

DECISION MAKING IN FUZZY REASONING TO SOLVE A BACKORDER ECONOMIC ORDER QUANTITY MODEL

SUJIT KUMAR DE^{1,*} AND GOUR CHANDRA MAHATA²

Abstract. Fuzzy reasoning is the subject of fuzzy system where the fuzzy set is characterized by the randomization of the variable associated in the fuzzy set itself. It is the first-time application of fuzzy reasoning over the backorder economic order quantity (EOQ) inventory management problem. We first define the fuzzy reasoning membership function through the use of L-fuzzy number and possibility theory on fuzzy numbers. Considering the holding cost, set up cost, backordering cost and demand rate as reasoning based fuzzy number, we have constructed a dual fuzzy mathematical problem. Then this problem has been solved over the dual feasible space which is associated to the aspiration level and the fuzzy approximation constant. Numerical study reveals the superiority of the proposed method with respect to the crisp solution as well as general fuzzy solution. Sensitivity analysis and graphical illustrations have also been done to justify the novelty of this article.

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1. LITERATURE REVIEW

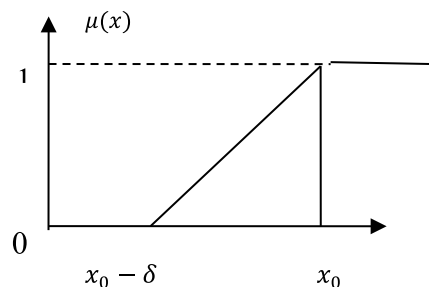
Traditionally, most of the inventory models are made in deterministic and stochastic environment. But if we wish to study the parameters associated to the inventory model assume fuzzy stochastic or the defined fuzzy membership function itself behave randomize number then it is noted that not a single article has been found in the literature. The related works of fuzzy reasoning is very much limited. Zadeh [1] invented fuzzy set theory to capture the problems of non-random uncertain world. Zadeh [2] applied the concept of Linguistic variable and its application to approximate reasoning. The role of fuzzy logic in management uncertainty was studied by Zadeh [3]. Toga and Watanadc [4] discussed an expert system on a chip for real-time approximate reasoning. Pal and Mandal [5, 6] studied a linguistic recognition system based on approximate reasoning. Gaines [7] invented the foundations of fuzzy reasoning. Buckley [8] solved a linear programming problem using possibilistic fuzzy sets. Negoita [9] discussed the current interest in fuzzy optimization. Carlsson [10] solved a multi criteria decision making problem using approximate reasoning. Delgado *et al.* [11, 12] studied on imprecise costs in mathematical programming and new ranking method of optimization problems. Beyond the above methods, some remarkable

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FIGURE 1. Membership function of $\tilde{A}$.

solution methodologies of fuzzy (non)linear programming problems may also be considered over here [13–18]. Lapuccl *et al.* [19] developed new optimality methods for cardinality constrained problems.

However, in the literature very few articles over fuzzy optimization problem are available in which duality principle has been employed. Ma *et al.* [20] studied an interval type fuzzy linear programming problem where the matrix inversion method was utilized and found no duality gap. Authors were basically involved to focus no duality gap and several solution procedures with the help of Lagrangian of the primal (non)linear problems, strong-weak duality principles, pareto optimality considering the fuzzy parameters as Triangular and point to set fuzzy numbers [21–26]. Fuller and Zimmerman [27] gave the concept of fuzzy reasoning in a theoretical manner but he did not apply this theory in any real-life problem. De [28] introduced a research article on inventory management problem using fuzzy reasoning and showed that a duality gap always exists.

Above discussion explores none of the research articles are made for solving the problems of backorder inventory models under fuzzy reasoning in which duality gap within the solution as well as in the feasible space exist. Thus, in this article we solve a backorder EOQ problem that definitely keeps a new footstep in the field of optimization itself.

The organization of this article as developed as follows: Section 2 is a preliminary section that discusses some useful concepts and definitions over fuzzy reasoning, Section 3 gives the formulation of the mathematical model, Section 4 explores the fuzzy mathematical model, Section 5 represents the numerical illustrations, Section 6 includes the sensitivity analysis of the proposed model, Section 7 covers the graphical illustrations and finally Section 8 gives a conclusion.

2. PRELIMINARIES

Here we shall discuss some basic concepts on fuzzy sets, fuzzy reasoning, defuzzification methods and dual feasible space.

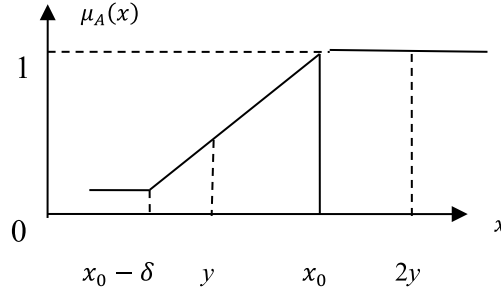
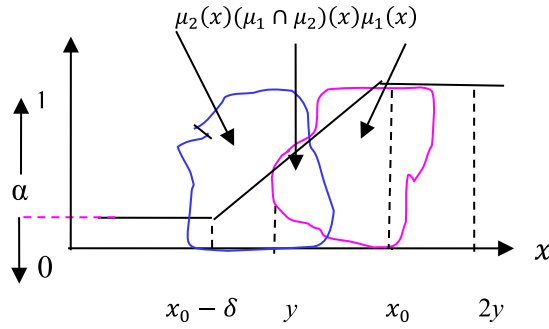
2.1. Fuzzy membership function under fuzzy reasoning

We consider the L-fuzzy number $\tilde{A}$ with the membership grade

$$\mu(x) = \begin{cases} 1, & \text{if } x \geq x_0 \\ 1 - \frac{x_0 - x}{\delta}, & \text{if } x_0 - \delta \leq x \leq x_0 \\ 0, & \text{if } x \leq x_0 - \delta \end{cases} \quad (1)$$

where x_0 refers initial value of the fuzzy number $\tilde{A}$ and δ gives tolerance level with the graphical representation defined at Figure 1.

Now as per De [28] the approximated fuzzy membership of $\tilde{A}$ can be stated in Figure 2 which is the better realization of Figure 3 obtained in (5) that came from the simplification of (2) using (3), (4) exclusively.

FIGURE 2. Membership function of fuzzy reasoning $\tilde{A}$.FIGURE 3. Approximating membership of $\tilde{A}$.

