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A NOVEL RANKING APPROACH WITH COMMON WEIGHTS: AN IMPLEMENTATION IN THE PRESENCE OF INTERVAL DATA AND FLEXIBLE MEASURES

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Abstract. In this paper a ranking method using common weights methodology is presented. The goal of the method is enhancing the decision maker (DM)'s influence in the ranking procedure. Although DM's preference information is an important element in our method, the approach can also be modified to be used in the absence of it. Since we aim to implement the approach on an empirical instance, the model is modified to deal with the properties of the sample, so it is developed in the presence of the interval data and flexible measures. Finally, the results are discussed.

Keywords: Data envelopment analysis, ranking, common set of weights, preference information.

Mathematics Subject Classification. 90C05, 90C29, 90C90

1. Introduction

Data envelopment analysis (DEA) uses mathematical programming based techniques to evaluate the performance of a set of homogeneous decision making units (DMUs). The first DEA model is known as CCR that was developed by Charnes et al. [8]. Classical DEA models calculate the maximum relative efficiency per DMU, and accordingly DMUs are classified in two groups of efficient and inefficient ones. So far, lots of ranking methods have been presented by researches that each can be applied regarding its conditions and the insight behind it. We mention some of them briefly as follows: Blas et al. [7] suggested a ranking method that

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used measures of dominance derived from social network analysis in combination with DEA. Oukil [38] suggested a new perspective for ranking DMUs under a DEA peer-evaluation framework. He exploited the property of multiple weighting schemes generated over the cross evaluation process in developing a methodology that yields not only robust ranking patterns but also more realistic sets of weights for the DMUs. Also see [45], [23], [25], and etc. Now, we focus on some methodologies more relevant to our study, in the followings:

1.1. Weight restrictions

Extra information or special conditions may impose some weight restrictions to DEA models. Assurance region (AR) and cone ratio are two well-known approaches in this field. AR approach was firstly introduced by Thompson et al. [50]. In the approach, the weights of some indicators are relatively restricted. Cone ratio approach is a more generalized version introduced by Charnes et al. [9] in which the input and output weights are restricted to belong to the related polyhedral convex cones each of which spanned by a finite number of admissible non-negative direction vectors. For further information see [19], [21], [39], [10], and [35].

1.2. Common weights methodology

In classical DEA models, the best relative efficiency of each DMU is obtained based on individual set of weights. Common weights methodology, introduced by Cook et al. [13] and developed by Roll et al. [41], can provide the opportunity to assess performance of DMUs fairly under common base. In this study, we propose a linear programming to generate a common set of weights. Before describing our method, we review some methods in the context. The idea behind the models generating a CSW can be different. A considerable number of the models, directly or indirectly, search a CSW with the purpose of maximizing the efficiencies (of all DMUs or a subset of them). See the following works in this regard: Chiang and Tzeng [11] presented a multi objective model to find a CSW such that the efficiencies of all DMUs become maximum. In order to solve the multi objective problem, they used max-min approach and presented a non-linear programming to choose a set of weights which its minimum efficiency is maximum in the feasible region. Kao and Hung [29] used compromise solution approach to determine a CSW. They solved the standard DEA models for DMUs and considered them as the ideal solution to achieve. Then, they searched a CSW that its vector of efficiency scores would be the closest to the ideal solution. Liu and Peng [34] searched one common set of weights to maximize group efficiencies. They proposed a linear programming to generate a CSW by using CWA-methodology. Some modifications on their work are presented by Ramezani et al. [40]. Chiang et al. [12] used a linear programming to generate a CSW with the aim of maximizing efficiencies. They applied their method to rank some countries based on the gained medals in 2008 Beijing Olympic Games. For further works, see [28], [62], [49], [31], [46], [64], [24] and etc. However, maximizing the efficiencies is not always the case to produce a common set of weights. Besides, it may be along with some difficulties, as follows:

- Inappropriate treatment
  Generating a CSW with the aim of increasing the efficiency scores of the DMUs may cause the unrealistic imagination of individual performance of the DMUs. So, it may lead to wrong judgments and policies to be made by DM.

- Diminution of distinction
  In addition to the importance of the purpose behind each ranking method, its power to distinct the performance of DMUs is important, too. The more the distinctive power of a ranking method, the easier DM can judge the performance of DMUs. In many of the models used to generate a CSW, the benchmark level for efficiency scores is 1, and generating a CSW with the aim of maximizing the efficiencies may increase the probability of having some DMUs with the same efficiency scores of 1. Hence, using more ranking criteria for more distinction may be required. (for DMUs with the same efficiency scores less than 1, it usually can be removed by increasing the decimal precision)

Common weights methodology is also utilized in concept of determining most efficient DMUs. Toloo et al. [58] presented an integrated model for determining most BCC-efficient DMU by solving one linear programming. Toloo [54] presented an epsilon-free basic integrated LP model to identify the most efficient candidate unit(s). In some cases the model can even find the most efficient DMU. Toloo [57] developed a supplier selection approach based on DEA for the case of suppliers with imprecise data. They presented an integrated mixed integer programming-data envelopment analysis (MIP-DEA) model for finding the most efficient suppliers in that condition. Toloo and Salahi [61] suggested a model that lets the efficiency score of only a single unit be strictly greater than one. Toloo and Mirbolouki [60] used common weights methodology in project selection problem. They developed a DEA approach with the aim of finding a composite project with the highest average CCR-efficiency score of indivisdual involved proposals. Selection-based problems are one of the most interesting applications of the common weights methodology in DEA. For example see, [59], [22], [52], [32].
For Further works in common weights methodology we refer to [16], [63], [33], [42], [2], [30], and [43].

1.2.1. *The motivation and the purpose of the new method*

In our approach a common set of weights is generated with a distinct perspective rather than the mentioned methods. In fact, the approach is to be utilized in the situation in which, for some environmental conditions or DM's objectives, some of the indicators may become significantly important for DM so that he/she prefers to influence their affect in evaluation of DMUs as much as possible. In other words, the adopted criterion to generate a common set of weights is based on a special category of DM's preference information. Thus, it makes us consider another category and another way to deal with DM's preferences.

1.2.2. *Overview of our approach*

We present a linear programming in which the objective function concentrates on the mentioned DM's preference information. Actually, the DM's preference information is divided into two distinct groups to be used in the body of the model. The CSW is obtained based on considering both of the information. Although attending to DM's preferences is a main element in this ranking approach, but the method can be also developed when no prior information is available. That is, this method can be used in either the presence or absence of DM's preference information. In the first case, DM's information is an essential element; in the second, the CSW is obtained indirectly for maximizing the efficiencies of DMUs.

1.2.3. *An implementation of the proposed approach*

At first, we briefly mention two necessary concepts in this regard:

- Imprecise data envelopment analysis (IDEA) extends DEA models to deal with imprecise data, that is when some input or output data are not exact. For more information, we refer to [15], [44], [47], [3], [17], [26], and [56].

- In some real problems, one may face with some variables which can be considered both as input and as output in evaluation of DMUs. They are known as flexible measures. For more information, we refer to [14], [53], [6], [5], [51], [1], [37], [55], [18], and [20].

We aim to implement our approach on a data set regarding to a number of bank branches. The case contains some properties: Firstly, some input and output data are not exact and lie in some bounded intervals. Secondly, there are some flexible measures in the evaluation. Hence, we developed our approach to deal with interval data and flexible measures. We also implement another method with different point of view, and finally the results are discussed.
The rest of this paper is as follows:
In Section 2, some preliminary concepts are stated. In Section 3, the suggested approach is presented. In Section 4, the method is considered in the presence of interval data and flexible measures, and the related modifications are conducted on the model. In Section 5, for comparison purpose, a model to generate a CSW in interval DEA is modified in presence of flexible measures. In Section 6, finally, we are ready to implement our method on the considered case, and the results are investigated. Section 7 presents the conclusion.

2. Preliminaries

Here, a brief mention of the elementary notions is presented.

2.1. CCR Model

Consider a set of n homogeneous DMUs that consume m inputs and produce s outputs. Let x_{ij} $(i = 1, \ldots, m)$ and y_{rj} $(r = 1, \ldots, s)$ denote, respectively, the amounts of the i^{th} input consumed and the r^{th} output produced by $DMU_j (j = 1, \ldots, n)$. The following properties are satisfied:

$$\begin{array}{ll} \forall j \in \{1, \ldots, n\}\ , & \exists i \in \{1, \ldots, m\}\ : x_{ij} > 0 \\ \forall j \in \{1, \ldots, n\}\ , & \exists r \in \{1, \ldots, s\}\ : y_{rj} > 0 \\ \forall i \in \{1, \ldots, m\}\ , & \exists i \in \{1, \ldots, n\}\ : x_{ij} > 0 \\ \forall r \in \{1, \ldots, s\}\ , & \exists j \in \{1, \ldots, n\}\ : y_{rj} > 0 \end{array} \tag{1}$$

Given non-negative input and output weights (U, V), the absolute efficiency of DMU_o $(o = 1, \ldots, n)$ is defined as $E_o = \frac{\sum_{r=1}^{s} u_r y_r o}{\sum_{i=1}^{m} v_i x_i o}$, and the relative efficiency of DMU_j is defined as follows:

$$RE_o = \frac{\frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}}{\max_{j=1,\ldots,n} \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}}$$

Model (2) shows the input oriented CCR model in multiplier form which determines the maximum relative efficiency of DMU_o $(o \in \{1, \ldots, n\})$ under constant return to scale:

$$\begin{array}{ll} \theta_o^* = max \sum_{r=1}^{s} u_r y_{ro} & \\ s.t. & \\ \sum_{i=1}^{m} v_i x_{io} = 1 & \\ \sum_{r=1}^{s} u_r y_{rj} - \sum_{r=1}^{s} u_r y_{rj} \leq 0 & j = 1, \ldots, n \\ v_i \geq 0 & i = 1, \ldots, m \\ u_r \geq 0 & r = 1, \ldots, s \end{array} \tag{2}$$

Definition 2.1. DMU_o is called efficient if $\theta_o^* = 1$, otherwise it's called inefficient.

