Identifying approximate proper efficiency in an infinite dimensional space

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Abstract

The main idea of this article is to characterize approximate proper efficiency that is a widely used optimality concept in multicriteria optimization problems that prevents solutions having unbounded trade-offs. We analyze a modification of approximate proper efficiency for problems with infinitely many objective functions. We obtain some necessary and sufficient optimality conditions for this modification of approximate proper efficiency. This modified version of approximation guarantees the general characterizations of approximate properly efficient points as solutions to weighted sum problems and modified weighted Tchebycheff norm problems, even if there is an infinite number of criteria. The provided proofs concerning the modified definition show that if the number of the objective functions is infinite, then these results become invalid under the primary definition of approximate proper efficiency.

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1 Introduction

The multicriteria optimization theory is one of the most exciting and rapidly growing areas of optimization theory, which deals with the simultaneous minimization or maximization of problems, when two or more than two objective functions are present. This approach is applicable in many other areas in practical issues such as medicine, engineering, environmental

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control, transportation, game theory, and economics [1, 3, 11, 15, 51]. In many cases, there is only a finite number of objective functions for such problems. In contrast, there exist essential applications in statistics, game theory, decision-making under uncertainty and control with infinitely many objective functions. For a good introduction to optimization with vector-valued maps as well as several of its mathematical and engineering applications in finite and infinite-dimensional settings we refer to the standard text book by Jahn [28].

Multiple solution concepts exist in multiobjective optimization problems with finite or infinite number of objectives, that often have a powerful influence on problems both in theory and computational experiment. Between the concepts of optimality that have fundamental importance for these problems, concepts such as efficiency, weak efficiency, and, in particular proper efficiency are essential. The notion of proper efficiency and its properties are the most critical concepts of optimality in the theoretical development and the numerical applications of multicriteria optimization. The primary purpose is that a disadvantage in one objective function can be paid by a controlled advantage in another objective function. Afterward, the points under consideration are compared componentwise. Kuhn and Tucker [35] presented the initial notion of proper efficiency, and after that, Geoffrion [17] refined this notion. Then, this concept was studied by many researchers, including [2, 11, 22, 26, 30, 42, 55]. Since then, Geoffrion's definition was generalized to vector optimization problems with other ordering cones and to infinite dimensional spaces by a couple of authors, such as Benson [5], Borwein [6] and Henig [25]. A comparisons of these different notions for infinite-dimensional spaces is given in [8]. Based on our knowledge, until now, two researchers considered proper efficiency in the Geoffrion's sense in an infinite-dimensional space. Winkler [54] extended Geoffrion's definition of proper efficiency utilizing its componentwise concept to an infinite-dimension. Also, Engau [13] considered multiobjective optimization problems with infinite number of objective functions and proposed a modification to the Geoffrion's definition of proper efficiency which maintained the fundamental properties of properly efficient points, when the number of criteria was countably infinite. He also obtained necessary and sufficient conditions for identifying these points.

As mentioned above, multiobjective optimization problems, especially problems with infinitely many criteria, have many applications. We briefly describe their application in stochastic optimization. Recently, the methodology and theory of multiobjective optimization have been applied for robust optimization and decision making under uncertainty ([27, 31, 41]). Klamroth et al. [34] states that "most of the classical solution concepts commonly used in multicriteria optimization have their equivalents in some approaches to handle uncertainty in the decision analysis". Based on the mathematical equivalence between expected values in stochastic optimization and weighted sums in multiobjective optimization, Engau [12] investigated the role of proper efficiency and its significance to scenario tradeoffs. He considered stochastic optimization and decision-making under uncertainty, in which the domain of a single real-valued function $f(x, \tau)$ consists of the decision vector x and the random vector τ. If the random vector

τ has a discrete probability distribution with a finite number of possible scenario realizations t_i and nonzero probability masses $w_i > 0$ for all $i = 1, ..., p$, then he set $f_i(x) = f(x, t_i)$ and replaced the objective vector $(f_1(x), ..., f_p(x))$ with its expectation $E(x) = \sum_{i=1}^{p} w_i f_i(x)$ and showed that a solution that maximizes an expected value is also properly efficient in a stochastic sense, i.e. there exists no other feasible decision whose gain-to-loss ratio from one possible outcome or scenario to another outcome or scenario is still infinitely large. Also, if τ has a general discrete distribution whose number of associated criterion functions $f(x, t)$ is countably infinite, then he replaced the criterion vector of possible outcomes $(f_1(x), ..., f_p(x), ...)$ with its expectation $E(x) = \sum_{i=1}^{\infty} w_i f_i(x)$ and showed that for multiobjective programs with infinitely many criteria, or similarly, for stochastic problems with discrete random variables, the available characterizations of proper efficiency can be retain after a small modification to the original definition of the Geoffrion's proper efficiency.
However, the (weakly, properly) efficient set of a multicriteria optimization problem with finite or infinite number of objectives might be empty in a noncompact case, whereas approximate solutions might exist under some weaker assumptions. In addition, in applied optimization, the models characterize simplified versions of real problems, and moreover, numerical algorithms usually generate only approximate solutions if we stop them after a finite number of steps. Furthermore, iterative methods give a sequence of approximate solutions, understanding properties of such a sequence would help researchers design efficient algorithms.

Therefore, it is necessary to define concepts of approximate solutions and establish their features. Kutateladze [36] introduced the first definition of approximate efficient solutions, and after that, Loridan [40] presented some characterizations of these solutions in multicriteria optimization. Subsequently, various notions of approximate efficiency in vector optimization and some applications of approximate efficiency were offered in [9, 15, 14, 18, 23, 24, 43, 45, 46]. In particular, necessary and sufficient optimality conditions for approximate Pareto solutions of a convex semi-infinite/infinite vector optimization problem under various kind of Farkas–Minkowski constraint qualifications were obtained in [37, 47].

