Solutions to four open problems on quorum colorings of graphs

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Abstract

A partition $\pi = \{V_1, V_2, ..., V_k\}$ of the vertex set V of a graph G into k color classes V_i, with $1 \leq i \leq k$ is called a quorum coloring if for every vertex $v \in V$, at least half of the vertices in the closed neighborhood $N[v]$ of v have the same color as v. The maximum cardinality of a quorum coloring of G is called the quorum coloring number of G and is denoted $\psi_q(G)$. In this paper, we give answers to four open problems stated in 2013 by Hedetniemi, Hedetniemi, Laskar and Mulder. In particular, we show that there is no good characterization of the graphs G with $\psi_q(G) = 1$ nor for those with $\psi_q(G) > 1$ unless $\mathcal{P} \neq \mathcal{NP} \cap co - \mathcal{NP}$. We also construct several new infinite families of such graphs, one of which the diameter $diam(G)$ of G is not bounded.

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1 Introduction

Let $G = (V, E)$ be a simple graph with order $n = |V|$. The graph induced in G by a subset S of V is denoted by $G[S]$. For every vertex $v \in V$, the *open neighborhood* $N_G(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in G is $d_G(v) = |N_G(v)|$. A vertex of G with degree one is a *leaf* of G. The maximum and minimum vertex degrees in G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. More generally, the degree of a vertex v in $G[S]$ is denoted by $d_S(v)$. The *diameter* of G, denoted $diam(G)$, is the greatest distance between any pair of vertices of V.

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The *join* of two graphs G and H, denoted $G + H$ is the graph consisting of the disjoint union of G and H together with all edges between the vertices in G and those in H. For every integer $k \geq 2$, a *k-partite* graph is a graph whose vertices can be partitioned into k different independent sets, that is, sets whose vertices are pairwise non adjacent. The *complete multipartite* graph, denoted $K_{n_1,n_2,...,n_\ell}$ is the graph $\overline{K_{n_1}} + \overline{K_{n_2}} + \cdots + \overline{K_{n_\ell}}$, where K_{n_i} denotes the complete graph of cardinality n_i, and $\overline{K_{n_i}}$ denotes the complement of K_{n_i} which consists of n_i isolated vertices, for each $i \in \{1, 2, \ldots, \ell\}$.

The concept of defensive alliances in graphs was introduced in [6] by Kristiansen, Hedetniemi and Hedetniemi as follows. A *defensive alliance* in a graph G is a subset S of $V(G)$ such that for every vertex $v \in S$, $|N_G[v] \cap S| \geq |N_G[v] \cap (V \setminus S)|$, or equivalently $d_S(v) + 1 \geq d_{(V \setminus S)}(v)$. The *defensive alliance number* of G, denoted $a(G)$ equals the minimum cardinality of a defensive alliance in G. A defensive alliance of cardinality $a(G)$ is called a *minimum defensive alliance*. The authors [4] also proved the following sharp upper bound on the alliance defensive number which is obtained by complete graphs.

Theorem 1. *[4] For any graph G of order n, $a(G) \leq \lceil n/2 \rceil$.*

The definition of a defensive alliance was mainly motivated by the study of alliances between nations so that, two adjacent vertices belonging to an alliance are considered as mutually protective allies against a threat, while all the vertices outside an alliance are potential enemies. This means that every vertex v of an alliance S is adjacent to at least as many allies as enemies, where v is allied with itself. Haynes and Lachniet initiated on their part in [5] the study of partitioning the vertex set of a graph into defensive alliances, where such partitions are called *alliance partitions*. This problem was further investigated in [3] by Eroh and Gera.

The concept of quorum colorings is closely related to that of defensive alliances in graphs. It was introduced in [7] by Hedetniemi, Hedetniemi, Laskar and Mulder in order to study the alliance partitions from the perspective of coloring theory. In fact, a partition $\pi = \{V_1, V_2, ..., V_k\}$ of the vertex set V of a graph G into k color classes V_i, with $i \in \{1, ..., k\}$ is called a *quorum coloring* if for every vertex $v \in V$, at least half of the vertices in the closed neighborhood $N_G[v]$ have the same color as v. The color classes V_i are called *quorum classes*. The maximum cardinalty of a quorum coloring of G is called the *quorum coloring number* of G and is denoted by $\psi_q(G)$. A quorum coloring of cardinality $\psi_q(G)$ is called a *ψ_q-coloring*. It can be seen from the definitions that every quorum class is a defensive alliance and consequently, a quorum coloring of a graph G is the same thing as an alliance partition of G. Nevertheless, we adopt in this paper the coloring notation and terminology given in [7]. It is worth noting that two other articles on quorum colorings of graphs were recently published ([10, 11]) containing answers to open problems posed in [7] including a Gaddum-Nordhaus inequality and complexity results.

