

New inverse DEA models for budgeting and planning

Tahere Sayar¹, Mojtaba Ghiyasi^{2*}, Jafar Fathali¹

¹Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran.

² Faculty of Industrial Engineering and Management Sciences, Shahrood University of Technology, Shahrood, Iran.

*Corresponding author

Abstract

Data envelopment analysis (DEA) measures the efficiency score of a set of homogeneous decision-making units (DMUs) based on observed input and output. Considering input-oriented, the inverse DEA models find the required input level for producing a given amount of production in the current efficiency level. This article proposes a new form of the inverse DEA model considering income (for planning) and budget (for finance and budgeting) constraints. In contrast with the classical inverse model, both input and output levels are variable in proposed models to meet income (or budget) constraints. Proposed models help decision-makers (DMs) to find the required value of each input and each output's income share to meet the income or budget constraint. We apply the proposed model in the efficiency analysis of 58 supermarkets belonging to the same chain. However, these methods are general and can be used in the budgeting and planning process of any production system, including business sectors and firms that provide services.

Keywords: DEA; Inverse DEA; MOLP; Budget constraint; Income constraint

1. Introduction

Data Envelopment Analysis (DEA) is a linear programming technique-based for evaluating the relative efficiency of a decision-making unit (DMU) by comparing it with other DMUs that first time as proposed by (Charnes, Cooper, & Rhodes, 1978), known as the CCR model. This technique has been used and developed by many scholars, see, e.g. (Cook & Seiford, 2009), (Hatami-Marbini, Emrouznejad, & Tavana, 2011). In recent years (Wei, Zhang, & Zhang, 2000) proposed the inverse DEA models that aim to answer this question: if among a group of DMUs, we increase certain inputs to a particular unit and assume that the DMU maintains its current efficiency level with respect to other units, how much more outputs could the unit produce? If the outputs need to be increased to a certain level and the unit's efficiency remains unchanged, how much more inputs should be provided to the unit? These types of questions are answered using Multiple Objectives Linear Programming (MOLP) in general in the inverse DEA literature. Different researchers introduce some extensions and modifications. After the initial work in inverse DEA by (Wei et al., 2000), it has been remarkably considered by some scholars in the DEA field, see, e.g., (Yan, Wei, & Hao, 2002), (G. Jahanshahloo, Hosseinzadeh Lotfi, Rostamy-Malkhalifeh, & Ghobadi, 2014; G. R. Jahanshahloo, Lotfi, Shoja, Tohidi, & Razavyan, 2004, 2005; G. R. Jahanshahloo, Soleimani-Damaneh, & Ghobadi, 2015; G. R. Jahanshahloo, Vencheh, Foroughi, & Matin, 2004), (Hadi-Vencheh & Foroughi, 2006), (Lin, 2010), (Lertworasirikul, Charnsethikul, & Fang, 2011), (Gattoufi, Amin, & Emrouznejad, 2014) and (Ghobadi & Jahangiri, 2015). Along the lines of (Wei et al., 2000), the inverse problem was investigated in inverse DEA filed by (Hadi-Vencheh, Foroughi, & Soleimani-damaneh, 2008). They used (weak) Pareto solutions of MOLP problems to estimate the desired inputs. Both Questions, input-estimation, and output-estimation, are investigated under

the inter-temporal dependence assumption by (G. R. Jahanshahloo et al., 2015). They are proposed a new optimality notion for multiple objective programming problems, periodic weak Pareto optimality. (G. Jahanshahloo et al., 2014) dealt with inverse DEA problem with non-radial input-output change. (Gattoufi et al., 2014) proposed an application of the inverse DEA models in merger and acquisition in banking. They developed an approach to realize the required level of the merged bank's inputs and outputs to reach a predetermined efficiency target. (Amin & Emrouznejad, 2007) applied the inverse optimization for forecasting and provided a streamlined approach to time series analysis using inverse linear programming. (Amin, Emrouznejad, & Gattoufi, 2017) proposed inverse DEA models for modeling generalized firms' restructuring and anticipating the minor and major consolidation for a merger in a market. (M. Ghiyasi, 2017) dealt with cost and revenue efficiency in the inverse DEA context. (Emrouznejad, Yang, & Amin, 2018; Mojtaba Ghiyasi, 2017) dealt with the inverse DEA models in the presence of undesirable outputs. (Kalantary and Saen, 2018) proposed an inverse dynamic network DEA model for assessing the sustainability of supply chains. In another research, (Kalantary et al., 2018) proposed an inverse version of network dynamic range adjusted measure for sustainability assessment of supply chains. (Chen et al., 2017) investigated the investment problem of sustainable development in China utilizing an inverse DEA model that is capable of dealing with undesirable outputs. (Ghiyasi, 2019) proposed an inverse DEA based method for emission utilization permission while taking the environmental efficiencies into consideration. (Emrouznejad et al., 2019) proposed an inverse DEA model for allocation of CO₂ emission for different chines regions. (Guijarro, Martínez-Gómez, & Visbal-Cadavid, 2020) dealt with technical inverse DEA and the merge problem that their model computes the global efficiency target by giving preference to merging DMUs over saving inputs. (Zhang & Cui, 2020) dealt with the inverse DEA based on non-redial DEA that call non-redial inverse DEA. (Amin & Boamah, 2020) dealt

with inverse data envelopment analysis (DEA) and cost efficiency model for estimating potential gains from mergers and restructuring scenarios for firms that want to minimize cost. The price information is important and valuable information in economic theory and in real-world applications. Taking the price information in hand, we can measure the revenue and cost and, consequently, the production system's profit. Thus, cost and revenue also play an essential role in real-world applications. Specifically, cost measure, some believe that the economics' art is managing the cost and revenue of a production system since available resources, including a budget, are usually limited.