We have

$$\mu_A(x) = \mu_1(x) + \mu_2(x) - \mu_1(x)\mu_2(x) \quad (2)$$

where

$$\mu_1(x) = \begin{cases} 1, & \text{if } x \geq 2y \\ \frac{x-y}{y}, & \text{if } y \leq x \leq 2y \\ 0, & \text{if } x \leq y \end{cases} \quad (3)$$

and

$$\mu_2(x) = \mu_1(x = x_0 - \delta) = \begin{cases} 0, & \text{if } x \geq 2y \\ \frac{x_0 - \delta}{y} - 1, & \text{if } y \leq x \leq 2y \\ 1, & \text{if } x \leq y \end{cases} \quad (4)$$

which gives after simplification

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \geq 2y \\ \left(\frac{2}{y} - \frac{x_0 - \delta}{y^2}\right)x + \frac{2(x_0 - \delta)}{y} - 3, & \text{if } y \leq x \leq 2y \\ \frac{x_0 - \delta}{y} - 1, & \text{if } x \leq y. \end{cases} \quad (5)$$

Now (5) can be simplified as

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \geq 2y \\ lx - m, & \text{if } y \leq x \leq 2y \\ \frac{x_0 - \delta}{y} - 1, & \text{if } x \leq y \end{cases} \quad (6)$$

where

$$\begin{cases} l = \left(\frac{2}{y} - \frac{x_0 - \delta}{y^2} \right) \\ m = 3 - \frac{2(x_0 - \delta)}{y}. \end{cases} \quad (7)$$

2.2. Choice of the probability density function of the random variable Y

For the random variable Y we assume the probability density function (8)

$$p(y) = \begin{cases} \frac{2}{\delta} \left(1 - \frac{x_0 - \delta}{y} \right), & \text{if } x_0 - \delta \leq x \leq x_0 \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

that satisfies

$$\int_{-\infty}^{\infty} p(y) dy = 1. \quad (9)$$

2.3. Constructions of expected value through α -cuts of $\mu_A(x)$

From (6) we have $lx - m \geq \alpha \Rightarrow x \geq \frac{m + \alpha}{l} = \frac{\alpha + 3 - \frac{2(x_0 - \delta)}{y}}{\left(\frac{2}{y} - \frac{x_0 - \delta}{y^2} \right)}$ giving

$$E(x) = \frac{(\alpha + 3)E(y^2) - 2(x_0 - \delta)E(y)}{2E(y) - 2(x_0 - \delta)} \quad (10)$$

and the values of $E(y)$ and $E(y^2)$ can be obtained from (11) and (12).

$$E(y) = \int_{-\infty}^{\infty} yp(y) dy = \frac{2}{\delta^2} \int_{x_0 - \delta}^{x_0} y[y - (x_0 - \delta)] dy = x_0 - \frac{\delta}{3} \quad (11)$$

and

$$E(y^2) = \int_{-\infty}^{\infty} y^2 p(y) dy = \left(x_0^2 - \frac{2}{3} \delta x_0 + \frac{\delta^2}{6} \right). \quad (12)$$

After substitutions (10) becomes

$$E(x) = \frac{\alpha \left(x_0^2 - \frac{2}{3} \delta x_0 + \frac{\delta^2}{6} \right) + \left(x_0^2 + \frac{2}{3} \delta x_0 - \frac{\delta^2}{6} \right)}{x_0 + \frac{\delta}{3}}. \quad (13)$$

Also, from (6) we get

$$\frac{x_0 - \delta}{y} - 1 \leq \alpha \Rightarrow x_0 \leq \delta + (\alpha + 1)y. \quad (14)$$

That appears as

$$x_0 \leq \delta + (\alpha + 1)E(y) \Rightarrow x_0 \leq \delta + (\alpha + 1) \left(x_0 - \frac{\delta}{3} \right). \quad (15)$$

2.4. Selection of dual feasible space under α -dual β cuts (Extension of De [28])

Here we shall set a dual feasible space as $0 < \alpha\beta < 1$ defined in Figure 4 where β and α are complementary to each other.

Now, according to the above feasible space, we may approximate the bounds of the crisp number x_0 by utilizing (15) and get (16) and the schematic diagram of model optimization is given in Figure 5.

$$x_0 \geq \delta \left(\frac{2 + \alpha\beta}{3\alpha\beta} \right) \text{ with } \alpha\beta < 1 \text{ and } \beta < 1. \quad (16)$$

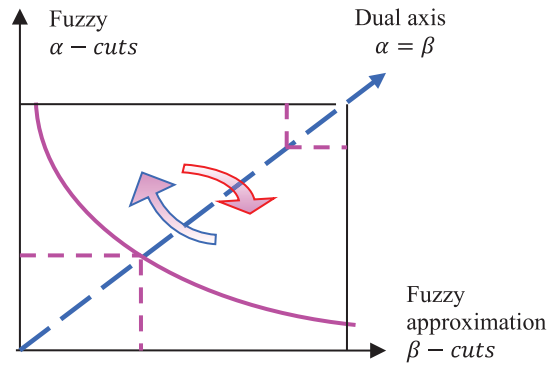


FIGURE 4. Dual feasible space.

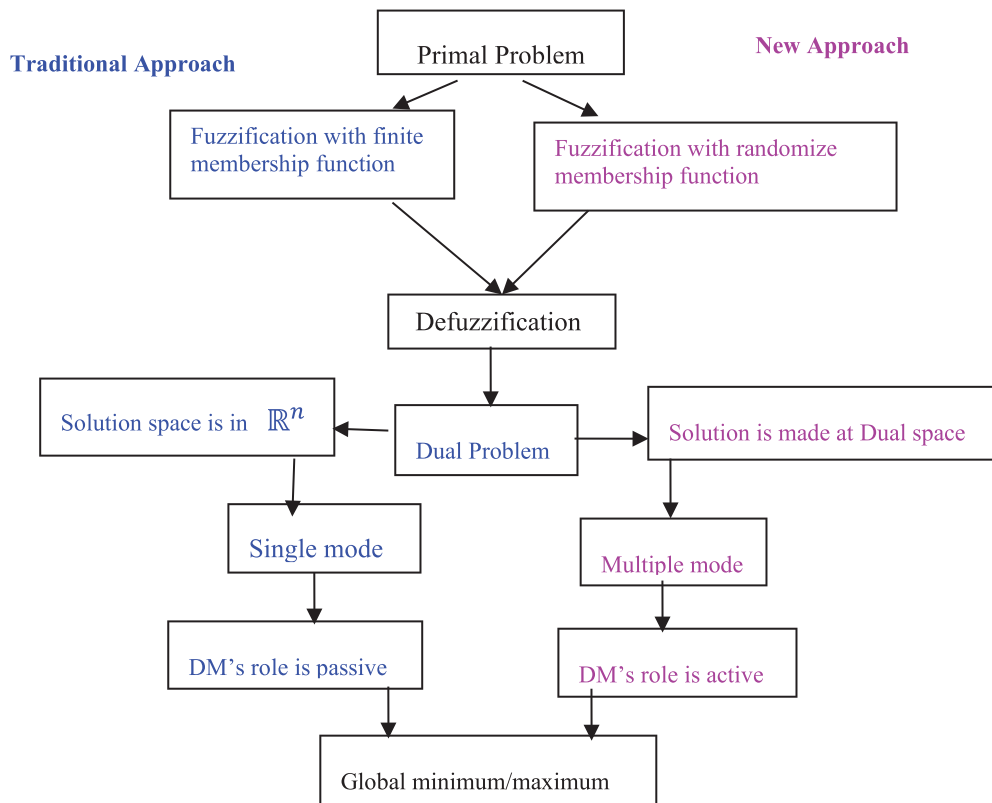


FIGURE 5. Schematic diagram of model optimization.