The definition of ϵ-proper efficiency in the sense of Geoffrion was proposed by Li and Wang [39], and they attained necessary optimality conditions for ϵ-proper efficiency by using scalarization approaches. Subsequently, Liu [38] applied the weighted sum method to derive a necessary and sufficient optimality condition for ϵ-properly efficient solutions of a multicriteria optimization problem with convex functions. Thereafter, Ghaznavi-ghosoni and Khorram [18] and Ghaznavi-ghosoni et al. [19] obtained necessary and sufficient optimality conditions for general multicriteria optimization problems with no convexity assumption. Afterward, many researchers utilized the scalarization approaches to attain necessary and sufficient optimality conditions for approximate proper efficiency in the Geoffrion sense [4, 16, 20, 21, 32, 33, 44, 48]. Also, new kinds of approximate proper efficiency in vector optimization were considered in [16, 56].

Based on the author's knowledge, until now, there is no result on approximate proper efficiency in an infinite-dimensional setting. If we apply the existing definition of approximate solutions [19, 39] for a problem with infinite objective functions, it can be observed that this definition can not fulfill the main intentions of the Geoffrion definition of (approximate) proper efficiency. Therefore, these observations motivate us to state a slight but notable change of approximate proper efficiency definition [19, 39] for problems with infinitely many objective functions. We distinguish this modified definition in the next sections. We will indicate that only this concept implies the characteristic properties of ϵ-properly efficient points as optimal solutions to the proposed scalarization approaches. Also, we derive some necessary and sufficient optimality conditions in case of infinite dimensional space. We use tow scalarization techniques: the weighted sum method [10] and the augmented weighted Tchebycheff norms [7, 29, 49, 50]. We utilize the weighted sum method and get necessary and sufficient optimality conditions for the modified ϵ-properly efficient solutions under a kind of subconvexity assumption for problems with infinitely many objective functions. Furthermore, we obtain suitable approximations to attain the ideal point by applying augmented weighted Tchebycheff norms without any convexity hypothesis. We derive necessary and sufficient optimality conditions for the modified ϵ-properly efficient points of an (even nonconvex) multiobjective optimization problem for infinitely many objectives. Therefore, we present appropriate proofs for these cases, which unlike those that are shown in the papers [18, 19, 38], the number of objective functions is not necessarily finite.

We organize this article into six sections. We provide some preliminaries and technical tools and characterize the modified ϵ-properly efficient solutions for problems with infinitely many criteria in Section 2. Necessary and sufficient optimality conditions for δ-optimal solutions of the weighted sum problem to be ϵ-properly efficient solutions for problems with infinitely many objectives are detailed in Section 3. In Section 4, we investigate optimality conditions for the augmented weighted Tchebycheff norm problem. We provide an example that involves infinitely many objective functions in Section 5. We have discussed the conclusions in the last section.

2 Basic definitions and technical tools

In this section, we recall some fundamental concepts in multiobjective optimization and continue our investigations by looking at their relationships. Let us consider the n-dimensional Euclidean space $\mathbb{R}^n$. For $x_i, y_i \in \mathbb{R}$, $x_i \geqslant y_i$ if and only if $x_i - y_i \geqslant 0$. Thus, for $x, y \in \mathbb{R}^n$, we obtain the following conventions:

$$x \geqq y \text{ if and only if } x_i \geqslant y_i \text{ for all } i \in \{1, \dots, n\}.$$

In fact, the preorder $\geqq$ in $\mathbb{R}^n$ demonstrates the standard order, component by component.

$$x \geq y \text{ if and only if } x \geqq y \text{ and } x \neq y,$$

in other words, $\geq$ represents that each component of x is greater than or equal to each component of y and $x \neq y$.

$$x > y \text{ if and only if } x_i > y_i \text{ for all } i,$$

so $>$ indicates the common strict inequality, component by component.

Definition 2.1. Consider a feasible set $X \subseteq \mathbb{R}^m$ and $f : X \to \mathbb{R}$. Let $\delta \geqslant 0$. A feasible solution x^* is a δ-optimal solution of the single objective optimization problem

$$\begin{array}{ll} \min & f(x) \\ s.t. & x \in X, \end{array}$$

if $f(x) \geqslant f(x^*) - \delta$ for all $x \in X$.

First, we investigate a multicriteria optimization problem (MOP) in a finite dimensional space, as follows:

$$MOP \quad \min_{x \in X} \quad F(x) = (f_1(x), f_2(x), \ldots, f_n(x)), \tag{2.1}$$

with $X \subseteq \mathbb{R}^m$ and $f_i : X \to \mathbb{R}$, for all $i = 1, 2, \ldots, n$. Let us now introduce the notions of ϵ-efficiency and ϵ-weak efficiency for MOP (2.1) as follows.

Definition 2.2. [40] Let $\epsilon \in \mathbb{R}^n_{\geqq}$. A feasible solution $x^* \in X$ is

(i) an ϵ-efficient (ϵ-Pareto optimal) solution of MOP (2.1) if there exists no $x \in X$ such that $F(x) \leq F(x^*) - \epsilon$, i.e., there is no $x \in X$ such that $f_i(x) \leqslant f_i(x^*) - \epsilon_i$ for all $i = 1, \ldots, n$ and $f_j(x) < f_j(x^*) - \epsilon_j$ for at least one index j.

(ii) an ϵ-weakly efficient (ϵ-weakly Pareto optimal) solution of MOP (2.1) if if there exists no $x \in X$ such that $f_i(x) < f_i(x^*) - \epsilon_i$ for all $i = 1, 2, \ldots, n$.

An ϵ-(weak) efficient solution with $\epsilon = 0$ is usually recognized as (weak) efficient solution. The study of ϵ-properly efficient solutions was first carried out by Li and Wang [39]. This concept has also been utilized in Ghaznavi et al. [19].