Quorum colorings have several real-world applications (cf. [7], [9] and [12]), including data clustering, the goal of which is to partition a dataset into homogeneous packets in the sense that the data in the same packet share more characteristics in common between them than with data outside of this packet. This problem can be modeled by a graph G in which each data is represented by a vertex so that two vertices are adjacent if the corresponding data share a fixed minimum number of common characteristics, and hence the objective is to color the vertex set of the resulting graph such that at least half of the neighbors of each vertex v have the same color as v, where v is counted itself as a neighbor. In other words, at least half of the vertices in the closed neighborhood of v must have the same color as v, that is, each color class is a quorum class. Therefore, the maximization of the number of color classes has as aim the refinement of the data classification as much as possible.

In [7], Hedetniemi et al. proved the following three propositions.

Proposition 2. [7] For the complete graph K_n of odd order, $\psi_q(K_n) = 1$, while for any complete graph K_n of even order, $\psi_q(K_n) = 2$.

Proposition 3. [7] If G is a graph of odd order n for which $a(G) = \lceil n/2 \rceil$, then $\psi_q(G) = 1$.

Proposition 4. [7] For any graph $G = K_r + \overline{K_s}$, where $r + s$ is odd, $\psi_q(G) = 1$.

Furthermore, the authors [7] raised the following open problems.

1. Can you characterize the class of graphs for which $\psi_q(G) = 1$ or the class of graphs for which $\psi_q(G) > 1$? In fact, can you find any infinite family of graphs other than those of the form K_{2n+1} or $K_r + \overline{K}_s$ for $r + s$ odd and $r \geq 2$, for which $\psi_q(G) = 1$?
2. Is $\psi_q(G) = 1$ if and only if $a(G) = \lceil n/2 \rceil$ and n is odd?
3. If $\psi_q(G) = 1$, is $diam(G) \leq 2$?
4. Is $\lfloor diam(G)/2 \rfloor \leq \psi_q(G)$? It is easy to prove the following.

   Proposition 5. For any tree T,
   $$\lfloor diam(T)/2 \rfloor \leq \psi_q(T).$$

In this paper, we answer Questions 1, 2, 3 and 4 as follows. In Sections 2 and 3 we give a first and a second answer to Question 1, respectively. In Section 4, we give a third answer to Question 1 which also answers Question 2 in negative, and in Section 5 we give a fourth answer to Question 1 which again answers Question 2 as well as Questions 3 and 4 in negative. We conclude our study by raising some open problems.

2 First answer to Question 1

In this section, we give a first answer to Question 1. We will first show that there exists no linear-time algorithm solving the following two complementary decision problems unless $\mathcal{P} \neq \mathcal{NP} \cap co\text{-}\mathcal{NP}$.

QUORUM-ONE
Instance: Graph $G = (V, E)$.
Question: Is $\psi_q(G) > 1$?

ONE QUORUM
Instance: Graph $G = (V, E)$.
Question: Is $\psi_q(G) = 1$?

Then, we provide three necessary and sufficient conditions for a graph G with $\psi_q(G) \geq 2$. Before establishing our first result which is related to the complexity aspect, we have to recall the following definition due to Edmonds [2].

A characterization $\mathscr{C}$ of a given class of graphs $\mathcal{G}$ is said to be *good* if the decision problem asking whether a given graph G satisfies the property $\mathscr{C}$ is both in $\mathcal{NP}$ and *co*-$\mathcal{NP}$, that is, if it belongs to $\mathcal{NP} \cap$ *co*-$\mathcal{NP}$. The author [2] also posed the following well-known conjecture.

Conjecture 6. *[2]* $\mathcal{P} = \mathcal{NP} \cap co - \mathcal{NP}$.