Model (2) is run separately for each DMU, and so the extracted set of optimal weights are individual per DMU. The optimal value of 1 for Model (2) means that DMU_o has the opportunity to gain the best relative efficiency among the group of DMUs. Moreover, if there is also an optimal solution with totally positive weights, then DMU_o is called a CCR-efficient DMU.

2.2. Multi-Objective Programming problem

A multi-objective programming problem (MOP) can be written as follows:

$$\begin{array}{l} \max \{f_1(X), \ldots, f_l(X)\} \\ s.t. \\ X \in S. \end{array} \tag{3}$$

, where $f_i(X),\ i = 1, \dots, l$, are real valued functions on S, and $S \subseteq \mathbb{R}^n$. Usually, there is no optimal solution to Model (3), and so a Pareto-optimal solution to Model (3), that is defined as follows, is searched instead.

Definition 2.2. $\overline{X}$ is said to be a Pareto-optimal solution to Model (3) whenever there is no $X \in S$ such that

$$f_i(X) \geq f_i(\overline{X})\ ,\ i = 1, \dots, l$$

$$(f_1(X), \dots, f_l(X)) \neq (f_1(\overline{X}), \dots, f_l(\overline{X}))$$

Theorem 2.3. *Let α_i $(i = 1, \dots, l)$ be positive parameters. Then each optimal solution to (4) is a Pareto-optimal solution to (3).*

$$\begin{array}{l} \max \sum_{i=1}^{l} \alpha_i f_i(X) \\ s.t. \\ X \in S. \end{array} \tag{4}$$

Proof. Refer to [27]. □

Model (3) is called a Multi Objective Linear programming problem (MOLP) when the objective functions are linear and S can be represented as $\{X|AX \leq b, X \geq 0\}$, where $A = [a_{ij}]_{m \times n}$ is a real matrix and $b \in \mathbb{R}^m$.

3. The approach

In this Section, at first, we explain how to consider DM's preference information, and then the proposed method to generate a CSW is presented.

3.1. Categorization of DM's preference information

The DM's preferences is an important element in the proposed method. The initial assumption of this method is that DM assumes more importance for some indicators, so that he/she tends these indicators to have relatively more impact on the assessment of DMUs. We call such indicators worthy indicators. The set of indices of worthy input indicators is denoted by Φ_I and that of worthy output indicators by Φ_O. The assumption of this method is $\Phi_I \cup \Phi_O \neq \emptyset$ (so, may $\Phi_I = \emptyset$ or $\Phi_O = \emptyset$). We call the information about the worthy indicators as preference information of type I. In other hand, DM may have more information about the relative importance of some indicators, too. We call such information, preference information of type II (it means the conventional weight restrictions which is not a new notion). In our method, there is no necessarily for DM to state the type II exactly, and it is sufficient to state only the range of their expected values. Now, we introduce the other notations used in this regard in our approach. let $I = \{1, \dots, m\}$ and $O = \{1, \dots, s\}$ where m and s are the number of input and output indicators, respectively. $\Gamma_I \subseteq I \times I$ denotes the set of ordered pairs of input indicators which there is some information about their relative importance. Then, Γ_I is assumed to satisfy the following conditions:

- $\forall i \in I\ ,\ (i,j) \notin \Gamma_I$
- $\forall i,k \in I, ((i,k) \in \Gamma_I \Rightarrow (k,i) \notin \Gamma_I)$
- If $(i,t) \in \Gamma_I$ then at least one of the maximum or minimum of expected value for relative importance of the i^{th} input indicator to the t^{th} indicator is given by DM (it's evident that if DM has determined an exact value, the minimum and maximum are the same).

Also, $\Gamma_O \subseteq O \times O$ is defined in a similar way. $\Gamma_{IO} \subseteq I \times O$ is a set of all (i,r) for which there is some information, about the relative importance of the i^{th} input indicator to the r^{th} output indicator, as mentioned above. $\Gamma_{OI} \subseteq O \times I$ is defined in a similar way. In our method, it's assumed that if $(i,r) \subseteq \Gamma_{IO}$ then $(r,i) \notin \Gamma_{OI}$. It is noticeable that all of Γ_I , Γ_O, Γ_{IO} and Γ_{OI} may be equal with $\emptyset$.

3.2. The suggested model

Here, we present our proposed model to generate a CSW. In the model, it is intended to generate a CSW satisfying the relative importance of the indicators in which the weights of the worthy indicators are maximum. Hence, DM's preference information of type I is used in the objective function, and the DM's preference information of type II is used as the weight restrictions, if there are any.

Suppose that Φ_I, Φ_O, Γ_I, Γ_O , Γ_{IO} and Γ_{OI} are defined as it was discussed in the previous section. Then, the initial model to generate a CSW, in general form, is considered as follows:

$$
\begin{array}{ll}
\max\{u_r|r \in \Phi_O\} \cup \{v_i|i \in \Phi_I\} & \\
s.t. & \\
\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 & j = 1,\dots,n \\
\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1 & \\
\alpha_{ik}^{v} \leq \frac{v_i}{v_k} \leq \beta_{ik}^{v} & \forall (i,k) \in \Gamma_I \\
\alpha_{rl}^{u} \leq \frac{u_r}{u_l} \leq \beta_{rl}^{u} & \forall (r,l) \in \Gamma_O \\
\alpha_{ir}^{vu} \leq \frac{v_i}{u_r} \leq \beta_{ik}^{v} & \forall (i,r) \in \Gamma_{IO} \\
\alpha_{ri}^{uv} \leq \frac{u_r}{v_i} \leq \beta_{ri}^{uv} & \forall (r,i) \in \Gamma_{OI} \\
v_i \geq \epsilon & i = 1,\dots,m \\
u_r \geq \epsilon & r = 1,\dots,s
\end{array}
\tag{5}
$$

The parameters α_{ik}^{v} and β_{ik}^{v} , $(i,k) \in \Gamma_I$, are used, respectively as the lower and the upper bounds for the relative importance of the i^{th} input indicator to the k^{th} input indicator. There are similar explanations about the parameters α_{rl}^{u} and β_{rl}^{u} , $(r,l) \in \Gamma_O$, α_{ir}^{vu} and β_{ir}^{vu} , $(i,r) \in \Gamma_{IO}$, and α_{ri}^{uv} and β_{ri}^{uv} , $(r,i) \in \Gamma_{OI}$.

It is evident that if $\Gamma_I \cup \Gamma_O \cup \Gamma_{IO} \cup \Gamma_{OI} = \emptyset$ then Model (5) contains no weight restrictions. It is an MOP problem. ϵ is a non-Archimedean infinitesimal constant which prevents the weights to become zero. In fact, as it is seen later, the absolute efficiencies calculated based upon the generated CSW have a pivotal affection on the ranking scores, and so we would like to involve all of the input and output indicators in the assessment. In other words, if some of the weights in the generated CSW is equal to zero, the role of the corresponding indicator would be ignored in the in the continuation of the ranking process. Also, since the last batch constraints guarantee that the input and output weights are positive, the model can be equivalently rewritten as the MOLP (6), because of the non-zero denominators:

$$\begin{array}{ll}
\max\{u_r | r \in \Phi_O\} \cup \{v_i | i \in \Phi_I\} & \\
s.t. & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 & \\
\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1 & j = 1, \ldots, n \\
\alpha_{it}^{v} v_t - v_i \leq 0 & \forall (i,t) \in \Gamma_I \\
v_i - \beta_{it}^{v} v_t \leq 0 & \forall (i,t) \in \Gamma_I \\
\alpha_{rl}^{u} u_l - u_r \leq 0 & \forall (r,l) \in \Gamma_O \\
u_r - \beta_{rl}^{u} u_l \leq 0 & \forall (r,l) \in \Gamma_O \\
\alpha_{ir}^{vu} u_r - v_i \leq 0 & \forall (i,r) \in \Gamma_{IO} \\
v_i - u_r \beta_{ir}^{vu} \leq 0 & \forall (i,r) \in \Gamma_{IO} \\
\alpha_{ri}^{uv} v_i - u_r \leq 0 & \forall (r,i) \in \Gamma_{OI} \\
u_r - v_i \beta_{ri}^{uv} \leq 0 & \forall (r,i) \in \Gamma_{OI} \\
v_i \geq \epsilon & i = 1, \ldots, m \\
u_r \geq \epsilon & r = 1, \ldots, s
\end{array} \quad (6)$$

According to Theorem 2.3, an optimal solution to the linear programming problem (7) is a Pareto-optimal solution to model (6).

$$\begin{array}{lll}
\max \sum_{r \in \Phi_O} \gamma_r^u u_r + \sum_{i \in \Phi_I} \gamma_i^v v_i & & \\
s.t. & & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 & j = 1, \ldots, n & (i) \\
\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1 & & (ii) \\
(U,V)A \leq 0 & & (iii) \\
v_i \geq \epsilon & i = 1, \ldots, m & (vi) \\
u_r \geq \epsilon & r = 1, \ldots, s & (v)
\end{array} \quad (7)$$

, where the parameters $\gamma_i^v > 0$ $(i \in \Phi_I)$ and $\gamma_r^u > 0$ $(r \in \Phi_O)$ specify the priority among the indicators of $\Phi_I \cup \Phi_O$. If DM assumes no priority among the indicators, then all of these parameters are set to 1. The constraint (i) makes the absolute efficiencies of the DMUs to be less than or equal to 1. The constraint (ii) is a normalizing constraint which, as we show later, also guarantees that Model (7) has a finite optimal value. If $\Gamma_I \cup \Gamma_O \cup \Gamma_{IO} \cup \Gamma_{OI} \neq \emptyset$, $A_{(s+m)q}$ is a matrix corresponding with the weight restrictions, where $0 < q \leq 2\ (|\Gamma_I| + |\Gamma_O| + |\Gamma_{IO}| + |\Gamma_{OI}|)$. If we have the weight restriction $\alpha_{ik}^{v} v_k - v_i \leq 0,\ (i,k) \in \Gamma_I$, in the model, there is corresponding column $\alpha_{ik}^{v} e_{s+k} - e_{s+i}$, and if we have the weight restriction $v_i - \beta_{ik}^{v} v_k \leq 0,\ (i,k) \in \Gamma_I$, in the model, then there is corresponding column $e_{s+i} - \beta_{ik}^{v} e_{s+k}$ in Matrix A. Other columns of Matrix A are also based upon the weight restrictions related to Γ_O, Γ_{IO} and Γ_{OI} similar to what was stated about Γ_I. It is trivial that in the absence of the weight restrictions, the constraints (iii) is omitted.