Definition 2.3. [39] Let $\epsilon \in \mathbb{R}^n_{\geqq}$. A vector $x^* \in X$ is called an ϵ-properly efficient (ϵ-properly Pareto optimal) solution of MOP (2.1) if it is ϵ-efficient and there exists a scalar $M > 0$ such that, for all $i \in \{1, 2, \ldots, n\}$ and $x \in X$ satisfying $f_i(x) < f_i(x^*) - \epsilon_i$, there is an index $j \in \{1, 2, \ldots, n\}$ such that $f_j(x^*) - \epsilon_j < f_j(x)$ and

$$\frac{f_i(x^*) - f_i(x) - \epsilon_i}{f_j(x) - f_j(x^*) + \epsilon_j} \leqslant M.$$

Let us consider a finite or countably infinite index set I. We take $F : X \to D$ with $X \subseteq U$, where U is a real vector space and D is an associated real linear vector or $\mathcal{L}^\infty$ sequence space, with $\|s\|_\infty = \sup_{i\in I} |s_i| < \infty$ for all $s \in D$. We get

$$D_> := \{s \in D : s_i > 0, \forall i \in I\},$$

$$D_\geq := \{s \in D : s_i \geqslant 0, \forall i \in I \text{ and } s \neq 0\}$$

and

$$D_\geqq := \{s \in D : s_i \geqslant 0, \forall i \in I\}.$$

Let $D_\geqq$ be a given cone. Consider the following vector optimization problem:

$$IMOP \quad \min_{x\in X} \quad F(x) \tag{2.2}$$

where $F : X \to D$.
If $|I| = n$ and $D = \mathbb{R}^n$, then the vector optimization problem (2.2) coincides with the multicriteria optimization problem (2.1). If I is infinite, then the definitions of ϵ-efficient and ϵ-weakly efficient solutions are related to ordering relations and can be described as follows.

Definition 2.4. Let $x^* \in X$ be a feasible solution and $\epsilon \in D_\geqq$ be given.

(i) The vector x^* is an ϵ-efficient solution of IMOP (2.2) if $x \in X$ does not exist such that $F(x) \leq F(x^*) - \epsilon$, i.e., there is no $x \in X$ such that $f_i(x) \leqslant f_i(x^*) - \epsilon_i$ for all $i \in I$ and $f_j(x) < f_j(x^*) - \epsilon_j$ for at least one index j.

(ii) The vector x^* is an ϵ-weakly efficient solution of IMOP (2.2) if there is no $x \in X$ such that $f_i(x) < f_i(x^*) - \epsilon_i$ for all $i \in I$.

Now, we define a generalized version of approximate proper efficiency for problems with infinitely many objective functions. Definition 2.5 is somewhat reworded formulation of the basic Definition 2.3, which appears better suited for its primary intention to prevent solutions with unbounded marginal rates of substitution.

Definition 2.5. A decision vector $x^* \in X$ is an ϵ-properly efficient solution of IMOP (2.2) if it is ϵ-efficient and, for each index $i \in I$, there is some positive real number M_i such that for each vector $x \in X$ satisfying $f_i(x) < f_i(x^*) - \epsilon_i$, there exists at least one index $j \in I$ such that $f_j(x^*) - \epsilon_j < f_j(x)$ and

$$\frac{f_i(x^*) - f_i(x) - \epsilon_i}{f_j(x) - f_j(x^*) + \epsilon_j} \leqslant M_i.$$

Note that if $|I| = n$ and we take $M = \max_{i\in I}\{M_i\}$ or $\{M_i\} = M$ for all $i \in I$, then Definition 2.5 is equal to Definition 2.3. A similar interpretation applies to $|I| = \infty$, if a vector x^* is an ϵ-properly efficient solution by Definition 2.3 with a single real number $M > 0$, then this vector is an ϵ-properly efficient solution that satisfies Definition 2.5 with $M_i = M$ for all $i \in I$. It is evident that Definition 2.5 does not necessarily imply Definition 2.3 for infinitely many criteria. Therefore, we have the following observation:

Theorem 2.6. *If I is finite, then Definitions 2.3 and 2.5 of approximate proper efficiency are equivalent. If I is infinite, then an approximate properly efficient solution by Definition 2.3 is also approximate properly efficient by Definition 2.5.*

It also can be seen that if $\epsilon = 0$, then Definition 2.5 coincides with the definition of properly efficiency in [13].
As mentioned by Geoffrion [17], all of the (approximate) efficient solutions may not have equally nice properties which a decision-maker may desire and he suggested a restricted definition of efficiency that (A): filter out efficient solutions with a specific anomalous type and keep the good ones; and (B) address acceptable theoretical characterization. By approximate proper efficient solutions also we are going to filter out bad approximate efficient solutions. However, the following example shows that Definition 2.3 of approximate proper efficiency, is not suitable for infinitely many criteria and may filter out many $\epsilon-$efficient solutions with no obvious anomalies, whereas Definition 2.5 accepts these ones as $\epsilon-$properly efficient solutions.

Example 2.7. *Consider the index set $I = \{0, 1, 2, \dots\}$. Consider the following optimization problem for infinitely many objectives:*

$$\min_{x\in[0,4]} \quad F(x) = (f_0(x), f_1(x), \dots) \tag{2.3}$$

where $F : [0, 4] \to D$, $f_0(x) = \frac{16}{49}(x - 2)$ and $f_i(x) = \min\{2, 2^i(2 - x)\}$ for all $i \geqslant 1$. We consider the feasible point $x^* = 2$. Since investigating all of the feasible solutions is rather time-consuming, we only consider the case $x^* = 2$, here. Let $\epsilon_0 = \frac{3}{7}$ and $\epsilon_i = \frac{1}{7}$ for all $i \geqslant 1$. The point $x^* = 2$ is ϵ-efficient for the optimization problem (5.1). Because f_0 is strictly decreasing, whereas every f_i for $i \geqslant 1$ is strictly increasing for each $x < 2$ and f_0 is strictly increasing whereas every f_i for $i \geqslant 1$ is strictly decreasing for all $x > 2$. We distinguish two cases $x < 2$ and $x > 2$, respectively.
We focus on the first case. If $2 > 2^i(2 - x)$ then $f_i(x) = 2^i(2 - x)$. At point $x^* = 2$, $f_i(x) < f_i(x^*) - \epsilon_i$, only if $i = 0$ and

$$\frac{f_0(2) - f_0(x) - \epsilon_0}{f_j(x) - f_j(2) + \epsilon_j} \leqslant \frac{0 - \frac{16}{49}(x - 2) - \frac{3}{7}}{2^j(2 - x) - 0 + \frac{1}{7}} \leqslant \frac{16}{49}2^{-j} \leqslant 1.$$