In [10], Sahbi showed that problem QUORUM-ONE is *NP*-complete.

Theorem 7. *[10] Problem QUORUM-ONE is NP-complete.*

Theorem 7 says that it is unlikely that polynomial-time algorithms exist to solve QUORUM-ONE or ONE QUORUM. Therefore, our first announced result follows by Conjecture 6.

Corollary 8. *There is no good characterization neither of the graphs G with $\psi_q(G) = 1$ nor of those with $\psi_q(G) > 1$ unless $\mathcal{P} \neq \mathcal{NP} \cap co - \mathcal{NP}$.*

Although it is likely that no good characterization exists for graphs G with $\psi_q(G) > 1$, we provide in the following three necessary and sufficient conditions, however. The statement of the first one is inspired by a theorem proved by Shafique and Dutton [13] on graphs admitting a satisfactory partition. Before presenting it, we need the following definitions.

An edge *cutset* of a connected graph G is a set $S \subseteq E(G)$ such that $G \setminus S$ is disconnected. If no proper subset of S is a cutset, then S is called *minimal cutset*. The edges of the cutset S which have one end vertex in V_1 and the other in V_2 is denoted by $e(V_1, V_2)$. A *critical* (-1)-*cutset* $e(V_1, V_2)$ of a connected graph G is a minimal cutset, such that $|V_i| \geq 2$, $i \in \{1, 2\}$ and moving any vertex from one set to the other decreases the size of $e(V_1, V_2)$ by at most one edge.

To prove our first result, we also need to use the following two propositions both due to Hedetniemi et al. [7].

Proposition 9. [7] Let $G = (V, E)$ be a graph without isolated vertices, and let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a quorum coloring of G. Then, for every color class V_i, if $|V_i| = 1$ then the only vertex in V_i is a leaf in G; otherwise $|V_i| \geq 2$.

Proposition 10. [7] Let G be a graph, and let $\pi = \{V_1, V_2, \ldots, V_k\}$ be any ψ_q-coloring of G. Then, for every i, $1 \leq i \leq k$, the induced subgraph $G[V_i]$ is connected.

For a connected graph admitting a bipartition as a quorum coloring, each vertex belonging to any of the two classes of the bipartition has at most one less neighbor in its class than outside it. Therefore, moving any vertex from a class to the other decreases the number of edges between the two classes by at most one. This leads us to state our first equivalence.

Theorem 11. *For a connected graph G without leaves, $\psi_q(G) \geq 2$ if and only if G has a critical (-1)-cutset.*

Proof. Let $e(V_1, V_2)$ be a critical (-1)-cutset of G. Then, for every $i \in \{1, 2\}$, moving any vertex from V_i to V_{3-i} decreases the size of $e(V_1, V_2)$ by at most one edge. Thus, by putting $V_i' = V_i \setminus \{v\}$ and $V_{3-i}' = V_{3-i} \cup \{v\}$ for some $i \in \{1, 2\}$ and some vertex $v \in V_i$, we have $|V_i'| \geq 1$ (that is, $V_i' \neq \emptyset$) and $|e(V_i', V_{3-i}')| = |e(V_i, V_{3-i})| - d_{V_{3-i}}(v) + d_{V_i}(v) \geq |e(V_i, V_{3-i})| - 1$. By eliminating $|e(V_i, V_{3-i})|$ on both sides of the inequality, we obtain that $d_{V_i}(v) + 1 \geq d_{V_{3-i}}(v)$, which means that $\{V_1, V_2\}$ is a quorum coloring of G of order 2. Consequently, $\psi_q(G) \geq 2$.