The optimal solution to Model (7) is accepted as the common set of weights. It is considerable to state a general issue encountering ranking methods using common weights: Since the ranking methods are usually dependent on the generated CSW, having alternative optimal solutions may lead to different ranking results. Thus, an extra model can be solved to ensure that only one of the optimal solutions is considered as the CSW. For example, see Liu and Peng [34], Sun et al. [48]. To this aim, we use the criterion presented by Sun et al. [48] to choose one of the optimal solutions of Model (7) as the CSW.

As it is discussed in the following, in the absence of DM's preference information of type II, Model (7) is feasible for a small enough ϵ. It is clear that the feasibility of Model (7) depends on the weight restrictions, too.

Lemma 3.1. *Let* $\sum_{r=1}^{s} y_{rj} - \sum_{i=1}^{m} x_{ij} \leq 0$ *,* $j = 1, \ldots, n$, *and* $\overline{U} = \frac{1}{m+s} 1_s^t$, $\overline{V} = \frac{1}{m+s} 1_m^t$, $\overline{\epsilon} = \frac{1}{m+s}$ *, where all of the components of* $1_s \in \overline{R}^s$ *and* $1_m \in \overline{R}^m$ *are equal to 1.*
Then $(\overline{U},\ \overline{V},\ \overline{\epsilon})$ *is a feasible solution to (7) in the absence of DM's preference information of type II.*

Proof. . $\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} = \frac{1}{m+s} \sum_{r=1}^{s} y_{rj} - \sum_{i=1}^{m} x_{ij} \leq 0$. It's easily seen that $(\overline{U}, \overline{V}, \overline{\epsilon})$ satisfies the rest of the constraints to Model (7) in the absence of DM's preference information of type II, and we are done. □

Lemma 3.2. *Let* $\exists j \in \{1, \ldots, n\} : \sum_{r=1}^{s} y_{rj} - \sum_{i=1}^{m} x_{ij} > 0$ *and* $\overline{U} = \frac{\alpha}{s\alpha+m} \mathbf{1}_s^t, \overline{sV} = \frac{1}{s\alpha+m} \mathbf{1}_m^t$, $\overline{\epsilon} = \frac{\alpha}{s\alpha+m}$ *, where* $\alpha = min_{j=1,\ldots,n} \frac{\sum_{i=1}^{m} x_{ij}}{\sum_{r=1}^{s} y_{rj}}$*, Then* $(\overline{U}, \overline{V}, \overline{\epsilon})$ *is a feasible solution to (7) in the absence of DM's preference information of type II.*

Proof. $\forall j \in \{1, \ldots, n\}$, $(0 < \alpha = min_{t=1,\ldots,n} \frac{\sum_{i=1}^{m} x_{it}}{\sum_{r=1}^{s} y_{rt}} \leq \frac{\sum_{i=1}^{m} x_{ij}}{\sum_{r=1}^{s} y_{rj}} \Rightarrow \frac{1}{s\alpha+m} \alpha \leq \frac{1}{s\alpha+m} \frac{\sum_{i=1}^{m} x_{ij}}{\sum_{r=1}^{s} y_{rj}} \Rightarrow \frac{1}{s\alpha+m} \alpha \sum_{r=1}^{s} y_{rj} \leq \frac{1}{s\alpha+m} \sum_{i=1}^{m} x_{ij} \Rightarrow \sum_{r=1}^{s} \frac{\alpha}{s\alpha+m} y_{rj} - \sum_{i=1}^{m} \frac{1}{s\alpha+m} x_{ij} \leq 0)$.
Also,

$$\sum_{r=1}^{s} \overline{u}_r + \sum_{i=1}^{m} \overline{v}_i = \sum_{r=1}^{s} \frac{\alpha}{s\alpha+m} + \sum_{i=1}^{m} \frac{1}{s\alpha+m} = s\frac{\alpha}{s\alpha+m} + m\frac{1}{s\alpha+m} = 1$$

Besides, according to the assumption,

$$\exists l \in \{1, \ldots, n\} : \sum_{r=1}^{s} y_{rl} - \sum_{i=1}^{m} x_{il} > 0 \Rightarrow \frac{\sum_{i=1}^{m} x_{il}}{\sum_{r=1}^{s} y_{rl}} < 1 \Rightarrow \alpha = min_{j=1,,n} \frac{\sum_{i=1}^{m} x_{ij}}{\sum_{r=1}^{s} y_{rj}} \leq \frac{\sum_{i=1}^{m} x_{il}}{\sum_{r=1}^{s} y_{rl}} < 1$$

Hence, $\overline{\epsilon} = \frac{\alpha}{s\alpha+m} < \frac{1}{s\alpha+m}$.
It's easily seen that $(\overline{U}, \overline{V}, \overline{\epsilon})$ satisfies the rest of the constraints to model (7) in the absence of DM's preference information of type II, and we are done. □

Theorem 3.3. *In the absence of DM's preference information of type II, model (7) is feasible for a small enough ϵ.*

Proof. According to Lemma 3.1 and Lemma 3.2, we consider two general cases, as follows:

Case1:

$$\sum_{r=1}^{s} y_{rj} - \sum_{i=1}^{m} x_{ij} \leq 0 \ , \ j = 1, \dots, n$$

According to Lemma 3.1, it's easily seen that:
(U_1, V_1, ϵ), where $U_1 = \frac{1}{m+s}\mathbf{1}_s^t, V_1 = \frac{1}{m+s}\mathbf{1}_m^t, 0 < \epsilon \leq \epsilon_1 = \frac{1}{m+s}$ is a feasible solution to the model.

Case2:

$$\exists j \in \{1, \dots, n\}\colon \sum_{r=1}^{s} y_{rj} - \sum_{i=1}^{m} x_{ij} > 0$$

According to Lemma 2, it's easily seen that:
(U_2, V_2, ϵ), where $U_2 = \frac{\alpha}{s\alpha+m}\mathbf{1}_s^t, V_2 = \frac{1}{s\alpha+m}\mathbf{1}_m^t, 0 < \epsilon \leq \epsilon_2 = \frac{\alpha}{s\alpha+m}$ is a feasible solution to the model.

Therefore, in the absence of DM's preference information of type II, in general, for each ϵ where, $0 < \epsilon \leq min\{\epsilon_1, \epsilon_2\}$, Model (7) is feasible, and we are done. □

For further discussions in this regards, one can see Mehrabian et al. [36], Amin and Toloo [4].

Theorem 3.4. *In feasibility, Model (7) has a finite optimal objective value.*

Proof. Let S be the feasible region of Model (7) and $(\overline{U}, \overline{V}) \in S$ be an arbitrary feasible solution to it, where $\overline{U}^t \in \mathbb{R}^s$ and $\overline{V}^t \in \mathbb{R}^m$. As a consequence of the constraint (ii) and the constraints (iv) and (v), and the assumption that at least one input indicator and one output indicator exist in evaluation process:

$$\begin{array}{ll} 0 < \overline{v}_i < 1 & i = 1, \dots, m \\ 0 < \overline{u}_r < 1 & r = 1, \dots, s \end{array} \tag{8}$$

According to (8) and the positivity of γ_r^u $(r \in \Phi_O)$ and γ_i^v $(i \in \Phi_I)$:

$$\sum_{i\in\Phi_I} \gamma_i^v \overline{v}_i + \sum_{r\in\Phi_O} \gamma_r^u \overline{u}_r < \sum_{i\in\Phi_I} \gamma_i^v + \sum_{r\in\Phi_O} \gamma_r^u$$

We set $M = \sum_{i\in\Phi_I} \gamma_i^v + \sum_{r\in\Phi_O} \gamma_r^u$. Thus, M is an upper bound for the set of objective values of Model (7). Therefore, in feasibility, model (7) has a finite optimal objective value. □ □

3.3. The developed model in the absence of DM's preference information

As it is mentioned earlier, model (7) is based upon a pivot assumption of the presence of DM's preference information of type I. However, in the absence of such information, the model can be slightly modified to be used indirectly with respect to the conventional objective of generating a CSW maximizing the efficiencies of DMUs. To this aim we set $\Phi_O = \{1, \ldots, s\}$ and $\Phi_I = \emptyset$. That is $\Phi = \Phi_o$. Model (9) is the modified model to deal with the circumstance lacking of any DM's preference information.

$$\begin{array}{ll}
\max\{u_r | r \in O\} & \\
s.t. & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 & j = 1, \ldots, n \\
\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1 & \\
v_i \geq \epsilon & i = 1, \ldots, m \\
u_r \geq \epsilon & r = 1, \ldots, s
\end{array} \tag{9}$$

With regard to Theorem 2.3, Model (9) can be rewritten as follows:

$$\begin{array}{ll}
\max \sum_{r=1}^{s} u_r & \\
s.t. & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 & j = 1, \ldots, n \\
\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1 & \\
v_i \geq \epsilon & i = 1, \ldots, m \\
u_r \geq \epsilon & r = 1, \ldots, s
\end{array} \tag{10}$$

The explanations about the constraints is similar to those of Model (7) so it is ignored. It is evident that if there are any weight restrictions, they can be added to the set of the constants to Model (8).

3.4. The ranking approach

In our method, DM's preference information is also regarded in the ranking procedure.

For ranking the DMUs, firstly, the absolute efficiency of each DMU is calculated based upon the generated CSW. Let (U^*, V^*) be the generated CSW. The absolute efficiency of DMU_j $(j = 1, \ldots, n)$, based upon the CSW is:

$$E_j^* = \frac{\sum_{r=1}^{s} u_r^* y_{rj}}{\sum_{i=1}^{m} v_i^* x_{ij}}$$

The greater value of E_j^*, the greater ranking score of DMUj. It is notable that in the case of existing two DMUs with the same efficiency of less than 1, it is usually removed through increasing decimal accuracy. However, we present the following general approach.