2.5. Problem definition under fuzzy duality principle

Let us take Primal fuzzy problem as

$$\begin{aligned}
 &\widetilde{\text{Minimize}} \quad \widetilde{z(x)} \\
 &\text{Subject to} \quad \widetilde{f(x)} \geq \widetilde{0}
 \end{aligned} \tag{17}$$

where $\widetilde{z(x)}$ and $\widetilde{f(x)}$ are both differentiable fuzzy objective and fuzzy constraint functions respectively defined on $\Omega \subseteq \mathbb{R}^n$. To construct a saddle point problem, we may use the fuzzy Lagrangian and get a fuzzy dual problem

$$\begin{aligned} & \text{Maximize } \widetilde{L(x, u)} = \widetilde{z(x)} + v^T \widetilde{f(x)} \\ & \text{Subject to } \widetilde{\nabla z(x)} + v^T \widetilde{\nabla f(x)} \geq \widetilde{0}, v(\geq 0) \in \mathbb{R}^n, x \in \Omega \end{aligned} \quad (18)$$

for the divergence operator ∇ .

2.6. Problems for defuzzification

Since α and β are complementary to each other so the above problem (9) can be viewed for a maximization and minimization problem under strong-weak cuts of fuzzy numbers simultaneously and they can be put in (19) and (20) respectively.

$$\left\{ \begin{array}{l} \text{Maximize } \beta \\ \text{Subject to } L(x, v, \beta, \alpha)^{-1} = z(x, \beta, \alpha)^{-1} + v^T f(x, \beta, \alpha)^{-1} \\ \quad \nabla z(x, \beta, \alpha)^{-1} + v^T \nabla f(x, \beta, \alpha)^{-1} \geq 0 \\ \quad \alpha\beta < 1, 0 < \beta < 1, 0 < \alpha < 1 \\ \quad v(\geq 0) \in \mathbb{R}^n, x \in \Omega \end{array} \right. \quad (19)$$

$$\left\{ \begin{array}{l} \text{Maximize } \alpha \\ \text{Subject to } L(x, v, \beta, \alpha)^{-1} = z(x, \beta, \alpha)^{-1} + v^T f(x, \beta, \alpha)^{-1} \\ \quad \nabla z(x, \beta, \alpha)^{-1} + v^T \nabla f(x, \beta, \alpha)^{-1} \geq 0 \\ \quad \alpha\beta < 1, 0 < \beta < 1, 0 < \alpha < 1 \\ \quad v(\geq 0) \in \mathbb{R}^n, x \in \Omega. \end{array} \right. \quad (20)$$

Note. For solving (19) and (20), we may go for an efficient solution algorithm otherwise some software like LINGO 8.0 may be employed.

3. FORMULATION OF BACKORDER INVENTORY MODEL

In this section we use some notations and assumptions (Sect. 3.1) and the development of crisp inventory model with backorder (Sect. 3.2).

3.1. The following notations and assumptions are used for developing the model

Notations

T	Length of cycle time/plan (months)
a	Holding cost for one unit per month (\$)
b	Backorder cost for one unit per month (\$)
c	Cost of placing an order (\$)
d	Total demand over the planning time period $[0, T]$ (Unit)
t_q	Length of a cycle (months)
q	Order quantity per cycle (Units) (a decision variable)
s	Shortage quantity per cycle (Units) (a decision variable)
$F(q, s)$	Crisp objective function (decision variable)
Z	Fuzzy objective function (decision variable)

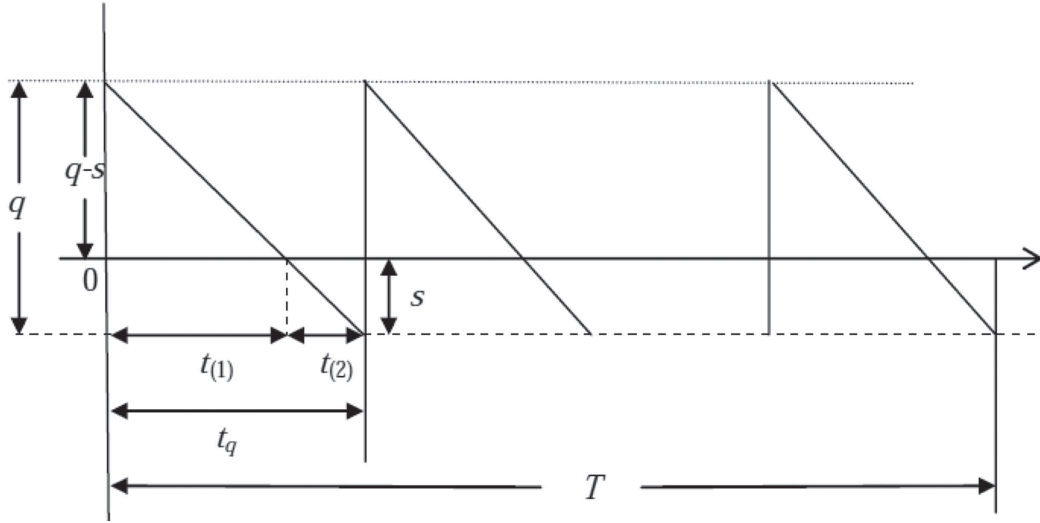


FIGURE 6. Inventory with backorder.

Assumptions

- (i) Replenishments are instantaneous.
- (ii) Backlogging is allowed.
- (iii) Lead time is zero.
- (iv) Demand rate is constant for crisp model but it is assumed to be fuzzy in the fuzzy model.
- (v) All cost components are constant for crisp model but assume fuzzy for fuzzy model.

3.2. The mathematical model

Let us consider the classical backorder inventory model (shown in Fig. 6) where the parameter T stands for the time horizon (months), h represents the holding cost per unit quantity per month. Similarly, the parameters b, c, d, t_q, q and s refer to the unit backorder cost, ordering cost, demand rate, inventory cycle time (months), the order quantity and the shortage quantity per cycle respectively. Our aim is to minimize the objective function for the optimum values of the decision variables q and s respectively.