Conversely, assume that $\psi_q(G) \geq 2$ and let $\pi = \{V_1, V_2\}$ be a quorum coloring of G of order 2. Then $d_{V_i}(v) + 1 \geq d_{V_{3-i}}(v)$, for every $i \in \{1, 2\}$ and $v \in V_i$. Furthermore, since G does not have leaves, then we have by Proposition 9 that $|V_i| \geq 2$ for every $i \in \{1, 2\}$. Let us show that $e(V_1, V_2)$ is a critical (-1)-cutset. For an arbitrarily chosen vertex $v \in V$, set $V_i' = V_i \setminus \{v\}$ and $V_{3-i}' = V_{3-i} \cup \{v\}$. Therefore, by moving any vertex v from V_i to V_{3-i}, we obtain that $|e\left(V_i', V_{3-i}'\right)| = |e(V_i, V_{3-i})| - d_{V_{3-i}}(v) + d_{V_i}(v)$. By using the inequality $d_{V_i}(v) + 1 \geq d_{V_{3-i}}(v)$, it follows that $|e\left(V_i', V_{3-i}'\right)| \geq |e(V_i, V_{3-i})| - 1$, which means that $|e(V_1, V_2)|$ decreased by at most 1. Moreover, since $G[V_i]$ is connected according to Proposition 10, then $e(V_1, V_2)$ is a minimal (-1)-cutset. As a result, π is a critical (-1)-cutset. ∎

The negation of Theorem 11 gives the following corollary as direct consequence.

Corollary 12. *If G is a connected graph without leaves, then $\psi_q(G) = 1$ if and only if G has no critical (-1)-cutset.*

Remark 13. Note that if G has a leaf v, then the bipartition $\pi = \{\{v\}, V(G) \setminus \{v\}\}$ is a quorum coloring of order two, that is, $\psi_q(G) \geq 2$.

To state the two other necessary and sufficient conditions, we need to recall a definition and a proposition both due to Stiebitz [14].

Definition 14. Let $G = (V, E)$ be a graph and $a, b : V \to \mathbb{N}$ two functions such that $d_G(x) \geq a(x) + b(x) + 1$, for every vertex $x \in V$. A pair (A, B) is said to be *feasible* if A and B are disjoint, non empty subsets of V such that:

(i) $d_A(x) \geq a(x)$ for all $x \in A$, and

(ii) $d_B(x) \geq b(x)$ for all $x \in B$.

Moreover, if $A \cup B = V$, then (A, B) is called a *feasible partition* of V.

Proposition 15. [14] If there exists a feasible pair, then there exists a feasible partition of $V(G)$, too.

Remark 16. Stiebitz pointed out that Proposition 15 remains valid under the weaker assumption that $d_G(x) \geq a(x) + b(x) - 1$ for all $x \in V(G)$.

Proposition 15 and Remark 16 will be used in the proof of our next result, which we can now state.

Theorem 17. *Let G be a connected graph with n_ϵ vertices of even degree at each of which we join a leaf, resulting in a graph G'. Then the following assertions are equivalent:*

(i) G admits two disjoint quorum classes.

(ii) $\psi_q(G) \geq 2$.

(iii) $\psi_q(G') \geq n_\epsilon + 2$.

Proof. We will prove the following implications loop: $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$**.** Let A and B be two disjoint quorum classes of G. Then

$$\text{for all } x \in A: \; d_A(x) + 1 \geq d_{(V \setminus A)}(x) = d_G(x) - d_A(x)$$

$$\Leftrightarrow d_A(x) \geq \left\lfloor \frac{d_G(x)}{2} \right\rfloor, \text{ and similarly}$$

$$\text{for all } x \in B: \; d_B(x) \geq \left\lfloor \frac{d_G(x)}{2} \right\rfloor.$$

By putting $a(x) = b(x) = \left\lfloor \frac{d_G(x)}{2} \right\rfloor$, we have

$$\text{for every } x \in V(G): \; a(x) + b(x) - 1 = 2\left\lfloor \frac{d_G(x)}{2} \right\rfloor - 1 \leq 2\left(\frac{d_G(x)}{2} + \frac{1}{2}\right) - 1 = d_G(x).$$

Therefore, (A, B) is a feasible pair and it follows by Remark 16 and Proposition 15 that G admits a quorum coloring of order 2, that is, $\psi_q(G) \geq 2$.

$(ii) \Rightarrow (iii)$. Let $G' = (V', E')$ be the graph obtained from G by joining a leaf to each vertex of G of even degree. Let $\{V_1, V_2\}$ be a quorum coloring of G. Let n_ϵ denote the number of vertices of G of even degree. Then, one can easily see that the partition $\pi = \big\{\{v\} \subset V' \mid d_{G'}(v) = 1\big\} \cup \{V_1, V_2\}$ is a quorum coloring of G' of order $n_\epsilon + 2$.