The current article proposes a new class of inverse DEA to the literature. It considers the budget constraint in the inverse DEA models and instead of increasing inputs and seeking for producible output with the same measure of efficiency. This paper considers a specific and limited budget that can be spent for buying different inputs (material vs. service or equipment vs. manpower), and the aim is estimating producible output with current efficiency measures. In contrast with classical inverse DEA models, we do not have any predetermined input or output level. Both input and output are decision variables in our model. The second model is planning-based and considers the total income, and sets a total income goal. It estimates the required value of each input and income share of each output to reach the goal. The rest of the paper is organized as follows. Section 2 provides preliminarily of DEA and inverse DEA models. Section 3 proposes two new classes of inverse DEA models that consider income and budget constraints. Section 4 applies proposed models for efficiency analysis of 58 supermarkets belonging to the same chain in Tehran city, Iran's capital.

2. Preliminarily

2.1. Classical DEA models

Suppose there are n DMUs as (DMU_1, \dots, DMU_n) in which DMU_j ; ($j = 1, \dots, n$), produces multiple positive outputs y_{rj} ($r = 1, \dots, s$), by utilizing multiple positive inputs x_{ij} ($i = 1, \dots, m$). Let input and output vector for DMU_j be denoted by $X_j = (x_{j1}, \dots, x_{jm})$ and $Y_j = (y_{j1}, \dots, y_{js})$ previously.

To discuss the inverse DEA problem, we use the generalized DEA models, which is defined as follows: (Suppose unit under assessment is DMU_o ; ($o \in \{1, \dots, n\}$)) Consider the following general model in terms of returns to scale for estimating the efficiency of DMU_o .

$$\begin{aligned} \theta_o &= \min \theta \\ \text{s. t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{i0} \quad i = 1, \dots, m \quad (1) \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0} \quad r = 1, \dots, s \\ & \lambda \in \Omega \end{aligned}$$

Where

$$\Omega = \left\{ \lambda \mid \lambda = (\lambda_1, \dots, \lambda_n), \sigma_1 \left(\sum_{j=1}^n \lambda_j = 1 + \sigma_2 (-1)^{\sigma_3 v} \right) = \sigma_1, v \geq 0, \lambda_j \geq 0, j = 1, \dots, n \right\}$$

In the above model σ_1 , σ_2 and σ_3 are parameters with 0-1 values. θ_o is called the input-oriented efficiency score of DMU_o and $\theta_o \leq 1$.

The output-oriented model for estimating the efficiency of DMU_o can be considered as follows :

$$\varphi_o = \max \varphi$$

$$\begin{aligned}
s. t. \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0} \quad i = 1, \dots, m \quad (2) \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq \varphi y_{r0} \quad r = 1, \dots, s \\
& \lambda \in \Omega
\end{aligned}$$

φ_o is called the output-oriented efficiency score of DMU_o and it is a well-known fact that $\varphi_o \geq 1$.

2.2. Classical inverse DEA

Now assume we perturb the output level of DMU_o from y_o to $y_o + \overline{\Delta y_o}$. The classical inverse DEA models find the level of the required inputs needed for producing a new level of output with the current efficiency level. The following model does this aim:

$$\begin{aligned}
& \min (\Delta x_{10}, \Delta x_{20}, \dots, \Delta x_{m0}) \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_o (x_{i0} + \Delta x_{i0}) \quad i = 1, \dots, m \quad (3) \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq (y_{r0} + \Delta y_{r0}) \quad r = 1, \dots, s \\
& \lambda \in \Omega
\end{aligned}$$

In an output orientation view, assume we perturb the inputs level of DMU_o from x_o to $x_o + \overline{\Delta x_o}$. The classical inverse DEA model seeks for producible outputs level with the current efficiency measure as follows:

$$\max (\Delta y_{10}, \Delta y_{20}, \dots, \Delta y_{s0})$$

$$\begin{aligned}
\sum_{j=1}^n \lambda_j x_{ij} &\leq (x_{i0} + \Delta x_{i0}) \quad i = 1, \dots, m \\
\sum_{j=1}^n \lambda_j y_{rj} &\geq \varphi_0 (y_{r0} + \Delta y_{r0}) \quad r = 1, \dots, s \\
\lambda &\in \Omega
\end{aligned} \tag{4}$$

3. The new inverse DEA for budgeting and planning

This section is devoted to studying a new class of inverse DEA models. The first subsection is an income-based model with input orientation, and the second subsection is the budget-based model and has an output orientation.

3.1. Income-based inverse DEA

Consider the case that DM aims at increasing the total income of DMU_0 to reach the minimum value of A . The question is how much input is required to get to this aim with the current efficiency level. The following model does this aim.

$$\begin{aligned}
&\min (\Delta x_{10}, \Delta x_{20}, \dots, \Delta x_{m0}) \\
&\sum_{r=1}^s a_r \Delta y_{r0} \geq A \\
&\sum_{j=1}^n \lambda_j x_{ij} \leq \theta_0 (x_{i0} + \Delta x_{i0}) \quad i = 1, \dots, m \\
&\sum_{j=1}^n \lambda_j y_{rj} \geq (y_{r0} + \Delta y_{r0}) \quad r = 1, \dots, s \\
&\lambda \in \Omega
\end{aligned} \tag{5}$$

, where a_r , $1 \leq r \leq s$ is the price of r -th output. The above model finds the minimum input level for producing the desired output level and reaching the minimum total income of DMU_o to A with the current efficiency level. Note that $\Delta x_0 \in \mathbb{R}_+^m$ and $\Delta y_0 \in \mathbb{R}_+^s$ and $\lambda \in \mathbb{R}_+^n$ are decision variables in the above model. It is important to point out that the associated classic inverse DEA model is as follows and can be considered as a special case of the model (5).

$$\begin{aligned} & \min (\Delta x_{10}, \Delta x_{20}, \dots, \Delta x_{m0}) \\ & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_0 (x_{i0} + \Delta x_{i0}) \quad i = 1, \dots, m \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq (y_{r0} + \overline{\Delta y_{r0}}) \quad r = 1, \dots, s \\ & \lambda \in \Omega \end{aligned} \tag{6}$$

In the above classic inverse DEA model, an expected individual output level is considered, while MOLP (5) is concern about a total income level with the minimum level of inputs. In fact, $\overline{\Delta y_{r0}}$ is the expected value of the output and therefore, it is a vector parameter in the model (6). Theoretical talking, both input, and output levels are variable in the model (5) while the output level is given and parameter in associated inverse model of (6). Model (5) has $m + s + n$ variable while its classical peer, model (6) has $m + n$ variable.