If $2 < 2^i(2-x)$ *then* $f_i(x) = 2$. *Relation* $f_i(x) < f_i(x^*) - \epsilon_i$ *holds, only if* $i = 0$ *and*

$$\frac{f_0(2) - f_0(x) - \epsilon_0}{f_j(x) - f_j(2) + \epsilon_j} \leqslant \frac{0 - \frac{16}{49}(x-2) - \frac{3}{7}}{2 - 0 + \frac{1}{7}} \leqslant 1.$$

Hence, if $x < 2$, *then we can consider* $M = M_0$ *and* M_i *arbitrarily for every* $i \in I$ *and both Definitions 2.3 and 2.5 of* ϵ*-proper efficiency hold for the feasible point* $x^* = 2$.
Now we focus on the second case. If $x > 2$ *then* $f_i(x) = 2^i(2-x)$, *for all* $i \geq 1$. *At point* $x^* = 2$, $f_i(x) < f_i(x^*) - \epsilon_i$, *only if* $i \geqslant 1$ *and* $2^i(2-x) < -\frac{1}{7}$ *thus*

$$\frac{f_i(2) - f_i(x) - \epsilon_i}{f_0(x) - f_0(2) + \epsilon_0} \leqslant \frac{0 - 2^i(2-x) - \frac{1}{7}}{\frac{16}{49}(x-2) - 0 + \frac{3}{7}} \leqslant \frac{49}{16}2^i.$$

It implies that we require $M_i \geqslant (49/16)2^i$, *that is a finite value, but might be very large. This case also follows that the feasible point* $x^* = 2$ *satisfies Definition 2.5 of* ϵ*-proper efficiency, but Definition 2.3 of* ϵ*-proper efficiency does not hold for the point* $x^* = 2$ *since there does no exist a constant* $M > 0$ *for all the objective functions.*
Thus, the above cases imply that Definition 2.5 of ϵ*-proper efficiency holds for feasible point* $x^* = 2$, *but Definition 2.3 does not hold for this point. Many other* ϵ*-efficient points can be considered that are* ϵ*-properly efficient by Definition 2.5 and are not* ϵ*-properly efficient with respect to Definition 2.3.*

In the next sections, we will show that Definition 2.3 can not fulfill intention (B) of the Geoffrion definition, for infinitely many objective functions, whereas Definition 2.5 will provide a satisfactory characterization for approximate properly efficient solutions.

Remark 2.8. In what follows, unless otherwise expressed, we will investigate ϵ-proper efficiency given in Definition 2.5.

3 The weighted sum approach for IMOP (2.2)

To obtain some results on ϵ-proper efficient solutions of the vector optimization problem IMOP (2.2), in this section, the relationships between this class of solutions and δ-optimal solutions of the weighted sum scalarization problem are investigated. Consider the following single objective optimization problem for infinitely many objectives,

$$\min_{x \in X} \sum_{i \in I} \gamma_i f_i(x). \tag{3.1}$$

Note that, if the index set I is countably infinite, then $\sum_{i\in I}\gamma_i f_i(x)$ is an infinite series. If this summation exists, then we normalize $\sum_{i\in I}\gamma_i$ to guarantee that the minimum or infimum value of the scalarized problem (3.1) is finite. To do that, we choose $\gamma_i \in D_{\geq}$ so that $\sum\limits_{i\in I}\gamma_i = 1$.

Lemma 3.1. *Let $\epsilon \geqq 0$ and $\delta \leqslant \sum_{i\in I}\gamma_i\epsilon_i$. If x^* is a δ-optimal point of the scalarized problem (3.1) with $\gamma > 0$, then the vector x^* is an ϵ-efficient solution of IMOP (2.2).*

Proof. Suppose that the vector x^* is not an ϵ-efficient solution of IMOP (2.2). Therefore, there is some $x \neq x^*$ such that $f_i(x)+\epsilon_i \leqslant f_i(x^*)$ for each $i \in I$ and $f_j(x)+\epsilon_j < f_j(x^*)$ for one index j, yielding

$$\sum_{i\in I}\gamma_i f_i(x^*) > \sum_{i\in I}\gamma_i f_i(x) + \sum_{i\in I}\gamma_i\epsilon_i \geqslant \sum_{i\in I}\gamma_i f_i(x) + \delta.$$

This yields a contradiction to x^* being a δ-optimal solution of the weighted sum problem. Thus, x^* is an ϵ-efficient solution of IMOP (2.2). □

Now, we derive a sufficient condition for δ-optimal solutions of the scalarized problem (3.1) to be ϵ-properly efficient solutions of IMOP (2.2). If the index set I is finite, then Theorem 3.2 is the same as Theorem 2 in [38], whose proof demonstrates that x^* is an ϵ-properly efficient solution of the multicriteria problem with $M = (n-1)\max_{i,j}\{\gamma_j/\gamma_i\}$, where n indicates the number of objective functions. If n tends to infinity, then it is obvious that M necessarily tends to infinity. Therefore, we can not choose M_i as that of Theorem 2 in [38]. Thus, M must be assigned uniformly and independent of index i. Consequently, we provide a more general proof that is like the one introduced in [38], but with different choices for the M_i, and is independent of the actual number of objective functions.

Theorem 3.2. *Let $\epsilon \in D_{\geqq}$ be given. If x^* is a δ-optimal solution of the scalarized problem (3.1) with $\delta \leqslant \sum_{i\in I}\gamma_i\epsilon_i$ and $\gamma_i > 0$ for all $i \in I$. Then, the vector x^* is an ϵ-properly efficient solution of IMOP (2.2).*

Proof. Due to Lemma 3.1, the vector x^* is an ϵ-efficient solution of IMOP (2.2). We only have to show that x^* is an ϵ-properly efficient solution. We consider appropriately large numbers M_i as follows:

$$0 < M_i = \frac{\sum_{j\in I, j\neq i}\gamma_j}{\gamma_i} = \frac{1-\gamma_i}{\gamma_i} < \infty, \quad \forall i \in I. \tag{3.2}$$