$(iii) \Rightarrow (i)$. Assume that $\psi_q(G') \geq n_\epsilon + 2$ and let π' be a ψ_q-coloring of G'. Let L denote the set of the leaves of G'. We define the following sets $W = \{S \in \pi' \mid S \subseteq V\}$, $Y = \{S \in \pi' \mid S \subseteq L\}$ and $X = \pi' - (Y \cup W)$. Observe that $|L| = n_\epsilon$ and $|Y| \leq |L| - |X| = n_\epsilon - |X|$. Using the facts that $|\pi'| = |X| + |Y| + |W|$ and $|\pi'| \geq n_\epsilon + 2$, we deduce that

$$n_\epsilon + 2 \leq |\pi'| = |X| + |Y| + |W| \leq |X| + |L| - |X| + |W| \leq n_\epsilon + |W|,$$

and so $|W| \geq 2$, which means that G admits at least two disjoint quorum classes.

The proof of Theorem 17 is complete. ■

Note that if the union of the two disjoint quorum classes of Assertion (i) of Theorem 17 is maximal, then these two classes form a partition of $V(G)$ and therefore a quorum coloring of G as shown by the author [14] in the proof of Proposition 15. Furthermore, Assertion (iii) of Theorem 17 comes from the fact that each vertex of even degree belonging to a quorum class has at least as many neighbors in its class as outside it and hence, each leaf joined to a vertex of even degree increases the quorum coloring number of G by one. On the other hand, if $\psi_q(G) = 1$ and since G is connected, then by assigning in G' the same color to all the vertices of $V(G)$ and n_ϵ new colors to the n_ϵ leaves of G', we obtain a quorum coloring of G' of order $n_\epsilon + 1$. Thus, by negating the equivalence of Assertions (ii) and (iii) of Theorem 17, we deduce the following corollary.

Corollary 18. *Let $G = (V, E)$ be a connected graph and $G' = (V', E')$ the graph obtained from G by joining a leaf to each of the n_ϵ vertices of G of even degree. Then $\psi_q(G) = 1$ if and only if $\psi_q(G') = n_\epsilon + 1$.*

3 Second answer to Question 1

Our aim in this section is to provide a second answer to Question 1 by showing how to construct some infinite families of join graphs $K_r + G$ satisfying the conditions of Proposition 3, and hence generalizing the family $K_r + \overline{K_s}$ of Proposition 4. Our result is stated as follows.

Theorem 19. *Let $r \geq 2$ be an integer and G a graph such that $r + |V(G)|$ is an odd integer, and G satisfies one of the following two conditions:*

1. *$\Delta(G) \leq r - 2$;*

2. G is connected, $\delta(G) \le r-2$ *and* $\Delta(G) \le r-1$.

Then we have that $\psi_q(K_r+G)=1$.

Proof. Let us show that $a(K_r+G) = \left\lceil \frac{r+|V(G)|}{2} \right\rceil$. Let A be a minimum defensive alliance of K_r+G. Observe that if A contains a vertex v of $V(K_r)$, then $|A| = a(K_r+G) \ge \left\lceil \frac{d_{(K_r+G)}(v)+1}{2} \right\rceil = \left\lceil \frac{|V(K_r+G)|}{2} \right\rceil = \left\lceil \frac{r+|V(G)|}{2} \right\rceil$. Therefore, we obtain by Theorem 1 that
$a(K_r+G) = \left\lceil \frac{r+|V(G)|}{2} \right\rceil$.
Now, we will show that if one of the conditions 1 and 2 is satisfied, then $V(G)$ can not contain A. Assume to the contrary that $A \subseteq V(G)$ and set $V(K_r) \cup V(G) = W$.
Case 1. $\Delta(G) \le r-2$. In this case, we have $r \le d_{(W\setminus A)}(v) \le d_A(v)+1 \le \Delta(G)+1 \le r-1$ for every vertex $v \in A$, which is absurd.
Case 2. G is connected, $\delta(G) \le r-2$ and $\Delta(G) \le r-1$. Since $\Delta(G) \le r-1$, we have $r \le d_{(W\setminus A)}(v) \le d_A(v)+1 \le \Delta(G)+1 \le r$ for every vertex $v \in A$, which is equivalent to $d_A(v) = r-1$, for every vertex $v \in A$. As consequence, no vertex u with $d_G(u) \le r-2$ (such a vertices exist since $\delta(G) \le r-2$), belongs to A. Moreover, since G is connected, then there exists necessarily a pair of adjacent vertices u and v of $V(G)$ such that $d_G(u) \le r-2$ and $v \in A$. Therefore, using the fact that $d_A(v) = r-1$ we obtain that $\Delta(G) \ge d_G(v) \ge d_A(v)+1 \ge r$, a contradiction.