Considering $\Phi \neq \emptyset$, if there are two DMUs with the same efficiencies, we utilize a second ranking criterion to help to prioritize of them. To this aim, we concentrate on the state of the units in terms of the worthy indicators for DM. Since the DM's preference information of type I may contain only some input indicators, only some output indicators, or both of them, the definition of the second criterion depends on the indicators existing in this type of information, as follows:

I. If $\Phi_I = \emptyset$ then $\Phi = \Phi_O \subset \{1, \ldots, s\}$. In this case, if it is required, the second criterion for assessment of DMU_j $(j = 1, \ldots, n)$ is defined as:

$$\eta_j = \sum_{r \in \Phi_O} u_r^* y_{rj}$$

II. If $\Phi_O = \emptyset$, $\Phi = \Phi_I \subseteq \{1, \ldots, m\}$. In this case, if it is required, the second criterion for assessment of DMU_j $(j = 1, \ldots, n)$ is defined as:

$$\eta_j = -\sum_{i \in \Phi_I} v_i^* x_{ij}.$$

III. If $\Phi_I \neq \emptyset$,$\Phi_O \neq \emptyset$, Φ contains some input and some output indicator indices. In this case, if it is required, the criterion for assessment of DMU_j $(j = 1, \ldots, n)$ is defined as:

$$\eta_j = \frac{\sum_{r \in \Phi_O} u_r^* y_{rj}}{\sum_{i \in \Phi_I} v_i^* x_{ij}}$$

In the last case, η_j is a ratio that can be interpreted as the absolute efficiency of DMU_j based on the worthy indicators. It is significant that, unlike E_j^*, 1 is not an upper bound for η_j. Besides, it is noticeable that $\sum_{i \in \Phi_I} v_i^* x_{ij}$ may be equal to zero. In this case, actually, DMU_j consumes none of the worthy inputs, that it is great. In our method, if this statement occurs for only one of two units under comparison, the unit is preferred to the other. If the situation occurs for both of them, they are compared in terms of the valuable output indicators, and the second criterion is defined as

$$\eta_j = \sum_{r \in \Phi_O} u_r^* y_{rj}.$$

Eventually, considering the above discussion, our ranking rules can be stated as follows: Let $i, j \in \{1, \ldots, n\}$ and R_i and R_j denotes the ranking scores of DMU_i and DMU_j, respectively.
Then:

1 If $E_i^* < E_j^*$ then $R_i < R_j$.
2 If $E_i^* = E_j^*$ then

$$\begin{cases} R_i < R_j & \Phi_I, \Phi_O \neq \emptyset, \sum_{k \in \Phi_I} x_{kj} = 0, \sum_{k \in \Phi_I} x_{ki} \neq 0 \\ \text{If } \eta_i < \eta_j \text{ then } R_i < R_j & \text{otherwise} \end{cases}$$

4. The development in encountering with the interval data and flexible measures

We aim to implement our approach on a sample presented in Section 6 To this aim, it is required that the approach is developed in encountering with interval data and flexible measures.

4.1. Dealing with interval data

Let $\{DMU_1, \ldots, DMU_n\}$ be n homogeneous DMUs each of which associated with m inputs and s outputs. The amounts of the i^{th} input and the r^{th} output for DMU_j $(j = 1, \ldots, n)$ are generally denoted by x_{ij} and y_{rj}, respectively. Besides, it is assumed that their value lie in the intervals of $[x_{ij}^l, x_{ij}^u]$ and $[y_{rj}^l, y_{rj}^u]$ $(i = 1, \ldots, m, r = 1, \ldots, s)$, respectively, satisfying the following conditions:

$$
\begin{array}{llll}
I. & \forall j \in \{1,\ldots,n\} & \forall i \in \{1,\ldots,m\} & (0 \leq x_{ij}^l \leq x_{ij}^u) \\
II. & \forall j \in \{1,\ldots,n\} & \forall r \in \{1,\ldots,s\} & (0 \leq y_{rj}^l \leq y_{rj}^u) \\
III. & \forall j \in \{1,\ldots,n\} & \exists i \in \{1,\ldots,m\} & (x_{ij}^l > 0) \\
IV. & \forall j \in \{1,\ldots,n\} & \exists r \in \{1,\ldots,s\} & (y_{rj}^l > 0) \\
V. & \forall i \in \{1,\ldots,m\} & \exists j \in \{1,\ldots,n\} & (x_{ij}^u > 0) \\
VI. & \forall r \in \{1,\ldots,s\} & \exists j \in \{1,\ldots,n\} & (y_{rj}^u > 0)
\end{array}
$$

Conditions III and IV conclude that per activity level of each DMU, the vectors of inputs and outputs are not equal to zero. Besides, given a positive set of weights, it guarantees that the absolute efficiency of each DMU always is definable.
Hint: The best (worst) activity level of DMU_j occurs when the amounts of input and output indicators for it are, respectively, minimum (maximum) and maximum (minimum).
(11) is the developed model to obtain a CSW in the presence of interval data:

$$
\begin{array}{ll}
\max\{u_r | r \in \Phi_O\} \cup \{v_i | i \in \Phi_I\} & \\
s.t. & \\
\sum_{r=1}^{s} u_r y_{rj}^u - \sum_{i=1}^{m} v_i x_{ij}^l \leq 0 & j = 1, \ldots, n \\
\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1 & \\
(U, V)A \leq 0 & \\
v_i \geq \epsilon & i = 1, \ldots, m \\
u_r \geq \epsilon & r = 1, \ldots, s
\end{array}
\tag{11}
$$

Regarding Theorem 2.3 each optimal solution to Model (12) is a Pareto-efficient solution to model (11).

$$
\begin{array}{lr}
\max \sum_{r\in\Phi_O} \gamma_r^u u_r + \sum_{i\in\Phi_I} \gamma_i^v v_i & \\
s.t. & \\
\sum_{r=1}^{s} u_r y_{rj}^u - \sum_{i=1}^{m} v_i x_{ij}^l \leq 0 \qquad j = 1,\ldots,n & \\
\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1 & (12)\\
(U,V)A \leq 0 & \\
v_i \geq \epsilon \qquad i = 1,\ldots,m & \\
u_r \geq \epsilon \qquad r = 1,\ldots,s &
\end{array}
$$

, where the parameters $\gamma_i^v > 0$ $(i \in \Phi_I)$ and $\gamma_r^u > 0$ $(r \in \Phi_O)$ are as already explained. Since, for any $0 < (U,V)$, we have:

$$\sum_{r=1}^{s} u_r y_{rj}^l - \sum_{i=1}^{m} v_i x_{ij}^u \leq \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq \sum_{r=1}^{s} u_r y_{rj}^u - \sum_{i=1}^{m} v_i x_{ij}^l$$

The constraint $\sum_{r=1}^{s} u_r y_{rj}^u - \sum_{i=1}^{m} v_i x_{ij}^l \leq 0$ $(j = 1,\ldots,n)$ allows the most extensive search for the weights placed on the objective function compared to other activity levels. It is also concluded that the absolute efficiencies of each DMU, considering all of its activity levels, based upon each feasible solution do not exceed one. Matrix A is defined similar to what was explained in model (7). For determining one CSW the criterion used in [48] is utilized on optimal solutions to Model (11).

In this case, the ranking criterion is the absolute efficiency of DMUs based upon the average data of them and extracted CSW. Hence we set:

$$\overline{X}_j = \frac{1}{2}(X_j^l + X_j^u)$$

$$\overline{Y}_j = \frac{1}{2}(Y_j^l + Y_j^u)$$

and calculate

$$\overline{E}_j = \frac{\sum_{r\in\Phi_O} u_r^* \overline{y}_{rj}}{\sum_{i\in\Phi_I} v_i^* \overline{x}_{ij}}$$

, where (U^*, V^*) is the generated CSW. In the case of $\Phi \neq \emptyset$, we can consider the same discussions as we had in 3.4 Here, $\overline{\eta}_j$ is calculated similar to η_j, but based on the average data.

Hence, the developed ranking method in presence of interval data can be presented as the following:

Let $i, j \in \{1,\ldots,n\}$ and R_i and R_j denote the ranking scores of DMU_i and DMU_j, respectively.

Then:

1. If $\overline{E}_i^* < \overline{E}_j^*$ then $R_i < R_j$.

2. If $\overline{E}_i^* = \overline{E}_j^*$ then

$$\begin{cases} R_i < R_j & \Phi_I, \Phi_O \neq \emptyset, \sum_{k \in \Phi_I} \overline{x}_{kj} = 0, \sum_{k \in \Phi_I} \overline{x}_{ki} \neq 0 \\ \text{If } \overline{\eta}_i < \overline{\eta}_j \text{ then } R_i < R_j & \text{otherwise} \end{cases}$$

In the absence of DM's preference information, there is a similar discussion as presented in 3.3, so we ignore the details.

4.2. Dealing with flexible measures

In the sample, which shall be discussed in section 6, we face with a bank assessment case. The deposit is considered in the assessment, but, in general, there is two points of view to involve it. First, since the deposit should be invested within special periods, it can be regarded as an input for a bank branch. In the other hand, considering the fact that deposit is gained through personnel and advertisement, it can also be considered as an output. In classic models, it is assumed that the nature of the indicators, including inputs and output ones are already clear; however, regarding empirical issues, in some circumstance, there are indicators in the assessment upon the nature of which there is disagreement. Therefore, it is important to consider "flexible measures" as well.