Assume that x^* is not an ϵ-properly efficient solution. Hence there is $i_\epsilon \in I$ and $x_\epsilon \in X$ such that $f_{i_\epsilon}(x_\epsilon) + \epsilon_{i_\epsilon} < f_{i_\epsilon}(x^*)$ and

$$f_{i_\epsilon}(x^*) - f_{i_\epsilon}(x_\epsilon) - \epsilon_{i_\epsilon} > M_{i_\epsilon}(f_j(x_\epsilon) - f_j(x^*) + \epsilon_j), \tag{3.3}$$

for all $j \in I$ such that $f_j(x^*) < f_j(x_\epsilon)+\epsilon_j$. Since $f_{i_\epsilon}(x^*)-f_{i_\epsilon}(x_\epsilon)-\epsilon_{i_\epsilon} > 0$ and $M_{i_\epsilon} > 0$, therefore the relation (3.3) is also true if $f_j(x^*) \geqslant f_j(x_\epsilon)+\epsilon_j$. Therefore, by using these inequalities and by multiplying any of them by its corresponding γ_j for all $j \neq i_\epsilon$ and summing them over $j \neq i_\epsilon$ we have

$$\gamma_j(f_{i_\epsilon}(x^*)-f_{i_\epsilon}(x_\epsilon)-\epsilon_{i_\epsilon}) > M_{i_\epsilon}\gamma_j(f_j(x_\epsilon)-f_j(x^*)+\epsilon_j),\ \forall j \neq i_\epsilon$$

$$\Rightarrow \sum_{j\in I, j\neq i_\epsilon} \gamma_j(f_{i_\epsilon}(x^*)-f_{i_\epsilon}(x_\epsilon)-\epsilon_{i_\epsilon}) > M_{i_\epsilon} \sum_{j\in I, j\neq i_\epsilon} \gamma_j(f_j(x_\epsilon)-f_j(x^*)+\epsilon_j). \tag{3.4}$$

Based on the relation (3.2), we have $\gamma_{i_\epsilon} = \sum_{j\in I, j\neq i_\epsilon}\gamma_j/M_{i_\epsilon}$. Hence, from relation (3.4) it follows

$$\gamma_{i_\epsilon}(f_{i_\epsilon}(x^*)-f_{i_\epsilon}(x_\epsilon)-\epsilon_{i_\epsilon}) > \sum_{j\in I, j\neq i_\epsilon} \gamma_j(f_j(x_\epsilon)-f_j(x^*)+\epsilon_j).$$

Therefore,

$$\gamma_{i_\epsilon} f_{i_\epsilon}(x^*) + \sum_{j\in I, j\neq i_\epsilon}\gamma_j f_j(x^*) - \gamma_{i_\epsilon}\epsilon_{i_\epsilon} - \sum_{j\in I, j\neq i_\epsilon}\gamma_j\epsilon_j > \sum_{j\in I, j\neq i_\epsilon}\gamma_j f_j(x_\epsilon) + \gamma_{i_\epsilon} f_{i_\epsilon}(x_\epsilon)$$

$$\Rightarrow \sum_{i\in I}\gamma_i f_i(x^*) - \sum_{i\in I}\gamma_i\epsilon_i > \sum_{i\in I}\gamma_i f_i(x_\epsilon)$$

$$\Rightarrow \sum_{i\in I}\gamma_i f_i(x^*) - \delta > \sum_{i\in I}\gamma_i f_i(x_\epsilon),$$

contradicting the δ-optimality of x^*. Thus x^* is an ϵ-properly efficient solution. □

The proof of Theorem 3.2 is true either for infinitely many objective functions ($|I| = \infty$) or for finitely many objective functions ($|I| = n$). Especially for ($|I| = n$), we obtain

$$\max_{i\in I}\{M_i\} = \max_{i\in I}\left\{\frac{\sum_{j\in I, j\neq i}\gamma_j}{\gamma_i}\right\} \leqslant \max_{i,j}\left\{\frac{(n-1)\gamma_j}{\gamma_i}\right\},$$

Hence, M_i for each $i \in I$ in proof of Theorem 3.2 are never larger than M in proof of Theorem 2 in [38]. If the number of objective functions is infinite, then $\lim \gamma_i = \inf \gamma_i = 0$. Therefore $\lim M_i = \sup M_i = \infty$ does not exist and, generally, we can not choose a uniform M to satisfy the statement of Theorem 3.2 for infinitely many objective functions, as postulated by Definition 2.3 of ϵ-properly efficient solutions. Note that if we set $\epsilon = 0$, then we obtain Theorem 3.1 in [13].

In what follows in this section, we attain a necessary condition for approximate proper efficient solutions of IMOP (2.2). To this end, we state Definition 3.3 and Theorem 3.4 that are efficient and useful to obtain this necessary condition.

Definition 3.3. The objective function $F : X \to D$ is named a generalized $D_{\geqq}$-subconvexlike function on X if for each $x^1, x^2 \in X$, $\beta \in (0,1)$ and $\eta > 0$ there are $\phi \in D_{>}$, $x^3 \in X$ and $\upsilon > 0$ such that

$$\eta\phi + \beta F(x^1) + (1-\beta)F(x^2) - \upsilon F(x^3) \in D_{\geqq}.$$

In the other words, $F : X \to D$ is a generalized $D_{\geqq}$-subconvexlike function if for each $x^1, x^2 \in X$, $\beta \in (0,1)$ and $\eta > 0$ there are $\phi \in D_{>}$, $x^3 \in X$ and $\upsilon > 0$ such that

$$\eta\phi_i + \beta f_i(x^1) + (1-\beta)f_i(x^2) - \upsilon f_i(x^3) \in \mathbb{R}_{\geqslant}, \quad \forall i \in I.$$

It is clear that a convex function F is a generalized $D_{\geqq}$-subconvexlike function, but the converse implication is not necessarily true. We refer to [53] for more details.
Next, we propose an alternative theorem for generalized $D_{\geqq}$-subconvexlike functions in infinite dimensional objective space. The following theorem extends Theorem 1 in [52] to an infinite number of generalized subconvexlike functions.

Theorem 3.4. *Consider $F : X \to D$. If $F : X \to D$ is a generalized $D_{\geqq}$-subconvexlike function, then one and only one of the following statements can happen:*

(1) There exists $\bar{x} \in X$ such that $-f_i(\bar{x}) > 0$ for all $i \in I$,

(2) There is $t \in D_{\geq}$ with $\sum_{i \in I} t_i = 1$ such that $\sum_{i \in I} t_i f_i(x) \geqslant 0$ for all $x \in X$.