Consequently, A contains necessarily a vertex of $V(K_r)$ and by the first part of the theorem, we deduce that $a(K_r+G) = \left\lceil \frac{r+|V(G)|}{2} \right\rceil$. The result follows by Proposition 3. ∎

Theorem 19 allows to construct infinite classes of graphs K_r+G with $r+|V(G)|$ odd and $\psi_q(K_r+G)=1$, as illustrated by the following corollaries.

Corollary 20. *For any integer* $r \ge 2$ *and every* k*-regular graph* G*, where* $k \le r-2$ *and* $r+|V(G)|$ *is odd,* $\psi_q(K_r+G)=1$.

For every integer $n \ge 3$, let C_n denote the cycle of order n.

Corollary 21. *For any integers* $r \ge 4$ *and* $n \ge 3$*, where* $r+n$ *is odd, we have* $\psi_q(K_r+C_n)=1$.

Corollary 22. *For every complete multipartite graph* $K_{n_1,n_2,\ldots,n_\ell}$*, with* $n_i \ge 1$ *for every* $i \in \{1,2,\ldots,\ell\}$*, and every integer* $r \ge 2 + \max_{1\le i\le \ell} \sum_{\substack{j=1 \\ j\neq i}}^{r} n_j$ *where* $r + \sum_{i=1}^{\ell} n_i$ *is odd, we have*

$\psi_q(K_r+K_{n_1,n_2,\ldots,n_\ell})=1$.

For every integer $n \ge 1$, let P_n denote the path of order n.

Corollary 23. *For any integers $r \geq 3$ and $n \geq 1$, where $r+n$ is odd, we have* $\psi_q(K_r + P_n) = 1$.

Corollary 24. *For every complete multipartite graph $K_{n_1,n_2,\ldots,n_\ell}$, with $n_i \geq 1$ for every $i \in \{1, 2, \ldots, \ell\}$ and $n_j \neq n_k$ for some $j \neq k$, and for every integer $r \geq 1 + \max_{1 \leq i \leq \ell} \sum_{\substack{j=1 \\ j \neq i}}^{r} n_j$, where $r + \sum_{i=1}^{\ell} n_i$ is odd, we have $\psi_q(K_r + K_{n_1,n_2,\ldots,n_\ell}) = 1$.*

Corollary 25. *For every tree T of order n and every integer $r \geq \Delta(T) + 1$, where $r+n$ is odd, we have $\psi_q(K_r + T) = 1$.*

Remark 26. Note first that Theorem 19 remains valid when G is disconnected and $\Delta(G) \leq r-2$. Therefore, we can take, for example, in Corollary 20 a disjoint union of regular graphs satisfying the condition 1 of Theorem 19. Furthermore, Corollary 22 (respectively Corollary 24) can be considered with general multipartite graphs satisfying the condition 1 (respectively the condition 2) of Theorem 19.

4 Answer to Questions 1 and 2

In this section, we extend an observation made in [1] by Bazgan et al. saying that $\psi_q(K_{3,3,3}) = 1$, by proving that the infinite family of $(2k+1)$-partite graphs $G = K_{3,3,\ldots,3}$ has $\psi_q(G) = 1$, which gives a third answer to Question 1. Moreover, the proof of this result shows that $a(G) < \left\lceil \frac{|V(G)|}{2} \right\rceil$, which therefore answers also Question 2 in negative. Before proving this generalization, we need first to recall the following result due to Olsen and Revsbæk [8].

Proposition 27. [8] Let $G = (V, E)$ be a regular graph of odd order n and $d_G(u) = n-3$ for all $u \in V$. Then $\psi_q(G) \geq 2$ if and only if G contains a clique with $\left\lfloor \frac{n}{2} \right\rfloor$ vertices. Checking whether such a clique exists can be done in polynomial time.