Remark 1. Classical inverse DEA model (6) is a special case of income-based model (5).

Proof. Considering $\Delta y_{r0} = \overline{\Delta y_{r0}}$ and ignoring the income constraint of $\sum_{r=1}^s a_r \Delta y_{r0} \geq A$ we get the classical inverse DEA model of (6). The above

discussion shows that the income-based model of (5) is generalized and more flexible than its classical peer.

Please note that both inputs and outputs are variable in the above models, in contrast with the classic inverse DEA model. The main constraint in the model is the targeted income that should be met. It is well-known that model (7) is linear programming and has an optimum solution, but model (5) is multiple objective linear programming (MOLP) and does not have an optimal solution; instead, we have an efficient solution for this model as follows:

Definition. Suppose $(\lambda, x_0 + \Delta x_0, y_0 + \Delta y_0)$ is a feasible solution model (5), if there is no feasible solution $(\bar{\lambda}, \bar{x}_0 + \bar{\Delta x}_0, \bar{y}_0 + \bar{\Delta y}_0)$ of this model such that $x_0 + \bar{\Delta x}_0, \bar{y}_0 < x_0 + \Delta x_0, y_0 + \Delta y_0$, then we say the $(\lambda, x_0 + \Delta x_0, y_0 + \Delta y_0)$ is a (weak) efficient solution of model (5).

Theorem 1. Income-based model (5) preserves the efficiency score of DMU_o , that is, if $(\lambda, x_0 + \Delta x_0, y_0 + \Delta y_0)$ is a weak efficient solution of income-based model (5), then the efficiency of $DMU_o = (x_0, y_0)$ and $DMU_o^+ = (x_0 + \Delta x_0, y_0 + \Delta y_0)$ are the same.

Proof. The following testing model checks the efficiency of DMU_o after perturbation.

$$\begin{aligned} \theta_o^+ &= \min \theta \\ \sum_{r=1}^s a_r \overline{\Delta y_{r0}} &\geq A \\ \sum_{j=1}^n \lambda_j x_{ij} &\leq \theta (x_{i0} + \overline{\Delta x_{i0}}) \quad i = 1, \dots, m \\ \sum_{j=1}^n \lambda_j y_{rj} &\geq (y_{r0} + \overline{\Delta y_{r0}}) \quad r = 1, \dots, s \\ \lambda &\in \Omega \end{aligned}$$

If $(\bar{\lambda}, x_0 + \overline{\Delta x_0}, y_0 + \overline{\Delta y_0})$ is a weak efficient solution of income based model (5), then it satisfies associated constraints as follows:

$$\sum_{r=1}^s a_r \overline{\Delta y_{r0}} \geq A$$

$$\sum_{j=1}^n \bar{\lambda}_j x_{ij} \leq \theta_0 (x_{i0} + \overline{\Delta x_{i0}}) \quad i = 1, \dots, m$$

$$\sum_{j=1}^n \bar{\lambda}_j y_{rj} \geq (y_{r0} + \overline{\Delta y_{r0}}) \quad r = 1, \dots, s$$

$$\bar{\lambda} \in \Omega$$

This means $(\bar{\lambda}, \theta_0)$ is a feasible solution of the testing model that implies $\theta_0^+ \leq \theta_0$. We just need to show that $\theta_0^+ \neq \theta_0$. By contradiction, assume that $\theta_0^+ > \theta_0$, which (λ^+, θ_0^+) is the optimal solution of the testing model. This means that $(\lambda^+, k * (x_0 + \overline{\Delta x_0}), y_0 + \overline{\Delta y_0})$ is a feasible solution of MOLP model (5) since

$$\sum_{r=1}^s a_r \overline{\Delta y_{r0}} \geq A$$

$$\sum_{j=1}^n \lambda_j x_{ij} \leq \theta_0^+ (x_{i0} + \overline{\Delta x_{i0}}) = k \theta_0 (x_{i0} + \overline{\Delta x_{i0}}) \quad i = 1, \dots, m$$

$$\sum_{j=1}^n \lambda_j y_{rj} \geq (y_{r0} + \overline{\Delta y_{r0}}) \quad r = 1, \dots, s$$

$$\lambda \in \Omega$$

, where $0 < k < 1$. This is a contradiction with weak efficiency of $(\lambda, x_0 + \overline{\Delta x_0}, y_0 + \overline{\Delta y_0})$. Thus, $\theta_0^+ \neq \theta_0$ and therefore $\theta_0^+ = \theta_0$. \square

Considering input's weight, we can use the weighted sum approach for solving the above MOLP and consequently reach the following linear programming.

$$\begin{aligned}
& \min \sum_{i=1}^m c_i \Delta x_{i0} \\
& \sum_{r=1}^s a_r \Delta y_{r0} \geq A \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_0(x_{i0} + \Delta x_{i0}) \quad i = 1, \dots, m \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq (y_{r0} + \Delta y_{r0}) \quad r = 1, \dots, s \\
& \lambda \in \Omega
\end{aligned} \tag{7}$$

, where $c_i \geq 0$, $1 \leq i \leq m$ is the price of i -th input.

Theorem 2. If $(\bar{\lambda}, \bar{\Delta x}, \bar{\Delta y})$ is an optimal solution of linear programming model (7) then it is a weak efficient solution of MOLP model (5).