Proof. The proof can be concluded from Theorem 1 in [52]. □

In the next theorem, we establish a necessary condition for a δ-optimal solution of the scalarized problem (3.1) to be an ϵ-properly efficient solution of IMOP (2.2). The proof of this theorem requires the alternative theorem, under the generalized subconvexlike assumption. We generalize Theorem 4.11 in [19] and provide a necessary condition. Also, remember that the proof in [19] demonstrates that there are weights $\gamma_i > 1$ for $i = 1, 2, \ldots, n$ such that these weights can be normalized to $\sum_{i=1}^{n} \gamma_i = 1$ only if the number of objective functions is not infinite. Thus, we provide a more popular proof. We make an additional set of positive weights to imply a vector $\gamma > 0$ with $\sum_{i \in I} \gamma_i = 1$.

Theorem 3.5. *Let $\epsilon \in D_{\geqq}$ be given and $F(\cdot) - F(x^*) + \epsilon$ be a generalized $D_{\geqq}$-subconvexlike function. If x^* is an ϵ-properly efficient solution, then there is $\gamma \in D_{>}$ such that x^* is a δ-optimal solution of the scalarized problem (3.1) with $\delta = \sum_{i \in I} \gamma_i \epsilon_i$.*

Proof. x^* is an ϵ-properly efficient solution. Therefore, by definition, there is a scalar $M_i > 0$ for each index $i \in I$ such that the system

$$\begin{aligned} &f_i(x) < f_i(x^*) - \epsilon_i, \\ &M_i(f_j(x) - f_j(x^*) + \epsilon_j) < f_i(x^*) - f_i(x) - \epsilon_i, \forall j \neq i \end{aligned} \tag{3.5}$$

has no solution in X. To see that, simply rearrange the inequalities in Definition 2.5. Since $F(x)-F(x^*)+\epsilon$ is a generalized $D_{\geqq}$-subconvexlike function, for any $x^1, x^2 \in X$, $\beta \in (0,1)$ and $\eta > 0$, there are $\phi \in D_{>}$, $x^3 \in X$ and $\upsilon > 0$ such that for all $t \in I$, we have

$$\eta\phi_t + \beta(f_t(x^1) - f_t(x^*) + \epsilon_t) + (1-\beta)(f_t(x^2) - f_t(x^*) + \epsilon_t) - \upsilon(f_t(x^3) - f_t(x^*) + \epsilon_t) \geqslant 0. \quad (3.6)$$

Therefore, for $t = i$ we have

$$\eta\phi_i + \beta(f_i(x^1) - f_i(x^*) + \epsilon_i) + (1-\beta)(f_i(x^2) - f_i(x^*) + \epsilon_i) - \upsilon(f_i(x^3) - f_i(x^*) + \epsilon_i) \geqslant 0, \quad (3.7)$$

and for $t = j,\ \forall j \neq i$ we have

$$\eta\phi_j + \beta(f_j(x^1) - f_j(x^*) + \epsilon_j) + (1-\beta)(f_j(x^2) - f_j(x^*) + \epsilon_j) - \upsilon(f_j(x^3) - f_j(x^*) + \epsilon_j) \geqslant 0. \quad (3.8)$$

Multiplying Relation (3.8) by M_i, for each $j \neq i$ and summing with Relation (3.7), we obtain

$$\begin{aligned} \eta(M_i\phi_j + \phi_i) + \beta\big(M_i(f_j(x^1) - f_j(x^*) + \epsilon_j) + f_i(x^1) - f_i(x^*) + \epsilon_i\big) + \\ (1-\beta)\big(M_i(f_j(x^2) - f_j(x^*) + \epsilon_j) + f_i(x^2) - f_i(x^*) + \epsilon_i\big) - \\ \upsilon\big(M_i(f_j(x^3) - f_j(x^*) + \epsilon_j) + f_i(x^3) - f_i(x^*) + \epsilon_i\big) \geqslant 0. \end{aligned} \quad (3.9)$$

Consider

$$\mho_{ij}(x) = \begin{cases} f_i(x) - f_i(x^*) + \epsilon_i, & i = j, \\ M_i(f_j(x) - f_j(x^*) + \epsilon_j) - f_i(x^*) + f_i(x) + \epsilon_i, & j \neq i \end{cases}$$

and

$$\vartheta_{ij} = \begin{cases} \phi_i, & i = j, \\ M_i\phi_j + \phi_i, & j \neq i. \end{cases}$$

According to relations (3.7) and (3.9), we have

$$\begin{aligned} &\eta\vartheta_{ii} + \beta\mho_{ii}(x^1) + (1-\beta)\mho_{ii}(x^2) \geqq \upsilon\mho_{ii}(x^3), \\ &\eta\vartheta_{ij} + \beta\mho_{ij}(x^1) + (1-\beta)\mho_{ij}(x^2) \geqq \upsilon\mho_{ij}(x^3), \quad \forall i,j:\ i \neq j. \end{aligned}$$

Based on the above results, the vector functions $(\mho_{i1}, \mho_{i2}, \ldots)$ are generalized $D_{\geqq}$-subconvexlike functions for all $i \in I$. By applying Theorem 3.4 and the relation (3.5) we have that for the ith system, there is $\alpha^i \in D_{\geq}$ with $\sum_{j\in I} \alpha^i_j = 1$ such that

$$\alpha^i_i(f_i(x) - f_i(x^*) + \epsilon_i) + \sum_{j\in I, j\neq i} \alpha^i_j(M_i(f_j(x) - f_j(x^*) + \epsilon_j) - f_i(x^*) + f_i(x) + \epsilon_i) \geqslant 0, \quad \forall x \in X.$$

Therefore

$$f_i(x) + M_i \sum_{j\in I, j\neq i} \alpha^i_j f_j(x) \geqslant f_i(x^*) + M_i \sum_{j\in I, j\neq i} \alpha^i_j f_j(x^*) - (\epsilon_i + M_i \sum_{j\in I, j\neq i} \alpha^i_j \epsilon_j), \quad \forall x \in X. \quad (3.10)$$

We pick any $\alpha_i^l > 0$ and define

$$0 < \zeta_i = \frac{\alpha_i^l}{1 + M_i(1 - \alpha_i^i)} < \infty, \quad \forall i \in I.$$