As mentioned by the authors [8], Proposition 27 shows among other things how to construct a d-regular graph that is not a clique and impossible to partition into quorum classes for any even integer $d \geq 6$. Our result, as a consequence of Proposition 27, confirms this statement.

Corollary 28. *For any integer $k \geq 1$, if G is the $(2k+1)$-partite graph $K_{3,3,\ldots,3}$, then $\psi_q(G) = 1$.*

Proof. For every $k \geq 1$, set $n = 6k+3$. Then, G is an $(n-3)$-regular graph of odd order n. Let us show that G contains no clique of order $\left\lfloor \frac{n}{2} \right\rfloor$. Assume to the contrary that G has a clique of order $\left\lfloor \frac{n}{2} \right\rfloor$, and let A be such a clique. Since G is the $(2k+1)$-partite graph $K_{3,3,\ldots,3}$, then G can be partitioned into $2k+1$ independent sets $V_1, V_2, \ldots, V_{2k+1}$, each set containing exactly 3 independent vertices. Hence A contains at most one vertex of each independent set V_i. Therefore, $|A| \leq 2k+1 = \frac{n}{3} < \lfloor \frac{n}{2} \rfloor = \lfloor \frac{6k+3}{2} \rfloor = 3k+1$, a contradiction. This implies by Proposition 27 that $\psi_q(G) = 1$. ■

5 Answer to Questions 1, 2, 3 and 4

In this section, we exhibit a new infinite family of graphs G_ℓ of even orders with $\psi_q(G_\ell) = 1$, for any $\ell \geq 1$ (Figure 1). Moreover, we show that $diam(G_\ell)$ and $\left(\left\lceil \frac{|V(G_\ell)|}{2} \right\rceil - a(G_\ell)\right)$ are both not bounded. On the one hand, this result gives an interesting fourth answer to Question 1 in the sense that $\{G_\ell\}_{\ell \geq 1}$ satisfies no condition of Proposition 3. On the other hand, it answers in the same time Questions 2, 3 and 4 in negative, where the diameter as well as the gap between half of the order of the graph and the alliance defensive number can both be as large as we want. Before stating this result, let us recall the following useful proposition due to Olsen and Revsbæk [8].

Proposition 29. [8] Let $G = (V, E)$ be a graph and $\{B_1, B_2, \ldots, B_\ell\}$ a partition of V for some $\ell \geq 2$, satisfying the following conditions:

$$|B_1 \cup B_2| \text{ is odd,} \tag{1}$$

$$\forall u \in B_1 \ : \ d(u) = d_{B_1 \cup B_2}(u) = |B_1 \cup B_2| - 1 \tag{2}$$

$$\forall i \geq 2, \ \forall u \in B_i \ : \ d_{B_{i-1}}(u) > d_{(V \setminus B_{i-1})}(u) + 1 \tag{3}$$

Then $\psi_q(G) = 1$.

The authors [8] described Proposition 29 as a recipe for constructing infinitely many examples of graphs that can not be partitioned into quorum classes. The $\{G_\ell\}_{\ell \geq 1}$ family is one of these examples as shown in the following corollary.

Corollary 30. *Let $\ell \geq 1$ be an integer, G_ℓ a graph and $\{B_1, B_2, \ldots, B_{4\ell}\}$ a partition of $V(G_\ell)$ such that:*

(i) $G_\ell[B_1] = K_{4\ell}$.

(ii) For any integer $i \in \{2, \ldots, 4\ell\}$, $G_\ell[B_i] = \overline{K_{4\ell-i+1}}$.

(iii) For every vertex $v \in B_1$, $N_{G_\ell}[v] = B_1 \cup B_2$.

(iv) For every vertex $v \in B_{4\ell}$, $N_{G_\ell}(v) = B_{4\ell-1}$.

(v) For any integer $i \in \{2, \ldots, 4\ell - 1\}$ *and every vertex* $v \in B_i$, $N_{G_\ell}(v) = B_{i-1} \cup B_{i+1}$.

Then $\psi_q(G_\ell) = 1$.