Proof. Since $(\bar{\lambda}, \bar{\Delta x}, \bar{\Delta y})$ is an optimal solution of linear programming model (7) then it satisfies all associated constraints as follow

$$\begin{aligned}
& \sum_{r=1}^s a_r \bar{\Delta y}_{r0} \geq A \\
& \sum_{j=1}^n \bar{\lambda}_j x_{ij} \leq \theta_0(x_{i0} + \bar{\Delta x}_{i0}) \quad i = 1, \dots, m \\
& \sum_{j=1}^n \bar{\lambda}_j y_{rj} \geq (y_{r0} + \bar{\Delta y}_{r0}) \quad r = 1, \dots, s \\
& \bar{\lambda} \in \Omega
\end{aligned}$$

This implies $(\bar{\lambda}, \bar{\Delta x}, \bar{\Delta y})$ as a feasible solution of the MOLP model (5). By contradiction, if we assume this solution is not a weak efficient solution for the MOLP problem of (5) then there exists a feasible solution of this model like

$(\hat{\lambda}, \hat{\Delta x}, \hat{\Delta y})$ such that $\widehat{\Delta x_{i0}} < \overline{\Delta x_{i0}}$. By multiplying both sides of this inequality by $c_i \geq 0$, $1 \leq i \leq m$ we get $c_i \widehat{\Delta x_{i0}} < c_i \overline{\Delta x_{i0}}$ and by summing up both sides of the manuscript over $1 \leq i \leq m$ we get $\sum_{i=1}^m \widehat{c_i \Delta x_{i0}} < \sum_{i=1}^m c_i \overline{\Delta x_{i0}}$ and this fact contradicts the optimality of $(\bar{\lambda}, \bar{\Delta x}, \bar{\Delta y})$. \square

Remark 2. The optimal solution of (7) preserves the efficiency score of the unit under evaluation.

Proof. This can be inferred by considering theorem 1 and theorem 2. In fact, if $(\bar{\lambda}, \bar{\Delta x}, \bar{\Delta y})$ is an optimal solution of linear programming model (7) then theorem 2 implies that it is a weak efficient solution of model (5). After that, we conclude the efficiency of DMU_o are the same before and after changing its input and output levels, that is, $DMU_o = (x_0, y_0)$ and $DMU_o^+ = (x_0 + \Delta x_0, y_0 + \Delta y_0)$.

3.2. Budget based inverse DEA model

In an output-oriented view, assume a given level of budget is available, and DM wants to increase its inputs. A classical case is spending a budget for improving equipment or hiring more manpower. The question is how to assign and append this budget within inputs and how much more output can be produced at the current efficiency level. The following model does this aim for us:

$$\begin{aligned}
& \max (\Delta y_{10}, \Delta y_{20}, \dots, \Delta y_{s0}) \\
& \sum_{i=1}^m b_i \Delta x_{i0} \leq B \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq (x_{i0} + \Delta x_{i0}) \quad i = 1, \dots, m \quad (8) \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq \varphi_0 (y_{r0} + \Delta y_{r0}) \quad r = 1, \dots, s
\end{aligned}$$

$$\sum_{j=1}^n \lambda_j = 1$$

$$\lambda_j \geq 0 \quad j = 1, \dots, n$$

, where b_i , $1 \leq i \leq m$ is the price of i -th input. In fact, the above model estimates the maximum level of output' increment of DMU_o provided with the budget constraint with the current efficiency level.

Like income based model, the classical inverse peer model of (8), that is, the inverse DEA model without the budget constraint, is as follows. It can be considered as a special case of a budget-based model. However, the budget-based model has $m + s + n$ variable, and its peer has only $s + n$ variable.

$$\max (\Delta y_{10}, \Delta y_{20}, \dots, \Delta y_{s0})$$

$$\sum_{j=1}^n \lambda_j x_{ij} \leq (x_{i0} + \overline{\Delta x_{i0}}) \quad i = 1, \dots, m \quad (9)$$

$$\sum_{j=1}^n \lambda_j y_{rj} \geq \varphi_0 (y_{r0} + \Delta y_{r0}) \quad r = 1, \dots, s$$

$$\sum_{j=1}^n \lambda_j = 1$$

$$\lambda_j \geq 0 \quad j = 1, \dots, n$$

Both input and output levels are unknown in the budget-based model, and therefore, they are variable. Still, its peer only considers the output levels as variable, and the input levels are presumed. In other words, the budget-based model presumes only the total budget while its peer presumes all individual input levels.

In the budget-based inverse DEA model, the goal is to get the maximum possible income, thus considering the output's price. Using the weighted sum approach for solving the above MOLP, we get the following linear programming.

$$\begin{aligned}
& \max \sum_{r=1}^s a_r \Delta y_{r0} \\
& \sum_{i=1}^m b_i \Delta x_{i0} \leq B \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq (x_{i0} + \Delta x_{i0}) \quad i = 1, \dots, m \quad (10) \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq \varphi_0 (y_{r0} + \Delta y_{r0}) \quad r = 1, \dots, s \\
& \sum_{j=1}^n \lambda_j = 1 \\
& \lambda_j \geq 0 \quad j = 1, \dots, n
\end{aligned}$$

Theorem 3. Budget based model (9) preserves the efficiency score of DMU_o , that is, if $(\lambda, x_0 + \Delta x_0, y_0 + \Delta y_0)$ is a weak efficient solution of income-based model (2-3), then the efficiency of $DMU_o = (x_0, y_0)$ and $DMU_o^+ = (x_0 + \Delta x_0, y_0 + \Delta y_0)$ are the same.

Proof. It is similar to the proof of theorem 1. \square

Theorem 4. If $(\bar{\lambda}, \bar{\Delta x}, \bar{\Delta y})$ is an optimal solution of linear programming model (7) then it is a weak efficient solution of MOLP model (5).

Proof. It is similar to the proof of theorem 2. \square

Remark 3. The optimal solution of (7) preserves the efficiency score of the unit under evaluation.

