathbb{R}$ is called sublinear in the third component, if for all $(x, u) \in B \times B$,
(i) $\mathcal{F}(x, u; b_1 + b_2) \leq \mathcal{F}(x, u; b_1) + \mathcal{F}(x, u; b_2),$ for all $b_1,\ b_2 \in \mathbb{R}^n$,
(ii) $\mathcal{F}(x, u; \gamma b) = \gamma \mathcal{F}(x, u; b),$ for all $\gamma \in \mathbb{R}_+$ and $b \in \mathbb{R}^n$.

Definition 1.3. Let $\mathcal{F} : B \times B \times \mathbb{R}^n \to \mathbb{R}$ be a sublinear functional in the third variable. Then the pair $(\psi,\ \phi)$ is called (strictly) higher order $(K \times Q) - (\mathcal{F}, \alpha, \rho, d)-$ type I convex at $\tilde{u} \in \mathbb{R}^n$ with respect to $L : B \times \mathbb{R}^n \to \mathbb{R}^k$ and $S : B \times \mathbb{R}^n \to \mathbb{R}^m$, if for all $x \in B,\ p,\ q \in \mathbb{R}^n$, there exist real valued function $\alpha(.\ ,\ .) : B \times B \to \mathbb{R}_+ \setminus \{0\}$, a function $d = (d_i^{(1)},\ d_j^{(2)}) : B \times B \to \mathbb{R} \times \mathbb{R}$ and a real number $\rho = (\rho_i^{(1)}, \rho_j^{(2)}) \in \mathbb{R} \times \mathbb{R},\ i = 1, 2, ..., k,\ j = 1, 2, ..., m$, such that

$$\begin{aligned}
&\Big(\psi_1(x) - \psi_1(\tilde{u}) - L_1(\tilde{u}, p_1) + p_1^T \nabla_{p_1} L_1(\tilde{u}, p_1) - \mathcal{F}_{x,\tilde{u}}\big[\alpha(x,\tilde{u})\big(\nabla_x \psi_1(\tilde{u}) + \nabla_{p_1} L_1(\tilde{u}, p_1)\big)\big] \\
&\quad - \rho_1^{(1)}\ \big(d_1^{(1)}(x,\tilde{u})\big)^2, ..., \psi_k(x) - \psi_k(\tilde{u}) - L_k(\tilde{u}, p_k) + p_k^T \nabla_{p_k} L_k(\tilde{u}, p_k) \\
&\quad - \mathcal{F}_{x,\tilde{u}}\big[\alpha(x,\tilde{u})\big(\nabla_x \psi_k(\tilde{u}) + \nabla_{p_k} L_k(\tilde{u}, p_k)\big)\big] - \rho_k^{(1)}\ \big(d_k^{(1)}(x,\tilde{u})\big)^2\Big) \in K\ (K \setminus \{0\}) \text{ and} \\
&\Big(- \phi_1(\tilde{u}) - S_1(\tilde{u}, q_1) + q_1^T \nabla_{q_1} S_1(\tilde{u}, q_1) - \mathcal{F}_{x,\tilde{u}}\big[\alpha(x,\tilde{u})\big(\nabla_x \phi_1(\tilde{u}) + \nabla_{q_1} S_1(\tilde{u}, q_1)\big)\big] \\
&\quad - \rho_1^{(2)}\ \big(d_1^{(2)}(x,\tilde{u})\big)^2, ..., -\phi_m(\tilde{u}) - S_m(\tilde{u}, q_m) + q_m^T \nabla_{q_m} S_m(\tilde{u}, q_m) \\
&\quad - \mathcal{F}_{x,\tilde{u}}\big[\alpha(x,\tilde{u})\big(\nabla_x \phi_m(\tilde{u}) + \nabla_{q_m} S_m(\tilde{u}, q_m)\big)\big] - \rho_m^{(2)}\ \big(d_m^{(2)}(x,\tilde{u})\big)^2\Big) \in Q\ (Q \setminus \{0\}).
\end{aligned}$$

Next, we will show a non-trivial example to illustrate the existence of such functions.

Example 1. In problem (MP), let the functions $\psi : \mathbb{R} \to \mathbb{R}^2,\ \phi : \mathbb{R} \to \mathbb{R}^2,\ L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2,\ S : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ be given by:

$$\begin{aligned}
&\psi = (\psi_1,\ \psi_2) = (x^2 + 1,\ x^2 - 1),\ \ \phi = (\phi_1,\ \phi_2) = (x^2 - 1,\ x^4 - 1), \\
&L(u,\ p) = (L_1(u,\ p_1),\ L_2(u,\ p_2)) = (-2p_1 u^2,\ -p_2(u + 1)), \\
&S(u,\ q) = (S_1(u,\ q_1),\ S_2(u,\ q_2)) = (-q_1 u^2 + \frac{5}{2},\ -4q_2 u^2).
\end{aligned}$$

Let $d_1^{(1)}(x,y) = d_1^{(2)}(x,y) = x - y,\ d_2^{(1)}(x,y) = d_2^{(2)}(x,y) = x + y$ and $\alpha(x,\ y) = x^2 + 1$. Let $\mathcal{F}_{x,u}(b) = b(x^2 - u^2),\ K = \{(x,\ y) \in \mathbb{R}^2\ :\ x \leq 0,\ y \geq x\}$ and $Q = \{(x,\ y) \in \mathbb{R}^2\ :\ y \geq 0,\ y \geq x\}$. Now, $-\phi(x) \in Q$ implies $x^2 \leq 1$, therefore $-1 \leq x \leq 1$. Hence, the feasible region of the problem (MP) is $B_0 = [-1,\ 1]$. Next, for all $x \in \mathbb{R},\ p_1,\ p_2,\ q_1,\ q_2 \in \mathbb{R}$ and for $\rho_1^{(1)} = 1,\ \rho_2^{(1)} = -1$, we have

$$\begin{aligned}
&\Big(\psi_1(x) - \psi_1(u) - L_1(u,p_1) + p_1^T \nabla_{p_1} L_1(u,p_1) - \mathcal{F}_{x,u}\big[\alpha(x,u)\big(\nabla_x \psi_1(u) + \nabla_{p_1} L_1(u,p_1)\big)\big] \\
&\quad - \rho_1^{(1)}\ (d_1^{(1)}(x,u))^2,\ \psi_2(x) - \psi_2(u) - L_2(u,p_2) + p_2^T \nabla_{p_2} L_2(u,p_2) \\
&\quad - \mathcal{F}_{x,u}\big[\alpha(x,u)\big(\nabla_x \psi_2(u) + \nabla_{p_2} L_2(u,p_2)\big)\big] - \rho_2^{(1)}\ (d_2^{(1)}(x,u))^2\Big) \\
&= \begin{cases} \big(0,\ 3x^2 + x^4\big) \in K & \text{at } u = 0, \\ \big(2x - 2,\ 2x(x+1)\big) \in K \setminus \{0\} & \text{at } u = 1 \end{cases}
\end{aligned}$$

and also for all $x \in \mathbb{R},\ \rho_1^{(2)} = 1,\ \rho_2^{(2)} = -1$, we obtain

$$\begin{aligned}
&\Big(- \phi_1(u) - S_1(u,q_1) + q_1^T \nabla_{q_1} S_1(u,q_1) - \mathcal{F}_{x,u}\big[\alpha(x,u)\big(\nabla_x \phi_1(u) + \nabla_{q_1} S_1(u,q_1)\big)\big] \\
&\quad - \rho_1^{(2)}\ (d_1^{(2)}(x,u))^2,\ -\phi_2(u) - S_2(u,q_2) + q_2^T \nabla_{q_2} S_2(u,q_2) \\
&\quad - \mathcal{F}_{x,u}\big[\alpha(x,u)\big(\nabla_x \phi_2(u) + \nabla_{q_2} S_2(u,q_2)\big)\big] - \rho_2^{(2)}\ (d_2^{(2)}(x,u))^2\Big) \\
&= \begin{cases} \Big(- x^2 - \dfrac{3}{2}, 1 + x^2\Big) \in Q & \text{at } u = 0, \\ \Big(- x^4 - x^2 + 2x - \dfrac{5}{2},\ (1+x)^2\Big) \in Q \setminus \{0\} & \text{at } u = 1. \end{cases}
\end{aligned}$$

Hence, the pair $(\psi,\ \phi)$ is higher order $(K\times Q)-(\mathcal{F},\alpha,\rho,d)-$ type$-I$ convex at $u=0$ and $(\psi,\ \phi)$ is strictly higher order $(K\times Q)-(\mathcal{F},\alpha,\rho,d)-$ type I convex at $u=1$.

Definition 1.4. [30] Let $f:\mathbb{R}^n\to\mathbb{R}$ be a convex function. Then, the subdifferential of f at $\tilde{x}$ is defined as

$$\partial f(\tilde{x})=\{\bar{\nu}\in\mathbb{R}^n\ :\ f(x)-f(\tilde{x})\geq\langle\bar{\nu},\ x-\tilde{x}\rangle,\ \text{for all } x\in\mathbb{R}^n\}.$$

Definition 1.5. [28] The support function of a compact convex set $\mathcal{A}\subseteq\mathbb{R}^n$ is defined as

$$\Omega(x|\mathcal{A})=\max\{x^Ty\ :\ y\in\mathcal{A}\}.$$

The subdifferential of support function $\Omega(x|\mathcal{A})$ at $\tilde{x}$ is given by

$$\partial\Omega(\tilde{x}|\mathcal{A})=\{\bar{\nu}\in\mathbb{R}^n\ :\ \Omega(\tilde{x}|\mathcal{A})=\bar{\nu}^T\tilde{x}\}.$$

Now, consider the semi-infinite multiobjective fractional programming problem as follows:

$$\textbf{(SIFP)}\qquad K-\min\ \frac{f(x)}{g(x)}=\left(\frac{f_1(x)+\Omega(x|C_1)}{g_1(x)-\Omega(x|D_1)},\dots,\frac{f_k(x)+\Omega(x|C_k)}{g_k(x)-\Omega(x|D_k)}\right)$$
$$\text{subject to}\ \ -[h_j(x,\ t)+\Omega(x|E_j)+\Omega(t|M_j)]\in Q,\quad \text{for all } t\in T$$

where $i\in\tilde{I}=\{1,2,...,k\}$, $j\in\tilde{J}=\{1,2,...,m\}$, $f:\mathbb{R}^n\to\mathbb{R}^k$, $g:\mathbb{R}^n\to\mathbb{R}^k$, $h:\mathbb{R}^n\to\mathbb{R}^m$ are continuously differentiable functions and for compact convex sets $C_i,\ D_i,\ E_j$ and M_j in $\mathbb{R}^n$, respective support functions are $\Omega(x|C_i),\ \Omega(x|D_i),\ \Omega(x|E_j)$ and $\Omega(x|M_j)$ for $i\in\tilde{I},\ j\in\tilde{J}$. Also, assume that $f_i(.)+\Omega((.)|C_i)\geq 0$ and $g_i(.)-\Omega((.)|D_i)>0$ for all feasible x and T is an infinite index set.
Suppose $S_0=\{x\in\mathbb{R}^n\ :-[h_j(x,\ t)+\Omega(x|E_j)+\Omega(t|M_j)]\in Q,\ \text{for all } t\in T\}$ represents the feasible region of the problem (SIFP). Let K^* and Q^* be positive dual cones of K and Q, respectively.

Following the lines of Debnath and Gupta [28], we now state the following necessary Karush-Kuhn Tucker condition for (SIFP):

Theorem 1.6. *Let $\tilde{x}\in B\subseteq\mathbb{R}^n$ be a weakly efficient point of (SIFP) and a suitable constraint qualification be fulfilled at $\tilde{x}$. Then, there exist $(\lambda,\ \mu)\in int\,K^*\times int\,Q^*$, $(\lambda,\mu)\neq(0,0)$ and $t\in T$ such that*

$$0\in\partial\left(\sum_{i=1}^{k}\lambda_i\left(\frac{f_i(\tilde{x})+\Omega(\tilde{x}|C_i)}{g_i(\tilde{x})-\Omega(\tilde{x}|D_i)}\right)+\sum_{j=1}^{m}\mu_j\Big(h(\tilde{x},t)+\Omega(\tilde{x}|E_j)+\Omega(t|M_j)\Big)\right)\ \textit{and}$$

$$\sum_{j=1}^{m}\mu_j\big(h(\tilde{x},t)+\Omega(\tilde{x}|E_j)+\Omega(t|M_j)\big)=0.$$

2. Mond-Weir Type Dual

For (SIFP) model, consider the following Mond-Weir type higher order dual:

$$\textbf{(MD)}\quad K-\max\left(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1},\dots,\frac{f_k(u)+u^Tz_k}{g_k(u)-u^Tv_k}\right)$$

subject to

$$\sum_{i=1}^{k}\lambda_i\Big[\nabla_x\Big(\frac{f_i(u)+u^Tz_i}{g_i(u)-u^Tv_i}\Big)+\nabla_{p_i}L_i(u,\ p_i)\Big]$$
$$+\sum_{j=1}^{m}\mu_j\Big[\nabla_x\big(h_j(u,\tau)+u^Tw_j^1+\tau^Tw_j^2\big)+\nabla_{q_j}S_j(u,\ q_j)\Big]=0, \tag{1}$$

$$\sum_{j=1}^{m}\mu_j\Big[h_j(u,\tau)+u^Tw_j^1+\tau^Tw_j^2+S_j(u,\ q_j)-q_j^T\nabla_{q_j}S_j(u,\ q_j)\Big]\geq 0, \tag{2}$$

$$\sum_{i=1}^{k}\lambda_i[L_i(u,\ p_i)-p_i^T\nabla_{p_i}L_i(u,\ p_i)]\geq 0, \tag{3}$$

$z_i\in C_i,\ v_i\in D_i,\ w_j^1\in E_j,\ w_j^2\in M_j,\ i\in\tilde{I},\ j\in\tilde{J},\ (\lambda,\ \mu)\in \operatorname{int}K^*\times\operatorname{int}Q^*,\ (\lambda,\ \mu)\neq(0,\ 0)$ and $\tau\in T$.

Theorem 2.1. [*Weak duality*] *Assume that x and $(u,\ v,\ w^1,\ w^2,\ \lambda,\ \mu,\ z,\ p,\ q)$ be feasible for the problems (SIFP) and (MD), respectively. Let a sublinear functional (in third variable) be $\mathcal{F}:B\times B\times\mathbb{R}^n\to\mathbb{R}$. Let*

(i) $\Big(\Big(\frac{f_1(.)+(.)^Tz_1}{g_1(.)-(.)^Tv_1},...,\frac{f_k(.)+(.)^Tz_k}{g_k(.)-(.)^Tv_k}\Big),\ \Big(h_1(.,\ \tau)+(.)^Tw_1^1,...,h_m(.,\ \tau)+(.)^Tw_m^1\Big)\Big)$ *be higher order* $(K\times Q)-(\mathcal{F},\alpha,\rho,d)$ *−type I convex at u with respect to L and S,*

(ii) $K\supseteq\mathbb{R}_+^k,\ Q\supseteq\mathbb{R}_+^m$ *and*

(iii) $\sum_{i=1}^{k}\lambda_i\rho_i^{(1)}\big(d_i^{(1)}(x,u)\big)^2+\sum_{j=1}^{m}\mu_j\{\rho_j^{(2)}\big(d_j^{(2)}(x,u)\big)^2-\tau^Tw_j^2\}\geq 0.$

Then

$$\left(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1},...,\frac{f_k(u)+u^Tz_k}{g_k(u)-u^Tv_k}\right)-\left(\frac{f_1(x)+\Omega(x|C_1)}{g_1(x)-\Omega(x|D_1)},...,\frac{f_k(x)+\Omega(x|C_k)}{g_k(x)-\Omega(x|D_k)}\right)\notin K\setminus\{0\}. \tag{4}$$

Proof. By hypothesis (i) and Definition 1.3 at u with respect to $L:B\times\mathbb{R}^n\to\mathbb{R}^k$ and $S:B\times\mathbb{R}^n\to\mathbb{R}^m$, we have

$$\Bigg(\frac{f_1(x)+x^Tz_1}{g_1(x)-x^Tv_1}-\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1}-L_1(u,p_1)+p_1^T\nabla_{p_1}L_1(u,p_1)-\mathcal{F}_{x,u}\Big[\alpha(x,u)\Big(\nabla_x\Big(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1}\Big)$$
$$+\nabla_{p_1}L_1(u,p_1)\Big)\Big]-\rho_1^{(1)}\big(d_1{}^{(1)}(x,u)\big)^2,...,\frac{f_k(x)+x^Tz_k}{g_k(x)-x^Tv_k}-\frac{f_k(u)+u^Tz_k}{g_k(u)-u^Tv_k}-L_k(u,p_k)+p_k^T\nabla_{p_k}L_k(u,p_k)$$
$$-\mathcal{F}_{x,u}\Bigg[\alpha(x,u)\Bigg(\nabla_x\Bigg(\frac{f_k(u)+u^Tz_k}{g_k(u)-u^Tv_k}\Bigg)+\nabla_{p_k}L_k(u,p_k)\Bigg)\Bigg]-\rho_k{}^{(1)}\big(d_k{}^{(1)}(x,u)\big)^2\Bigg)\in K \tag{5}$$

and

$$\Big(-\big(h_1(u,\tau)+u^Tw_1^1\big)-S_1(u,q_1)+q_1^T\nabla_{q_1}S_1(u,q_1)-\mathcal{F}_{x,u}\big[\alpha(x,u)\big(\nabla_x(h_1(u,\tau)+u^Tw_1^1)$$
$$+\nabla_{q_1}S_1(u,q_1)\big)\big]-\rho_1^{(2)}\ (d_1^{(2)}(x,u))^2,...,-\big(h_m(u,\tau)+u^Tw_m^1\big)-S_m(u,q_m)+q_m^T\nabla_{q_m}S_m(u,q_m)$$
$$-\mathcal{F}_{x,u}\big[\alpha(x,u)\big(\nabla_x(h_m(u,\tau)+u^Tw_m^1)+\nabla_{q_m}S_m(u,q_m)\big)\big]-\rho_m^{(2)}\ \big(d_m^{(2)}(x,u)\big)^2\Big)\in Q. \tag{6}$$