Multiplying each of relations (3.10) by its corresponding ζ_i and summing them over $i \in I$ follows for all $x \in X$

$$\sum_{i \in I}(\zeta_i + \sum_{j \in I, j \neq i} M_j \zeta_j \alpha_i^j) f_i(x) \geqslant \sum_{i \in I}(\zeta_i + \sum_{j \in I, j \neq i} M_j \zeta_j \alpha_i^j) f_i(x^*) - \sum_{i \in I}(\zeta_i + \sum_{j \in I, j \neq i} M_j \zeta_j \alpha_i^j) \epsilon_i. \quad (3.11)$$

We consider $\gamma_i = \zeta_i + \sum_{j \in I, j \neq i} M_j \zeta_j \alpha_i^j$. It is obvious that $\gamma_i > 0$ for each $i \in I$ and $\sum_{i \in I} \gamma_i = 1$ since

$$\sum_{i \in I} \gamma_i = \sum_{i \in I}(\zeta_i + \sum_{j \in I, j \neq i} M_j \zeta_j \alpha_i^j) = \sum_{i \in I} \zeta_i + \sum_{j \in I}(M_j \zeta_j \sum_{j \in I, j \neq i} \alpha_i^j) = \sum_{i \in I} \zeta_i (1 + M_i(1 - \alpha_i^i)) = \sum_{i \in I} \alpha_i^l = 1.$$

Hence, from (3.11) we immediately get

$$\sum_{i \in I} \gamma_i f_i(x) \geqslant \sum_{i \in I} \gamma_i f_i(x^*) - \delta.$$

where $\delta = \sum_{i \in I} \gamma_i \epsilon_i$. □

Note that the proof of Theorem 3.5 works for infinitely many objective functions, with the same value M_i for all $i \in I$. It is also evident that the normalization given in the proof of Theorem 4.11 in [19] is not applicable for infinitely many criteria. Hence, we prove the above theorem with a new normalization that can also be used for problems in a finite dimensional space.
It is important to emphasize that every convex function is a generalized subconvexlike function. Therefore, the following corollary can be concluded from Theorem 3.5. This corollary extends Theorem 3 in [38] from a finite number of criteria to an infinite number of objectives.

Corollary 3.6. *Let $\epsilon \in D_{\geqq}$ be given and $F(\cdot)$ be a convex function over the convex feasible set X. If x^* is an ϵ-properly efficient solution to IMOP (2.2), then $\gamma \in D_{>}$ exists such that x^* is a δ-optimal solution of the scalarized problem (3.1) with $\delta = \sum_{i \in I} \gamma_i \epsilon_i$.*

4 The augmented weighted Tchebycheff approach

In the previous section, we obtained a necessary optimality condition for approximate properly efficient solutions of the scalarized problem (3.1) under the generalized subconvexlike assumption. Now, in this section, we state a scalarization technique called the augmented weighted

Tchebycheff approach for obtaining necessary and sufficient optimality conditions in general and without any convexity assumption. The augmented weighted Tchebycheff problem related to the vector optimization problem IMOP (2.2) is formulated as

$$\min \sup_{i\in I}\{\lambda_i(f_i(x)-u_i)\}+\varphi\sum_{i\in I}\gamma_i f_i(x), \tag{4.1}$$

where $\varphi>0$, $\lambda\in D_{\geq}$ and $\gamma\in D_{\geq}$ are weights with $\sum_{i\in I}\gamma_i=1$. In the scalarized problem (4.1) it is necessary to have a utopia vector $u\in D$, that its components are described by

$$u_i=\inf_{x\in X}\{f_i(x)\}-\beta,\ \ \forall i\in I, \tag{4.2}$$

for a small scalar $\beta>0$. If $\inf_{x\in X}\{f_i(x)\}$ for each $i\in I$ exist and are finite, then the utopia vector $u\in D$ is well defined.

Lemma 4.1. *Let $\epsilon\in D_{\geqq}$ be given. If a feasible vector $x^*\in X$ is a δ-optimal solution of the scalarized problem (4.1) with $\lambda\geq 0$ and $\gamma>0$ such that $\delta\leqslant\inf_{i\in I}\{\lambda_i\epsilon_i\}+\varphi\sum_{i\in I}\gamma_i\epsilon_i$, then x^* is ϵ-efficient solution of IMOP (2.2).*

Proof. Suppose that there is a feasible solution $x\neq x^*$ such that $f_i(x)\leqslant f_i(x^*)-\epsilon_i$ for all $i\in I$ and $f_j(x)<f_j(x^*)-\epsilon_j$ for at least one index j. Then, since $\varphi>0,\gamma_i>0\ \forall i$, and $\lambda\geq 0$ we have

$$\varphi\sum_{i\in I}\gamma_i(f_i(x)+\epsilon_i)<\varphi\sum_{i\in I}\gamma_i f_i(x^*) \tag{4.3}$$

and

$$\lambda_i(f_i(x)-u_i+\epsilon_i)\leqslant\lambda_i(f_i(x^*)-u_i),\ \ \forall i\in I. \tag{4.4}$$

Therefore, from relations (4.3)-(4.4) and the fact that for any two sequences $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$, one has $\sup_{i\in I}\{a_i+b_i\}\geq\sup_{i\in I}\{a_i\}+\inf_{i\in I}\{b_i\}$, we obtain

$$\begin{aligned}\sup_{i\in I}\{\lambda_i(f_i(x^*)-u_i)\}+\varphi\sum_{i\in I}\gamma_i f_i(x^*) &> \sup_{i\in I}\{\lambda_i(f_i(x)-u_i+\epsilon_i)\}+\varphi\sum_{i\in I}\gamma_i(f_i(x)+\epsilon_i)\\ &\geqslant \sup_{i\in I}\{\lambda_i(f_i(x)-u_i)\}+\varphi\sum_{i\in I}\gamma_i f_i(x)+\inf_{i\in I}\{\lambda_i\epsilon_i\}+\varphi\sum_{i\in I}\gamma_i\epsilon_i\\ &\geqslant \sup_{i\in I}\{\lambda_i(f_i(x)-u_i)\}+\varphi\sum_{i\in I}\gamma_i f_i(x)+\delta.\end{aligned}$$

This contradiction means that x^* is not a δ-optimal solution of the scalarized problem (4.1). □

In the following theorem, we prove a sufficient optimality condition for a δ-optimal solution of the scalarized problem (4.1) to be an ϵ-properly efficient solution of IMOP (2.2). If we choose M_i for all $i \in I$ dependent on the number of criteria, then if the number of objective functions increases, these values tend to infinity. Hence, it is mandatory to introduce scalars M_i so that are not dependent on the finiteness or infiniteness of I.