Then we have

$$\frac{q-s}{t_1} = \frac{q}{t_q} = \frac{s}{t_2} = \frac{d}{T}. \quad (21)$$

The average total inventory cost in the planning period $[0, T]$ is given by

$$F(q, s) = \left[at_1 \frac{q-s}{2} + bt_2 \frac{s}{2} + c \right] \frac{d}{q} = \frac{a(q-s)^2 T}{2q} + \frac{bs^2 T}{2q} + \frac{cd}{q}, (0 < s < q). \quad (22)$$

The crisp optimal solutions are

$$\begin{cases} \text{Optimal order quantity } q^* = \sqrt{\frac{2(a+b)cd}{abT}} \\ \text{Optimal backorder quantity } s^* = \sqrt{\frac{2acd}{b(a+b)T}} \\ \text{Minimal total cost } F(q^*, s^*) = \sqrt{\frac{2abcdT}{a+b}}. \end{cases} \quad (23)$$

4. REPRESENTING MATHEMATICAL MODEL OF FUZZY REASONING

Let all the cost components and the demand rate assume fuzzy flexibilities *via* fuzzy reasoning. Then the fuzzy problem of (10) can be expressed as

$$\widetilde{\text{Minimize } Z} \cong \frac{\widetilde{a}(q-s)^2T}{2q} + \frac{\widetilde{b}s^2T}{2q} + \frac{\widetilde{c}\widetilde{d}}{q}. \quad (24)$$

Here the tilt bar ($\sim$) denotes the fuzzification of the parameters.

Moreover to construct the membership functions of the corresponding fuzzy parameters we may initialize each parameter along with their fuzzy deviation in the following: holding cost assumes value within (a_0, δ_1) , that for unit backordering cost per cycle is (b_0, δ_2) , the ordering cost per order is (c_0, δ_3) , the demand rate is (d_0, δ_4) and for the objective function is (Z_0, δ_0) respectively. Hence the corresponding membership functions are given by:

For the parameter a , let the membership function be

$$\mu_a(\widetilde{a}) = \begin{cases} 1 & \text{if } a \geq a_0 \\ 1 - \frac{a_0 - a}{\delta_1} & \text{if } a_0 - \delta_1 \leq a \leq a_0 \\ 0 & \text{if } a \leq a_0 - \delta_1. \end{cases} \quad (25)$$

For b , let the membership function be

$$\mu_b(\widetilde{b}) = \begin{cases} 1 & \text{if } b \geq b_0 \\ 1 - \frac{b_0 - b}{\delta_2} & \text{if } b_0 - \delta_2 \leq b \leq b_0 \\ 0 & \text{if } b \leq b_0 - \delta_2. \end{cases} \quad (26)$$

For c , let the membership function be

$$\mu_c(\widetilde{c}) = \begin{cases} 1 & \text{if } c \geq c_0 \\ 1 - \frac{c_0 - c}{\delta_3} & \text{if } c_0 - \delta_3 \leq c \leq c_0 \\ 0 & \text{if } c \leq c_0 - \delta_3. \end{cases} \quad (27)$$

For d , let the membership function be

$$\mu_d(\widetilde{d}) = \begin{cases} 1 & \text{if } d \geq d_0 \\ 1 - \frac{d_0 - d}{\delta_4} & \text{if } d_0 - \delta_4 \leq d \leq d_0 \\ 0 & \text{if } d \leq d_0 - \delta_4. \end{cases} \quad (28)$$

And for the fuzzy objective function, we have

$$\mu_{\widetilde{Z}}(Z) = \begin{cases} 1 & \text{if } Z \geq Z_0 \\ 1 - \frac{Z_0 - Z}{\delta_0} & \text{if } Z_0 - \delta_0 \leq Z \leq Z_0 \\ 0 & \text{if } Z \leq Z_0 - \delta_0. \end{cases} \quad (29)$$

Now to carry out the fuzzy problem under fuzzy reasoning, we shall utilize the estimated values of the approximated fuzzy parameters developed at (13) and (16) the following relationship would be arrived

$$E(a_\alpha) = \frac{\alpha \left(a_0^2 - \frac{2}{3} \delta_1 a_0 + \frac{\delta_1^2}{6} \right) + \left(a_0^2 + \frac{2}{3} \delta_1 a_0 - \frac{\delta_1^2}{6} \right)}{a_0 + \frac{\delta_1}{3}}, a_0 \geq \delta_1 \left(\frac{2 + k\alpha}{3k\alpha} \right) \quad (30)$$

$$E(b_\alpha) = \frac{\alpha \left(b_0^2 - \frac{2}{3} \delta_2 b_0 + \frac{\delta_2^2}{6} \right) + \left(b_0^2 + \frac{2}{3} \delta_2 b_0 - \frac{\delta_2^2}{6} \right)}{b_0 + \frac{\delta_2}{3}}, b_0 \geq \delta_2 \left(\frac{2 + k\alpha}{3k\alpha} \right) \quad (31)$$

$$E(c_\alpha) = \frac{\alpha \left(c_0^2 - \frac{2}{3} \delta_3 c_0 + \frac{\delta_3^2}{6} \right) + \left(c_0^2 + \frac{2}{3} \delta_3 c_0 - \frac{\delta_3^2}{6} \right)}{c_0 + \frac{\delta_3}{3}}, c_0 \geq \delta_3 \left(\frac{2 + k\alpha}{3k\alpha} \right) \quad (32)$$

$$E(d_\alpha) = \frac{\alpha \left(d_0^2 - \frac{2}{3} \delta_4 d_0 + \frac{\delta_4^2}{6} \right) + \left(d_0^2 + \frac{2}{3} \delta_4 d_0 - \frac{\delta_4^2}{6} \right)}{d_0 + \frac{\delta_4}{3}}, d_0 \geq \delta_4 \left(\frac{2 + k\alpha}{3k\alpha} \right) \quad (33)$$

$$\begin{aligned} E(Z_\alpha) = & \left\{ \frac{\alpha \left(a_0^2 - \frac{2}{3} \delta_1 a_0 + \frac{\delta_1^2}{6} \right) + \left(a_0^2 + \frac{2}{3} \delta_1 a_0 - \frac{\delta_1^2}{6} \right)}{a_0 + \frac{\delta_1}{3}} \right\} \frac{(q-s)^2 T}{2q} \\ & + \left\{ \frac{\alpha \left(b_0^2 - \frac{2}{3} \delta_2 b_0 + \frac{\delta_2^2}{6} \right) + \left(b_0^2 + \frac{2}{3} \delta_2 b_0 - \frac{\delta_2^2}{6} \right)}{b_0 + \frac{\delta_2}{3}} \right\} \frac{s^2 T}{2q} \\ & + \frac{1}{q} \left\{ \frac{\alpha \left(c_0^2 - \frac{2}{3} \delta_3 c_0 + \frac{\delta_3^2}{6} \right) + \left(c_0^2 + \frac{2}{3} \delta_3 c_0 - \frac{\delta_3^2}{6} \right)}{c_0 + \frac{\delta_3}{3}} \right\} \\ & \times \left\{ \frac{\alpha \left(d_0^2 - \frac{2}{3} \delta_4 d_0 + \frac{\delta_4^2}{6} \right) + \left(d_0^2 + \frac{2}{3} \delta_4 d_0 - \frac{\delta_4^2}{6} \right)}{d_0 + \frac{\delta_4}{3}} \right\} \end{aligned} \quad (34)$$