It follows from $\lambda \in \operatorname{int} K^*$ and (5) that

$$\sum_{i=1}^{k} \lambda_i \Bigg[\frac{f_i(x)+x^T z_i}{g_i(x)-x^T v_i} - \frac{f_i(u)+u^T z_i}{g_i(u)-u^T v_i} - L_i(u,p_i) + p_i^T \nabla_{p_i} L_i(u,p_i) \\ - \mathcal{F}_{x,u}\Bigg[\alpha(x,u) \Bigg(\nabla_x \Bigg(\frac{f_i(u)+u^T z_i}{g_i(u)-u^T v_i} \Bigg) + \nabla_{p_i} L_i(u,p_i) \Bigg) \Bigg] - \rho_i^{(1)} \big(d_i^{(1)}(x,u) \big)^2 \Bigg] \geq 0.$$

Using the sublinearity property of $\mathcal{F}$, $\lambda \in \operatorname{int} K^* \subseteq \operatorname{int} \mathbb{R}_+^k$ and dual constraint (3), we get

$$\sum_{i=1}^{k} \lambda_i \Bigg[\frac{f_i(x)+x^T z_i}{g_i(x)-x^T v_i} - \frac{f_i(u)+u^T z_i}{g_i(u)-u^T v_i} \Bigg] \\ \geq \mathcal{F}_{x,u}\Bigg[\alpha(x,u) \sum_{i=1}^{k} \lambda_i \Bigg\{ \nabla_x \Bigg(\frac{f_i(u)+u^T z_i}{g_i(u)-u^T v_i} \Bigg) + \nabla_{p_i} L_i(u,p_i) \Bigg\} \Bigg] + \sum_{i=1}^{k} \lambda_i \rho_i^{(1)} \big(d_i^{(1)}(x,u) \big)^2. \tag{7}$$

Now, from hypothesis (ii), $\mu \in \operatorname{int} Q^* \subseteq \operatorname{int} \mathbb{R}_+^m$, it follows from (6) that

$$\sum_{j=1}^{m} \mu_j \Bigg[- \big(h_j(u,\tau) + u^T w_j^1 \big) - S_j(u,q_j) + q_j^T \nabla_{q_j} S_j(u,q_j) \\ - \mathcal{F}_{x,u}\big[\alpha(x,u) \big(\nabla_x (h_j(u,\tau) + u^T w_j^1) + \nabla_{q_j} S_j(u,q_j) \big) \big] - \rho_j^{(2)} \ \big(d_j^{(2)}(x,u) \big)^2 \Bigg] \geq 0.$$

Using $\mu > 0$, along with sublinearity of $\mathcal{F}$, the above inequality gives

$$\sum_{j=1}^{m} \mu_j \Big[- \big(h_j(u,\tau) + u^T w_j^1 \big) - S_j(u,q_j) + q_j^T \nabla_{q_j} S_j(u,q_j) \Big] \\ \geq \mathcal{F}_{x,u}\Bigg[\alpha(x,u) \sum_{j=1}^{m} \mu_j \big\{ \nabla_x (h_j(u,\tau) + u^T w_j^1) + \nabla_{q_j} S_j(u,q_j) \big\} \Bigg] + \sum_{j=1}^{m} \mu_j \rho_j^{(2)} \ \big(d_j^{(2)}(x,u) \big)^2. \tag{8}$$

Further, using inequality (2) in the addition of (7) and (8), we obtain

$$\sum_{i=1}^{k} \lambda_i \Bigg[\frac{f_i(x)+x^T z_i}{g_i(x)-x^T v_i} - \frac{f_i(u)+u^T z_i}{g_i(u)-u^T v_i} \Bigg] \\ \geq \mathcal{F}_{x,u}\Bigg[\alpha(x,u) \sum_{i=1}^{k} \lambda_i \Bigg\{ \nabla_x \Bigg(\frac{f_i(u)+u^T z_i}{g_i(u)-u^T v_i} \Bigg) + \nabla_{p_i} L_i(u,p_i) \Bigg\} \Bigg] + \mathcal{F}_{x,u}\big[\alpha(x,u) \sum_{j=1}^{m} \mu_j \big\{ \nabla_x (h_j(u,\tau) \\ + u^T w_j^1) + \nabla_{q_j} S_j(u,q_j) \big\} \big] + \sum_{i=1}^{k} \lambda_i \rho_i^{(1)} \big(d_i^{(1)}(x,u) \big)^2 + \sum_{j=1}^{m} \mu_j \{ \rho_j^{(2)} \big(d_j^{(2)}(x,u) \big)^2 - \tau^T w_j^2 \}.$$

It follows from assumption (iii), dual constraint (1), sublinearity of $\mathcal{F}$ and $\mathcal{F}_{x,u}(0) = 0$ that

$$\sum_{i=1}^{k} \lambda_i \left[\frac{f_i(x)+x^T z_i}{g_i(x)-x^T v_i} - \frac{f_i(u)+u^T z_i}{g_i(u)-u^T v_i} \right] \geq 0. \tag{9}$$

Now, on the contrary, suppose that (4) is not correct. Then, $\lambda \in \text{int } K^*$ implies

$$\sum_{i=1}^{k} \lambda_i \left[\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} - \frac{f_i(x) + \Omega(x|C_i)}{g_i(x) - \Omega(x|D_i)} \right] > 0.$$

Finally, since $x^T z_i \leq \Omega(x|C_i)$, $x^T v_i \leq \Omega(x|D_i)$ and $\lambda_i > 0$, for all i, therefore

$$\sum_{i=1}^{k} \lambda_i \left[\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} - \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \right] > 0$$

which contradicts the inequality (9). This completes the proof. □

Theorem 2.2. [*Strong duality*] *Let $\breve{x} \in B$ be a weakly efficient solution of (SIFP) and the suitable constraint qualification holds at $\breve{x}$. Then, for $L_i(\breve{x}, 0) = 0$, $S_j(\breve{x}, 0) = 0$, $\nabla_{\breve{p}_i} L_i(\breve{x}, 0) = 0$ and $\nabla_{\breve{q}_j} S_j(\breve{x}, 0) = 0$, $i \in \tilde{I}$, $j \in \tilde{J}$, there exist $\check{\lambda} = (\check{\lambda}_1, ..., \check{\lambda}_k) \in$ int K^*, $\breve{\mu} = (\breve{\mu}_1, ..., \breve{\mu}_m) \in$ int Q^*, with $(\check{\lambda}, \breve{\mu}) \neq (0, 0)$ such that ($\breve{x}$, $\breve{v}$, $\breve{w}^1$, $\breve{w}^2$, $\check{\lambda}$, $\breve{\mu}$, $\breve{z}$, $\breve{p} = 0$, $\breve{q} = 0$) is a feasible point of (MD) and the objective function values of (SIFP) and (MD) are equal. Moreover, if all the assumptions of Theorem 2.1 are satisfied for every feasible point $\breve{x}$ of (SIFP) and ($\breve{u}$, $\breve{v}$, $\breve{w}^1$, $\breve{w}^2$, $\check{\lambda}$, $\breve{\mu}$, $\breve{z}$, $\breve{p}$, $\breve{q}$) of (MD), then ($\breve{x}$, $\breve{v}$, $\breve{w}^1$, $\breve{w}^2$, $\check{\lambda}$, $\breve{\mu}$, $\breve{z}$, $\breve{p} = 0$, $\breve{q} = 0$) is an efficient solution of (MD).*

Proof. For the weakly efficient solution $\breve{x} \in S_0$ of (SIFP), from Theorem 1.6, there exist $\check{\lambda} \in \text{int}K^*$, $\breve{\mu} \in \text{int}Q^*$, $(\check{\lambda}, \breve{\mu}) \neq (0, 0)$ and $\breve{t} \in T$ such that

$$0 \in \partial \left(\sum_{i=1}^{k} \check{\lambda}_i \left(\frac{f_i(\breve{x}) + \Omega(\breve{x}|C_i)}{g_i(\breve{x}) - \Omega(\breve{x}|D_i)} \right) + \sum_{j=1}^{m} \breve{\mu}_j \Big(h_j(\breve{x},\ \breve{t}) + \Omega(\breve{x}|E_j) + \Omega(\breve{t}|M_j) \Big) \right) \text{ and}$$

$$\sum_{j=1}^{m} \breve{\mu}_j \big(h(\breve{x}, \breve{t}) + \Omega(\breve{x}|E_j) + \Omega(\breve{t}|M_j) \big) = 0$$

which implies

$$0 \in \left(\sum_{i=1}^{k} \check{\lambda}_i \partial \left(\frac{f_i(\breve{x}) + \Omega(\breve{x}|C_i)}{g_i(\breve{x}) - \Omega(\breve{x}|D_i)} \right) + \sum_{j=1}^{m} \breve{\mu}_j \partial \big(h_j(\breve{x},\ \breve{t}) + \Omega(\breve{x}|E_j) + \Omega(\breve{t}|M_j) \big) \right).$$

For $\breve{z}_i \in \partial\Omega(\breve{x}|C_i)$, $\breve{v}_i \in \partial\Omega(\breve{x}|D_i)$, $\breve{w}_j^1 \in \partial\Omega(\breve{x}|E_j)$ and $\breve{w}_j^2 \in \partial\Omega(\breve{x}|M_j)$, we have

$$\Omega(\breve{x}|C_i) = \breve{x}^T \breve{z}_i \ \ \Omega(\breve{x}|D_i) = \breve{x}^T \breve{v}_i, \ \Omega(\breve{x}|E_j) = \breve{x}^T \breve{w}_j^1 \text{ and } \Omega(\breve{t}|M_j) = \breve{t}^T \breve{w}_j^2.$$

It further follows that

$$\sum_{i=1}^{k} \check{\lambda}_i \nabla \left(\frac{f_i(\breve{x}) + \breve{x}^T \breve{z}_i}{g_i(\breve{x}) - \breve{x}^T \breve{v}_i} \right) + \sum_{j=1}^{m} \breve{\mu}_j \nabla \big(h_j(\breve{x},\ \breve{t}) + \breve{x}^T \breve{w}_j^1 + \breve{t}^T \breve{w}_j^2 \big) = 0 \text{ and}$$

$$\sum_{j=1}^{m} \breve{\mu}_j \big(h(\breve{x}, \breve{t}) + \breve{x}^T \breve{w}_j^1 + \breve{t}^T \breve{w}_j^2 \big) = 0.$$

The above equations with $L_i(\breve{u}, 0) = 0$, $S_i(\breve{x}, 0) = 0$, $\nabla_{\breve{p}_i} L_i(\breve{x}, 0) = 0$, $\nabla_{\breve{q}_j} S_j(\breve{x}, 0) = 0$ imply that ($\breve{x}$, $\breve{v}$, $\breve{w}^1$, $\breve{w}^2$, $\check{\lambda}$, $\breve{\mu}$, $\breve{z}$, $\breve{p} = 0$, $\breve{q} = 0$) is feasible for (MD) and respective values of objective functions are equal.

Now, suppose that $(\check{x},\ \check{v},\ \check{w}^1,\ \check{w}^2,\ \check{\lambda},\ \check{\mu},\ \check{z},\ \check{p}=0,\ \check{q}=0)$ is not an efficient solution of (MD), then there exists a feasible point $(u,\ v,\ w^1,\ w^2,\ \lambda,\ \mu,\ p,\ q)$ of (MD) such that

$$\left(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1},...,\frac{f_k(u)+u^Tz_k}{g_k(u)-u^Tv_k}\right)-\left(\frac{f_1(\check{x})+\check{x}^T\check{z}_1}{g_1(\check{x})-\check{x}^T\check{v}_1},...,\frac{f_k(\check{x})+\check{x}^T\check{z}_k}{g_k(\check{x})-\check{x}^T\check{v}_k}\right)\in K\setminus\{0\}.$$

This further gives

$$\left(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1},...,\frac{f_k(u)+u^Tz_k}{g_k(u)-u^Tv_k}\right)-\left(\frac{f_1(\check{x})+\Omega(\check{x}|C_1)}{g_1(\check{x})-\Omega(\check{x}|D_1)},...,\frac{f_k(\check{x})+\Omega(\check{x}|C_k)}{g_k(\check{x})-\Omega(\check{x}|D_k)}\right)\in K\setminus\{0\}$$

which contradicts Theorem 2.1. Hence proved. ☐

Theorem 2.3. [*Strict converse duality*] *Let $\tilde{x}$ and $(\tilde{u},\ \tilde{v},\ \tilde{w}^1,\ \tilde{w}^2,\ \tilde{\lambda},\ \tilde{\mu},\ \tilde{z},\ \tilde{p},\ \tilde{q})$ be feasible solutions of the problems (SIFP) and (MD), respectively. Let a sublinear functional (in third variable) be $\mathcal{F}:B\times B\times\mathbb{R}^n\to\mathbb{R}$ such that*

(i) $\left(\frac{f_1(\tilde{u})+\tilde{u}^T\tilde{z}_1}{g_1(\tilde{u})-\tilde{u}^T\tilde{v}_1}-\sum_{j=1}^m\tilde{\mu}_j\{h_j(\tilde{u},\tilde{\tau})+\tilde{u}^T\tilde{w}_j^1+\tilde{\tau}^T\tilde{w}_j^2+S_j(\tilde{u},\tilde{q}_j)-\tilde{q}_j^T\nabla_{\tilde{q}_j}S_j(\tilde{u},\tilde{q}_j)\},...,\frac{f_k(\tilde{u})+\tilde{u}^T\tilde{z}_k}{g_k(\tilde{u})-\tilde{u}^T\tilde{v}_k}-\sum_{j=1}^m\tilde{\mu}_j\{h_j(\tilde{u},\tilde{\tau})+\tilde{u}^T\tilde{w}_j^1+\tilde{\tau}^T\tilde{w}_j^2+S_j(\tilde{u},\tilde{q}_j)-\tilde{q}_j^T\nabla_{\tilde{q}_j}S_j(\tilde{u},\tilde{q}_j)\}\right)-\left(\frac{f_1(\tilde{x})+\tilde{x}^T\tilde{z}_1}{g_1(\tilde{x})-\tilde{x}^T\tilde{v}_1},...,\frac{f_k(\tilde{x})+\tilde{x}^T\tilde{z}_k}{g_k(\tilde{x})-\tilde{x}^T\tilde{v}_k}\right)\in K,$

(ii) $\left(\left(\frac{f_1(.)+(.)^Tz_1}{g_1(.)-(.)^Tv_1},...,\frac{f_k(.)+(.)^Tz_k}{g_k(.)-(.)^Tv_k}\right),\ \left(h_1(.,\ \tau)+(.)^Tw_1^1,...,h_m(.,\ \tau)+(.)^Tw_m^1\right)\right)$ *be strictly higher order $(K\times Q)-(\mathcal{F},\alpha,\rho,d)-$type I convex at $\tilde{u}$ with respect to L and S,*

(iii) $K\supseteq\mathbb{R}_+^k$, $Q\supseteq\mathbb{R}_+^m$ *and*

(iv) $\sum_{i=1}^k\tilde{\lambda}_i\rho_i^{(1)}\big(d_i^{(1)}(\tilde{x},\tilde{u})\big)^2+\sum_{j=1}^m\tilde{\mu}_j\{\rho_j^{(2)}\big(d_j^{(2)}(\tilde{x},\tilde{u})\big)^2-\tilde{\tau}^T\tilde{w}_j^2\}\geq 0.$

Then $\tilde{x}=\tilde{u}$.