Proof. This can be inferred by considering theorem 3 and theorem 4. \square

4. A real world application

This section utilized our proposed approach for performance assessment, budgeting, and future planning analysis of 58 supermarkets belonging to the same chain in Tehran. Two inputs and two outputs are considered in this analysis. The size of supermarkets (in square meter) and man-hours (in hours) are considered as inputs, and sales (in Rial) and the number of loyal customers are considered as outputs. A loyal customer is considering a customer that makes a minimum level of purchase in a specific period of time. The average of labor salary and the average of one square meter supermarket's price is considered input prices used in the budget-based method. The average price of total sale and the average price of loyal customers purchased are considered as output prices. Table 1 reports the input-out data, input-oriented efficiency, and output efficiency of all supermarkets.

Table 1: Input-output data and efficiency scores

Supermarket	Size $m^2(I_1)$	Man-hours $1^3(I_2)$	Sales $10^5 R^*$ (O_1)	customer loyalty O_2	Input efficiency score θ_j	Output efficiency score φ_j
s_1	62.75	2428.77	490.58	32	0.41	3.45
s_2	59.87	3670.84	244.18	43	0.43	3.72
s_3	81.14	3369.33	333.971	38	0.75	2.00
s_4	36.05	1975.45	169.363	96	0.72	1.89
s_5	48.3	2934.29	1010.478	45	0.54	2.15
s_6	68.69	4876.08	813.492	24	0.40	2.53
s_7	87.29	4443.59	1329.862	64	0.36	2.96
s_8	26.86	2572.35	911.163	44	1	1.33
s_9	54.4	149.71	1155.215	42	1	1
s_{10}	44.58	4274.74	551.987	11	0.60	2.02
s_{11}	41.07	4091.6	592.308	52	0.65	1.83
s_{12}	38.23	2633.98	296.208	53	0.68	2.01
s_{13}	51.22	2008	604.715	45	0.52	2.28
s_{14}	24.49	1082.48	908.383	53	1	1.22
s_{15}	30.53	698.22	1004.34	23	1	1
s_{16}	42.65	6306.56	599.694	64	0.74	1.36
s_{17}	43.55	102.6	149.201	74	1	1
s_{18}	40.68	4111.14	289.36	42	0.60	3.37
s_{19}	39.36	1323.11	1775.222	22	1	1
s_{20}	78.15	4074.09	648.628	93	0.32	5.43
s_{21}	53.79	4242.11	522.897	22	0.47	3.74
s_{22}	39.87	3132.16	170.862	34	0.67	1.74
s_{23}	37.35	1905.89	175.923	54	1	1
s_{24}	46.7	2359.07	429.785	88	0.59	1.72
s_{25}	49.3	476.3	630.533	64	0.67	2.08
s_{26}	35.7	3509.22	218.039	16	0.77	1.32
s_{27}	54.18	839.22	582.938	64	0.56	2.89
s_{28}	60.21	2621.24	854.275	33	0.49	2.22
s_{29}	80.26	3350.62	913.588	86	0.32	3.96
s_{30}	55.97	2984.62	1408.201	46	0.65	1.54
s_{31}	54.49	503.31	666.841	46	0.66	1.53
s_{32}	46.6	6405.53	120.539	34	0.54	3.24
s_{33}	64.19	5276.79	157.234	65	0.43	2.33
s_{34}	44.93	2177.51	404.547	55	0.60	1.93

S ₃₅	61.14	3063.37	733.55	34	0.41	1.76
S ₃₆	36.44	1732.57	1058.357	78	0.91	1.10
S ₃₇	76.49	3492	1018.737	87	0.48	2.08
S ₃₈	27.63	641.42	517.371	65	1	1
S ₃₉	37.6	2206.1	572.871	34	0.70	1.77
S ₄₀	43.89	452.6	959.774	25	0.74	2.62
S ₄₁	53.19	725.34	1545.296	54	0.62	1.46
S ₄₂	92.59	6511.8	575.228	94	0.47	1.63
S ₄₃	71.35	2827.96	1469.73	44	0.53	1.93
S ₄₄	33.89	3535.79	1670.08	11	0.72	1.76
S ₄₅	83.82	4193.17	870.44	63	0.36	3.14
S ₄₆	62.58	2478.18	944.269	33	0.40	4.48
S ₄₇	65.26	1937.47	397.232	71	0.46	2.17
S ₄₈	52.42	4403.41	480.11	71	0.57	1.74
S ₄₉	32.99	306.03	685.991	40	1	1
S ₅₀	53.99	3744.04	833.011	33	0.47	3.86
S ₅₁	74.83	3343.58	349.108	93	0.53	1.90
S ₅₂	54.04	4867.48	106.05	14	0.72	1.40
S ₅₃	61.11	4243.02	1737.879	63	0.52	2.22
S ₅₄	58.98	4066.9	1239.36	22	0.41	3.84
S ₅₅	76.01	2367.89	691.951	54	0.35	3.32
S ₅₆	94.06	3176.95	404.965	65	0.33	3.42
S ₅₇	79.83	5292.8	1090.803	103	0.57	1.83
S ₅₈	81.29	2708.43	232.962	86	0.45	2.23