$$E(Z_\alpha) \geq Z_0 - (1 - \alpha)\delta_0 \quad (35)$$

$$k\alpha < 1, 0 < \alpha < 1, 0 < k < 1. \quad (36)$$

Now utilizing (19) and (20) the equivalent crisp problem (34)–(36) can be developed as follows:

$$\left\{ \begin{array}{l} \text{Maximize } \alpha \\ \text{Subject to } E(Z_\alpha) \leq Z_0 - (1 - \alpha)\delta_0 \\ E(Z_\alpha) = \left\{ \frac{\alpha \left(a_0^2 - \frac{2}{3} \delta_1 a_0 + \frac{\delta_1^2}{6} \right) + \left(a_0^2 + \frac{2}{3} \delta_1 a_0 - \frac{\delta_1^2}{6} \right)}{a_0 + \frac{\delta_1}{3}} \right\} \frac{(q-s)^2 T}{2q} + \left\{ \frac{\alpha \left(b_0^2 - \frac{2}{3} \delta_2 b_0 + \frac{\delta_2^2}{6} \right) + \left(b_0^2 + \frac{2}{3} \delta_2 b_0 - \frac{\delta_2^2}{6} \right)}{b_0 + \frac{\delta_2}{3}} \right\} \frac{s^2 T}{2q} \\ \quad + \frac{1}{q} \left\{ \frac{\alpha \left(c_0^2 - \frac{2}{3} \delta_3 c_0 + \frac{\delta_3^2}{6} \right) + \left(c_0^2 + \frac{2}{3} \delta_3 c_0 - \frac{\delta_3^2}{6} \right)}{c_0 + \frac{\delta_3}{3}} \right\} \left\{ \frac{\alpha \left(d_0^2 - \frac{2}{3} \delta_4 d_0 + \frac{\delta_4^2}{6} \right) + \left(d_0^2 + \frac{2}{3} \delta_4 d_0 - \frac{\delta_4^2}{6} \right)}{d_0 + \frac{\delta_4}{3}} \right\} \\ a_0 \geq \delta_1 \left(\frac{2+k\alpha}{3k\alpha} \right), b_0 \geq \delta_2 \left(\frac{2+k\alpha}{3k\alpha} \right), c_0 \geq \delta_3 \left(\frac{2+k\alpha}{3k\alpha} \right), d_0 \geq \delta_4 \left(\frac{2+k\alpha}{3k\alpha} \right) \\ k\alpha < 1, 0 < \alpha < 1, 0 < k < 1. \end{array} \right. \quad (37)$$

Otherwise

$$\left\{ \begin{array}{l} \text{Maximize } k \\ \text{Subject to } E(Z_\alpha) \leq Z_0 - (1 - \alpha)\delta_0 \\ E(Z_\alpha) = \left\{ \frac{\alpha \left(a_0^2 - \frac{2}{3} \delta_1 a_0 + \frac{\delta_1^2}{6} \right) + \left(a_0^2 + \frac{2}{3} \delta_1 a_0 - \frac{\delta_1^2}{6} \right)}{a_0 + \frac{\delta_1}{3}} \right\} \frac{(q-s)^2 T}{2q} + \left\{ \frac{\alpha \left(b_0^2 - \frac{2}{3} \delta_2 b_0 + \frac{\delta_2^2}{6} \right) + \left(b_0^2 + \frac{2}{3} \delta_2 b_0 - \frac{\delta_2^2}{6} \right)}{b_0 + \frac{\delta_2}{3}} \right\} \frac{s^2 T}{2q} \\ \quad + \frac{1}{q} \left\{ \frac{\alpha \left(c_0^2 - \frac{2}{3} \delta_3 c_0 + \frac{\delta_3^2}{6} \right) + \left(c_0^2 + \frac{2}{3} \delta_3 c_0 - \frac{\delta_3^2}{6} \right)}{c_0 + \frac{\delta_3}{3}} \right\} \left\{ \frac{\alpha \left(d_0^2 - \frac{2}{3} \delta_4 d_0 + \frac{\delta_4^2}{6} \right) + \left(d_0^2 + \frac{2}{3} \delta_4 d_0 - \frac{\delta_4^2}{6} \right)}{d_0 + \frac{\delta_4}{3}} \right\} \\ a_0 \geq \delta_1 \left(\frac{2+k\alpha}{3k\alpha} \right), b_0 \geq \delta_2 \left(\frac{2+k\alpha}{3k\alpha} \right), c_0 \geq \delta_3 \left(\frac{2+k\alpha}{3k\alpha} \right), d_0 \geq \delta_4 \left(\frac{2+k\alpha}{3k\alpha} \right) \\ k\alpha < 1, 0 < \alpha < 1, 0 < k < 1. \end{array} \right. \quad (38)$$

Note. Problem (37) represents the problem of strong fuzzy and the problem (38) gives a weak fuzzy optimization problem.

TABLE 1. Optimal solution.

Problems	q^*	s^*	Z^*	α^*	k^*
Crisp	100	33.333	8000	–	–
General fuzzy	100.033	33.3443	7602.5		
Fuzzy reasoning for α^*	135.959	45.747	7931.82	0.91	0.20
Fuzzy reasoning for k^*	110.136	36.650	7400.00	0.20	0.91

TABLE 2. Optimal solution for maximum α .

α^*	k^*	q^* (Units)	s^* (Units)	Z^* (\$)
0.91	0.20	135.9589	45.7468	7931.818
0.73	0.25	129.8258	43.59	7795.454
0.61	0.30	125.5714	42.09	7704.545
0.52	0.35	122.4428	40.9928	7639.610
0.45	0.40	120.0427	40.1469	7590.909
0.40	0.45	118.1427	39.4770	7553.030
0.36	0.50	116.6004	38.9330	7522.727
0.33	0.55	115.3233	38.4823	7497.934
0.30	0.60	114.2484	38.1029	7477.273
0.28	0.65	113.3308	37.779	7459.79
0.26	0.70	112.5385	37.499	7444.805
0.24	0.75	111.847	37.255	7431.818
0.23	0.80	111.239	37.0401	7420.455
0.21	0.85	110.699	36.8495	7410.428
0.20	0.90	110.2176	36.6792	7401.515
0.19	0.95	109.7847	36.5262	7393.541

TABLE 3. Optimal solution for maximum k .

k^*	α^*	q^* (Units)	s^* (Units)	Z^* (\$)
0.91	0.20	110.1355	36.6502	7400.00
0.73	0.25	112.1501	37.3620	7437.50
0.61	0.30	114.1295	38.0610	7475.00
0.52	0.35	116.0754	38.7478	7512.50
0.45	0.40	117.9893	39.4229	7550.00
0.40	0.45	118.1427	39.4770	7553.03
0.36	0.50	123.5545	41.3842	7662.50
0.33	0.75	124.2545	41.9738	7672.04
0.30	0.60	125.3548	42.0182	7700.00
0.28	0.65	128.8807	43.2589	7737.50
0.26	0.70	130.6081	43.8664	7775.00
0.24	0.75	130.6081	43.8664	7812.50
0.23	0.80	132.3130	44.4658	7850.000
0.21	0.85	133.9953	45.0565	7887.50
0.20	0.90	135.6589	45.6414	7925.00
0.19	0.95	137.5277	46.2977	7967.70

TABLE 4. Sensitivity analysis of the fuzzy reasoning backorder EOQ Model.