Proof. Let $\tilde{x}$ and $(\tilde{u},\ \tilde{v},\ \tilde{w}^1,\ \tilde{w}^2,\ \tilde{\lambda},\ \tilde{\mu},\ \tilde{z},\ \tilde{p},\ \tilde{q})$ be feasible solutions of the problems (SIFP) and (MD), respectively. On the contrary, suppose that $\tilde{x}\neq\tilde{u}$. Then, by $\tilde{\lambda}\in\text{int}K^*\subseteq\text{int }\mathbb{R}_+^k$, and $\tilde{\mu}\in\text{int }Q^*\subseteq\text{int }\mathbb{R}_+^m$, hypothesis (ii) and Definition 1.3, we obtain

$$\sum_{i=1}^k\tilde{\lambda}_i\Bigg[\frac{f_i(\tilde{x})+\tilde{x}^T\tilde{z}_i}{g_i(\tilde{x})-\tilde{x}^T\tilde{v}_i}-\frac{f_i(\tilde{u})+\tilde{u}^T\tilde{z}_i}{g_i(\tilde{u})-\tilde{u}^T\tilde{v}_i}-L_i(\tilde{u},\tilde{p}_i)+\tilde{p}_i^T\nabla_{\tilde{p}_i}L_i(\tilde{u},\tilde{p}_i)$$
$$-\mathcal{F}_{\tilde{x},\tilde{u}}\bigg[\alpha(\tilde{x},\tilde{u})\bigg\{\nabla_{\tilde{x}}\bigg(\frac{f_i(\tilde{u})+\tilde{u}^T\tilde{z}_i}{g_i(\tilde{u})-\tilde{u}^T\tilde{v}_i}\bigg)+\nabla_{\tilde{p}_i}L_i(\tilde{u},\tilde{p}_i)\bigg\}\bigg]-\rho_i^{(1)}(d_i^{(1)}(\tilde{x},\tilde{u}))^2\Bigg]>0. \tag{10}$$

$$\sum_{j=1}^m\tilde{\mu}_j\Bigg[-(h_j(\tilde{u},\tilde{\tau})+\tilde{u}^T\tilde{w}_j^1)-S_j(\tilde{u},\tilde{q}_j)+\tilde{q}_j^T\nabla_{\tilde{q}_j}S_j(\tilde{u},\tilde{q}_j)$$
$$-\mathcal{F}_{\tilde{x},\tilde{u}}\big[\alpha(\tilde{x},\tilde{u})\big\{\nabla_{\tilde{x}}(h_j(\tilde{u},\tilde{\tau})+\tilde{u}^T\tilde{w}_j^1)+\nabla_{\tilde{q}_j}S_j(\tilde{u},\tilde{q}_j)\big\}\big]-\rho_j^{(2)}\ (d_j^{(2)}(\tilde{x},\tilde{u}))^2\Bigg]>0. \tag{11}$$

Further, using the sublinearity of $\mathcal{F}$ and inequality (3) in (10), we get

$$\begin{aligned}
&\sum_{i=1}^{k} \tilde{\lambda}_i \left[\frac{f_i(\tilde{x}) + \tilde{x}^T \tilde{z}_i}{g_i(\tilde{x}) - \tilde{x}^T \tilde{v}_i} - \frac{f_i(\tilde{u}) + \tilde{u}^T \tilde{z}_i}{g_i(\tilde{u}) - \tilde{u}^T \tilde{v}_i} \right] \\
&> \mathcal{F}_{\tilde{x},\tilde{u}} \Big[\alpha(\tilde{x},\tilde{u}) \sum_{i=1}^{k} \tilde{\lambda}_i \Big\{ \nabla_{\tilde{x}} \Big(\frac{f_i(\tilde{u}) + \tilde{u}^T \tilde{z}_i}{g_i(\tilde{u}) - \tilde{u}^T \tilde{v}_i} \Big) + \nabla_{\tilde{p}_i} L_i(\tilde{u},\tilde{p}_i) \Big\} \Big] + \sum_{i=1}^{k} \tilde{\lambda}_i \rho_i^{(1)} (d_i^{(1)}(\tilde{x},\tilde{u}))^2. \qquad (12)
\end{aligned}$$

It follows from (2), (11)-(12) and sublinearity of $\mathcal{F}$ that

$$\begin{aligned}
&\sum_{i=1}^{k} \tilde{\lambda}_i \left[\frac{f_i(\tilde{x}) + \tilde{x}^T \tilde{z}_i}{g_i(\tilde{x}) - \tilde{x}^T \tilde{v}_i} - \frac{f_i(\tilde{u}) + \tilde{u}^T \tilde{z}_i}{g_i(\tilde{u}) - \tilde{u}^T \tilde{v}_i} \right] \\
&> \mathcal{F}_{\tilde{x},\tilde{u}} \Bigg[\alpha(\tilde{x},\tilde{u}) \Bigg(\sum_{i=1}^{k} \tilde{\lambda}_i \Big\{ \nabla_{\tilde{x}} \Big(\frac{f_i(\tilde{u}) + \tilde{u}^T \tilde{z}_i}{g_i(\tilde{u}) - \tilde{u}^T \tilde{v}_i} \Big) + \nabla_{\tilde{p}_i} L_i(\tilde{u},\tilde{p}_i) \Big\} + \sum_{j=1}^{m} \tilde{\mu}_j \big\{ \nabla_{\tilde{x}} (h_j(\tilde{u},\tilde{\tau}) + \tilde{u}^T \tilde{w}_j^1) \\
&\qquad + \nabla_{\tilde{q}_j} S_j(\tilde{u},\tilde{q}_j) \big\} \Bigg) \Bigg] + \sum_{i=1}^{k} \tilde{\lambda}_i \rho_i^{(1)} (d_i^{(1)}(\tilde{x},\tilde{u}))^2 + \sum_{j=1}^{m} \tilde{\mu}_j \{ \rho_j^{(2)} \, (d_j^{(2)}(\tilde{x},\tilde{u}))^2 - \tilde{\tau}^T \tilde{w}_j^2 \}.
\end{aligned}$$

Further, using (1) and hypothesis (iv), we obtain

$$\sum_{i=1}^{k} \tilde{\lambda}_i \left[\frac{f_i(\tilde{x}) + \tilde{x}^T \tilde{z}_i}{g_i(\tilde{x}) - \tilde{x}^T \tilde{v}_i} - \frac{f_i(\tilde{u}) + \tilde{u}^T \tilde{z}_i}{g_i(\tilde{u}) - \tilde{u}^T \tilde{v}_i} \right] > 0. \qquad (13)$$

Now, using dual constraint (2) and $\lambda_i \in \text{int} K^*$ in hypothesis (i), we get

$$\sum_{i=1}^{k} \tilde{\lambda}_i \left[\frac{f_i(\tilde{x}) + \tilde{x}^T \tilde{z}_i}{g_i(\tilde{x}) - \tilde{x}^T \tilde{v}_i} - \frac{f_i(\tilde{u}) + \tilde{u}^T \tilde{z}_i}{g_i(\tilde{u}) - \tilde{u}^T \tilde{v}_i} \right] \leq 0.$$

This contradicts the inequality (13). Hence the result. ☐

Example 2. Let in the problem (SIFP), $f : \mathbb{R} \to \mathbb{R}^2$, $g : \mathbb{R} \to \mathbb{R}^2$ and $h : \mathbb{R} \times [-1,\ 0] \to \mathbb{R}^2$ be given as:

$$\begin{aligned}
f(x) &= (f_1(x),\ f_2(x)) = (x^2 + 2,\ x^2(x^2 + 1)), \\
g(x) &= (g_1(x),\ g_2(x)) = (x^4 + 4,\ (x^2 + 1)^2) \text{ and} \\
h(x,\ t) &= (h_1(x,\ t),\ h_2(x,\ t)) = (x^3 t - 2,\ x^2 t).
\end{aligned}$$

Let the cones be $K = \{(x,\ y) \in \mathbb{R}^2\ :\ x \geq 0,\ y \geq -17x\}$ and $Q = \{(x,\ y) \in \mathbb{R}^2\ :\ y \geq 0,\ 2y \geq -x\}$. Also, suppose

$$\begin{aligned}
L(u,\ p) &= (L_1(u,\ p_1),\ L_2(u,\ p_2)) = (p_1 u,\ -p_2 u), \\
S(u,\ q) &= (S_1(u,\ q_1),\ S_2(u,\ q_2)) = (q_1 u^2,\ q_2(u - 1)) \text{ and} \\
d_1^{(1)}(x,y) &= d_1^{(2)}(x,y) = 1 - xy,\ d_2^{(1)}(x,y) = d_2^{(2)}(x,y) = y^2 + 1.
\end{aligned}$$

Let the sublinear functional be $\mathcal{F}_{x,u}(b) = bxu$ and $\alpha(x,\ y) = 1 + x^2 y^2$. Let

$$C_1 = \{0\},\ C_2 = [0,\ 1] = D_1 = E_2 = M_2, \text{ and } D_2 = [-1,\ 0] = E_1 = M_1.$$

Thus, their support functions will be

$$\Omega(x|C_1) = \{0\},\ \Omega(x|D_2) = \Omega(x|E_1) = \frac{|x| - x}{2},\ \Omega(t|M_1) = \frac{|t| - t}{2},$$

$$\Omega(x|C_2) = \Omega(x|D_1) = \Omega(x|E_2) = \frac{x+|x|}{2} \text{ and } \Omega(t|M_2) = \frac{|t|+t}{2}.$$

The feasible region $S_0 = \Big\{x \in \mathbb{R} \ : \ \Big(2 - xt - \frac{(|x|-x)}{2} - \frac{(|t|-t)}{2},\ -x^2t - \frac{(x+|x|)}{2} - \frac{(|t|+t)}{2}\Big) \in Q,$ for all $t \in [-1,0]\Big\}$.

Clearly, $0,\ -1 \in S_0$. Also, $\beta = \big(u,\ v,\ w^1,\ w^2,\ \lambda,\ \mu,\ z,\ p,\ q\big) = \Big(0,\ \big(0,\ 0\big),\ \big(0,\ 1\big),\ \Big(-1,\ 1\Big),\ \Big(\frac{1}{8},\ 2\Big),\ (1,\ 1),\ \big(0,\ 0\big),\ \big(1,\ 0\big),\ \big(0,\ 1\big)\Big)$ is feasible for (MD).

Validation of Theorem 2.1:
First we will show that all the hypothesis of Theorem 2.1 are satisfied.
At $u = 0$, for $v = (0,\ 0),\ z = (0,\ 0),\ \rho_1^{(1)} = -2$ and $\rho_2^{(1)} = \frac{1}{8}$, we get

$$\begin{aligned}&\Bigg(\frac{f_1(x)+x^Tz_1}{g_1(x)-x^Tv_1} - \frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1} - L_1(u,p_1) + p_1^T\nabla_{p_1}L_1(u,p_1) - \mathcal{F}_{x,u}\Big[\alpha(x,u)\Big(\nabla_x\Big(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1}\Big)\\
&\quad + \nabla_{p_1}L_1(u,p_1)\Big)\Big] - \rho_1^{(1)}\big(d_1^{(1)}(x,u)\big)^2,\ \frac{f_2(x)+x^Tz_2}{g_2(x)-x^Tv_2} - \frac{f_2(u)+u^Tz_2}{g_2(u)-u^Tv_2} - L_2(u,p_2)\\
&\quad + p_2^T\nabla_{p_2}L_2(u,p_2) - \mathcal{F}_{x,u}\Big[\alpha(x,u)\Big(\nabla_x\Big(\frac{f_2(u)+u^Tz_2}{g_2(u)-u^Tv_2}\Big) + \nabla_{p_2}L_2(u,p_2)\Big)\Big] - \rho_2^{(1)}\big(d_2^{(1)}(x,u)\big)^2\Bigg)\\
&\qquad = \left(\frac{x^2+2}{x^4+4} + \frac{3}{2},\ \frac{x^2}{x^2+1} - \frac{1}{8}\right) \in K \setminus \{0\},\ \text{for all } x \in \mathbb{R}.\end{aligned}$$

Further, for $w_1^1 \in E_1,\ w_2^1 \in E_2,\ w_1^2 \in M_1,\ w_2^2 \in M_2,\ \rho_1^{(2)} = 1,\ \rho_2^{(2)} = -1$ and for all $\tau \in [-1,0]$, we obtain

$$\begin{aligned}&\Big(-(h_1(u,\tau)+u^Tw_1^1) - S_1(u,q_1) + q_1^T\nabla_{q_1}S_1(u,q_1) - \mathcal{F}_{x,u}\big[\alpha(x,u)\big(\nabla_x(h_1(u,\tau)+u^Tw_1^1)\\
&\quad + \nabla_{q_1}S_1(u,q_1)\big)\big] - \rho_1^{(2)}\ \big(d_1^{(2)}(x,u)\big)^2,\ -(h_2(u,\tau)+u^Tw_2^1) - S_2(u,p_2) + q_2^T\nabla_{q_2}S_2(u,q_2)\\
&\quad - \mathcal{F}_{x,u}\big[\alpha(x,u)\big(\nabla_x(h_2(u,\tau)+u^Tw_2^1) + \nabla_{q_2}S_2(u,q_2)\big)\big] - \rho_2^{(2)}\big(d_2^{(2)}(x,u)\big)^2\Big) = \Big(1,\ 1\Big) \in Q \setminus \{0\}.\end{aligned}$$

Hence, the hypothesis (i) of Theorem 2.1 is satisfied. Moreover, $\mathbb{R}_+^2 \subset K,\ \mathbb{R}_+^2 \subset Q$ and

$$\sum_{i=1}^{2}\lambda_i\rho_i^{(1)}\big(d_i^{(1)}(x,u)\big)^2 + \sum_{j=1}^{2}\mu_j\{\rho_j^{(2)}\big(d_j^{(2)}(x,u)\big)^2 - \tau^Tw_j^2\} = 0.$$

Thus, hypotheses $(ii)-(iii)$ of Theorem 2.1 also hold. Now, for β, the expression

$$\begin{aligned}&\left(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1},\ \frac{f_2(u)+u^Tz_2}{g_2(u)-u^Tv_2}\right) - \left(\frac{f_1(x)+\Omega(x|C_1)}{g_1(x)-\Omega(x|D_1)},\ \frac{f_2(x)+\Omega(x|C_2)}{g_2(x)-\Omega(x|D_2)}\right)\\
&\qquad = \begin{cases}(0,\ 0) \notin K/\{0\} & \text{for } x = 0,\\ \Big(-\frac{1}{10},\ -\frac{2}{3}\Big) \notin K/\{0\} & \text{for } x = -1.\end{cases}\end{aligned}$$

Hence, the Theorem 2.1 is verified at $x = 0,\ -1 \in S_0$ and the point β feasible for (MD).

Validation of Theorem 2.3:
For the points β and $x = 0$, it has been shown above that the assumptions (ii), (iii) and (iv) of Theorem 2.3 hold true. Also, the value of the expression

$$\Bigg(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1}-\sum_{j=1}^{2}\mu_j\{h_j(u,\tau)+u^Tw_j^1+\tau^Tw_j^2+S_j(u,q_j)-q_j^T\nabla_{q_j}S_j(u,q_j)\},\ \frac{f_2(u)+u^Tz_2}{g_2(u)-u^Tv_2}$$

$$-\sum_{j=1}^{2}\mu_j\{h_j(u,\tau)+u^Tw_j^1+\tau^Tw_j^2+S_j(u,q_j)-q_j^T\nabla_{q_j}S_j(u,q_j)\}\Bigg)-\left(\frac{f_1(x)+x^Tz_1}{g_1(x)-x^Tv_1},\ \frac{f_2(x)+x^Tz_2}{g_2(x)-x^Tv_2}\right)$$

$$=(0,0)\in K.$$

Thus, the assumption (i) of Theorem 2.3 also holds. Hence verified. $\square$

Next, by demonstrating the following example, we will show that if the assumption of higher order $K \times Q - (F, \alpha, \rho, d)$ type I convexity does not hold, then the result of Theorem 2.1 may not hold:

Example 3. Let in the problem (SIFP), $f : \mathbb{R} \to \mathbb{R}^2,\ g : \mathbb{R} \to \mathbb{R}^2$, and $h : \mathbb{R} \times [2,\ 4] \to \mathbb{R}^2$ be given as:

$$f(x) = (f_1(x),\ f_2(x)) = (x^2+4,\ x^4(x^4-1)),$$
$$g(x) = (g_1(x),\ g_2(x)) = (2x^2+1,\ (x^4+1)^2) \text{ and}$$
$$h(x,\ t) = (h_1(x,\ t),\ h_2(x,\ t)) = (-x^3t+2,\ -x^2t^2).$$

Let the cones be $K = \{(x,\ y) \in \mathbb{R}^2\ :\ x \geq 0,\ y \geq -2x\}$ and $Q = \{(x,\ y) \in \mathbb{R}^2\ :\ x \geq 0,\ y \geq 0\}$. Also, suppose

$$L(u,\ p) = (L_1(u,\ p_1),\ L_2(u,\ p_2)) = (-p_1u,\ p_2u),$$
$$S(u,\ q) = \big(S_1(u,\ q_1),\ S_2(u,\ q_2)\big) = \Big(q_1u,\ q_2u+4\Big) \text{ and}$$
$$d_1^{(1)}(x,y) = d_1^{(2)}(x,y) = y(x+1),\ d_2^{(1)}(x,y) = d_2^{(2)}(x,y) = x^2y^2.$$

Let the sublinear functional be $\mathcal{F}_{x,u}(b) = b^2xu$ and $\alpha(x,\ y) = 2 + x^2y^2$. Consider

$$C_1 = \{0\},\ C_2 = [-1,\ 0] = E_1 = M_2 \text{ and } D_1 = D_2 = [0,\ 1] = E_2 = M_1.$$

Thus, their support functions will be

$$\Omega(x|C_1) = \{0\},\ \Omega(x|C_2) = \Omega(x|E_1) = \frac{|x|-x}{2},\ \Omega(t|M_1) = \frac{|t|+t}{2},$$
$$\Omega(x|D_1) = \Omega(x|D_2) = \Omega(x|E_2) = \frac{x+|x|}{2} \text{ and } \Omega(t|M_2) = \frac{|t|-t}{2}.$$

One can easily verify that $x = 1$ is feasible for (SIFP) and $\beta_1 = \big(u,\ v,\ w^1,\ w^2,\ \lambda,\ \mu,\ z,\ p,\ q\big) = \Big(0,\ \big(0,\ 0\big),\ \big(-1,\ 1\big),\ \Big(\frac{1}{2},\ -1\Big),\ \Big(2,\ 1\Big),\ (1,\ 1),\ \big(0,\ 0\big),\ \big(2,\ 0\big),\ \big(0,\ 2\big)\Big)$ is feasible for (MD). At β_1, for any $\rho_1^{(1)},\ \rho_2^{(1)},\ \rho_1^{(2)},\ \rho_2^{(2)} \in \mathbb{R}$, since $\mathbb{R}^2_+ \subset K,\ \mathbb{R}^2_+ \subset Q$ and

$$\sum_{i=1}^{2}\lambda_i\rho_i^{(1)}\big(d_i^{(1)}(x,u)\big)^2+\sum_{j=1}^{2}\mu_j\{\rho_j^{(2)}\big(d_j^{(2)}(x,u)\big)^2-\tau^Tw_j^2\} = \frac{\tau}{2} \geq 0.$$

Thus, hypotheses (ii) and (iii) of Theorem 2.1 are satisfied but

$$\begin{aligned}&\frac{f_1(x)+x^Tz_1}{g_1(x)-x^Tv_1}-\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1}-L_1(u,p_1)+p_1^T\nabla_{p_1}L_1(u,p_1)-\mathcal{F}_{x,u}\Big[\alpha(x,u)\Big(\nabla_x\Big(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1}\Big)\\&+\nabla_{p_1}L_1(u,p_1)\Big)\Big]-\rho_1^{(1)}\big(d_1^{(1)}(x,u)\big)^2\\&=\frac{x^2+4}{2x^2+1}-4<0 \text{ at } x=1.\end{aligned}$$

That is, $(K\times Q)-(\mathcal{F},\alpha,\rho,d)-$ type I convexity of $\left(\left(\frac{f_1(.)+(.)^Tz_1}{g_1(.)-(.)^Tv_1}\ ,\frac{f_2(.)+(.)^Tz_2}{g_2(.)-(.)^Tv_2}\right),\ (h_1(.\ ,\ \tau)+(.)^Tw_1^1,\ h_2(.\ ,\ \tau)+(.)^Tw_2^1)\right)$ is not satisfied. Also,

$$\left(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1},\ \frac{f_2(u)+u^Tz_2}{g_2(u)-u^Tv_2}\right)-\left(\frac{f_1(x)+\Omega(x|C_1)}{g_1(x)-\Omega(x|D_1)},\ \frac{f_2(x)+\Omega(x|C_2)}{g_2(x)-\Omega(x|D_2)}\right)=\Big(\frac{3}{2},\ 0\Big)\in K \text{ at } x=1.$$

Hence, the result of Theorem 2.1 does not hold. □

3. Wolfe type dual

Consider the following Wolfe type higher order dual model for (SIFP):

$$\textbf{(WD)}\quad K-\max\left[\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1}+L_1(u,\ p_1)-p_1^T\nabla_{p_1}L_1(u,\ p_1),...,\right.$$
$$\left.\frac{f_k(u)+u^Tz_k}{g_k(u)-u^Tv_k}+L_k(u,\ p_k)-p_k^T\nabla_{p_k}L_k(u,\ p_k)\right]$$

subject to

$$\sum_{i=1}^{k}\lambda_i\left[\nabla_x\left(\frac{f_i(u)+u^Tz_i}{g_i(u)-u^Tv_i}\right)+\nabla_{p_i}L_i(u,\ p_i)\right]+\sum_{j=1}^{m}\mu_j\Big[\nabla_x(h_j(u,\tau)+u^Tw_j^1+\tau^Tw_j^2)+\nabla_{q_j}S_j(u,\ q_j)\Big]=0, \tag{14}$$

$$\sum_{j=1}^{m}\mu_j\Big[h_j(u,\tau)+u^Tw_j^1+\tau^Tw_j^2+S_j(u,\ q_j)-q_j^T\nabla_{q_j}S_j(u,\ q_j)\Big]\geq 0, \tag{15}$$

$z_i\in C_i,\ v_i\in D_i,\ w_j^1\in E_j,\ w_j^2\in M_j,\ i\in\tilde{I},\ j\in\tilde{J},\ \tau\in T$ and $(\lambda,\mu)\in \text{int}\ K^*\times \text{int}Q^*,\ (\lambda,\ \mu)\neq(0,\ 0)$.