Now, assume that there are m determined input indicators and s determined output indicators. Also, assume that f shows the number of the indicators which are flexible and each can be used as an input or output indicator in the evaluation process. In Model (7), the role assignment of the flexible indicators is done in such a way that the best result is obtained for the target weights in the objective function. These indicators can appear in the preference information, so the new symbols are defined as follows:
Assume that $F = \{1, \ldots, f\}$, where $f \in \mathbb{N}$, is the set of flexible indicators. We define:
Φ_F: The set of valuable indicators which are important for the DM in assessment but their natures are not determined.
We call them valuable flexible indicators.
Furthermore, $\Gamma_F \subseteq F \times F$, $\Gamma_{IF} \subseteq I \times F$, $\Gamma_{FI} \subseteq F \times I$, $\Gamma_{OF} \subseteq O \times F$ and $\Gamma_{FO} \subseteq F \times O$ have the same condition similar to what stated about Γ_I, Γ_O, Γ_{IO} and Γ_{OI}. Model (13) is to generate a CSW with the aim of maximizing the weights of the indicators whose indices belong to $\Phi = \Phi_I \cup \Phi_O \cup \Phi_F$ considering the various

states of flexible indicators:

$$
\begin{array}{ll}
\max\{u_r|r\in\Phi_O\}\cup\{v_i|i\in\Phi_I\}\cup\{w_k|k\in\Phi_F\} & \\
s.t. & \\
\sum_{r=1}^{s}u_ry_{rj}+\sum_{k=1}^{f}\sigma_kw_kz_{kj}-\sum_{i=1}^{m}v_ix_{ij}-\sum_{k=1}^{f}(1-\sigma_k)w_kz_{kj}\leq 0 & j=1,\dots,n\\
\sum_{i=1}^{m}v_i+\sum_{r=1}^{s}u_r+\sum_{k=1}^{f}w_k=1 & \\
(U,V)B\leq 0 & \\
v_i\geq\epsilon & i=1,\dots,m\\
u_r\geq\epsilon & r=1,\dots,s\\
w_k\geq\epsilon & k=1,\dots,f\\
\sigma_k\in\{0,1\} & k=1,\dots,f
\end{array}
\qquad (13)
$$

Model (14) is solved to find a Pareto-efficient solution to Model (13).

$$
\begin{array}{ll}
\max\sum_{r\in\Phi_O}\gamma_r^uu_r+\sum_{i\in\Phi_I}\gamma_i^vv_i+\sum_{k\in\Phi_F}\gamma_k^ww_k & \\
s.t. & \\
\sum_{r=1}^{s}u_ry_{rj}+\sum_{k=1}^{f}\sigma_kw_kz_{kj}-\sum_{i=1}^{m}v_ix_{ij}-\sum_{k=1}^{f}(1-\sigma_k)w_kz_{kj}\leq 0 & j=1,\dots,n\\
\sum_{i=1}^{m}v_i+\sum_{r=1}^{s}u_r+\sum_{k=1}^{f}w_k=1 & \\
(U,V)B\leq 0 & \\
v_i\geq\epsilon & i=1,\dots,m\\
u_r\geq\epsilon & r=1,\dots,s\\
w_k\geq\epsilon & k=1,\dots,f\\
\sigma_k\in\{0,1\} & k=1\dots,f
\end{array}
\qquad (14)
$$

, where the parameters $\gamma_r^u>0$ $(r\in\Phi_O)$,$\gamma_i^v>0$ $(i\in\Phi_I)$ and $\gamma_k^w>0$ $(k=1,\dots,f)$ are positive, and there is a similar explanation as before. In the first constraint, the value 0 for the binary variable $\sigma_k(k=1,\dots,f)$ causes the k^{th} flexible indicator to be considered as an input indicator, and the value 1 for this variable causes that the indicator is considered as an output indicator. In the normalizing constraint $\sum_{i=1}^{m}v_i+\sum_{r=1}^{s}u_r+\sum_{k=1}^{f}w_k=1$, regardless of the role of the flexible indicators, the weights of all indicators are considered. In the presence of preference information of type II, $B_{(m+s)g}$ is the matrix related to the weight restrictions where $0<g\leq 2(|\Gamma_I|+|\Gamma_O|+|\Gamma_F|+|\Gamma_{IO}|+|\Gamma_{OI}|+|\Gamma_{IF}|+|\Gamma_{FI}|+|\Gamma_{OF}|+|\Gamma_{FO}|)$. Explanations about columns of B are similar to what was stated in Model (7), for matrix A.

After obtaining a CSW and the corresponding σ^* , the state of each flexible measure is determined as an input or output indicator. Afterwards, accordingly, the ranking approach presented in section 3 can be used.

Also, in this case, in the absence of DM's preference information, we can use Model

(15).

$$
\begin{array}{ll}
\max \sum_{r=1}^{s} u_r + \sum_{k=1}^{f} \sigma_k w_k & \\
s.t. & \\
\sum_{r=1}^{s} u_r y_{rj} + \sum_{k=1}^{f} \sigma_k w_k z_{kj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{k=1}^{f} (1-\sigma_k) w_k z_{kj} \leq 0 & j=1,\dots,n \\
\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r + \sum_{k=1}^{f} w_k = 1 & \\
v_i \geq \epsilon & i=1,\dots,m \\
u_r \geq \epsilon & r=1,\dots,s \\
w_k \geq \epsilon & k=1,\dots,f \\
\sigma_k \in \{0,1\} & k=1\dots,f
\end{array} \tag{15}
$$

The repetitive matters are ignored for abridgment.

4.3. The developed approach dealing with flexible measures and interval data, simultaneously

Here, eventually, regarding with 4.1 and 4.2, the developed versions of the models (7) and (10) in simultaneous presence of interval data and flexible measures are presented. The models (16) and (17) show, respectively, the developed models, as follows:

$$
\begin{array}{ll}
\max \sum_{r\in\Phi_O} \gamma_r^u u_r + \sum_{i\in\Phi_I} \gamma_i^v v_i + \sum_{k\in\Phi_F} \gamma_k^w w_k & \\
s.t. & \\
\sum_{r=1}^{s} u_r y_{rj}^u + \sum_{k=1}^{f} \sigma_k w_k z_{kj}^u - \sum_{i=1}^{m} v_i x_{ij}^l - \sum_{k=1}^{f} (1-\sigma_k) w_k z_{kj}^l \leq 0 & j=1,\dots,n \\
\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r + \sum_{k=1}^{f} w_k = 1 & \\
(U,V)B \leq 0 & \\
v_i \geq \epsilon & i=1,\dots,m \\
u_r \geq \epsilon & r=1,\dots,s \\
w_k \geq \epsilon & k=1,\dots,f \\
\sigma_k \in \{0,1\} & k=1\dots,f
\end{array} \tag{16}
$$

The explanations about the previously mentioned notations and the constraints to these models are ignored.

$$
\begin{array}{ll}
\max \sum_{r=1}^{s} u_r + \sum_{k=1}^{f} (\sigma_k w_k) & \\
s.t. & \\
\sum_{r=1}^{s} u_r y_{rj}^u + \sum_{k=1}^{f} \sigma_k w_k z_{kj}^u - \sum_{i=1}^{m} v_i x_{ij}^l - \sum_{k=1}^{f} (1-\sigma_k) w_k z_{kj}^l \leq 0 & j=1,\dots,n \\
\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r + \sum_{k=1}^{f} w_k = 1 & \\
v_i \geq \epsilon & i=1,\dots,m \\
u_r \geq \epsilon & r=1,\dots,s \\
w_k \geq \epsilon & k=1,\dots,f \\
\sigma_k \in \{0,1\} & k=1\dots,f
\end{array} \tag{17}
$$

After obtaining a CSW and the corresponding σ^*, the flexible measures are classified as input or output indicators, accordingly. Afterwards, the ranking approach presented in 4.1 can be used in this regard.

5. A model for comparison

Shirdel et al. [47] proposed a method using common weights methodology in interval DEA. They presented a linear programming whose aim was generally to maximize possible efficiencies of DMUs. We aim to utilize the model in Section 6, for comparison purpose, so it is developed to deal with flexible measures, too:

$$
\begin{array}{ll}
min \sum_{j=1}^{n} \Delta_j & \\
s.t. & \\
\sum_{r=1}^{s} u_r y_{rj}^u + \sum_{k=1}^{f} \sigma_k w_k z_{kj}^u - \sum_{i=1}^{m} v_i x_{ij}^l - \sum_{k=1}^{f} (1-\sigma_k) w_k z_{kj}^l \leq 0 & j=1,\dots,n \\
\sum_{r=1}^{s} u_r y_{rj}^l + \sum_{k=1}^{f} \sigma_k w_k z_{kj}^l - \sum_{i=1}^{m} v_i x_{ij}^u - \sum_{k=1}^{f} (1-\sigma_k) w_k z_{kj}^u + \Delta_j = 0 & j=1,\dots,n \\
\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i + \sum_{k=1}^{f} w_k = 1 & \\
v_i \geq \epsilon & i=1,\dots,m \\
u_r \geq \epsilon & r=1,\dots,s \\
w_k \geq \epsilon & k=1,\dots,f \\
\Delta_j \geq 0, & j=1\dots,n
\end{array}
\tag{18}
$$

, where z_{kj}^l and z_{kj}^u $(k=1,\dots,f,\ j=1,\dots,n)$ are respectively the lower and the upper bounds for possible values of the k^{th} flexible indicator for DMU_j. In general, Shirdel et al.'s model [47] searches a set of weights for maximizing the minimum possible efficiencies of DMUs. The utilized criterion to the aim, is reducing the sum of the differences between the virtual input and the virtual output of each DMU. Considering the binary variables, Model (18) results in the role of the flexible indicators such that the best value of the objective function is obtained in that regard.

6. An Implementation

In this section, we implement our method on a data set regarding with 20 bank branches. In this assessment, there are 5 indicators such that one of which, the number of employees, is considered as the input indicator and two of which, overdue claims and facilities are considered as output indicators. However, the nature of the indicators called long term deposit and short term deposit are not certainly determined in the evaluation and each can be assumed as either input or output indicator. As Table 1 and Table 2 show, all data, except the number of employees, lie on intervals. In this assessment, the relative importance of the indicators are as follows:
The relative importance of the overdue claims rather than the bank facilities is considered between 1 to 1.5, that of the short term deposit rather than the bank facilities is between 2 to 3, and that of the bank facilities rather than the long

term deposit is between 0.5 to 1.
The overdue claims indicator has an important role in bank performance assessment because a bank branch can improve its resources and capital by collecting overdue claims, and so it would be able to give better and more facilities and services to its customers. Regarding the given information in this instance, we use Model (16). The weight restrictions to be applied in the model are considered as follows:

$$u_2 \leq u_1 \leq 1.5u_2 \ , \ 2u_2 \leq w_2 \leq 3u_2 \ , \ 0.5w_1 \leq u_2 \leq w_1$$

Hence, we can set $\Gamma_I, \Gamma_F, \Gamma_{IF}, \Gamma_{FI}, \Gamma_{IO}, \Gamma_{OI} = \emptyset, \Gamma_O = \{(1,2)\}, \Gamma_{FO} = \{(2,2)\}, \Gamma_{OF} = \{(2,1)\}$. Also, $\Phi_I = \emptyset$ and $\Phi_O = \{1\}$.
One can refer to Table 1 and Table 2 for getting the information about the lower and upper bounds of the input and the output data per DMU.