*Iranian Rial

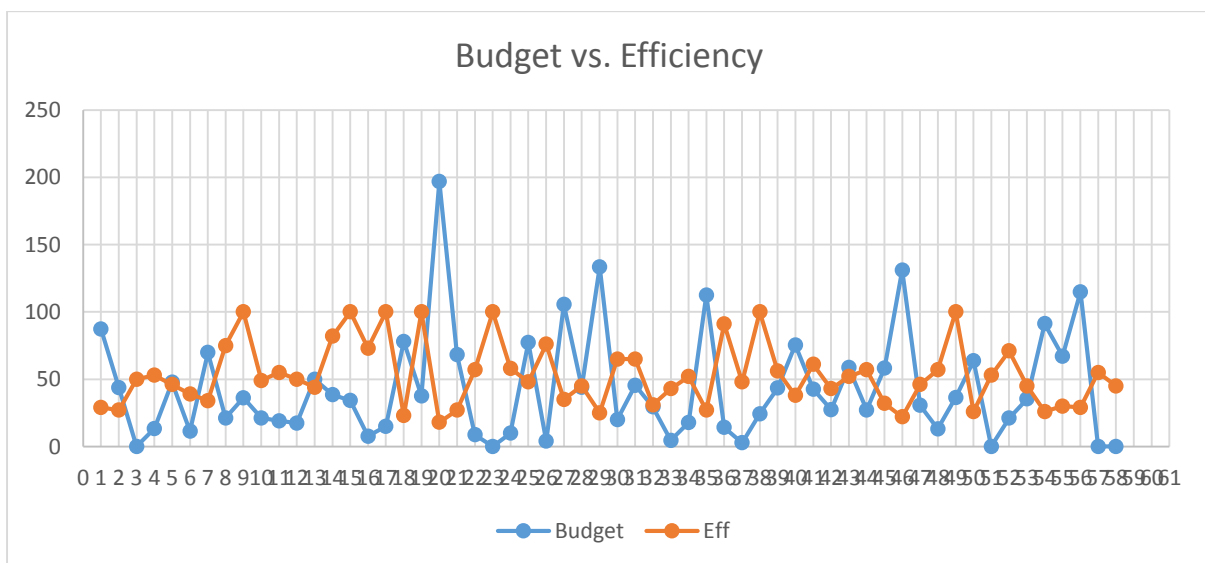
In the next run of analysis, we perform proposed income-based and budget-based models for going one step forward to the efficiency analysis. In the first analysis, we consider a given income level, let us say 200 million Rial that should be reached, and then perform the income-based model for all supermarkets. Table 2 reports the results of this analysis. Different supermarkets behave differently in this analysis. For instance, S3 needs no more labor but should not expect more loyal customers. S4 needs no more labor while having more loyal customers, and S10-S12 also keeps the same behavior. S23 that is an efficient supermarket need no extra inputs including size and labor but can reach more sale. S51, S57, and S58 also have the same behavior.

Table 2: result of income based model for all supermarkets

Supermarket	Δx_1	Δx_2	Δy_1	Δy_2	Required Budget	Input efficiency score θ_j
s_1	65.69445	1888.967	1189.426	45.01186	87.32781	0.29
s_2	58.54212	309.6543	1197.794	37.54188	43.69583	0.27
s_{10}	34.49361	0	1195.545	39.54940	21.21357	0.49
s_{18}	100.8074	645.0443	1178.360	54.89065	78.02075	0.23
s_{25}	33.08849	2293.239	1153.100	77.44061	77.31806	0.48
s_{28}	40.58636	767.0901	1191.486	43.17296	44.01666	0.45
s_{31}	8.603349	1617.611	1182.830	50.90023	45.47575	0.65
s_{39}	46.08572	607.0457	1153.895	76.73050	43.42295	0.56
s_{48}	21.26112	0	1208.221	28.23364	13.07559	0.57
s_{56}	79.51032	2657.720	1179.433	53.93271	114.9219	0.29

We see the behavior of the total required budget and all supermarkets' efficiency in figure 1. we observe various behaviors of different supermarkets that show no unique pattern for all supermarkets. Thus, we investigate different budgeting plans for individual supermarkets; we report it for one efficient and one inefficient supermarket.

Figure 1: The required input vs. efficiency of all supermarkets



Considering a given gradual level of income, we look for required input levels that are capable of gaining aforesaid income level. Consider supermarket #1 (S1) for this analysis; for instance, that is an inefficient supermarket using its resources to produce its outputs. Let us consider the different amounts of income are considered, and required input levels are found for gaining those income levels, using model (7). Table 3 reports the result of this analysis. It assumes a gradual expected income from 50 to 700 million Rial income. To reach 50 million Rial income in this supermarket, we do not need to much resources, not the extra size or man-hour in this case. Recall that supermarket is inefficient there exist possible performance improvement for this supermarket. This means that more income is possible for this supermarket with its current input levels. However, increasing expected output requires more resources. In the next analysis, we consider S9 that is an efficient supermarket and perform the same analysis for this efficient supermarket. The result is reported in the table 4. The result shows no extra size for the expanding plans up to 200 million Rial income, instead more labor is found that is require for this supermarket. Figure 2 and figure 3 depict the pattern of income vs. cost of these plans for S1 and S2 as an efficient and inefficient supermarket. An overview of these figure suggest an expanding income plan for S1 up to 50 million Rial at most and a income expanding plan up to 700 million Rial for S9 as an efficient supermarket.

Table 3: Gradual income analysis for S1

Expected income	Δx_1	Δx_2	Δy_1	Δy_2	Required resources
50	0.0041	0	309.9622	0	0.00256
100	20.92555	384.0339	603.8656	14.33561	22.40938
200	65.69445	1888.967	1189.426	45.01186	87.32781
300	110.4633	3393.900	1774.987	75.68810	152.2462
400	155.2322	4898.834	2360.548	106.3644	217.1646

700	289.5389	9413.633	4117.230	198.3931	411.9199
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Table 3: Gradual income analysis for

Expected income	Δx_1	Δx_2	Δy_1	Δy_2	Required resources
50	0	363.9278	281.4231	25.4	9.040693
100	0	727.8555	562.8462	50.9	18.08139
200	0	1455.711	1125.692	101.9	36.16277
300	9.0004	1981.535	1704.286	138.8	54.76060
400	21.9834	2417.966	2289.846	169.4	73.58695
700	60.9324	3727.258	4046.528	261.5	130.0660