Parameters	% change	q^*	s^*	Z^*	α^*	k^*	$\frac{Z^* - Z_s}{Z_s} \times 100\%$
δ_0	+20%	68.264	22.798	7437.398	0.3749	0.485	+0.50
	+10%	68.776	22.968	7482.2	0.3724	0.488	+1.11
	-10%	69.815	23.313	7573.045	0.3675	0.495	+2.33
	-20%	70.341	23.488	7619.007	0.3650	0.498	+2.96
δ_1	+20%	69.360	23.129	7528.197	0.3709	0.4901	+1.73
	+10%	69.325	23.134	7527.831	0.3704	0.4908	+1.73
	-10%	69.264	23.145	7527.071	0.3694	0.4922	+1.72
	-20%	69.238	23.151	7526.676	0.3689	0.4929	+1.71
δ_2	+20%	67.894	22.754	7561.085	0.4147	0.5357	+2.18
	+10%	68.568	22.934	7544.466	0.3926	0.5141	+1.95
	-10%	70.076	23.374	7509.981	0.3466	0.4678	+1.49
	-20%	70.926	23.642	7491.948	0.3226	0.4428	1.24
δ_2	+20%	61.301	23.056	7533.174	0.3775	0.4815	+1.8
	+10%	64.926	23.095	7530.520	0.3740	0.4861	+1.76
	-10%	74.654	23.192	7523.878	0.3651	0.4978	+1.67
	-20%	81.391	23.254	7519.649	0.3595	0.5057	+1.62
δ_4	+20%	69.178	23.103	7529.989	0.3733	0.4870	+1.76
	+10%	69.235	23.121	7528.721	0.3716	0.4892	+1.74
	-10%	69.350	23.158	7526.193	0.3682	0.4937	+1.71
	-20%	69.408	23.177	7524.934	0.3666	0.496	+1.69
a_0	+20%	61.301	23.056	7533.174	0.3776	0.4815	+1.8
	+10%	64.926	23.095	7530.520	0.3740	0.4861	+1.76
	-10%	74.655	23.192	7523.878	0.3652	0.4979	+1.67
	-20%	81.391	23.255	7519.649	0.3595	0.5057	+1.62
b_0	+20%	66.579	19.586	7500.896	0.3345	0.4462	+1.36
	+10%	67.806	21.206	7513.188	0.3509	0.4672	+1.53
	-10%	71.129	25.490	7544.291	0.3924	0.5201	+1.95
	-20%	73.454	28.416	7564.572	0.4194	0.5544	+2.22
c_0	+20%	68.904	23.014	7536.045	0.3814	0.4767	+1.84
	+10%	69.085	23.073	7532.0360	0.3760	0.4835	+1.78
	-10%	69.535	23.217	7522.1770	0.3629	0.5010	+1.65
	-20%	69.818	23.308	7516.035	0.3547	0.5126	+1.57
d_0	+20%	70.226	23.439	7507.321	0.3431	0.5299	+1.45
	+10%	69.784	23.297	7516.780	0.3557	0.5111	+1.58
	-10%	73.256	24.411	7446.123	0.2615	0.6953	+0.62
	-20%	72.817	24.270	7454.623	0.2728	0.6664	+0.74
Z_0	+20%	85.533	28.544	9101.589	0.3354	0.5420	+22.99
	+10%	77.328	25.815	8314.541	0.3527	0.5155	+12.36
	-10%	61.433	20.521	6740.052	0.3867	0.4701	-8.92
	-20%	53.746	17.959	5952.024	0.4027	0.4515	-19.57

5. ILLUSTRATIVE NUMERICAL EXAMPLE

To perform a numerical study, we shall consider the parametric values in the following: Let, $a = \$10$ per unit, $b = \$20$ per unit, $c = \$200$ per cycle, $d = 2000$ units per months, $T = 12$ months, then using LINGO 8.0 software we get the optimal solution as $q^* = 100$, $s^* = 33.3333$ and the minimal total cost $F(q^*, s^*) = 8000$ for the crisp model. However, for fuzzy model we may take the initial values and their corresponding tolerances as $(Z_0, \delta_0) = (8000, 750)$, $(a_0, \delta_1) = (10, 2)$, $(b_0, \delta_2) = (20, 5)$, $(c_0, \delta_3) = (200, 40)$ and $(d_0, \delta_4) = (2000, 250)$ respectively. The optimal results are put in Table 1.

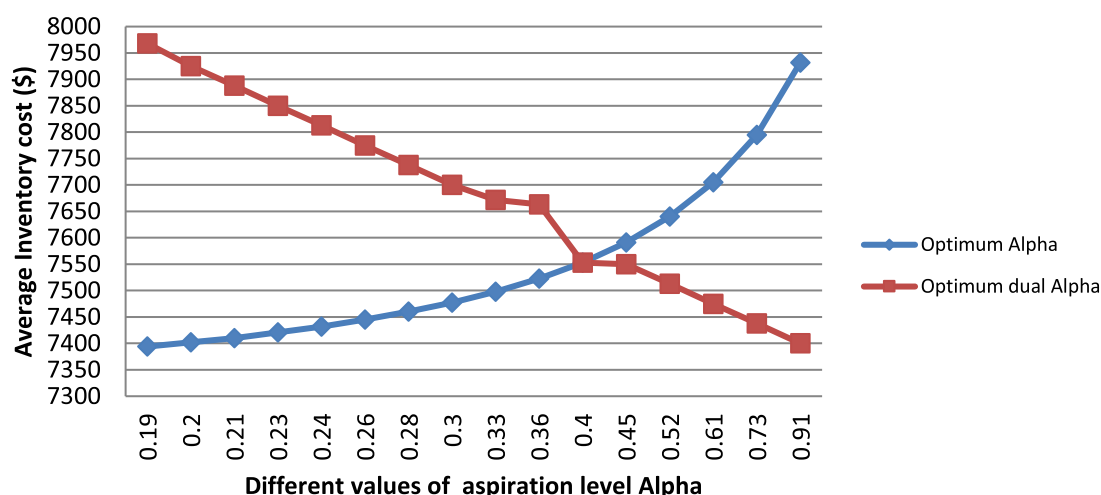


FIGURE 7. Variation of cost function under fuzzy reasoning.