Table 1. The lower bounds of the input and the output data for 20 bank branches.

	Input indicator	Flexible indicators		Output indicators	
Branch	Number of employees	Long term deposit	Short term deposit	Overdue claims	Bank facilities
Branch1	15.000000	424085.145762	157283.362123	39843.721262	322747.771492
Branch2	9.000000	374045.099068	89838.121176	12340.569773	144391.329348
Branch3	6.000000	272441.286265	119993.069296	204031.942946	381015.800768
Branch4	20.000000	986544.089113	340478.751349	3702565.232068	7855541.444198
Branch5	9.000000	507890.702777	104023.162299	2879.933429	108510.930129
Branch6	14.000000	543741.711707	209541.407905	75497.965243	198680.975304
Branch7	8.000000	250750.353931	57335.476633	8721.794733	599950.459124
Branch8	7.000000	426046.527359	110196.428917	85315.111789	218010.809051
Branch9	8.000000	172666.090163	42151.649983	44195.022432	637241.633455
Branch10	10.000000	289024.173029	115741.614866	819.690216	150535.604490
Branch11	6.000000	348117.122916	62261.426017	4745.379920	71511.933991
Branch12	10.000000	263567.349128	98449.128721	4425.208192	127006.847451
Branch13	8.000000	340278.263721	89317.266523	2340.818321	99012.127951
Branch14	11.000000	475521.480993	102739.128638	4737.110679	111719.796293
Branch15	6.000000	300718.771310	77939.902709	1940.101841	70078.601997
Branch16	8.000000	388112.347310	46204.948147	111.025198	46433.402147
Branch17	9.000000	378903.708892	85088.934728	1421.888185	62713.114504
Branch18	11.000000	233398.510066	100612.697638	3294.043292	141923.094758
Branch19	6.000000	218132.410225	56077.736958	408.734792	33962.239739
Branch20	15.000000	1525958.154615	173734.450856	5726.595924	436243.245887

Beside applying our approach, we also intend to survey another situation when there is no DM's preference information available. To this aim, our approach in the absence of DM's preference information is used, too. Plus, for comparison intention in the situation, we use the 3^{rd} method. Actually, both of the methods can be considered in the category of the methods that generate a CSW with the aim of maximizing efficiencies, but with two different points of view in this regard. There are two flexible measures in this sample, both of which are of deposit type. Hence, we aim to assign them the same roles, both as input indicators or as output indicators. So, regarding with this sample, only one binary variable is required in the models (16), (17) and (18).
For convenience, we call the utilized methods as follows:
The 1^{st} method: our approach in the presence of DM's preference information.

Table 2. The upper bounds of the input and the output data for 20 bank branches.

	Input indicator	Flexible indicators		Output indicators	
Branch	Number of employees	Long term deposit	Short term deposit	Overdue claims	Bank facilities
Branch1	15.000000	513218.960257	246751.097358	46014.906812	380647.724870
Branch2	9.000000	507880.692272	112144.333464	15169.482476	170797.706519
Branch3	6.000000	345765.305758	169371.921043	212258.431484	602334.105548
Branch4	20.000000	3255357.187514	662096.096749	4069155.094653	8458770.581239
Branch5	9.000000	613579.906768	164669.802395	16927.582344	140574.940105
Branch6	14.000000	658280.396228	298792.587823	89539.120659	221737.368505
Branch7	8.000000	389601.654217	98689.046695	11090.053546	815060.471614
Branch8	7.000000	541544.762292	149453.065229	92495.821203	228890.463141
Branch9	8.000000	199340.893142	63856.637432	48352.358231	815610.260412
Branch10	10.000000	364793.973455	168390.149732	2088.316747	224130.875836
Branch11	6.000000	395331.519273	87326.731582	6858.647854	75497.553719
Branch12	10.000000	417103.606316	130339.974772	46948.350587	138794.333309
Branch13	8.000000	417103.606316	123253.202218	5814.743180	104049.003938
Branch14	11.000000	562853.835688	179030.514377	8128.585775	171476.380506
Branch15	6.000000	367470.244517	107904.785829	3008.068711	85733.601060
Branch16	8.000000	484971.334800	67054.990235	10439.689177	50824.355730
Branch17	9.000000	519401.981617	125058.328545	4528.278757	77166.840050
Branch18	11.000000	368774.150951	155932.487456	5000.808529	160655.559531
Branch19	6.000000	248647.472568	82108.714539	1763.042838	40639.270943
Branch20	15.000000	1965361.397460	265254.947543	16911.645353	471942.371179

The 2^{nd} method: our approach in the absence of DM's preference information.
The 3^{rd} method: the developed approach based on what is presented at [47].
At first, we implement the methods on the set of normalized data, and then they are implemented on the original data set.

6.1. Implementing the methods on the set of normalized data

Since the indicators has significantly noted in our method, in order to have a better sense of the impact of them, we also consider the set of normalized data. Besides, for the purpose of comparison, we consider both the original data and the normalized data to be used in the implementation of our method, separately. Here, for normalizing the vectors, for example the first output's, we act as the following:

$$M_1^O = \max_{j=1,\ldots,n} \{\max\{y_{1j}^l, y_{1j}^u\}\} = \max_{j=1,\ldots,n} \{y_{1j}^u\} = \|Y_1^u\|_\infty$$

where, $Y_1^l \in \mathbb{R}^n$ and $Y_1^u \in \mathbb{R}^n$ are, respectively, the vector of the lower bounds and the vector of the upper bounds for the first output indicator's values, and $\|.\|_\infty$ is the infinity norm. We set:

$$\tilde{y}_{1j}^l = \frac{y_{1j}^l}{M_1^o} \ , \ \tilde{y}_{1j}^u = \frac{y_{1j}^u}{M_1^o} \ , \ j = 1, \ldots, n$$

Thus, after the normalization, $[\tilde{y}_{1j}^l, \tilde{y}_{1j}^u]$ is considered as the interval of the possible normalized values of the first output of DMU_j. See Table 3 and Table 4.

For each method, the generated CSW and the optimal value of the binary variable, indicating the state of the flexible measures in the optimality, are shown in Table 5.

TABLE 3. The lower bounds of the normalized input and output data for 20 bank branches.

	Input indicator	Flexible indicators		Output indicators	
Branch	Number of employees	Long term deposit	Short term deposit	Overdue claims	Bank facilities
Branch1	0.75	0.130273	0.237553677	0.009791645	0.0381554
Branch2	0.45	0.1149014	0.135687435	0.003032711	0.017070014
Branch3	0.3	0.083690136	0.181232105	0.050141107	0.045043875
Branch4	1	0.303052486	0.514243707	0.909910079	0.928685956
Branch5	0.45	0.15601689	0.15711188	0.000707747	0.012828215
Branch6	0.7	0.167029816	0.316481866	0.01855372	0.023488162
Branch7	0.4	0.077026986	0.086596911	0.002143392	0.070926437
Branch8	0.35	0.130875509	0.166435702	0.020966296	0.025773345
Branch9	0.4	0.053040597	0.063663946	0.010860983	0.075335018
Branch10	0.5	0.088784166	0.1748109	0.00020144	0.017796393
Branch11	0.3	0.10693669	0.094036842	0.001166183	0.008454176
Branch12	0.5	0.080964187	0.148693111	0.0010875	0.015014812
Branch13	0.4	0.104528703	0.13490076	0.000575259	0.011705262
Branch14	0.55	0.146073519	0.155172533	0.001164151	0.013207569
Branch15	0.3	0.092376582	0.117716904	0.000476782	0.008284727
Branch16	0.4	0.119222661	0.069785864	2.73E-05	0.00548938
Branch17	0.45	0.116393897	0.128514479	0.000349431	0.007413975
Branch18	0.55	0.071696744	0.151960868	0.000809515	0.016778218
Branch19	0.3	0.067007212	0.084697278	0.000100447	0.004015033
Branch20	0.75	0.46875291	0.262400657	0.001407318	0.05157289

TABLE 4. The upper bounds of the normalized input and output data for 20 bank branches.

	Input indicator	Flexible indicators		Output indicators	
Branch	Number of employees	Long term deposit	Short term deposit	Overdue claims	Bank facilities
Branch1	0.75	0.157653655	0.372681698	0.011308221	0.04500036
Branch2	0.45	0.156013814	0.169377729	0.003727919	0.020191788
Branch3	0.3	0.106214245	0.255811689	0.052162777	0.071208233
Branch4	1	1	1	1	1
Branch5	0.45	0.188483128	0.248709822	0.004159975	0.016618838
Branch6	0.7	0.202214491	0.451282811	0.022004352	0.0262139
Branch7	0.4	0.119680155	0.149055473	0.002725395	0.096356848
Branch8	0.35	0.166354944	0.22572715	0.022730965	0.027059543
Branch9	0.4	0.061234722	0.096446177	0.011882653	0.096421844
Branch10	0.5	0.112059584	0.254328866	0.000513206	0.026496862
Branch11	0.3	0.121440289	0.131894346	0.001685521	0.008925358
Branch12	0.5	0.128128369	0.196859603	0.011537616	0.016408334
Branch13	0.4	0.128128369	0.186156062	0.00142898	0.012300724
Branch14	0.55	0.172900792	0.270399592	0.00199761	0.020272022
Branch15	0.3	0.112881697	0.162974508	0.000739237	0.010135468
Branch16	0.4	0.148976382	0.101276825	0.002565567	0.00600848
Branch17	0.45	0.159552993	0.188882443	0.00111283	0.009122702
Branch18	0.55	0.113282239	0.235513377	0.001228955	0.018992779
Branch19	0.3	0.076381011	0.124013289	0.00043327	0.004804395
Branch20	0.75	0.603731414	0.400629076	0.004156058	0.055793258

Table 5. The produced CSWs by using the normalized data.