Theorem 3.1. [*Weak duality*] *Let x and $(u,\ v,\ w^1,\ w^2,\ \lambda,\ \mu,\ z,\ p,\ q)$ be feasible for the problems (SIFP) and (WD), respectively. Let a sublinear functional (in third variable) be $\mathcal{F}: B\times B\times\mathbb{R}^n\to\mathbb{R}$. Also, assume that*

(i) $\left(\left(\frac{f_1(.)+(.)^Tz_1}{g_1(.)-(.)^Tv_1},...,\frac{f_k(.)+(.)^Tz_k}{g_k(.)-(.)^Tv_k}\right),\ \Big(h_1(.,\ \tau)+(.)^Tw_1^1,...,h_m(.,\ \tau)+(.)^Tw_m^1\Big)\right)$ *is higher order $(K\times Q)-(\mathcal{F},\alpha,\rho,d)-$type I convex at u with respect to L and S,*

(ii) $K \supseteq \mathbb{R}_+^k$, $Q \supseteq \mathbb{R}_+^m$ *and*

(iii) $\sum_{i=1}^{k} \lambda_i \rho_i^{(1)} \big(d_i^{(1)}(x,u)\big)^2 + \sum_{j=1}^{m} \mu_j \{\rho_j^{(2)} \big(d_j^{(2)}(x,u)\big)^2 - \tau^T w_j^2\} \geq 0.$

Then

$$\left(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1} + L_1(u,\ p_1) - p_1^T \nabla_{p_1} L_1(u,\ p_1), ..., \frac{f_k(u)+u^Tz_k}{g_k(u)-u^Tv_k} + L_k(u,\ p_k) - p_k^T \nabla_{p_k} L_k(u,\ p_k)\right)$$
$$- \left(\frac{f_1(x)+\Omega(x|C_1)}{g_1(x)-\Omega(x|D_1)}, ..., \frac{f_k(x)+\Omega(x|C_k)}{g_k(x)-\Omega(x|D_k)}\right) \notin K \setminus \{0\}. \tag{16}$$

Proof. It follows from hypotheses $(i)-(ii)$ and sublinearity of functional $\mathcal{F}$ that

$$\sum_{i=1}^{k} \lambda_i \left[\frac{f_i(x)+x^Tz_i}{g_i(x)-x^Tv_i} - \frac{f_i(u)+u^Tz_i}{g_i(u)-u^Tv_i} - L_i(u,p_i) + p_i^T \nabla_{p_i} L_i(u,p_i)\right]$$
$$\geq \mathcal{F}_{x,u}\Big[\alpha(x,u) \sum_{i=1}^{k} \lambda_i \Big\{\nabla_x \Big(\frac{f_i(u)+u^Tz_i}{g_i(u)-u^Tv_i}\Big) + \nabla_{p_i} L_i(u,p_i)\Big\}\Big] + \sum_{i=1}^{k} \lambda_i \rho_i^{(1)} \big(d_i^{(1)}(x,u)\big)^2. \tag{17}$$

$$\sum_{j=1}^{m} \mu_j \left[(-(h_j(u,\tau)+u^Tw_j^1) - S_j(u,q_j) + q_j^T \nabla_{q_j} S_j(u,q_j)\right]$$
$$\geq \mathcal{F}_{x,u}\Big[\alpha(x,u) \sum_{j=1}^{m} \mu_j \big\{\nabla_x (h_j(u,\tau)+u^Tw_j^1) + \nabla_{q_j} S_j(u,q_j)\big\}\Big] + \sum_{j=1}^{m} \mu_j \rho_j^{(2)}\ \big(d_j^{(2)}(x,u)\big)^2. \tag{18}$$

Using (15) in (18) and then adding with (17), we get

$$\sum_{i=1}^{k} \lambda_i \left[\frac{f_i(x)+x^Tz_i}{g_i(x)-x^Tv_i} - \frac{f_i(u)+u^Tz_i}{g_i(u)-u^Tv_i} - L_i(u,p_i) + p_i^T \nabla_{p_i} L_i(u,p_i)\right]$$
$$\geq \mathcal{F}_{x,u}\Big[\alpha(x,u) \sum_{i=1}^{k} \lambda_i \Big\{\nabla_x \Big(\frac{f_i(u)+u^Tz_i}{g_i(u)-u^Tv_i}\Big) + \nabla_{p_i} L_i(u,p_i)\Big\}\Big] + \mathcal{F}_{x,u}\Big[\alpha(x,u) \sum_{j=1}^{m} \mu_j \big\{\nabla_x (h_j(u,\tau)$$
$$+ u^Tw_j^1) + \nabla_{q_j} S_j(u,q_j)\big\}\Big] + \sum_{i=1}^{k} \lambda_i \rho_i^{(1)} \big(d_i^{(1)}(x,u)\big)^2 + \sum_{j=1}^{m} \mu_j \{\rho_j^{(2)}\ \big(d_j^{(2)}(x,u)\big)^2 - \tau^T w_j^2\}.$$

It follows from the hypothesis (iii) and sublinearity of $\mathcal{F}$ that

$$\sum_{i=1}^{k} \lambda_i \left[\frac{f_i(x)+x^Tz_i}{g_i(x)-x^Tv_i} - \frac{f_i(u)+u^Tz_i}{g_i(u)-u^Tv_i} - L_i(u,p_i) + p_i^T \nabla_{p_i} L_i(u,p_i)\right]$$
$$\geq \mathcal{F}_{x,u}\left[\alpha(x,u)\Big\{\sum_{i=1}^{k} \lambda_i \Big\{\nabla_x \Big(\frac{f_i(u)+u^Tz_i}{g_i(u)-u^Tv_i}\Big) + \nabla_{p_i} L_i(u,p_i)\Big\}\right.$$
$$\left. + \sum_{j=1}^{m} \mu_j \big\{\nabla_x (h_j(u,\tau)+u^Tw_j^1) + \nabla_{q_j} S_j(u,q_j)\big\}\Big\}\right].$$

Further, applying inequality (14) and using $\mathcal{F}_{x,u}(0) = 0$, we get

$$\sum_{i=1}^{k} \lambda_i \left[\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} - L_i(u, p_i) + p_i^T \nabla_{p_i} L_i(u, p_i) \right] \geq 0. \tag{19}$$

Now, if possible, suppose that

$$\Bigg(\frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1} + L_1(u,\ p_1) - p_1^T \nabla_{p_1} L_1(u,\ p_1), ..., \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} + L_k(u,\ p_k)$$
$$- p_k^T \nabla_{p_k} L_k(u,\ p_k) \Bigg) - \left(\frac{f_1(x) + \Omega(x|C_1)}{g_1(x) - \Omega(x|D_1)}, ..., \frac{f_k(x) + \Omega(x|C_k)}{g_k(x) - \Omega(x|D_k)} \right) \in K \setminus \{0\}.$$

From $\lambda \in \text{int } K^*$, we get

$$\sum_{i=1}^{k} \lambda_i \left[\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + L_i(u,\ p_i) - p_i^T \nabla_{p_i} L_i(u,\ p_i) - \frac{f_i(x) + \Omega(x|C_i)}{g_i(x) - \Omega(x|D_i)} \right] > 0. \tag{20}$$

Since $x^T z_i \leq \Omega(x|C_i)$, $x^T v_i \leq \Omega(x|D_i)$ and $\lambda \in \text{int } K^* \subseteq \text{int}\mathbb{R}_+^k$, therefore

$$\sum_{i=1}^{k} \lambda_i \left(\frac{f_i(x) + \Omega(x|C_i)}{g_i(x) - \Omega(x|D_i)} \right) \geq \sum_{i=1}^{k} \lambda_i \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \right). \tag{21}$$

Finally, using (21) in (20), we obtain

$$\sum_{i=1}^{k} \lambda_i \left[\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + L_i(u,\ p_i) - p_i^T \nabla_{p_i} L_i(u,\ p_i) - \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \right] > 0$$

which contradicts (19). This proves the theorem. □

Theorem 3.2. [*Strong duality*] *Let* $\check{x} \in B$ *be a weakly efficient solution of (SIFP) and the suitable constraint qualification be satisfied at* $\check{x}$. *Then, there exist* $\check{\lambda} = (\check{\lambda}_1, ..., \check{\lambda}_k) \in int\, K^*$, $\check{\mu} = (\check{\mu}_1, ..., \check{\mu}_m) \in int\, Q^*$ *and* $(\check{\lambda}, \check{\mu}) \neq (0, 0)$ *such that* $(\check{x},\ \check{v},\ \check{w}^1,\ \check{w}^2,\ \check{\lambda},\ \check{\mu},\ \check{z},\ \check{p} = 0,\ \check{q} = 0)$ *is a feasible solution of (WD) and respective objective function values are same, provided* $L_i(\check{x}, 0) = 0$, $S_i(\check{x}, 0) = 0$, $\nabla_{\check{p}_i} L_i(\check{x}, 0) = 0$ *and* $\nabla_{\check{q}_j} S_j(\check{x}, 0) = 0$, $i \in \tilde{I}$, $j \in \tilde{J}$. *Moreover, if all assumptions of Theorem 3.1 are satisfied for every feasible point* $\check{x}$ *of (SIFP) and* $(\check{u},\ \check{v},\ \check{w}^1,\ \check{w}^2,\ \check{\lambda},\ \check{\mu},\ \check{z},\ \check{p},\ \check{q})$ *of (WD), then* $(\check{x},\ \check{v},\ \check{w}^1,\ \check{w}^2,\ \check{\lambda},\ \check{\mu},\ \check{z},\ \check{p} = 0,\ \check{q} = 0)$ *is an efficient solution of (WD).*

Proof. From the subdifferentiability of support functions, for $\check{z}_i \in \partial\Omega(\check{x}|C_i)$, $\check{v}_i \in \partial\Omega(\check{x}|D_i)$, $\check{w}_j^1 \in \partial\Omega(\check{x}|E_j)$ and $\check{w}_j^2 \in \partial\Omega(\check{x}|M_j)$, we get

$$\Omega(\check{x}|C_i) = \check{x}^T \check{z}_i\ \Omega(\check{x}|D_i) = \check{x}^T \check{v}_i,\ \Omega(\check{x}|E_j) = \check{x}^T \check{w}_j^1 \text{ and } \Omega(\check{t}|M_j) = \check{t}^T \check{w}_j^2. \tag{22}$$

It further follows from Theorem 1.6 that

$$\sum_{i=1}^{k} \check{\lambda}_i \nabla \left(\frac{f_i(\check{x}) + \check{x}^T \check{z}_i}{g_i(\check{x}) - \check{x}^T \check{v}_i} \right) + \sum_{j=1}^{m} \check{\mu}_j \nabla \big(h_j(\check{x},\ \check{t}) + \check{x}^T \check{w}_j^1 + \check{t}^T \check{w}_j^2 \big) = 0 \text{ and}$$

$$\sum_{j=1}^{m} \check{\mu}_j \big(h(\check{x}, \check{t}) + \check{x}^T \check{w}_j^1 + \check{t}^T \check{w}_j^2 \big) = 0$$

where $(\check{\lambda},\ \check{\mu}) \in \text{int } K^* \times \text{int } Q^*$, $(\check{\lambda},\ \check{\mu}) \neq (0,0)$. Clearly $L_i(\check{x},0) = 0$, $S_i(\check{x},0) = 0$, $\nabla_{\check{p}_i} L_i(\check{x},0) = 0$, $\nabla_{\check{q}_j} S_j(\check{x},0) = 0$, imply that $(\check{x},\ \check{v},\ \check{w}^1,\ \check{w}^2,\ \check{\lambda},\ \check{\mu},\ \check{z},\ \check{p}=0,\ \check{q}=0)$ is feasible for (WD) and respective values of objective functions values of (SIFP) and (WD) are equal.
Now, on contrary, suppose that $(\check{x},\ \check{v},\ \check{w}^1,\ \check{w}^2,\ \check{\lambda},\ \check{\mu},\ \check{z},\ \check{p}=0,\ \check{q}=0)$ is not a weak efficient solution of (WD), then there exists a feasible point $\big(u,\ v,\ w^1,\ w^2,\ \lambda,\ z,\ \mu,\ p,\ ,q\big)$ of (WD) such that

$$\begin{aligned}&\Bigg[\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1} + L_1(u,\ p_1) - p_1^T\nabla_{p_1}L_1(u,\ p_1), ..., \frac{f_k(u)+u^Tz_k}{g_k(u)-u^Tv_k} + L_k(u,\ p_k)\\ &- p_k^T\nabla_{p_k}L_k(u,\ p_k)\Bigg] - \Bigg[\frac{f_1(\check{x})+\check{x}^T\check{z}_1}{g_1(\check{x})-\check{x}^T\check{v}_1} + L_1(\check{x},\ \check{p}_1) - \check{p}_1^T\nabla_{\check{p}_1}L_1(\check{x},\ \check{p}_1), ...,\\ &\frac{f_k(\check{x})+\check{x}^T\check{z}_k}{g_k(\check{x})-\check{x}^T\check{v}_k} + L_k(\check{x},\ \check{p}_k) - \check{p}_k^T\nabla_{\check{p}_k}L_k(\check{x},\ \check{p}_k)\Bigg] \in K\setminus\{0\}.\end{aligned}$$

Further, using $\check{p}=0$ and (22), we get

$$\begin{aligned}&\Bigg[\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1} + L_1(u,\ p_1) - p_1^T\nabla_{p_1}L_1(u,\ p_1), ..., \frac{f_k(u)+u^Tz_k}{g_k(u)-u^Tv_k} + L_k(u,\ p_k) - p_k^T\nabla_{p_k}L_k(u,\ p_k)\Bigg]\\ &- \Bigg[\frac{f_1(\check{x})+\Omega(\check{x}^T|\check{z}_1)}{g_1(\check{x})-\Omega(\check{x}^T|\check{v}_1)}, ..., \frac{f_k(\check{x})+\Omega(\check{x}^T|\check{z}_k)}{g_k(\check{x})-\Omega(\check{x}^T|\check{v}_k)}\Bigg] \in K\setminus\{0\}.\end{aligned}$$

This contradicts the result of the Theorem 3.1. Hence proved. ☐

Theorem 3.3. [*Strict converse duality*] *Let* $\tilde{x}$ *and* $(\tilde{u},\ \tilde{v},\ \tilde{w}^1,\ \tilde{w}^2,\ \tilde{\lambda},\ \tilde{\mu},\ \tilde{z},\ \tilde{p},\ \tilde{q})$ *be feasible for the problems (SIFP) and (WD), respectively. Let a sublinear functional (in third variable) be* $\mathcal{F} : B\times B\times\mathbb{R}^n \to \mathbb{R}$ *such that*

(i) $$\begin{aligned}&\Bigg(\frac{f_1(\tilde{u})+\tilde{u}^T\tilde{z}_1}{g_1(\tilde{u})-\tilde{u}^T\tilde{v}_1} + L_1(\tilde{u},\ \tilde{p}_1) - \tilde{p}_1^T\nabla_{\tilde{p}_1}L(\tilde{u},\ \tilde{p}_1) - \sum_{j=1}^m \tilde{\mu}_j\{h_j(\tilde{u},\tilde{\tau}) + \tilde{u}^T\tilde{w}_j^1 + \tilde{\tau}^T\tilde{w}_j^2 + S_j(\tilde{u},\tilde{q}_j) -\\ &\tilde{q}_j^T\nabla_{\tilde{q}_j}S_j(\tilde{u},\tilde{q}_j)\}, ..., \frac{f_k(\tilde{u})+\tilde{u}^T\tilde{z}_k}{g_k(\tilde{u})-\tilde{u}^T\tilde{v}_k} + L_k(\tilde{u},\ \tilde{p}_k) - \tilde{p}_k^T\nabla_{\tilde{p}_k}L(\tilde{u},\ \tilde{p}_k) - \sum_{j=1}^m \tilde{\mu}_j\{h_j(\tilde{u},\tilde{\tau}) + \tilde{u}^T\tilde{w}_j^1 + \tilde{\tau}^T\tilde{w}_j^2 +\\ &S_j(\tilde{u},\tilde{q}_j) - \tilde{q}_j^T\nabla_{\tilde{q}_j}S_j(\tilde{u},\tilde{q}_j)\}\Bigg) - \left(\frac{f_1(\tilde{x})+\tilde{x}^T\tilde{z}_1}{g_1(\tilde{x})-\tilde{x}^T\tilde{v}_1}, ..., \frac{f_k(\tilde{x})+\tilde{x}^T\tilde{z}_k}{g_k(\tilde{x})-\tilde{x}^T\tilde{v}_k}\right) \in K,\end{aligned}$$

(ii) $\left(\left(\frac{f_1(.)+(.)^Tz_1}{g_1(.)-(.)^Tv_1}, ..., \frac{f_k(.)+(.)^Tz_k}{g_k(.)-(.)^Tv_k}\right),\ \left(h_1(.,\ \tau) + (.)^Tw_1^1, ..., h_m(.,\ \tau) + (.)^Tw_m^1\right)\right)$ *be strictly higher order* $(K\times Q) - (\mathcal{F},\alpha,\rho,d)-$*type I convex at* $\tilde{u}$ *with respect to* L *and* S,

(iii) $K \supseteq \mathbb{R}_+^k$, $Q \supseteq \mathbb{R}_+^m$ *and*

(iv) $\sum_{i=1}^k \tilde{\lambda}_i\rho_i^{(1)}\big(d_i^{(1)}(\tilde{x},\tilde{u})\big)^2 + \sum_{j=1}^m \tilde{\mu}_j\{\rho_j^{(2)}\big(d_j^{(2)}(\tilde{x},\tilde{u})\big)^2 - \tilde{\mu}_j\tilde{\tau}^T\tilde{w}_j^2\} \geq 0.$

Then $\tilde{x} = \tilde{u}$.