Figure 2: Expected income vs. required budget for S1

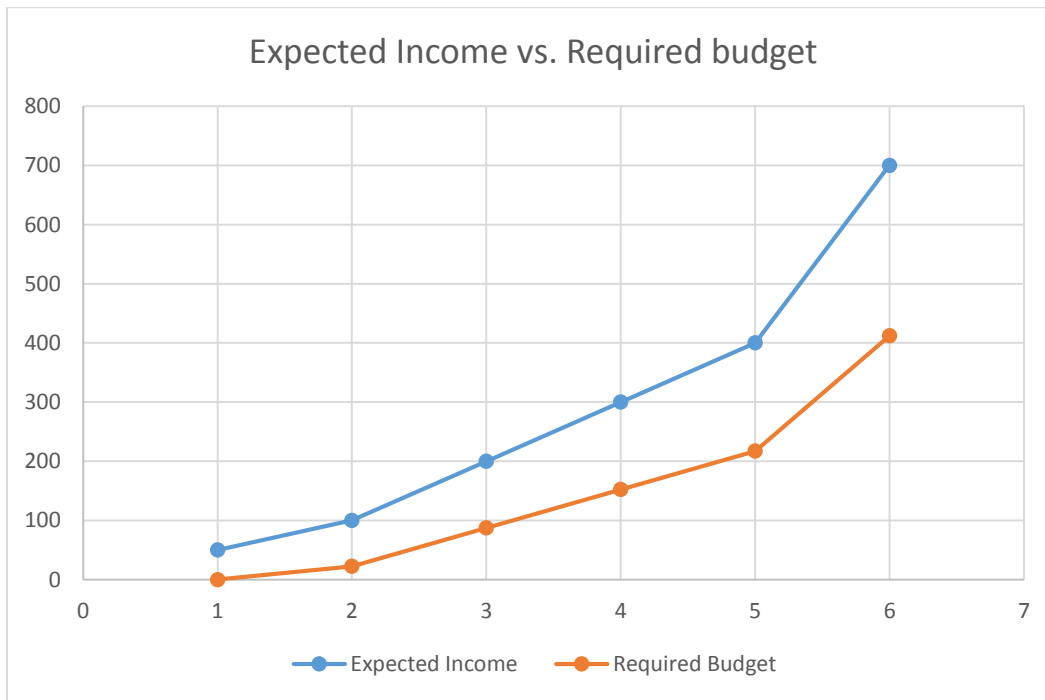
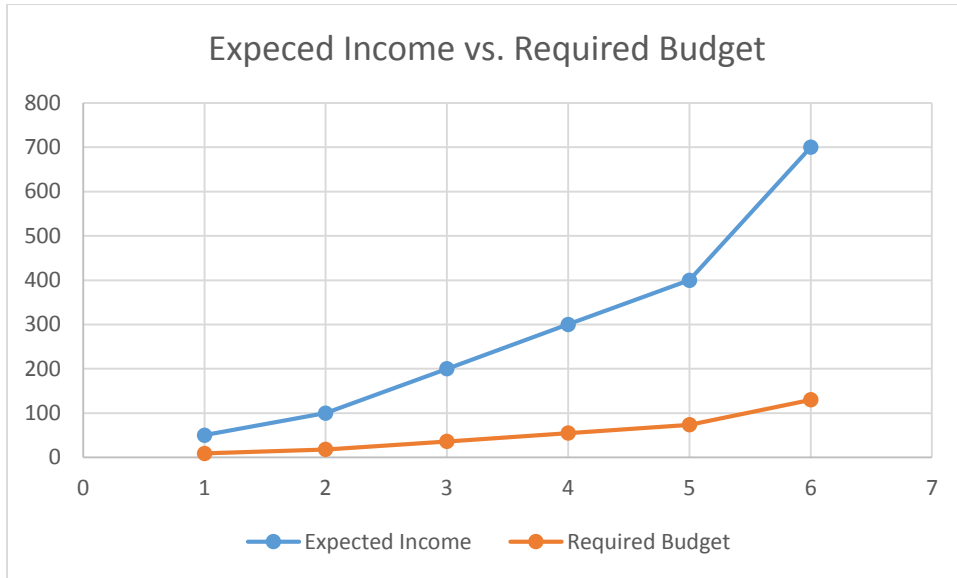


Figure 3: Expected income vs. required budget for S9



In the second part of the analysis, namely, the budget-based analysis, we consider an available and equal amount of budget. Let us say 10 million Rials for each supermarket and investigate for the maximum level of income gained. Now we are interested in finding the maximum level of income that can be gain by this level of budget for each supermarket. The results are reported in table 4. We see not only producible income in this result but also associated input share and output share for getting to this aim.

Table 7: The result of the budget-based model for all supermarkets

Supermarket	Δx_1	Δx_2	Δy_1	Δy_2	Gained income	Output efficiency score φ_j
s_1	12.36782	96.36060	491.4426	8.4	80.80080	3.45
s_2	16.26016	0	678.8406	10.35503	111.3749	3.72
s_{10}	16.26016	0	806.4409	19.16507	133.5501	2.02
s_{18}	16.26016	0	472.6951	17.92240	79.48903	3.37
s_{25}	0	402.5441	149.0833	13.80490	26.54317	2.08
s_{28}	16.26016	0	699.3157	17.38922	115.9489	2.22
s_{31}	0	402.5441	463.6542	0	74.79206	1.53
s_{39}	16.26016	0	412.6616	37.89890	73.41477	1.77

s_{48}	16.26016	0	1094.253	22.26312	180.5370	1.74
s_{56}	7.154292	225.4291	243.9881	4.926790	40.24800	3.42

Like the previous run, we observe different behavior from various supermarkets. However, most supermarkets' main observation is expanding the available for size expansion rather than hiring more labor. In fact, 60 percent of supermarkets prefer the aforementioned pattern.

Now we go one step forward and investigate the effect of the different available budgets for individual supermarkets and report this for one efficient and one inefficient supermarket. Take S1 into consideration that it is an inefficient supermarket. The budget-based model for this supermarket for having different budget levels is reported in table 8. It is considering 5 million Rial extra budget yields to almost 71 million Rial more income, in total. This requires almost 8 percentages more size, and no more labor is needed in this case. We perform this analysis for an efficient supermarket, too, that is S9. The result is reported in table 9. In contrast with S1, no extra size is required, and instead, more labor is needed in different budget levels for S9.