	The CSW					The binary variable in the optimality
The used method	v_1^*	w_1^*	w_2^*	u_1^*	u_2^*	σ^*
The 1^{st} method	0.385566	0.175524	0.0001	0.263286	0.175524	0
The 2^{nd} method	0.5	0.0001	0.0001	0.324276	0.175524	1
The 3^{rd} method	0.0001	0.767159	0.0001	0.0001	0.232541	0
The 1^{st} method with ($\sigma = 1$)	0.500000	0.142829	0.0001	0.214243	0.142829	–

As Table 5 shows, in our approach, the results are obtained while the deposits are considered as input of the bank branches. The CSWs extracted from the 1^{st} and 3^{rd} methods are obtained under the conditions when the flexible measures are assumed as input ones whereas the mentioned measures have a different role, as output, in extracting the CSW of the 2^{nd} method. In the 1^{st} method, satisfying the weight restrictions, the weight of the first output indicator, as the only worthy indicator in this instance, is 0.263286, that is greater compared to when the mentioned indicators function as output indicators in evaluation process, that is 0.214243. Although, u_1^* is greater in the generated CSW from the 2^{nd} method, one should envisage that this weight has been obtained under more restrictions in 1^{th} method rather than the 2^{nd} method. In other words, the CSW extracted through the 2^{nd} method is not even a feasible solution to the model (16).

By fixing the role of the flexible measures according to the results form Table 5, the minimum, the maximum and the average of the possible absolute efficiencies based upon the generated CSW are shown in Table 6.

Table 6. The efficiencies.

	The 1^{st} method			The 2^{nd} method			The 3^{rd} method		
Branch	E_j^l	E_j^u	$\overline{E}_j$	E_j^l	E_j^u	$\overline{E}_j$	E_j^l	E_j^u	$\overline{E}_j$
Branch1	0.02927	0.034851	0.032061	0.013156	0.015222	0.014189	0.0733	0.104614	0.088957
Branch2	0.018888	0.023366	0.021127	0.006855	0.008433	0.007644	0.033151	0.053237	0.043194
Branch3	0.157124	0.201204	0.179164	0.167243	0.174061	0.170652	0.128523	0.257798	0.19316
Branch4	0.717357	1	0.858678	0.909713	1	0.954857	0.281548	1	0.640774
Branch5	0.0118	0.019971	0.015886	0.001717	0.009441	0.005579	0.020621	0.032275	0.026448
Branch6	0.029491	0.034736	0.032114	0.026635	0.03161	0.029122	0.035194	0.047552	0.041373
Branch7	0.074257	0.105096	0.089677	0.005472	0.006991	0.006231	0.179533	0.37888	0.279206
Branch8	0.06118	0.067967	0.064573	0.060052	0.065146	0.062599	0.046957	0.062663	0.05481
Branch9	0.09748	0.122616	0.110048	0.027232	0.029817	0.028524	0.372545	0.550435	0.46149
Branch10	0.01495	0.022967	0.018958	0.000514	0.001183	0.000848	0.048096	0.090375	0.069235
Branch11	0.013072	0.014953	0.014013	0.004024	0.005791	0.004908	0.021093	0.025288	0.023191
Branch12	0.013572	0.028587	0.02108	0.002273	0.023199	0.012736	0.035498	0.061384	0.048441
Branch13	0.012481	0.01469	0.013586	0.001562	0.003734	0.002648	0.027675	0.035649	0.031662
Branch14	0.010827	0.017181	0.014004	0.002229	0.003799	0.003014	0.023142	0.042042	0.032592
Branch15	0.011659	0.014963	0.013311	0.001735	0.002653	0.002194	0.022236	0.033238	0.027737
Branch16	0.00538	0.009878	0.007629	0.000165	0.006539	0.003352	0.011164	0.01527	0.013217
Branch17	0.006913	0.009767	0.00834	0.000887	0.002631	0.001759	0.014078	0.023744	0.018911
Branch18	0.013615	0.016279	0.014947	0.001559	0.002367	0.001963	0.044855	0.080198	0.062526
Branch19	0.005663	0.007511	0.006587	0.000437	0.001579	0.001008	0.015922	0.021716	0.018819
Branch20	0.023844	0.029308	0.026576	0.002084	0.005821	0.003952	0.025887	0.03607	0.030978

Finally, Table 7 shows the ranking scores of all DMUs obtained from the considered approaches.

As Table 6 shows, the amount of the average of the lower and the upper efficiencies per branch is quite distinct, so there is no need to use extra ranking criterion (The efficiencies have been calculated with six decimal places).

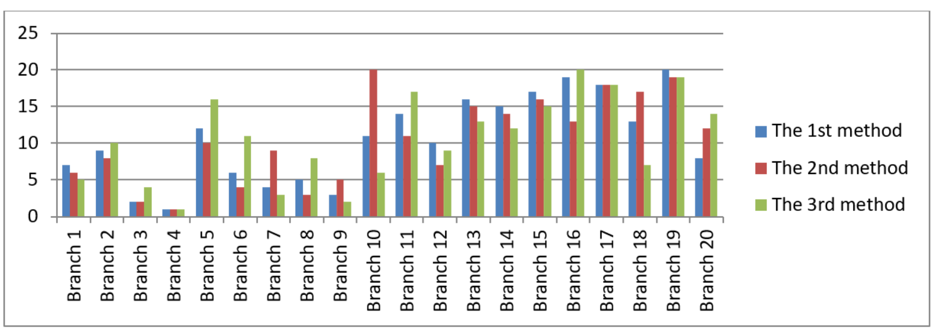
The ranking results have been compared visually in Figure 1; both the ranks of

TABLE 7. The obtained ranking scores by using the normalized data.

Branch	The 1^{st} method	The 2^{nd} method	The 3^{rd} method	The 1^{st} method with ($\sigma = 1$)
Branch1	7	6	5	16
Branch2	9	8	10	11
Branch3	2	2	4	3
Branch4	1	1	1	1
Branch5	12	10	16	6
Branch6	6	4	11	12
Branch7	4	9	3	5
Branch8	5	3	8	4
Branch9	3	5	2	8
Branch10	11	20	6	19
Branch11	14	11	17	7
Branch12	10	7	9	17
Branch13	16	15	13	15
Branch14	15	14	12	14
Branch15	17	16	15	9
Branch16	19	13	20	10
Branch17	18	18	18	13
Branch18	13	17	7	20
Branch19	20	19	19	18
Branch20	8	12	14	2

different branches in one method and the rank of one branch in different methods have been illustrated.

FIGURE 1. The Comparison of the obtained ranking scores from all methods by using the normalized.



6.2. Implementing the methods on the original data set

Here, the 1^{th}, 2^{nd}, and 3^{rd} methods were implemented on the original data set. Table 8, Table 9, and Table 10 have similar description as those for Table 5, Table 6, and Table 7 respectively. For brevity, the repetitive explanations are ignored.

Table 8. The produced CSWs by using the original data set.

	The CSW					The binary variable in the optimality
The used method	v_1^*	w_1^*	w_2^*	u_1^*	u_2^*	σ^*
The 1^{st} method	0.0001	0.048216	0.891413	0.036162	0.024108	0
The 2^{nd} method	0.0001	0.804784	0.0001	0.194916	0.0001	0
The 3^{rd} method	0.996369	0.0001	0.003331	0.0001	0.0001	0

Table 9. The efficiencies.

	The 1^{st} method			The 2^{nd} method			The 3^{rd} method		
Branch	E_j^l	E_j^u	$\overline{E}_j$	E_j^l	E_j^u	$\overline{E}_j$	E_j^l	E_j^u	$\overline{E}_j$
Branch1	0.037685	0.067479	0.052582	0.01888	0.02639	0.022635	0.040821	0.073398	0.057109
Branch2	0.031556	0.047557	0.039556	0.00592	0.009879	0.007899	0.036169	0.053803	0.044986
Branch3	0.098799	0.184821	0.14181	0.143045	0.188959	0.166002	0.096739	0.188151	0.142445
Branch4	0.432671	1	0.716335	0.275761	1	0.637881	0.453072	1	0.726536
Branch5	0.015423	0.034135	0.024779	0.001159	0.008106	0.004633	0.017999	0.038767	0.028383
Branch6	0.025227	0.040298	0.032763	0.027813	0.039932	0.033873	0.025502	0.040618	0.03306
Branch7	0.147698	0.317257	0.232478	0.008721	0.011115	0.009918	0.168234	0.368746	0.26849
Branch8	0.054245	0.074621	0.064433	0.048561	0.052647	0.050604	0.055408	0.077132	0.06627
Branch9	0.25492	0.46648	0.3607	0.054091	0.068408	0.06125	0.283194	0.521546	0.40237
Branch10	0.021818	0.046784	0.034301	0.000595	0.001846	0.001221	0.024919	0.0533	0.039109
Branch11	0.019562	0.028611	0.024086	0.00293	0.004799	0.003864	0.022668	0.033182	0.027925
Branch12	0.024075	0.050204	0.037139	0.002972	0.043205	0.023089	0.027339	0.050989	0.039164
Branch13	0.019015	0.028312	0.023664	0.001389	0.004177	0.002783	0.022021	0.032357	0.027189
Branch14	0.015341	0.038668	0.027005	0.002063	0.004185	0.003124	0.017548	0.044816	0.031182
Branch15	0.015448	0.025908	0.020678	0.001302	0.002458	0.00188	0.017907	0.030012	0.02396
Branch16	0.01351	0.026758	0.020134	0.000067	0.006531	0.003299	0.016632	0.030525	0.023579
Branch17	0.012049	0.021506	0.016777	0.000929	0.00292	0.001925	0.013838	0.024733	0.019286
Branch18	0.023564	0.040162	0.031863	0.003493	0.005275	0.004384	0.026225	0.044837	0.035531
Branch19	0.009958	0.017246	0.013602	0.000473	0.001981	0.001227	0.011407	0.019759	0.015583
Branch20	0.034591	0.052482	0.043536	0.000944	0.002723	0.001833	0.042046	0.065505	0.053775

In the absence of DM's preference information, the 2^{nd} and 3^{rd} methods almost have similar purposes of generating a CSW for maximizing efficiencies. We calculate the average for the lower, upper, and average efficiencies in both the methods; the ratio of the obtained averages in the 3^{rd} method compared to the 2^{nd} method is 0.974169, 1.701211, and 1.364175, respectively.