Proof. Let $\tilde{x}$ and $(\tilde{u},\ \tilde{v},\ \tilde{w}^1,\ \tilde{w}^2,\ \tilde{\lambda},\ \tilde{\mu},\ \tilde{z},\ \tilde{p},\ \tilde{q})$ be feasible solutions of the problems (SIFP) and (WD), respectively. Let $\tilde{x} \neq \tilde{u}$. Then, by the Definition 1.3 and supposition (ii), we have

$$\begin{aligned}
&\sum_{i=1}^{k}\tilde{\lambda}_i\left[\frac{f_i(\tilde{x})+\tilde{x}^T\tilde{z}_i}{g_i(\tilde{x})-\tilde{x}^T\tilde{v}_i}-\frac{f_i(\tilde{u})+\tilde{u}^T\tilde{z}_i}{g_i(\tilde{u})-\tilde{u}^T\tilde{v}_i}-L_i(\tilde{u},\tilde{p}_i)+\tilde{p}_i^T\nabla_{\tilde{p}_i}L_i(\tilde{u},\tilde{p}_i)\right]\\
&>\mathcal{F}_{\tilde{x},\tilde{u}}\left[\alpha(\tilde{x},\tilde{u})\sum_{i=1}^{k}\tilde{\lambda}_i\Big\{\nabla_{\tilde{x}}\Big(\frac{f_i(\tilde{u})+\tilde{u}^T\tilde{z}_i}{g_i(\tilde{u})-\tilde{u}^T\tilde{v}_i}\Big)+\nabla_{\tilde{p}_i}L_i(\tilde{u},\tilde{p}_i)\Big\}\right]+\sum_{i=1}^{k}\tilde{\lambda}_i\rho_i^{(1)}\big(d_i^{(1)}(\tilde{x},\tilde{u})\big)^2.
\end{aligned}\tag{23}$$

$$\begin{aligned}
&\sum_{j=1}^{m}\tilde{\mu}_j\Big[-(h_j(\tilde{u},\tilde{\tau})+\tilde{u}^T\tilde{w}_j^1)-S_j(\tilde{u},\tilde{q}_j)+\tilde{q}_j^T\nabla_{\tilde{q}_j}S_j(\tilde{u},\tilde{q}_j)\Big]\\
&>\mathcal{F}_{\tilde{x},\tilde{u}}\left[\alpha(\tilde{x},\tilde{u})\sum_{j=1}^{m}\tilde{\mu}_j\big\{\nabla_{\tilde{x}}(h_j(\tilde{u},\tilde{\tau})+\tilde{u}^T\tilde{w}_j^1)+\nabla_{\tilde{q}_j}S_j(\tilde{u},\tilde{q}_j)\big\}\right]+\sum_{j=1}^{m}\tilde{\mu}_j\rho_j^{(2)}\ \big(d_j^{(2)}(\tilde{x},\tilde{u})\big)^2
\end{aligned}\tag{24}$$

where $\tilde{\lambda}\in \text{int } K^* \subseteq \text{int } \mathbb{R}_+^k$ and $\tilde{\mu}\in \text{int } Q^* \subseteq \text{int } \mathbb{R}_+^m$. Adding (23)–(24) and using supposition (iv), we get

$$\begin{aligned}
&\sum_{i=1}^{k}\tilde{\lambda}_i\left[\frac{f_i(\tilde{x})+\tilde{x}^T\tilde{z}_i}{g_i(\tilde{x})-\tilde{x}^T\tilde{v}_i}-\frac{f_i(\tilde{u})+\tilde{u}^T\tilde{z}_i}{g_i(\tilde{u})-\tilde{u}^T\tilde{v}_i}-L_i(\tilde{u},\tilde{p}_i)+\tilde{p}_i^T\nabla_{\tilde{p}_i}L_i(\tilde{u},\tilde{p}_i)\right]\\
&\quad+\sum_{j=1}^{m}\tilde{\mu}_j\left[-(h_j(\tilde{u},\tilde{\tau})+\tilde{u}^T\tilde{w}_j^1+\tilde{\tau}^T\tilde{w}_j^2)-S_j(\tilde{u},\tilde{q}_j)+\tilde{q}_j^T\nabla_{\tilde{q}_j}S_j(\tilde{u},\tilde{q}_j)\right]\\
&>\mathcal{F}_{\tilde{x},\tilde{u}}\Bigg[\alpha(\tilde{x},\tilde{u})\Big\{\sum_{i=1}^{k}\tilde{\lambda}_i\Big\{\nabla_{\tilde{x}}\Big(\frac{f_i(\tilde{u})+\tilde{u}^T\tilde{z}_i}{g_i(\tilde{u})-\tilde{u}^T\tilde{v}_i}\Big)+\nabla_{\tilde{p}_i}L_i(\tilde{u},\tilde{p}_i)\Big\}\\
&\quad+\sum_{j=1}^{m}\tilde{\mu}_j\big\{\nabla_{\tilde{x}}(h_j(\tilde{u},\tilde{\tau})+\tilde{u}^T\tilde{w}_j^1)+\nabla_{\tilde{q}_j}S_j(\tilde{u},\tilde{q}_j)\big\}\Big\}\Bigg].
\end{aligned}$$

It further follows from (14)–(15), sublineairty of $\mathcal{F}$ and $\mathcal{F}_{x,u}(0)=0$ that

$$\sum_{i=1}^{k}\tilde{\lambda}_i\left[\frac{f_i(\tilde{x})+\tilde{x}^T\tilde{z}_i}{g_i(\tilde{x})-\tilde{x}^T\tilde{v}_i}-\frac{f_i(\tilde{u})+\tilde{u}^T\tilde{z}_i}{g_i(\tilde{u})-\tilde{u}^T\tilde{v}_i}-L_i(\tilde{u},\tilde{p}_i)+\tilde{p}_i^T\nabla_{\tilde{p}_i}L_i(\tilde{u},\tilde{p}_i)\right]>0. \tag{25}$$

But hypothesis (i), dual constraint (15) and $\tilde{\lambda}>0$ yield

$$\sum_{i=1}^{k}\tilde{\lambda}_i\left[\frac{f_i(\tilde{x})+\tilde{x}^T\tilde{z}_i}{g_i(\tilde{x})-\tilde{x}^T\tilde{v}_i}-\frac{f_i(\tilde{u})+\tilde{u}^T\tilde{z}_i}{g_i(\tilde{u})-\tilde{u}^T\tilde{v}_i}-L_i(\tilde{u},\tilde{p}_i)+\tilde{p}_i^T\nabla_{\tilde{p}_i}L_i(\tilde{u},\tilde{p}_i)\right]\leq 0.$$

This contradicts the inequality (25). Hence proved. $\square$

Example 4. Let in the problem (SIFP), $f:\mathbb{R}\to\mathbb{R}^2,\ g:\mathbb{R}\to\mathbb{R}^2$, and $h:\mathbb{R}\times[-0.5,\ 0]\to\mathbb{R}^2$ be given as:

$$\begin{aligned}
&f(x)=(f_1(x),\ f_2(x))=(4(x^4-1)^2,\ 3x^2),\\
&g(x)=(g_1(x),\ g_2(x))=(x^4+2,\ x^2+2)\text{ and}\\
&h(x,\ t)=(h_1(x,\ t),\ h_2(x,\ t))=(x+2t,\ x-t-1).
\end{aligned}$$

Let the cones be $K = \{(x,\ y) \in \mathbb{R}^2\ :\ x \geq 0,\ y \geq 0\}$ and $Q = \{(x,\ y) \in \mathbb{R}^2\ :\ y \geq 0,\ 3y \geq -x\}$. Also, suppose

$$\begin{aligned}
&L(u,\ p) = (L_1(u,\ p_1),\ L_2(u,\ p_2)) = (-p_1u^2,\ -2p_2(u+1)-1),\\
&S(u,\ q) = \big(S_1(u,\ q_1),\ S_2(u,\ q_2)\big) = \big(-5q_1u+2,\ q_2u-4\big),\\
&d_1^{(1)}(x,\ y) = d_2^{(1)}(x,\ y) = (1-y)(x^2-y^2) \text{ and}\\
&d_1^{(2)}(x,\ y) = x(y-1),\ d_2^{(2)}(x,\ y) = (1-y^2).
\end{aligned}$$

Let the sublinear functional be $\mathcal{F}_{x,u}(b) = -b(x^2-u^2)^2$ and $\alpha(x,\ y) = 1$. Let $C_1 = \{0\},\ C_2 = [-2,0]$, $D_2 = E_1 = [-1,\ 0]$, and $D_1 = E_2 = M_1 = M_2 = [0,\ 1]$ then the support functions be given by

$$\begin{aligned}
&\Omega(x|C_1) = \{0\},\ \Omega(x|C_2) = |x|-x,\ \Omega(x|D_2) = \Omega(x|E_1) = \frac{|x|-x}{2},\\
&\Omega(x|D_1) = \Omega(x|E_2) = \frac{|x|+x}{2} \text{ and } \Omega(t|M_1) = \Omega(t|M_2) = \frac{|t|+t}{2}.
\end{aligned}$$

Now, the feasible region for (SIFP) is

$$S_0 = \Big\{x \in \mathbb{R}\ :\ \Big(-x-2t-\frac{(|x|-x)}{2}+t,\ -x+t+1-\frac{(x+|x|)}{2}\Big) \in Q,\ \text{for all } t \in [-0.5,\ 0]\Big\}.$$

Clearly, $0,\ -1 \in S_0$. Also, one can easily verify that for the dual model (WD), $\beta_2 = \big(u,\ v,\ w^1,\ w^2,\ \lambda,\ \mu,\ z,\ p,\ q\big) = \Big(-1,\ (0,\ 0),\ (-1,\ 1),\ (0,\ 0),\ \Big(6,\ \frac{3}{4}\Big),\ (1,\ 2),\ (0,\ 0),\ (2,\ 0),\ (0,\ 2)\Big)$ and $\beta_3 = \big(u,\ v,\ w^1,\ w^2,\ \lambda,\ \mu,\ z,\ p,\ q\big) = \Big(0,\ (0,\ 0),\ (-1,\ 0),\ (0,\ 0),\ \Big(1,\ 2\Big),\ (1,\ 2),\ (0,\ 0),\ (1,\ 1),\ (0,\ 1)\Big)$ are feasible solutions.

Validation of Theorem 3.1:
At $u = -1$, for $z = (0,0),\ v = (0,0)$ and $\rho_1^{(1)} = \frac{1}{8},\ \rho_2^{(1)} = -\frac{1}{2}$, we obtain

$$\begin{aligned}
&\Bigg(\frac{f_1(x)+x^Tz_1}{g_1(x)-x^Tv_1} - \frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1} - L_1(u,p_1) + p_1^T\nabla_{p_1}L_1(u,p_1) - \mathcal{F}_{x,u}\Big[\alpha(x,u)\Big(\nabla_x\Big(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1}\Big)\\
&\quad + \nabla_{p_1}L_1(u,p_1)\Big)\Big] - \rho_1^{(1)}\big(d_1^{(1)}(x,u)\big)^2,\ \frac{f_2(x)+x^Tz_2}{g_2(x)-x^Tv_2} - \frac{f_2(u)+u^Tz_2}{g_2(u)-u^Tv_2} - L_2(u,p_2)\\
&\quad + p_2^T\nabla_{p_2}L_2(u,p_2) - \mathcal{F}_{x,u}\Big[\alpha(x,u)\Big(\nabla_x\Big(\frac{f_2(u)+u^Tz_2}{g_2(u)-u^Tv_2}\Big) + \nabla_{p_2}L_2(u,p_2)\Big)\Big] - \rho_2^{(1)}\big(d_2^{(1)}(x,u)\big)^2\Bigg)\\
&\quad = \Bigg(\frac{(x^2-1)^2(5x^4+16x^2+2)}{2(x^4+2)},\ \frac{3x^2}{x^2+2} + \frac{2}{3}(x^2-1)^2\Bigg) \in K \setminus \{0\},\ \text{for all } x \in \mathbb{R}.
\end{aligned}$$

Also, for all $x \in \mathbb{R},\ w_1^1 \in E_1,\ w_2^1 \in E_2,\ w_1^2 \in M_1,\ w_2^2 \in M_2,\ \tau \in T$ and $\rho_1^{(2)} = 0,\ \rho_2^{(2)} = 1$, we get

$$\begin{aligned}
&\Big(-(h_1(u,\tau)+u^Tw_1^1) - S_1(u,q_1) + q_1^T\nabla_{q_1}S_1(u,q_1) - \mathcal{F}_{x,u}\big[\alpha(x,u)\big(\nabla_x(h_1(u,\tau)+u^Tw_1^1)\\
&\quad + \nabla_{q_1}S_1(u,q_1)\big)\big] - \rho_1^{(2)}\ \big(d_1^{(2)}(x,u)\big)^2,\ -(h_2(u,\tau)+u^Tw_2^1) - S_2(u,p_2) + q_2^T\nabla_{q_2}S_2(u,q_2)
\end{aligned}$$

$$\begin{aligned}&-\mathcal{F}_{x,u}\big[\alpha(x,u)\big(\nabla_x(h_2(u,\tau)+u^Tw_2^1)+\nabla_{q_2}S_2(u,q_2)\big)\big]-\rho_2^{(2)}\big(d_2^{(2)}(x,u)\big)^2\Big)\\&=\Big(-1+w_1^1-2\tau+(6+w_1^1)(x^2-1)^2,\ 2+\tau+w_2^1(1+(x^2-1)^2)\Big)\in Q\setminus\{0\}.\end{aligned}$$

Hence, the hypothesis (i) of Theorem 3.1 hold.

Also, $\mathbb{R}^2_+=K$, $\mathbb{R}^2_+\subset Q$ and $\sum_{i=1}^{2}\lambda_i\rho_i^{(1)}\big(d_i^{(1)}(x,u)\big)^2+\sum_{j=1}^{2}\mu_j\{\rho_j^{(2)}\big(d_j^{(2)}(x,u)\big)^2-\tau^Tw_j^2\}=\frac{3}{2}(x^2-1)^2\geq 0.$

Thus, all the hypotheses of Theorem 3.1 are satisfied. Now for the feasible point (of (WD)) β_2, we get

$$\begin{aligned}&\left(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1}-L_1(u,\ p_1)+p_1^T\nabla_{p_1}L_1(u,p_1),\frac{f_2(u)+u^Tz_2}{g_2(u)-u^Tv_2}-L_2(u,\ p_2)+p_2^T\nabla_{p_2}L_2(u,p_2)\right)\\&\quad-\left(\frac{f_1(x)+\Omega(x|C_1)}{g_1(x)-\Omega(x|D_1)},\ \frac{f_2(x)+\Omega(x|C_2)}{g_2(x)-\Omega(x|D_2)}\right)=\begin{cases}\left(0,\ -\dfrac{1}{2}\right)\notin K/\{0\} & \text{at } x=-1,\\ \big(-2,\ 1\big)\notin K/\{0\} & \text{at } x=0.\end{cases}\end{aligned}$$

This validates the result of Theorem 3.1 for $x=0,-1\in S_0$ and β_2 feasible for (WD).