Proof. First, the order of G_ℓ equals $\sum_{i=1}^{4\ell}(4\ell-i+1) = 16\ell^2+4\ell-2\ell(4\ell+1) = 2\ell(4\ell+1) \geq 10$, which is even for every $\ell \geq 1$. Now, let us verify that all the graphs of the $\{G_\ell\}_{\ell\geq 1}$ family satisfy Conditions 1, 2 and 3 of Proposition 29.

- Firstly, we have by Hypothesis (i) and (ii) that

  $$|B_1 \cup B_2| = 4\ell + 4\ell - 1 = 8\ell - 1,$$

  which is odd. Hence, Condition 1 is satisfied.

- Secondly, for every vertex $v \in B_1$, we have by Hypothesis (iii) that

  $$d_{G_\ell}(v) = |N_{G_\ell}(v)| = |N_{G_\ell}[v] \setminus \{v\}| = |N_{G_\ell}[v]| - 1 = |B_1 \cup B_2| - 1,$$

  which means that Condition 2 is satisfied.

- Finally, using Hypotheses (iv) and (v), we obtain that

  for all $v \in B_{4\ell}$, $d_{B_{4\ell-1}}(v) = |N_{G_\ell}(v) \cap B_{4\ell-1}| = |B_{4\ell-1}| = 4\ell-1 > d_{V(G_\ell)\setminus B_{4\ell-1}}(v)+1 = 1,$

  and

  for all $i \in \{2,\ldots,4\ell-1\}$ and all $v \in B_i$, $d_{B_{i-1}}(v) = |N_{G_\ell}(v) \cap B_{i-1}| = |B_{i-1}| = 4\ell-i+2$

  $$> d_{V(G_\ell)\setminus B_{i-1}}(v) + 1 = d_{B_{i+1}}(v) + 1 = 4\ell - i,$$

  which means that Condition 3 is satisfied.

The proof of Corollary 30 is complete. ∎

Remark 31. For every $\ell \geq 1$, we have clearly $diam(G_\ell) = 4\ell - 1$. Hence, we deduce that

$$\lim_{\ell\to+\infty} diam(G_\ell) = +\infty,$$

which answers Questions 3 and 4 in negative. Furthermore, we have $\left\lceil \frac{|B_1\cup B_2|}{2} \right\rceil = \left\lceil \frac{8\ell-1}{2} \right\rceil = 4\ell = |B_1|$, so it is easy to see that B_1 is a defensive alliance of G_ℓ (the set of gray vertices in Figure 1). Consequently,

$$a(G_\ell) \leq |B_1| = 4\ell.$$

It follows that

$$\lim_{\ell\to+\infty} \lceil |V(G_\ell)|/2 \rceil - a(G_\ell) \geq \lim_{\ell\to+\infty} \ell(4\ell+1) - 4\ell = \lim_{\ell\to+\infty} 4\ell^2 = +\infty.$$

Finally,

$$\lim_{\ell\to+\infty} \lceil |V(G_\ell)|/2 \rceil - a(G_\ell) = +\infty,$$

which shows that $\{G_\ell\}_{\ell\geq 1}$ is another counter-example for Question 2.

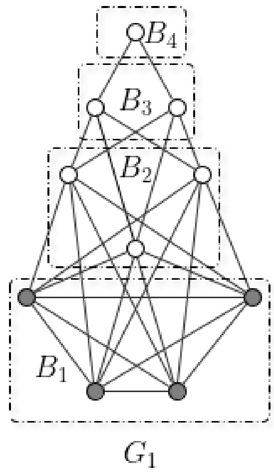

Figure 1: The partition $\{B_i\}_{1\leq i\leq 4}$ of the graph G_1

6 Open problems

The following open problems are suggested from our present study.

1. Characterize the family of graphs $K_r + G$ for which $\psi_q(K_r + G) = 1$.
2. Characterize the family of graphs $G + H$ for which $\psi_q(G + H) = 1$.
3. Can you find complete multipartite graphs G other than those of Corollary 28 such that $\psi_q(G) = 1$?
4. Characterize the class of graphs G, with $diam(G) = 2$ for which $\psi_q(G) = 1$.
5. Since every tree T satisfies $\lfloor diam(T)/2\rfloor \leq \psi_q(T)$ (see Proposition 5), characterize the graphs G for which $\lfloor diam(G)/2\rfloor \leq \psi_q(G)$.

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