6.3. Some further discussion on the results

In this sample, although the first output indicator, overdue claims, gets involved in the weight restrictions, it's desired to influence its affect in the performance assessment as much as possible. As it is seen in Table 5, u_1^* in the 1^{st} method is significantly greater than the 3^{rd} method, but u_1^* has gained the largest value in the 2^{nd} method. The results were obtained when the long term deposit and the short term deposit are considered as inputs in the 1^{st} and 3^{rd} methods and as outputs in the 2^{nd} method. It is considerable that, although, here, the purpose

TABLE 10. The obtained ranking scores by using the original data.

Branch	The 1^{st} method	The 2^{nd} method	The 3^{rd} method
Branch1	6	7	6
Branch2	8	9	8
Branch3	4	2	4
Branch4	1	1	1
Branch5	14	10	14
Branch6	11	5	12
Branch7	3	8	3
Branch8	5	4	5
Branch9	2	3	2
Branch10	10	20	10
Branch11	15	12	15
Branch12	9	6	9
Branch13	16	15	16
Branch14	13	14	13
Branch15	17	17	17
Branch16	18	13	18
Branch17	19	16	19
Branch18	12	18	11
Branch19	20	19	20
Branch20	7	18	7

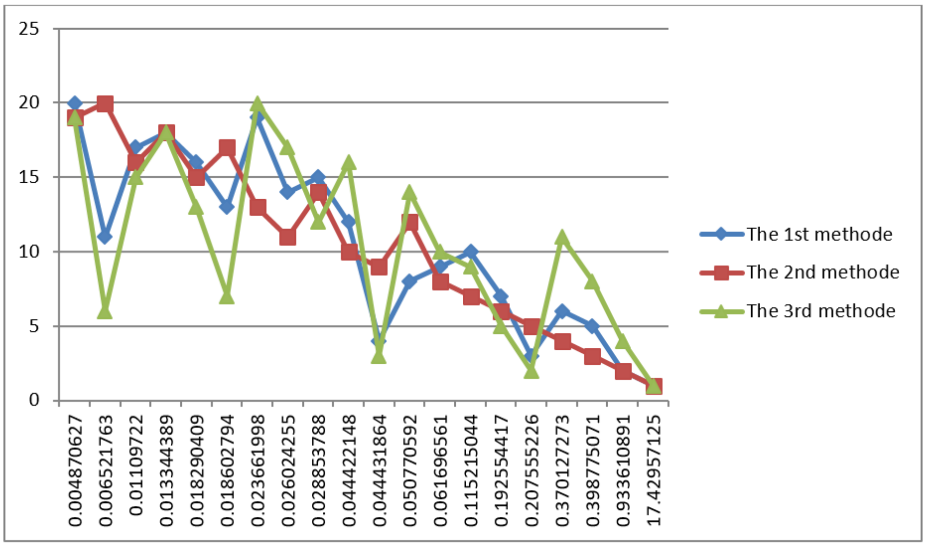
of the 1^{st} method is to produce a CSW maximizing u_1, the weight restrictions of the model is more than those of the 2^{nd} method. In Table 8, there is a similar discussion, but the long term deposit and the short term deposit are considered as inputs in all of the methods. As seen, the values of u_1^* obtained by using the normalized data are generally greater than those obtained by using the original data.

Now, we are interested in surveying the state of the branches in terms of the overdue claims indicator and their ranking scores. To this aim, we use a roughly criterion as follows: At first, the average of the lower and upper bounds of overdue claims is calculated per branch. The quantity for DMU_j is indicated by $AVEO1_j,\ j = 1, \ldots, 20$. Then, the average of $AVEO1_j,\ j = 1, \ldots, 20$, is calculated. It is indicated by $AVEO1$. We set $\tau_j = \frac{AVEO1_j}{AVEO1}\ , j = 1, \ldots, 20$.

Figure 2 illustrates the trends of changes in the ranking scores (obtained by using the normalized data) by increasing τ_j. The improvement in the ranking scores by increasing τ_j is more perceptible in the 2^{nd} method rather that of the others. The values of $\tau_j\ , j = 1, \ldots, 20$, are arranged in ascending order.

As Figure 2 illustrates, the Branch19 has adopted the worst ranking score in all of the methods, and τ_{19} is also minimum. Besides, the Branch4, has gained the greatest ranking score in all of the methods, and τ_{20} is maximum, too. However, it is not a general rule.

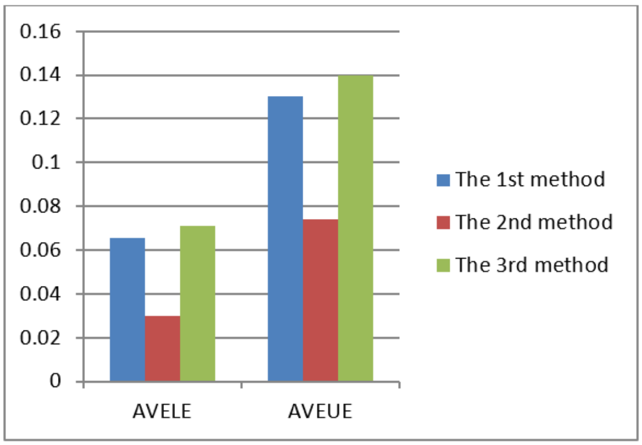
Figure 2. The trends of changes in the ranking scores (using the normalized data) by increasing τ_j.



Now, we concentrate on the original data, and compare the obtained efficiencies from the methods, specially the 2^{nd} and the 3^{rd} method. To this aim, we consider the columns of Table 9. Given a method, the average of the obtained lower efficiencies and the average of the obtained upper efficiencies for the branches are indicated, respectively, by AVELE and AVEUE. As Figure 2 illustrates, in the absence of the DM's preference information of type II, the 3^{rd} method has a better state than the 2^{nd} method in terms of maximizing the efficiencies. According to Table 8, for both methods, the efficiencies have been obtained under the situation that the flexible indicators are considered as inputs. The output weights from the 2^{nd} method are greater than or equal to the corresponding ones in the 3^{rd} method. However, it is not a contradiction and confirms that the magnitude of the obtained efficiencies is not only dependent on the magnitude of the output weights. In fact, as expected, the summation of the input weights (that includes 3 indicators in the optimality) in the 3^{rd} method is greater than that of the 2^{nd} method. But, in the 3^{nd} method, v_1^* is significantly larger than the other weights whereas the values of the first input indicator, number of employees, are considerably less than the others'. However, in the 2^{nd} method, w_1^* (that in the optimality acts as the weight of the second input indicator) is significantly larger than the other weights, and the values of the corresponding indicator is also considerable rather than the others'. Thus, the large value of w_1^* in the 2^{nd} method has been generally more effective in the obtained amounts of the efficiencies rather than the value of v_1^* in the 2^{nd} method.

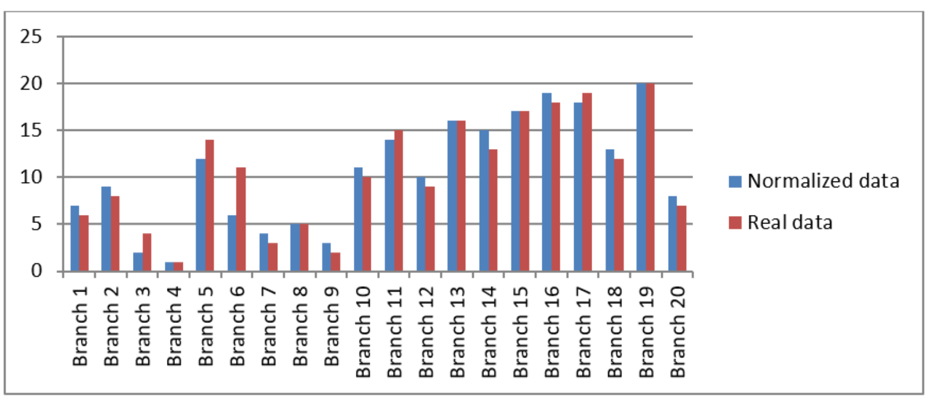
As figure 3 illustrates, the 1^{st} method has a relatively good state in terms of maximizing the efficiencies. Of course, this is not the concern of the 1^{st} method but shows that concentrating on the weights may also lead to relatively acceptable results in terms of maximizing the efficiencies.

FIGURE 3. Comparison of the averages of the efficiencies (by using the original data).



Ultimately, the ranking scores obtained from the 1^{st} method (as the main approach) in two cases of considering the normalized data and the original data are compared visually in Figure 4 .

FIGURE 4. The ranking scores obtained from the 1^{st} method using the normalized and the original.



7. Conclusion

In this paper, we propose a new ranking method using common weights with a different point of view as it is conventional. DM's preference information has a substantial role in this method. The information is categorized into two groups to be used in the model structure. Albeit, the model can be also developed to be used in the absence of the preference information through some modifications. In fact, we study another perspective to generate a CSW. The goal of our approach is to emphasize on the influence of some valuable indicators (in DM's viewpoint) as much as possible in the evaluation process. The approach is implemented on a data set regarding with a number of bank branches. The instance includes interval data and some flexible measures. Hence, the proposed approach is developed to encounter with the issues. Considering two cases of the original data and the normalized data, the results are discussed. It is noticeable that the structure of the feasible region of the proposed model is not a directly consequence of the considered criterion to select a CSW. Hence, the model can not be interpreted as one that makes the feasible region originally regarding its objective.

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