From the Table 1 it is seen that; the crisp solution is maximum with respect to the general fuzzy and dual space fuzzy reasoning problems. In crisp model, the order quantity and the backorder quantity are 100 units and 33.333 units respectively but for the case of fuzzy reasoning, the case of highest aspiration level they are 135.959 units and 45.747 units respectively and that of dual aspiration level they are 110.136 units and 36.650 units alone. The model minimum becomes \$7400.0.

Table 2 shows the optimum value of the decision variable whenever the higher aspiration level is considered. For the ranges of 0.19–0.91 of the aspiration levels the objective value assumes the range \$7393.541–\$7931.818 with respect to the range of order quantity 109.7847–135.9589 units and that for backorder quantity 36.5262–45.9589 units exclusively. The lower aspiration level giving lower minimum inventory cost.

Table 3 reveals the values of the objective function under optimum dual aspiration level giving the minimum value \$7400.00. The higher dual aspiration level gives the lower average inventory cost. The range of the order quantity and backorder quantity are respectively obtained as 110–138 units and 37–47 units respectively.

6. SENSITIVITY OF THE MODEL

Here we shall perform the sensitivity analysis of the proposed model for each of the fuzzy deviation parameters as well as their initial approximations of the fuzzy cost components and fuzzy demand parameters on and from (–20% to +20%), the outputs are put in Table 4.

Table 4 gives the values of the objective functions that have low positive deviation respect to crisp solution. All the initial values of the cost parameters and the demand parameters including their fuzzy deviations are poorly sensitive. The cost function assumes the range on and from 0.50% to +2.96% but for the change of the initial value (approximation) objective function this range shifted to –19.57% to +22.99% exclusively. However, the amount of order quantity and the backorder quantity keep values around 70 units and 23 units respectively in most of the cases. Another study reveals that, the values of the aspiration levels (in dual space) which are frequently arrived belongs to the weaker family of decision maker's (DM's) satisfactions. Moreover, some cases have arrived where the strong fuzzy/cut sets are very much supportive to have minimum objective value.

7. GRAPHICAL ILLUSTRATIONS

Here we shall draw several graphs of the proposed model using the data set obtained from sensitivity analysis Table 4.

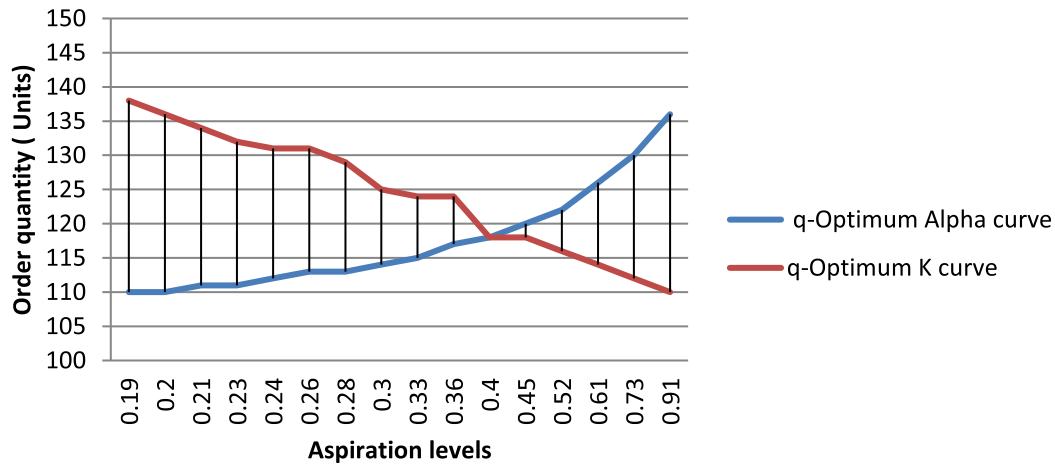


FIGURE 8. Variation of order quantity over dual space.

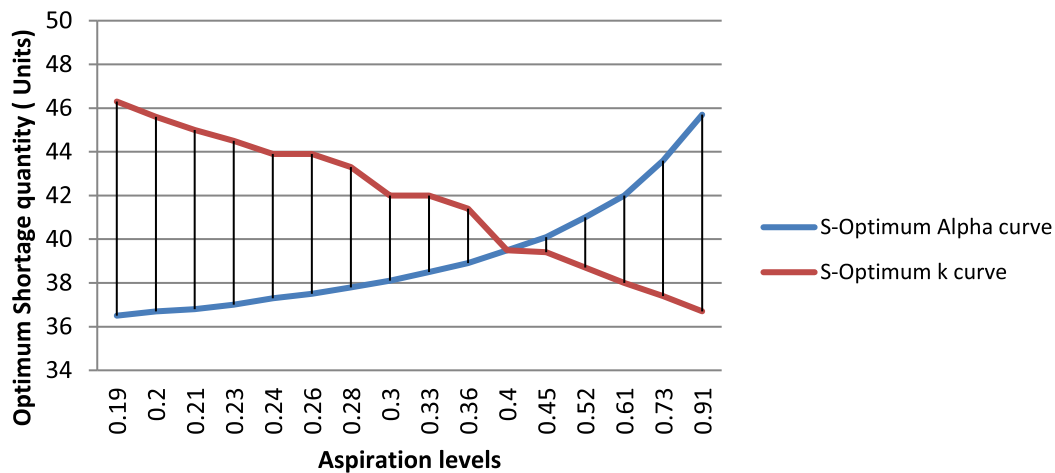


FIGURE 9. Variation of optimum shortage quantity over dual space.

Figure 7 reveals the average inventory cost variation with respect to the several aspiration and dual aspiration levels of the fuzzy reasoning model. At the value 0.4, the average inventory costs are equal for both the cases (dual spaces) of fuzzy reasoning model. For increasing aspiration level causes the incremental values of the objective function but for dual aspiration level the results become reverse.

Figure 8 states the optimum order quantity curves due to the variation of dual aspiration levels. The order quantity becomes fixed at 117 units for the fixed dual space point 0.4 alone. Beyond this fixed point the values of the order quantity getting diffracted.

Figure 9 indicates the optimum shortage quantity curve due to the changes of the dual spaces. The backorder quantity remains fixed at 39 units with respect to the aspiration level 0.4, but beyond this value the values of the backorder quantity are getting scattered.

Figure 10 showing the nature of dual axis of the dual space $k\alpha < 1, 0 < \alpha < 1, 0 < k < 1$. The aspiration level curve assumes a concave like shape but its dual assumes almost a straight line.