Validation of Theorem 3.3:
For the points β_2 and $x=-1$, it has been proved above that the assumptions (ii), (iii) and (iv) of Theorem 3.3 are satisfied. Further,

$$\begin{aligned}&\Bigg(\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1}+L_1(u,\ p_1)-p_1^T\nabla_{p_1}L(u,\ p_1)-\sum_{j=1}^{2}\mu_j\{h_j(u,\tau)+u^Tw_j^1+\tau^Tw_j^2+S_j(u,q_j)-q_j^T\nabla_{q_j}S_j(u,q_j)\},\\&\frac{f_2(u)+u^Tz_2}{g_2(u)-u^Tv_2}+L_2(u,\ p_2)-p_2^T\nabla_{p_2}L(u,\ p_2)-\sum_{j=1}^{2}\mu_j\{h_j(u,\tau)+u^Tw_j^1+\tau^Tw_j^2+S_j(u,q_j)-q_j^T\nabla_{q_j}S_j(u,q_j)\}\Bigg)\\&-\left(\frac{f_1(x)+\tilde{x}^Tz_1}{g_1(x)-x^Tv_1},\ \frac{f_2(x)+x^Tz_2}{g_2(x)-x^Tv_2}\right)=(0,\ -1)\notin K.\end{aligned}$$

Hence, the assumption (i) also holds at $x=u=-1$. This completes the validation of Theorem 3.3.

Without the assumption (i) of Theorem 3.1:
For any $\rho_1^{(1)},\ \rho_2^{(1)}\ \rho_1^{(2)},\ \rho_2^{(2)}\in\mathbb{R}$, at β_3, for (WD),

$$\sum_{i=1}^{2}\lambda_i\rho_i^{(1)}\big(d_i^{(1)}(x,u)\big)^2+\sum_{j=1}^{2}\mu_j\{\rho_j^{(2)}\big(d_j^{(2)}(x,u)\big)^2-\tau^Tw_j^2\}=0$$

and $\mathbb{R}^2_+=K$, $\mathbb{R}^2_+\subset Q$. Therefore, assumption $(ii)-(iii)$ of Theorem 3.1 are satisfied but at $x=0$,

$$\begin{aligned}&-(h_1(u,\tau)+u^Tw_1^1)-S_1(u,q_1)+q_1^T\nabla_{q_1}S_1(u,q_1)-\mathcal{F}_{x,u}\big[\alpha(x,u)\big(\nabla_x(h_1(u,\tau)+u^Tw_1^1)\\&+\nabla_{q_1}S_1(u,q_1)\big)\big]-\rho_1^{(2)}\ \big(d_1^{(2)}(x,u)\big)^2\\&=-7-2\tau<0,\ \text{for all } \tau\in[-0.5,\ 0].\end{aligned}$$

Hence, the pair $\left(\left(\frac{f_1(.)+(.)^T z_1}{g_1(.)-(.)^T v_1},\ \frac{f_2(.)+(.)^T z_2}{g_2(.)-(.)^T v_2}\right),\ (h_1(.\ ,\ \tau)+(.)^T w_1^1,\ h_2(.\ ,\ \tau)+(.)^T w_2^1)\right)$ is not $(K\times Q)-(\mathcal{F},\alpha,\rho,d)$ −type I convex. On the other hand, at $x=0$ and β_3,

$$\left(\frac{f_1(u)+u^T z_1}{g_1(u)-u^T v_1}-L_1(u,\ p_1)+p_1^T\nabla_{p_1}L_1(u,p_1),\ \frac{f_2(u)+u^T z_2}{g_2(u)-u^T v_2}-L_2(u,\ p_2)+p_2^T\nabla_{p_2}L_2(u,p_2)\right)$$
$$-\left(\frac{f_1(x)+\Omega(x|C_1)}{g_1(x)-\Omega(x|D_1)},\ \frac{f_2(x)+\Omega(x|C_2)}{g_2(x)-\Omega(x|D_2)}\right)=\Big(0,\ 1\Big)\in K\setminus\{0\}.$$

This shows the significance of assumption (i) in Theorem 3.1, without which the result may not satisfy. $\square$

4. SCHAIBLE TYPE DUAL

For the (SIFP) model, consider the following Schaible type higher order dual:

(SD) $K-\max\big(\eta_1,\ \eta_2,\ \ldots,\ \eta_k\big)$
subject to

$$\sum_{i=1}^{k}\lambda_i\nabla_x\Big(f_i(u)+u^T z_i-\eta_i(g_i(u)-u^T v_i)\Big)+\sum_{j=1}^{m}\mu_j\nabla_x\Big(h_j(u,\tau)+u^T w_j^1+\tau^T w_j^2\Big)$$
$$+\sum_{i=1}^{k}\lambda_i\nabla_{p_i}\Big[G_i(u,\ p_i)-\eta_i L_i(u,\ p_i)\Big]+\sum_{j=1}^{m}\mu_j\nabla_{q_j}S_j(u,\ q_j)=0, \tag{26}$$

$$\sum_{i=1}^{k}\lambda_i\Big\{f_i(u)+u^T z_i-\eta_i(g_i(u)-u^T v_i)+G_i(u,p_i)-\eta_i L_i(u,p_i)$$
$$-p_i^T\nabla_{p_i}\big(G_i(u,p_i)-\eta_i L_i(u,p_i)\big)\Big\}\geq 0, \tag{27}$$

$$\sum_{j=1}^{m}\mu_j\Big\{h_j(u,\tau)+u^T w_j^1+\tau^T w_j^2+S_j(u,\ q_j)-q_j^T\nabla_{q_j}S_j(u,\ q_j)\Big\}\geq 0, \tag{28}$$

$z_i\in C_i,\ v_i\in D_i,\ w_j\in E_j,\ \eta\in\mathbb{R}^k_+,\ i\in\tilde{I},\ j\in\tilde{J},\ (\lambda,\ \mu)\in \text{int}\ K^*\times\text{int}\ Q^*,\ (\lambda,\ \mu)\neq(0,0)$ and $\tau\in T$. Throughout this section, we have used $\rho_i^{(1)}=\big(\rho_{f_i}^{(1)},\ \rho_{g_i}^{(1)}\big)\in\mathbb{R}\times\mathbb{R}$ and $d_i^{(1)}=\big(d_{f_i}^{(1)},\ d_{g_i}^{(1)}\big)\in\mathbb{R}\times\mathbb{R}$.

Theorem 4.1. [*Weak duality*] *Let for every feasible solution x of (SIFP) and $(u,\ v,\ w^1,\ w^2,\ \lambda,\ \mu,\ z,\ \eta,\ p,\ q)$ of (SD), $\mathcal{F}:B\times B\times\mathbb{R}^n\to\mathbb{R}$ be a sublinear functional (in third variable) and*

(i) $\big((f_1(.)+(.)^T z_1,...,f_k(.)+(.)^T z_k),\ (h_1(.,\tau)+(.)^T w_1^1,...,h_m(.,\tau)+(.)^T w_m^1)\big)$ *be higher order* $(K\times Q)-(\mathcal{F},\ \alpha,\ \rho,\ d)-$ *type* I *convex with respect to* G *and* S*,*

(ii) $\big((-\eta_1(g_1(.)-(.)^T v_1),...,-\eta_k(g_k(.)-(.)^T v_k),\ (h_1(.,\tau)+(.)^T w_1^1,...,h_m(.,\tau)+(.)^T w_m^1)\big)$ *be higher order* $(K\times Q)-(\mathcal{F},\ \alpha,\ \rho,\ d)-$ *type* I *convex with respect to* $-\eta L$ *and* S*,*

(iii) $\sum_{i=1}^{k}\lambda_i\Big\{\rho_{f_i}^{(1)}\big(d_{f_i}^{(1)}(x,u)\big)^2+\rho_{g_i}^{(1)}\big(d_{g_i}^{(1)}(x,u)\big)^2\Big\}+\sum_{j=1}^{m}\mu_j\{\rho_j^{(2)}\big(d_j^{(2)}(x,u)\big)^2-\tau^T w_j^2\}\geq 0$ *and*

(iv) $K \supseteq \mathbb{R}^k_+$, $Q \supseteq \mathbb{R}^m_+$.

Then

$$\left[(\eta_1,...,\eta_k) - \left(\frac{f_1(x)+\Omega(x|C_1)}{g_1(x)-\Omega(x|D_1)},...,\frac{f_k(x)+\Omega(x|C_k)}{g_k(x)-\Omega(x|D_k)}\right)\right] \notin K \setminus \{0\}. \tag{29}$$

Proof. Let if possible (29) be not true. Then by hypothesis (iv), $\lambda \in \text{int}K^* \subseteq \text{int } \mathbb{R}^k_+$, we have

$$\sum_{i=1}^{k} \lambda_i \left[\eta_i - \frac{f_i(x)+\Omega(x|C_i)}{g_i(x)-\Omega(x|D_i)}\right] > 0. \tag{30}$$

Since $x^T z_i \leq \Omega(x|C_i)$ and $x^T v_i \leq \Omega(x|D_i)$, we obtain

$$\frac{f_i(x)+\Omega(x|C_i)}{g_i(x)-\Omega(x|D_i)} - \frac{f_i(x)+x^T z_i}{g_i(x)-x^T v_i} \geq 0.$$

It follows from $\lambda_i > 0$, for all i, that

$$\sum_{i=1}^{k} \lambda_i \left(\frac{f_i(x)+\Omega(x|C_i)}{g_i(x)-\Omega(x|D_i)} - \frac{f_i(x)+x^T z_i}{g_i(x)-x^T v_i}\right) \geq 0. \tag{31}$$

Further, adding (30) and (31), we get

$$\sum_{i=1}^{k} \lambda_i \Big[\eta_i (g_i(x) - x^T v_i) - (f_i(x) + x^T z_i)\Big] > 0. \tag{32}$$

Now, for $\lambda \in \text{int } K^*$ and $\mu \in \text{int } Q^*$, hypotheses (i) and (ii) imply

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i \Big[& f_i(x) + x^T z_i - (f_i(u) + u^T z_i) - G_i(u,p_i) + p_i^T \nabla_{p_i} G_i(u,p_i) \\ & - \mathcal{F}_{x,u}\big(\alpha(x,u)(\nabla_x (f_i(u) + u^T z_i) + \nabla_{p_i} G_i(u,p_i))\big) - \rho^{(1)}_{f_i}\big(d^{(1)}_{f_i}(x,u)\big)^2\Big] \geq 0.\end{aligned} \tag{33}$$

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i \Bigg[& \eta_i(-(g_i(x) - x^T v_i) + (g_i(u) - u^T v_i)) + \eta_i L_i(u,p_i) - \eta_i p_i^T \nabla_{p_i} L_i(u,p_i) \\ & - \mathcal{F}_{x,u}\Big(- \eta_i \alpha(x,u)\big(\nabla_x (g_i(u) - u^T v_i) + \nabla_{p_i} L_i(u,p_i)\big)\Big) - \rho^{(1)}_{g_i}(d^{(1)}_{g_i}(x,u))^2\Bigg] \geq 0.\end{aligned} \tag{34}$$

$$\begin{aligned}\sum_{j=1}^{m}\mu_j \Bigg[& - (h_j(u) + u^T w^1_j) - S_j(u,q_j) + q_j^T \nabla_{q_j} S_j(u,q_j)\Big\} \\ & - \mathcal{F}_{x,u}\Big(\alpha(x,u)\big(\nabla_x (h_j(u) + u^T w^1_j) + \nabla_{q_j} S_j(u,q_j)\big)\Big) - \sum_{j=1}^{m} \mu_j \rho^{(2)}_j \big(d^{(2)}_j(x,u)\big)^2 \geq 0.\end{aligned} \tag{35}$$

From $\lambda_i > 0$, for all i and sublinearity of $\mathcal{F}$, the inequalities (33) and (34) yield

$$\begin{aligned}
&\sum_{i=1}^{k}\lambda_i\Big\{f_i(x)+x^Tz_i-(f_i(u)+u^Tz_i)-G_i(u,p_i)+p_i^T\nabla_{p_i}G_i(u,p_i)\\
&\quad -\eta_i(g_i(x)-x^Tv_i)+\eta_i(g_i(u)-u^Tv_i)+\eta_iL_i(u,p_i)-\eta_ip_i^T\nabla_{p_i}L_i(u,p_i)\Big\}\\
&\quad\geq\sum_{i=1}^{k}\lambda_i\mathcal{F}_{x,u}\Big(\alpha(x,u)\big(\nabla_x(f_i(u)+u^Tz_i)+\nabla_{p_i}G_i(u,p_i)-\eta_i(\nabla_x(g_i(u)-u^Tv_i)\\
&\qquad\qquad +\nabla_{p_i}L_i(u,p_i))\big)\Big)+\sum_{i=1}^{k}\lambda_i\Big\{\rho_{f_i}^{(1)}\big(d_{f_i}^{(1)}(x,u)\big)^2+\rho_{g_i}^{(1)}\big(d_{g_i}^{(1)}(x,u)\big)^2\Big\}.
\end{aligned}$$

Further using (27), we get

$$\begin{aligned}
&\sum_{i=1}^{k}\lambda_i\Big\{f_i(x)+x^Tz_i-\eta_i(g_i(x)-x^Tv_i)\Big\}\\
&\quad\geq\sum_{i=1}^{k}\lambda_i\mathcal{F}_{x,u}\Big(\alpha(x,u)\big(\nabla_x(f_i(u)+u^Tz_i)+\nabla_{p_i}G_i(u,p_i)-\eta_i(\nabla_x(g_i(u)-u^Tv_i)\\
&\qquad\qquad +\nabla_{p_i}L_i(u,p_i))\big)\Big)+\sum_{i=1}^{k}\lambda_i\Big\{\rho_{f_i}^{(1)}\big(d_{f_i}^{(1)}(x,u)\big)^2+\rho_{g_i}^{(1)}\big(d_{g_i}^{(1)}(x,u)\big)^2\Big\}.
\end{aligned}\tag{36}$$

The inequality (35) together with sublinearity of $\mathcal{F}$ yield

$$\begin{aligned}
&\sum_{j=1}^{m}\mu_j\Big\{-(h_j(u)+u^Tw_j^1)-S_j(u,q_j)+q_j^T\nabla_{q_j}S_j(u,q_j)\Big\}\\
&\geq\mathcal{F}_{x,u}\Big(\alpha(x,u)\sum_{j=1}^{m}\mu_j\big(\nabla_x(h_j(u)+u^Tw_j^1)+\nabla_{q_j}S_j(u,q_j)\big)\Big)+\sum_{j=1}^{m}\mu_j\rho_j^{(2)}\big(d_j^{(2)}(x,u)\big)^2.
\end{aligned}\tag{37}$$

It follows from (36), (37) and hypothesis (iii) that

$$\begin{aligned}
&\sum_{i=1}^{k}\lambda_i\Big\{f_i(x)+x^Tz_i+\eta_i(-(g_i(x)-x^Tv_i))\Big\}\\
&\quad+\sum_{j=1}^{m}\mu_j\Big\{-(h_j(u)+u^Tw_j^1+\tau^Tw_j^2)-S_j(u,q_j)+q_j^T\nabla_{q_j}S_j(u,q_j)\Big\}\\
&\geq\mathcal{F}_{x,u}\Big[\alpha(x,u)\Big(\sum_{i=1}^{k}\lambda_i\big\{\big(\nabla_x(f_i(u)+u^Tz_i)+\nabla_{p_i}G_i(u,p_i)\big)-\eta_i(\nabla_x(g_i(u)-u^Tv_i)\\
&\quad+\nabla_{p_i}L_i(u,p_i))\big\}+\sum_{j=1}^{m}\mu_j\big(\nabla_x(h_j(u)+u^Tw_j^1)+\nabla_{q_j}S_j(u,q_j)\big)\Big)\Big].
\end{aligned}$$

Finally, using (26) and (28), we get

$$\sum_{i=1}^{k}\lambda_i\Big[\eta_i(g_i(x)-x^Tv_i)-(f_i(x)+x^Tz_i)\Big]\leq 0.\tag{38}$$

This contradicts the inequality (32). Hence the result. □

Theorem 4.2. [*Strong duality*] *Let* $\check{x}$ *be a weakly efficient solution of (SIFP) and the suitable constraint qualification be satisfied at* $\check{x}$. *Then for* $L_i(\check{x},0) = G_i(\check{x},0) = S_j(\check{x},0) = 0,\ \nabla_{\check{p}_i} L_i(\check{x},0) = \nabla_{\check{q}_j} S_j(\check{x},0) = \nabla_{\check{p}_i} G_i(\check{x},0) = 0,\ i = 1,2,...,k,\ j = 1,2,...,m,$ *there exist* $(0,0) \neq (\check{\lambda},\check{\mu}) \in \mathit{int}\, K^* \times \mathit{int}\, Q^*,\ \check{\eta} \in \mathbb{R}^k_+,\ \check{z}_i \in C_i,\ \check{v}_i \in D_i,\ \check{w}^1_j \in E_j,\ \check{w}^2_j \in M_j,\ i \in \tilde{I},\ j \in \tilde{J}$ *such that* $(\check{x},\ \check{v},\ \check{w}^1,\ \check{w}^2,\ \check{\lambda},\ \check{\mu},\ \check{z},\ \check{\eta},\ \check{p} = 0,\ \check{q} = 0)$ *is feasible point for (SD) and the objective function values are same for both problems. Moreover, if for every feasible point* $\check{x}$ *and* $(\check{u},\ \check{v},\ \check{w}^1,\ \check{w}^2,\ \check{\lambda},\ \check{\mu},\ \check{z},\ \check{\eta},\ \check{p},\ \check{q})$ *of (SIFP) and (SD) respectively, all assumptions of Theorem 4.1 are satisfied then* $(\check{x},\ \check{v},\ \check{w}^1,\ \check{w}^2,\ \check{\lambda},\ \check{\mu},\ \check{z},\ \check{\eta},\ \check{p} = 0,\ \check{q} = 0)$ *is an efficient solution of (SD).*

Proof. It follows on the lines of Theorem 3.2. □

Theorem 4.3. [*Strict converse duality*] *Let* $\check{x}$ *be feasible for (SIFP) and* $(\check{u},\ \check{v},\ \check{w}^1,\ \check{w}^2,\ \check{\lambda},\ \check{\mu},\ \check{z},\ \check{\eta},\ \check{p},\ \check{q})$ *be feasible for (SD). Let a sublinear functional (in third variable) be* $\mathcal{F} : B \times B \times \mathbb{R}^n \to \mathbb{R}$ *such that*

(i) $(\check{\eta}_1, ..., \check{\eta}_k) - \left(\dfrac{f_1(\check{x}) + \check{x}^T \check{z}_1}{g_1(\check{x}) - \check{x}^T \check{v}_1}, ..., \dfrac{f_k(\check{x}) + \check{x}^T \check{z}_k}{g_k(\check{x}) - \check{x}^T \check{v}_k}\right) \in K,$

(ii) $\big((f_1(.)+(.)^T \check{z}_1, ..., f_k(.)+(.)^T \check{z}_k),\ (h_1(.,\check{\tau})+(.)^T \check{w}^1_1, ..., h_m(.,\check{\tau})+(.)^T \check{w}^1_m)\big)$ *be strictly higher order* $(K \times Q) - (\mathcal{F},\ \alpha,\ \rho,\ d)-$*type I convex with respect to* G *and* S,

(iii) $\big((-\check{\eta}_1(g_1(.)-(.)^T \check{v}_1)), ..., (-\check{\eta}_k(g_k(.)-(.)^T \check{v}_k)), (h_1(.,\check{\tau})+(.)^T \check{w}^1_1, ..., h_m(.,\check{\tau})+(.)^T \check{w}^1_m)\big)$ *be strictly higher order* $(K \times Q) - (\mathcal{F},\ \alpha,\ \rho,\ d)-$*type I convex with respect to* $-\check{\eta}L$ *and* S,

(iv) $\sum_{i=1}^{k} \check{\lambda}_i \Big\{\rho^{(1)}_{f_i}\big(d^{(1)}_{f_i}(\check{x},\check{u})\big)^2 + \rho^{(1)}_{g_i}\big(d^{(1)}_{g_i}(\check{x},\check{u})\big)^2\Big\} + \sum_{j=1}^{m} \check{\mu}_j \{\rho^{(2)}_j \big(d^{(2)}_j(\check{x},\check{u})\big)^2 - \check{\tau}^T \check{w}^1_j\} \geq 0$ *and* $K \supseteq \mathbb{R}^k_+,\ Q \supseteq \mathbb{R}^m_+.$

Then $\check{x} = \check{u}$.