Table 8: Different budget setting and producible income for S1

Available budget	Δx_1	Δx_2	Δy_1	Δy_2	Gained income
5	8.130081	0	436.0422	5.5	71.33971
10	12.36782	96.36060	491.4426	8.4	80.80080
20	19.26400	328.1797	581.5970	13.1	96.19705
50	39.95253	1023.637	852.0602	27.3	142.3858

Table 9: Different budget setting and producible income for S9

Available budget	Δx_1	Δx_2	Δy_1	Δy_2	Gained income
5	0	201.2720	155.6424	14	27.65275
10	0	402.5441	311.2849	28.1	55.30549
20	0	805.0882	622.5697	56.3	110.6110
50	5.717492	1871.176	1556.215	131.0	274.7131

Figure 4 and figure 5 depict the result of the budget-based method for S1 and S9. As can be seen

Figure 4: Available budget vs. gain income based of income model for S1

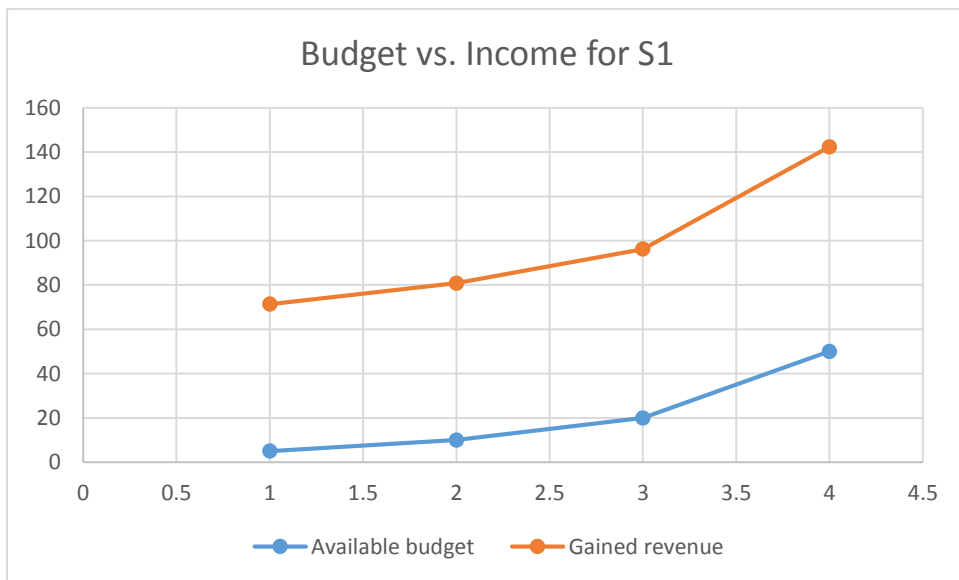
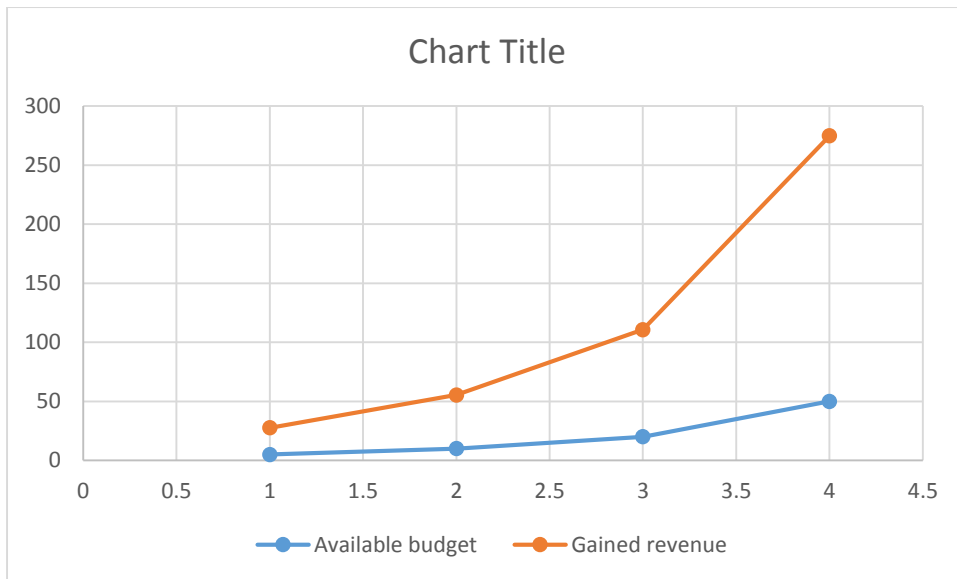


Figure 5: Available budget vs. gain income based of income model for S9



While the available budget and gained income increase with almost the same pattern, we observe a high potential of income gaining by more budget, specifically when we increase the budget level from 3 million Rial to 4 million Rial. In this case, when we increase the budget level by just one percent, we observe more than 2.5 income increments. Such information is vital for decision-makers when budgeting and future planning in the retailer sectors, including supermarkets. Such an analysis can be performed for all supermarkets separately, and the decision-maker's preferences and policies can set more proposed models on budgeting and future planning. In any production system, we face budgeting or planning regarding the limitation of resources. On the other hand, price availability provides the possibility of more deep analysis to get more insight from the production process. Thus, propose models help decision-makers in the procedure of input-output analysis taking the budgeting or planning (each one that is required) into account. Therefore, proposed models can be used for begetting and planning procedures of not only the business sectors but also for any business and production system while taking the efficiency status into consideration.

5. Conclusion

There has always been an interesting and important question for DMs that how to invest the budget within different segments. On the other side, estimating required input and producible output are also vital issues in future planning. This paper proposes two inverse DEA-based models to assist DMs on the budgeting and planning issue. This yields to a new class of inverse DEA problem that their strength and applicability is shown in a real-life problem. In any input-output analysis in real-world problems, we can use proposed models in the current paper when price information is available. Proposed models are general and can be used in any production system that is concerned about the budgeting and planning in the process of input-output estimation.

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