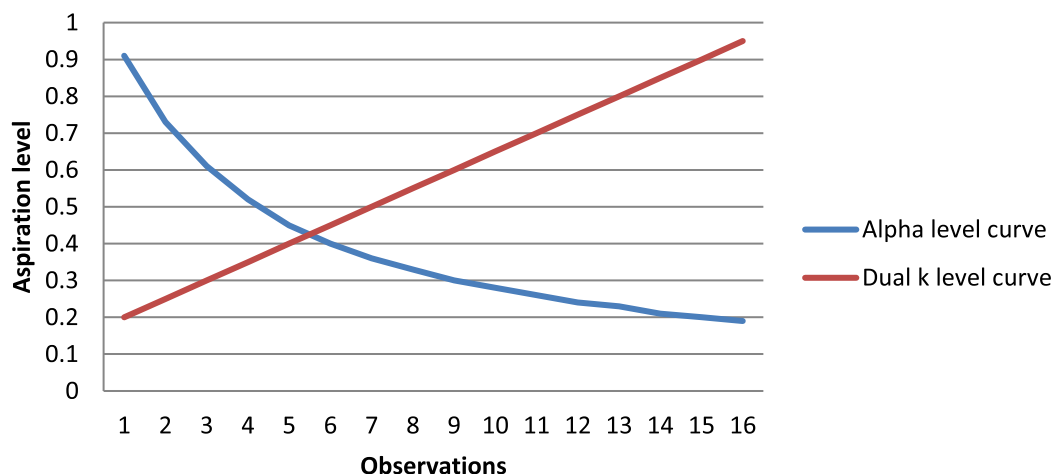


FIGURE 10. Nature of optimum Dual axes.

8. CONCLUSION

This study has developed for classical backorder inventory model of fuzzy approximate reasoning. The most fundamental concept of fuzzy reasoning is that the fuzzy membership functions are assumed to be randomized. By this way we are able to get finer optimal solution than the traditional fuzzy alpha cut approach. Here we have introduced a new solution space named dual aspiration level space. In this new feasible region, all the decision variables become finer than the values so obtained from some other existing methods. The managerial insights are

- (a) The decision maker can order/back order more quantities than (s)he wishes.
- (b) By this approach the DM can look the model for strong and weak aspiration level simultaneously.
- (c) The DM has many options to set several strategic plans once at a time.
- (d) By this method the DM might be able to know the upper limit of the inventory cost that would incur in the inventory process itself.

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REFERENCES

- [1] L.A. Zadeh, Fuzzy sets. *Inf. Control* **8** (1965) 338–356.
- [2] L.A. Zadeh, The concept of Linguistic variable and its application to approximate reasoning. Parts I, II, III. *Inf. Sci.* **8** (1975) 199–251; **8** (1975) 301–357; **9** (1975) 43–80.
- [3] L.A. Zadeh, The role of fuzzy logic in the management of uncertainty in expert systems. *Fuzzy Sets Syst.* **11** (1983) 199–223.
- [4] M. Toga and H. Watanadc, Expert systems on a chip: an engine for real-time approximate reasoning. *IEEE Expert.* **1** (1986) 55–62.
- [5] S.K. Pal and D.P. Mandal, Fuzzy logic and approximate reasoning: an overview. *IETE J. Res.* **37** (1991) 548–560.
- [6] S.K. Pal and D.P. Mandal, Linguistic recognition system based on approximate reasoning. *Btform Sci.* **61** (1992) 135–161.
- [7] B.R. Gaines, Foundations of fuzzy reasoning. *Int. J. Man-Mach. Stud.* **8** (1976) 623–668.
- [8] J.J. Buckley, Solving possibilistic linear programming problems. *Fuzzy Sets Syst.* **31** (1989) 329–341.
- [9] C.V. Negoita, The current interest in fuzzy optimization. *Fuzzy Sets Syst.* **6** (1981) 261–269.
- [10] C. Carlsson, Approximate reasoning for solving fuzzy MCDM problems. *Cybern. Syst.* **18** (1987) 35–48.

- [11] M. Delgado, J.L. Verdegay and M.A. Vila, Imprecise costs in mathematical programming problems. *Control Cybern.* **16** (1987) 114–121.
- [12] M. Delgado, J.L. Verdegay and M.A. Vila, A procedure for ranking fuzzy numbers using fuzzy relations. *Fuzzy Sets Syst.* **26** (1988) 49–62.
- [13] S.A. Orlovski, On formalization of a general fuzzy mathematical problem. *Fuzzy Sets Syst.* **3** (1980) 311–321.
- [14] H. Tanaka and K. Asai, Fuzzy solutions in fuzzy linear programming problems. *IEEE Trans. Syst. Man Cybern.* **SMC-14** (1984) 325–328.
- [15] M. Roubens and P. Vincke, Preference Modeling. Springer-Verlag, Berlin (1985).
- [16] H.J. Zimmermann, Description and optimization of fuzzy systems. *Int. J. General Syst.* **2** (1976) 209–215.
- [17] H.J. Zimmermann, Application of fuzzy set theory to mathematical programming. *Inf. Sci.* **36** (1985) 29–58.
- [18] J. Ramik and J. Rimanek, Inequality relation between fuzzy numbers and its use in fuzzy optimization. *Fuzzy Sets Syst.* **16** (1985) 123–138.
- [19] M. Lapuccl, T. Levato and M. Scalandrone, Convergent inexact penalty decomposition methods for cardinality-constrained problems. *J. Opt. Theory Appl.* **188** (2021) 473–496.
- [20] M. Ma, M. Friedman and A. Kandel, Duality in fuzzy linear systems. *Fuzzy Sets Syst.* **109** (2000) 55–58.
- [21] H.C. Wu, Duality theorems in fuzzy mathematical programming problems based on the concept of necessity. *Fuzzy Sets Syst.* **139** (2003) 363–377.
- [22] C. Zhang, X.H. Yung and E.S. Lee, Duality theory in fuzzy mathematical programming problems with fuzzy coefficients. *Comput. Math. Appl.* **49** (2005) 1709–1730.
- [23] H.C. Wu, Duality theorems and saddle point optimality conditions in fuzzy nonlinear programming problems based on different solution concepts. *Fuzzy Sets Syst.* **158** (2007) 1588–1607.
- [24] S.H. Nasser and A. Ebrahimnejad, A new approach to duality in fuzzy linear programming, in Fuzzy Engineering and Operations Research. Advances in Intelligent and Soft Computing, edited by B.Y. Cao and X.J. Xie. Vol. 147. Springer, Berlin, Heidelberg (2012).
- [25] H. Mishmast Nehi and A. Drayab, Duality theorems in fuzzy optimization problems. *Fuzzy Inf. Eng.* **5** (2013) 87–98.
- [26] T. Zou, Extended duality in fuzzy optimization problems. *Math. Prob. Eng.* (2015). DOI: [10.1155/2015/826752](https://doi.org/10.1155/2015/826752).
- [27] R. Fuller and H.J. Zimmermann, Fuzzy reasoning for solving fuzzy mathematical problems. *Fuzzy Sets Syst.* **60** (1993) 121–133.
- [28] S.K. De, Solving an EOQ model under fuzzy reasoning. *Appl. Soft Comput.* **99** (2021) 106892.



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