Proof. We will show the proof by contradiction. Let $\check{x} \neq \check{u}$. Then, by hypotheses (ii) and (iii), we have

$$\sum_{i=1}^{k} \check{\lambda}_i \Bigg[f_i(\check{x}) + \check{x}^T \check{z}_i - (f_i(\check{u}) + \check{u}^T \check{z}_i) - G_i(\check{u},\check{p}_i) + \check{p}_i^T \nabla_{\check{p}_i} G_i(\check{u},\check{p}_i)$$

$$- \mathcal{F}_{\check{x},\check{u}}\big(\alpha(\check{x},\check{u})(\nabla_{\check{x}}(f_i(\check{u}) + \check{u}^T \check{z}_i) + \nabla_{\check{p}_i} G_i(\check{u},\check{p}_i))\big) - \rho^{(i)}_{f_i}\big(d^{(i)}_{f_i}(\check{x},\check{u})\big)^2 \Bigg] > 0. \tag{39}$$

$$\sum_{i=1}^{k} \check{\lambda}_i \Bigg[\check{\eta}_i(-(g_i(\check{x}) - \check{x}^T \check{v}_i) + g_i(\check{u}) - \check{u}^T \check{v}_i) + \check{\eta}_i L_i(\check{u},\check{p}_i) - \check{\eta}_i \check{p}_i^T \nabla_{\check{p}_i} L_i(\check{u},\check{p}_i)$$

$$- \mathcal{F}_{\check{x},\check{u}}\Big(\alpha(\check{x},\check{u})\big(- \check{\eta}_i(\nabla_{\check{x}}(g_i(\check{u}) - \check{u}^T \check{v}_i) + \nabla_{\check{p}_i} L_i(\check{u},\check{p}_i))\big)\Big) - \rho^{(i)}_{g_i}(d^{(i)}_{g_i}(\check{x},\check{u}))^2 \Bigg] > 0. \tag{40}$$

$$\sum_{j=1}^{m} \check{\mu}_j \Bigg[- (h_j(\check{u}) + \check{u}^T \check{w}^1_j) - S_j(\check{u},\check{q}_j) + \check{q}_j^T \nabla_{\check{q}_j} S_j(\check{u},\check{q}_j)$$

$$- F_{\check{x},\check{u}}\Big(\alpha(\check{x},\check{u})\big(\nabla_{\check{x}}(h_j(\check{u}) + \check{u}^T \check{w}^1_j) + \nabla_{\check{q}_j} S_j(\check{u},\check{q}_j)\big)\Big) - \rho^{(2)}_j \big(d^{(2)}_j(\check{x},\check{u})\big)^2 \Bigg] > 0 \tag{41}$$

where $\check{\lambda} \in \text{int } K^*$ and $\check{\mu} \in \text{int } Q^*$. From (39), (40), hypothesis (iv) and sublinearity of $\mathcal{F}$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{k} \check{\lambda}_i \Big\{ f_i(\check{x}) + \check{x}^T \check{z}_i - (f_i(\check{u}) + \check{u}^T \check{z}_i) - G_i(\check{u}, \check{p}_i) + \check{p}_i^T \nabla_{\check{p}_i} G_i(\check{u}, \check{p}_i) \\
&+ \check{\eta}_i(-(g_i(\check{x}) - \check{x}^T \check{v}_i)) + \check{\eta}_i(g_i(\check{u}) - \check{u}^T \check{v}_i) + \check{\eta}_i L_i(\check{u}, \check{p}_i) - \check{\eta}_i \check{p}_i{}^T \nabla_{\check{p}_i} L_i(\check{u}, \check{p}_i) \Big\} \\
&> \mathcal{F}_{\check{x},\check{u}} \Big(\alpha(\check{x}, \check{u}) \sum_{i=1}^{k} \check{\lambda}_i \Big\{ \big(\nabla_{\check{x}}(f_i(\check{u}) + \check{u}^T \check{z}_i) + \nabla_{\check{p}_i} G_i(\check{u}, \check{p}_i) - \check{\eta}_i (\nabla_{\check{x}}(g_i(\check{u}) - \check{u}^T \check{v}_i) \\
&\quad + \nabla_{\check{p}_i} L_i(\check{u}, \check{p}_i))) \Big\} \Big) + \sum_{i=1}^{k} \check{\lambda}_i \Big\{ \rho_{f_i}^{(1)} \big(d_{f_i}^{(1)}(\check{x}, \check{u}) \big)^2 + \rho_{g_i}^{(1)} \big(d_{g_i}^{(1)}(\check{x}, \check{u}) \big)^2 \Big\}.
\end{aligned}$$

Now, using (27) in the above inequality, we get

$$\begin{aligned}
&\sum_{i=1}^{k} \check{\lambda}_i \Big\{ f_i(\check{x}) + \check{x}^T \check{z}_i + \check{\eta}_i(-(g_i(\check{x}) - \check{x}^T \check{v}_i)) \Big\} \\
&> \mathcal{F}_{\check{x},\check{u}} \Big(\alpha \sum_{i=1}^{k} \check{\lambda}_i \Big\{ (\check{x}, \check{u}) \big(\nabla_{\check{x}}(f_i(\check{u}) + \check{u}^T \check{z}_i) + \nabla_{\check{p}_i} G_i(\check{u}, \check{p}_i) - \check{\eta}_i (\nabla_{\check{x}}(g_i(\check{u}) - \check{u}^T \check{v}_i) \\
&\quad + \nabla_{\check{p}_i} L_i(\check{u}, \check{p}_i))) \Big\} \Big) + \sum_{i=1}^{k} \check{\lambda}_i \Big\{ \rho_{f_i}^{(1)} \big(d_{f_i}^{(1)}(\check{x}, \check{u}) \big)^2 + \rho_{g_i}^{(1)} \big(d_{g_i}^{(1)}(\check{x}, \check{u}) \big)^2 \Big\}. \qquad (42)
\end{aligned}$$

Since $\check{\mu} \in \text{int}Q^* \subseteq \text{int } \mathbb{R}^m_+$, and $\mathcal{F}$ is sublinear, the inequality (41) further yields

$$\begin{aligned}
&\sum_{j=1}^{m} \check{\mu}_j \Big\{ -(h_j(\check{u}) + \check{u}^T \check{w}_j^1) - S_j(\check{u}, \check{q}_j) + \check{q}_j^T \nabla_{\check{q}_j} S_j(\check{u}, \check{q}_j) \Big\} \\
&> \mathcal{F}_{\check{x},\check{u}} \Big(\alpha(\check{x}, \check{u}) \sum_{j=1}^{m} \check{\mu}_j \big(\nabla_{\check{x}}(h_j(\check{u}) + \check{u}^T \check{w}_j^1) + \nabla_{\check{q}_j} S_j(\check{u}, \check{q}_j) \big) \Big) + \sum_{j=1}^{m} \check{\mu}_j \rho_j^{(2)} \big(d_j^{(2)}(\check{x}, \check{u}) \big)^2. \qquad (43)
\end{aligned}$$

Applying (28) in (43) and then adding with (42), we obtain

$$\begin{aligned}
&\sum_{i=1}^{k} \check{\lambda}_i \Big\{ f_i(\check{x}) + \check{x}^T \check{z}_i + \check{\eta}_i(-(g_i(\check{x}) - \check{x}^T \check{v}_i)) \Big\} \\
&> \mathcal{F}_{\check{x},\check{u}} \Big(\alpha(\check{x}, \check{u}) \sum_{i=1}^{k} \check{\lambda}_i \big(\nabla_{\check{x}}(f_i(\check{u}) + \check{u}^T \check{z}_i) + \nabla_{\check{p}_i} G_i(\check{u}, \check{p}_i) - \check{\eta}_i (\nabla_{\check{x}}(g_i(\check{u}) - \check{u}^T \check{v}_i) \\
&\quad + \nabla_{\check{p}_i} L_i(\check{u}, \check{p}_i)) \big) \Big) + \mathcal{F}_{\check{x},\check{u}} \Big(\alpha(\check{x}, \check{u}) \sum_{j=1}^{m} \check{\mu}_j \big(\nabla_{\check{x}}(h_j(\check{u}) + \check{u}^T \check{w}_j^1) + \nabla_{\check{q}_j} S_j(\check{u}, \check{q}_j) \big) \Big) \\
&+ \sum_{i=1}^{k} \check{\lambda}_i \Big\{ \rho_{f_i}^{(1)} \big(d_{f_i}^{(1)}(\check{x}, \check{u}) \big)^2 + \rho_{g_i}^{(1)} \big(d_{g_i}^{(1)}(\check{x}, \check{u}) \big)^2 \Big\} + \sum_{j=1}^{m} \check{\mu}_j \{ \rho_j^{(2)} \big(d_j^{(2)}(\check{x}, \check{u}) \big)^2 - \check{\tau}^T \check{w}_j^1 \}. \qquad (44)
\end{aligned}$$

Further, using hypothesis (iv), dual constraint (26) and $\mathcal{F}_{x,u}(0)=0$, we have

$$\sum_{i=1}^{k}\check{\lambda}_i\Big[\check{\eta}_i(g_i(\check{x})-\check{x}^T\check{v}_i)-(f_i(\check{x})+\check{x}^T\check{z}_i)\Big]<0. \tag{45}$$

But from hypothesis (i) and $\check{\lambda}\in \operatorname{int} K^*$, we have

$$\sum_{i=1}^{k}\check{\lambda}_i[f_i(\check{x})+\check{x}^T\check{z}_i-\check{\eta}_i(g_i(\check{x})-\check{x}^T\check{v}_i)]\leq 0.$$

which contradicts (45). Hence the result. □

Example 5. Let in the problem (SIFP), $f:\mathbb{R}\to\mathbb{R}^2,\ g:\mathbb{R}\to\mathbb{R}^2$, and $h:\mathbb{R}\times[-2,\ -1]\to\mathbb{R}^2$ be given as:

$$\begin{aligned}
&f(x)=(f_1(x),\ f_2(x))=(e^{x-1},\ x^2+12),\\
&g(x)=(g_1(x),\ g_2(x))=(x^2+1,\ 13(2+x^2))\text{ and}\\
&h(x,\ t)=(h_1(x,\ t),\ h_2(x,\ t))=(x+3t,\ x+t-1).
\end{aligned}$$

Let

$$\begin{aligned}
&K=\{(x,\ y)\in\mathbb{R}^2\ :\ x\geq 0,\ y\geq -2x\},\\
&Q=\{(x,\ y)\in\mathbb{R}^2\ :\ x\geq 0,\ y\geq 0\},\\
&G(u,\ p)=(G_1(u,\ p_1),\ G_2(u,\ p_2))=(p_1-u^4,\ -(p_2u^2+5)),\\
&L(u,\ p)=(L_1(u,\ p_1),\ L_2(u,\ p_2))=(p_1u^2+5,\ e^{-p_2u}),\\
&S(u,\ q)=\big(S_1(u,\ q_1),\ S_2(u,\ q_2)\big)=\Big(q_1(u+1)^2+2,\ q_2(u-1)+3\Big).
\end{aligned}$$

Further, let $\alpha(x,\ y)=(2-y)^2$ and $\mathcal{F}_{x,u}(b)=-bx^2u$. Consider

$$C_1=[0,\ 2]=M_2,\ C_2=D_2=E_2=[-1,\ 1]\text{ and }D_1=E_1=M_1=[-2,\ 0].$$

Then, their support functions will be

$$\begin{aligned}
&\Omega(x|C_1)=x+|x|,\ \Omega(x|C_2)=\Omega(x|D_2)=\Omega(x|E_2)=|x|,\ \Omega(t|M_1)=|t|-t,\\
&\Omega(x|D_1)=\Omega(x|E_1)=|x|-x\text{ and }\Omega(t|M_2)=t+|t|.
\end{aligned}$$

The feasible set of the given problem:

$$S_0=\Big\{x\in\mathbb{R}\ :\ \Big(-2t-|x|-|t|,\ 1-x-|x|-2t-|t|\Big)\in Q,\ \text{for all }t\in[-2,\ -1]\Big\}=[-1,\ 1].$$

Also the point $\beta_4=\big(u,\ v,\ w^1,\ w^2,\ \lambda,\ \mu,\ z,\ \eta,\ p,\ q\big)=\Big(1,\ (0,\ 0),\ (-1,\ 1),\ (-2,\ 0),\ (1,\ 1),\ \Big(\frac{1}{2},\ \frac{1}{2}\Big),$ $(0,\ 0),\ (1,\ 0),\ (1,\ 2),\ (1,\ 2)\Big)$ satisfies the dual constraints (26)–(28) and hence feasible for (SD).

Validation for Theorem 4.1:
For $u=1$, taking $z_1=z_2=0,\ \rho^{(1)}_{f_1}=\rho^{(1)}_{f_2}=1$ and $d^{(1)}_{f_1}(x,y)=xy,\ d^{(1)}_{f_2}(x,y)=y+1$, we obtain

$$\begin{aligned}
&\Big(f_1(x)+x^Tz_1-(f_1(u)+u^Tz_1)-G_1(u,p_1)+p_1^T\nabla_{p_1}G_1(u,p_1)-\mathcal{F}_{x,u}\big(\alpha(x,u)(\nabla_x(f_1(u)+u^Tz_1)\\
&\quad+\nabla_{p_1}(G_1(u,p_1))\big)-\rho^{(1)}_{f_1}\big(d^{(1)}_{f_1}(x,u)\big)^2,\ f_2(x)+x^Tz_2-(f_2(u)+u^Tz_2)-G_2(u,p_2)+p_2^T\nabla_{p_2}G_2(u,p_2)
\end{aligned}$$

$$
\begin{aligned}
&- \mathcal{F}_{x,u}\big(\alpha(x,u)(\nabla_x(f_2(u) + u^T z_2) + \nabla_{p_2}(G_2(u,p_2)))\big) - \rho_{f_2}^{(1)}\big(d_{f_2}^{(1)}(x,u)\big)^2\Big) \\
&\quad = \big(e^{x-1} + x^2,\ 2x^2\big) \in K \setminus \{0\},\ \text{for all } x \in \mathbb{R}.
\end{aligned}
$$

Also, for $\eta_1 = 1,\ \eta_2 = 0,\ v_1 = v_2 = 0,\ \rho_{g_1}^{(1)} = 1,\ \rho_{g_2}^{(1)} = 0$ and $d_{g_1}^{(1)}(x,y) = x(y-1),\ d_{g_2}^{(1)}(x,y) = x+y$, we get

$$
\begin{aligned}
&\Big(\eta_1(-(g_1(x) - x^T v_1) + g_1(u) - u^T v_1) + \eta_1 L_1(u,p_1) - \eta_1 p_1^T \nabla_{p_1} L_1(u,p_1) - \mathcal{F}_{x,u}\big(-\eta_1\alpha(x,u)\big(\nabla(g_1(u) \\
&\quad - u^T v_1) + \nabla_{p_1} L_1(u,p_1)\big)\big) - \rho_{g_1}^{(1)}(d_{g_1}^{(1)}(x,u))^2,\ \eta_2(-(g_2(x) - x^T v_2) + g_2(u) - u^T v_2) + \eta_2 L_2(u,p_2) \\
&\quad - \eta_2 p_2^T \nabla_{p_2} L_2(u,p_2) - \mathcal{F}_{x,u}\big(-\eta_2\alpha(x,u)(\nabla(g_2(u) - u^T v_2) + \nabla_{p_2} L_2(u,p_2)) - \rho_{g_2}^{(1)}(d_{g_2}^{(1)}(x,u))^2\big)\Big) \\
&\quad = \big(2x^2 + 6,\ 0\big) \in K \setminus \{0\},\ \text{for all } x \in \mathbb{R}
\end{aligned}
$$

and for $\rho_1^{(2)} = 0,\ \rho_2^{(2)} = -1,\ d_1^{(2)}(x,y) = xy,\ d_2^{(2)}(x,y) = y+1$, we have

$$
\begin{aligned}
&\Big(-(h_1(u,\ \tau) + u^T w_1^1) - S_1(u,q_1) + q_1^T \nabla_{q_1} S_1(u,q_1) - \mathcal{F}_{x,u}\Big(\alpha(x,u)\big(\nabla_x(h_1(u,\ \tau) + u^T w_1^1) + \nabla_{q_1} S_1(u,q_1)\big)\Big) \\
&- \rho_1^{(2)}\big(d_1^{(2)}(x,u)\big)^2,\ -(h_2(u,\ \tau) + u^T w_2^1) - S_2(u,q_2) + q_2^T \nabla_{q_2} S_2(u,q_2) - \mathcal{F}_{x,u}\Big(\alpha(x,u)\big(\nabla_x(h_2(u,\ \tau) \\
&+ u^T w_2^1) + \nabla_{q_2} S_2(u,q_2)\big)\Big) - \rho_2^{(2)}\big(d_2^{(2)}(x,u)\big)^2\Big) \\
&= \big(-3(1+\tau) - w_1^1 + x^2(5 + w_1^1),\ -\tau + 1 - w_2^1 + x^2(1 + w_2^1)\big) \in Q \setminus \{0\},
\end{aligned}
$$

for all $\tau \in [-2,\ -1],\ w_1^1 \in E_1,\ w_2^1 \in E_2$ and $x \in \mathbb{R}$. Hence, hypotheses (i) and (ii) of Theorem 4.1 are satisfied. Moreover, $\mathbb{R}_+^2 \subset K,\ \mathbb{R}_+^2 = Q$ and

$$
\sum_{i=1}^{2} \lambda_i\{\rho_{f_i}^{(1)}\big(d_{f_i}^{(1)}(x,u)\big)^2 + \rho_{g_i}^{(1)}\big(d_{g_i}^{(1)}(x,u)\big)^2\} + \sum_{j=1}^{2} \mu_j\{\rho_j^{(2)} d_j^{(2)}(x,u)\big)^2 - \tau^T w_j^2\} = x^2 + 2 + \tau \geq 0.
$$

Thus, suppositions (iii) and (iv) of Theorem 4.1 are also true. Now for point β_4, we get

$$
\begin{aligned}
&(\eta_1, \eta_2) - \left(\frac{f_1(x) + \Omega(x|C_1)}{g_1(x) - \Omega(x|D_1)}, \frac{f_2(x) + \Omega(x|C_2)}{g_2(x) - \Omega(x|D_2)}\right) \\
&\quad = \begin{cases} \left(1 - e^{-1},\ -\dfrac{6}{13}\right) \notin K \setminus \{0\} & \text{at } x = 0, \\ \left(-\dfrac{1}{2},\ -\dfrac{7}{19}\right) \notin K \setminus \{0\} & \text{at } x = 1. \end{cases}
\end{aligned}
$$

Hence, the Theorem 4.1 has been verified for $x = 0, 1 \in S_0$ and β_4 feasible for (SD).

Validation for Theorem 4.3:
For β_4 and $x = 1$, the assumptions (ii), (iii) and (iv) hold true (shown above). Moreover,

$$
\left(\eta_1 - \frac{f_1(x) + x^T z_1}{g_1(x) - x^T v_1},\ \eta_2 - \frac{f_2(x) + x^T z_2}{g_2(x) - x^T z_2}\right) = \left(\frac{1}{2},\ -\frac{1}{3}\right) \notin K \setminus \{0\}
$$

which implies that the assumption (i) of Theorem 4.3 also holds. This validates Theorem 4.3. □

Next, in the following example, we will discuss the case if the assumption of higher order $(K \times Q)-(\mathcal{F},\ \alpha,\ \rho,\ d)-$type I convexity fails, then the result of Theorem 4.1 may not hold.

Example 6. Let in the problem (SIFP), $f : \mathbb{R} \to \mathbb{R}^2,\ g : \mathbb{R} \to \mathbb{R}^2$, and $h : \mathbb{R} \times [0,\ 2] \to \mathbb{R}^2$ be given as:

$$\begin{aligned}
&f(x) = (f_1(x),\ f_2(x)) = (\sin^2 x,\ e^x),\\
&g(x) = (g_1(x),\ g_2(x)) = ((x+1)^2,\ e^x + 1) \text{ and}\\
&h(x,\ t) = (h_1(x,\ t),\ h_2(x,\ t)) = (2x - 3t,\ x + t^2).
\end{aligned}$$

Let the cones be $K = Q = \{(x,\ y) \in \mathbb{R}^2\ :\ x \geq 0,\ y \geq 0\}$ and

$$\begin{aligned}
&G(u,\ p) = (G_1(u,\ p_1),\ G_2(u,\ p_2)) = (p_1 u^2,\ -p_2 + 4u),\\
&L(u,\ p) = (L_1(u,\ p_1),\ L_2(u,\ p_2)) = (p_1 u^2,\ -p_2 \cos u),\\
&S(u,\ q) = \big(S_1(u,\ q_1),\ S_2(u,\ q_2)\big) = (q_1 + u + 5,\ q_2 u),\\
&d_{f_1}^{(1)}(x,y) = d_{g_1}^{(1)}(x,y) = d_2^{(2)}(x,y) = x(2y - \pi), \text{ and}\\
&d_{f_2}^{(1)}(x,y) = d_{g_2}^{(1)}(x,y) = d_1^{(2)}(x,y) = \cos y.
\end{aligned}$$

Further, let $\alpha(x,\ y) = x^2(4 + y^2)$ and $\mathcal{F}_{x,u}(b) = -b(x + u)$. Consider

$C_1 = [0,\ 2] = M_2, C_2 = D_2 = E_2 = [-1,\ 1]$ and $D_1 = E_1 = M_1 = [-2,\ 0]$.

Then, $\Omega(x|C_1) = x + |x|,\ \Omega(x|C_2) = \Omega(x|D_2) = \Omega(x|E_2) = |x|,\ \Omega(t|M_1) = |t| - t,$

$\Omega(x|D_1) = \Omega(x|E_1) = |x| - x$ and $\Omega(t|M_2) = |t| + t$.

Then, the feasible region of the primal problem (SIFP) is $S_0 = \{0\}$ and $\beta_5 = \big(u,\ v,\ w^1,\ w^2,\ \lambda,\ \mu,\ z,\ \eta,\ p,\ q\big)$ $= \Big(\dfrac{\pi}{2},\ (1,\ 1),\ (-2,\ 1), (0,\ 0),\ (1,\ 6),\ \Big(3,\ 2\Big), (0,\ -1),\ (1,\ 1),\ (0,\ 2),\ (1,\ 0)\Big)$ is feasible for dual problem (SD). For any $\rho_{f_1}^{(1)},\ \rho_{f_2}^{(1)},\ \rho_{g_1}^{(1)},\ \rho_{g_2}^{(1)},\ \rho_1^{(2)},\ \rho_2^{(2)} \in \mathbb{R}$, at β_5,

$$\sum_{i=1}^{2} \lambda_i \Big\{\rho_{f_i}^{(1)} \big(d_{f_i}^{(1)}(x,u)\big)^2 + \rho_{g_i}^{(1)} \big(d_{g_i}^{(1)}(x,u)\big)^2\Big\} + \sum_{j=1}^{2} \mu_j \{\rho_j^{(2)} \big(d_j^{(2)}(x,u)\big)^2 - \tau^T w_j^2\} = 0.$$

Moreover, $K = Q = \mathbb{R}_+^2$. Hence, the assumptions (iii) and (iv) of Theorem 4.1 are satisfied but

$$\begin{aligned}
&f_1(x) + x^T z_1 - (f_1(u) + u^T z_1) - G_1(u,p_1) + p_1^T \nabla_{p_1} G_1(u,p_1) - \mathcal{F}_{x,u}\big(\alpha(x,u)(\nabla_x (f_1(u) + u^T z_1)\\
&\quad + \nabla_{p_1}(G_1(u,p_1)))\big) - \rho_{f_1}^{(1)} \big(d_{f_1}^{(1)}(x,u)\big)^2 = \sin^2 x + \frac{\pi^2}{4}\Big(4 + \frac{\pi^2}{4}\Big)\Big(x + \frac{\pi}{2}\Big)x^2 - 1 < 0, \text{ for } x = 0.
\end{aligned}$$

Therefore, assumption (i) of Theorem 4.1 does not hold. Moreover,

$$(\eta_1, \eta_2) - \left(\frac{f_1(x) + \Omega(x|C_1)}{g_1(x) - \Omega(x|D_1)}, \frac{f_2(x) + \Omega(x|C_2)}{g_2(x) - \Omega(x|D_2)}\right) = \Big(1,\ \frac{1}{2}\Big) \in K/\{0\} \text{ at } x = 0.$$

Hence, without having the condition of higher order $(K \times Q) - (\mathcal{F}, \alpha, \rho, d)-$type I convexity on functions, the result of Theorem 4.1 may not hold. □

5. CONCLUSION

To the best of our knowledge, the class of conic non-smooth semi-infinite multiobjective fractional programming problem has not been studied so far. In this article, semi-infinite model with multiple fractional type objective function is formulated. Further, introducing the idea of higher order

$(K \times Q) - (\mathcal{F}, \alpha, \rho, d)-$type I convex function, the duality relations for Mond-Weir, Wolfe and Schaible type dual models have been developed. Validation of various results obtained have also been shown by demonstrating non trivial examples. Further, it has been shown by giving examples that considering the assumptions of higher order $(K \times Q) - (\mathcal{F}, \alpha, \rho, d)-$type I convexity is significant since without this, the duality results obtained may not hold. Exploring optimality relations and duality theorems for (SIFP) over space of symmetric matrices by using $E-$convexity in objective functions be an enthralling future work in this direction.

Acknowledgements. We would like to thanks the anonymous referees for their helpful comments and suggestions.

References

[1] A. Charnes, W. W. Cooper and K. O. Kortanek, Duality, Haar programs and finite sequences spaces. *Proc. National Acad Sci USA*. **48** (1962) 783–789.

[2] A. Charnes, W. W. Cooper and K. O. Kortanek, On the theory of semi-infinite programming and a generalization of the Kuhn-Tucker saddle point theorem for arbitrary convex functions. *Naval Res Log Quart.* **16(1)** (1969) 41–52.

[3] D. F. Karney, A duality theorem for semi-infinite convex programs and their finite subprograms. *Math. Prog.* **27** (1983) 75–82.

[4] S. Ito, Y. Liu and K. L. Teo, A dual parametrization method for convex semi-infinite programming. *Ann. Oper. Res.* **98** (2000) 189–213.

[5] Z. A. Liang, H. X. Huang and P. M. Pardalos, Optimality conditions and duality for a class of nonlinear fractional programming problem. *J. Optim. Theory Appl.* **110** (2001) 611–619.

[6] Y. Liu, K. L. Teo and S. Y. WU, A new quadratic semi-infinite programming algorithm based on dual parametrization. *J. Global Optim.* **29** (2004) 401–413.

[7] A. Shapiro, On duality theory of convex semi-infinite programming. *Optimization* **54(6)** (2005) 535–543.

[8] R. P. Hettich, H. Th. Jongen, Semi-infinite programming: Conditions of optimality and applications.*In: Stoer J. (eds) Optimization Techniques. Lecture Notes in Control and Information Sciences, Springer, Berlin, Heidelberg* **7** (1978). `https://doi.org/10.1007/BFb00065021--11`.

[9] G. J. Zalami, Q. Zhang, Nonparametric duality models for semi-infinite discrete minmax fractional programming problems involving generalized $(\eta,\ \rho)-$invex functions. *Numer. Funct. Anal. Optim.* **28** (2007) 211–243.

[10] V. Jeyakumar, A note on strong duality in convex semidefinite optimization: necessary and sufficient conditions. *Optim. Lett.*. **2** (2008) 15–25.

[11] I. Ahmad, Z. Husain, S. Sharma, Higher-order duality in nondifferentiable minimax programming with generalized type I functions. *J. Optim. Theory Appl.* **141** (2009) 1–12.

[12] A. Shapiro, Semi-infinite programming, duality, discretization and optimality conditions. *Optimization* **58(2)** (2009) 133–161.

[13] A. Ismael F. Vaz, E. C. Ferreira, Air pollution control with semi-infinite programming. *Appl. Math. Model.* **33(4)** (2009) 1957–1969.

[14] S. K. Mishra, M. Jaiswal and H. A. Le Thi, Nonsmooth semi-infinite programming problem using Limiting subdifferentials. *J. Global Optim.* **53** (2012) 285–296.

[15] G. J. Zalmai, Second-order parameter-free duality models in semi-infinite minmax fractional programming. *Numer. Funct. Anal. Optim.* **34(11)** (2013) 1265–1298.

[16] A. Basu, K. Martin and C.T. Ryan, On the sufficiency of finite support duals in semi-infinite linear programming. *Oper. Res. Lett.* **42(1)** (2014) 16–20.

[17] Q. Lin, R. Loxton, K. L. Teo, Y. H. Wu and C. Yu, A new exact penalty method for semi-infinite programming problems. *J. Comput. Appl. Math.* **261** (2014) 271–286.

[18] T. Antczak, G. J. Zalmai, Second order $(\Phi, \rho) - V-$invexity and duality for semi-infinite minimax fractional programming. *Appl. Math. Comp.* **227** (2014) 831–856.

[19] S. K. Mishra, M. Jaiswal, Optimality conditions and duality for semi-infinite mathematical programming problem with equilibrium constraints. *Numer. Funct. Anal. Optim.* **36(4)** (2015) 460–480.

[20] Y. Pandey, S. K. Mishra, Duality for nonsmooth optimization problems with equilibrium constraints, using convexificators. *J. Optim. Theory Appl.* **171** (2016) 694–707.

[21] N. Kanzi, M. Soleimani-damaneh, Slater CQ, optimality and duality for quasiconvex semi-infinite optimization problems. *J. Math. Anal. Appl.* **434(1)** (2016) 638–651.

[22] T. R. Patel, R. D. Patel, Duality for semi- infinite multiobjective fractional programming problems involving generalized $(H_p, R)-$invexity. *Inter. Ref. J. Eng. Sci.* **5(4)** (2016) 7–15.

[23] I. M. Stancu-Minasian, K. Kummari and A. Jayswal, Duality for semi-infinite minimax fractional programming problem involving higher-order $(\Phi, \rho) - V-$Invexity. *Numer. Funct. Anal. Optim.* **38(7)** (2017) 926–950.

[24] R. Gupta, M. Srivastava, Optimality and duality in multiobjective programming involving support functions. *RAIRO: OR* **51(2)** (2017) 433–446.

[25] Y. Pandey, S. K. Mishra, Optimality conditions and duality for semi-infinite mathematical programming problems with equilibrium constraints, using convexificators. *Ann. Oper. Res.* **269** (2018) 549–564.

[26] L. T. Tung, Strong Karush-Kuhn-Tucker optimality conditions for multiobjective semi-infinite programming via tangential subdifferential. *RAIRO: OR* **52(4-5)** (2018) 1019–1041.

[27] R. U. Verma, G. J. Zalmai, Parameter-free duality models and applications to semi-infinite minmax fractional programming based on second-order $(\phi,\ \eta,\ \rho,\ \theta,\ \tilde{m})-$sonvexities. *OPSEARCH* **55** (2018) 381–410.

[28] I. P. Debnath, S. K. Gupta, Higher-order duality relations for multiobjective fractional problems involving support functions. *Bul. Mala. Math. Sci. Soci.* **42** (2019) 1255–1279.

[29] J. Zeng, P. Xu and H. Fu, On robust approximate optimal solutions for fractional semiinfinite optimization with uncertainty data. *J. Inequal. Appl.* **45** (2019). DOI : `10.1186/s13660-019-1997-7`.

[30] X. Sun, K. L. Teo and L. Tang, Dual approaches to characterize robust optimal solution sets for a class of uncertain optimization problems. *J. Optim. Theory Appl.* **182** (2019) 984–1000.

[31] L. Jiao, D. S. Kim and Y. Zhou, Quasi $\varepsilon-$solutions in semi-infinite programming problem with locally Lipschitz data. *Optim. Lett.* (2019). DOI : `10.1007/s11590-019-01457-2`.

[32] Sonali, V. Sharma and N. Kailey, Higher order non-symmetric duality for nondifferentiable minimax fractional programs with square root functions. *Acta. Math. Sci.* **40** (2020) 127–140.

[33] X. Sun, K. L. Teo, J. Zeng and L. Liu, Robust approximate optimal solutions for nonlinear semi-infinite programming with uncertainty. *Optimization* **69(9)** (2020) 2109—2129.

[34] T. Antczak, Y. Pandey, V. Singh and S. K. Mishra, On approximate efficiency for nonsmooth robust vector optimization problems. *Acta. Math. Sci.* **40B(3)** (2020) 887–902.

[35] L. T. Tung, Karush-Kuhn-Tucker optimality conditions and duality for multiobjective semi-infinite programming via tangential subdifferentials. *Numer. Funct. Anal. Optim.* **41(6)** (2020) 659–684.

[36] A. Aboussoror, S. Adly and S. Salim, An extended conjugate duality for generalized semi-infinite programming problems via a convex decomposition. *Optimization* **69(7-8)** (2020) 1635–1654.

[37] B. C. Joshi, Optimality and duality for nonsmooth semi-infinite mathematical program with equilibrium constraints involving generalized invexity of order $\sigma > 0$. *RAIRO: OR* **55** (2021) S2221–S2240.

[38] T. Emam, Nonsmooth semi-infinite $E-$convex multi-objective programming with support function. *J. Inf. and Optim. Sci.* **42(1)** (2021) 193–209.