Theorem 4.2. *Let $\epsilon \in D_{\geqq}$ and $\delta \leqslant \inf_{i\in I}\{\lambda_i\epsilon_i\} + \varphi \sum_{i\in I} \gamma_i\epsilon_i$. If $x^* \in X$ is a δ-optimal solution of the scalarized problem (4.1) with $\lambda > 0$ and $\gamma > 0$, then the vector x^* is an ϵ-properly efficient solution of IMOP (2.2).*

Proof. Let x^* be a δ-optimal solution of the scalarized problem (4.1) with $\inf_{i\in I}\{\lambda_i\epsilon_i\}+\varphi\sum_{i\in I}\gamma_i\epsilon_i \geqslant \delta$. Then,

$$
\begin{aligned}
&\sup_{i\in I}\{\lambda_i(f_i(x^*) - u_i)\} + \varphi\sum_{i\in I}\gamma_i f_i(x^*) - \delta \leqslant \sup_{i\in I}\{\lambda_i(f_i(x) - u_i)\} + \varphi\sum_{i\in I}\gamma_i f_i(x)\\
\Rightarrow 0 \quad &\leqslant \sup_{i\in I}\{\lambda_i(f_i(x) - u_i)\} + \varphi\sum_{i\in I}\gamma_i f_i(x) - \sup_{i\in I}\{\lambda_i(f_i(x^*) - u_i)\}\\
&-\varphi\sum_{i\in I}\gamma_i f_i(x^*) + \inf_{i\in I}\{\lambda_i\epsilon_i\} + \varphi\sum_{i\in I}\gamma_i\epsilon_i\\
&\leqslant \sup_{i\in I}\{\lambda_i(f_i(x) - f_i(x^*))\} + \inf_{i\in I}\{\lambda_i\epsilon_i\} + \varphi\sum_{i\in I}\gamma_i f_i(x) + \varphi\sum_{i\in I}\gamma_i\epsilon_i - \varphi\sum_{i\in I}\gamma_i f_i(x^*),
\end{aligned}
$$

for all $x \in X$. Because $\sup_{i\in I}\{\lambda_i(f_i(x) - f_i(x^*))\} + \inf_{i\in I}\{\lambda_i\epsilon_i\} \leqslant \sup_{i\in I}\{\lambda_i(f_i(x) - f_i(x^*) + \epsilon_i)\}$, from the above relation we have

$$
\sup_{i\in I}\{\lambda_i(f_i(x) - f_i(x^*) + \epsilon_i)\} + \varphi\sum_{i\in I}\gamma_i(f_i(x) - f_i(x^*) + \epsilon_i) \geqslant 0, \quad \forall x \in X. \tag{4.5}
$$

By Lemma 4.1, x^* is an ϵ-efficient solution of IMOP (2.2). We only have to show that x^* is ϵ-properly efficient. To demonstrate that x^* is ϵ-properly efficient solution of IMOP (2.2), we choose $K = \sup_{i\in I}\{\lambda_i\}$ and

$$
0 < M_i = \frac{1}{\gamma_i}(\frac{K}{\varphi} + (1 - \gamma_i)) < \infty, \quad \forall i \in I.
$$

Assume that an index $t \in I$ exists such that $f_t(\tilde{x}) < f_t(x^*) - \epsilon_t$ for some $\tilde{x} \in X$, now we prove that another index $j \in I$ exists such that

$$
f_j(x^*) - \epsilon_j < f_j(\tilde{x}) \tag{4.6}
$$

and

$$(f_t(x^*) - f_t(\tilde{x}) - \epsilon_t) \leqslant M_t(f_j(\tilde{x}) - f_j(x^*) + \epsilon_j). \tag{4.7}$$

For arbitrary $\alpha > 1$, take

$$\theta = \frac{\sup_{i\in I}\{f_i(\tilde{x}) - f_i(x^*) + \epsilon_i\}}{\alpha}, \tag{4.8}$$

which is finite, because f_i is bounded for all $i \in I$. Since x^* is an ϵ-efficient solution and $f_t(\tilde{x}) < f_t(x^*) - \epsilon_t$, it implies that we must have $f_j(x^*) - \epsilon_j < f_j(\tilde{x})$ for at least one index $j \in I$. This follows $\theta > 0$. According to relation (4.8) and $\alpha > 1$, there is $j \in I$ such that

$$f_j(\tilde{x}) - f_j(x^*) + \epsilon_j \geqslant \theta. \tag{4.9}$$

From relation (4.8) and $K = \sup_{i\in I}\{\lambda_i\}$, we also obtain

$$\sup_{i\in I}\{\lambda_i(f_i(\tilde{x}) - f_i(x^*) + \epsilon_i)\} \leqslant K\theta\alpha. \tag{4.10}$$

Isolating the term $f_t(x^*) - f_t(x) - \epsilon_t$ for $x = \tilde{x}$ in (4.5) and combining the obtained equation with relations (4.9) and (4.10), we obtain

$$\begin{aligned} f_t(x^*) - f_t(\tilde{x}) - \epsilon_t &\leqslant \tfrac{1}{\varphi\gamma_t}\left(\sup_{i\in I}\{\lambda_i(f_i(\tilde{x}) - f_i(x^*) + \epsilon_i)\} + \varphi\sum_{i\neq t}\gamma_i(f_i(\tilde{x}) - f_i(x^*) + \epsilon_i)\right) \\ &\leqslant \tfrac{\alpha}{\gamma_t}\left(\tfrac{K}{\varphi} + (1-\gamma_t)\right)\theta \\ &\leqslant \tfrac{\alpha}{\gamma_t}\left(\tfrac{K}{\varphi} + (1-\gamma_t)\right)(f_j(\tilde{x}) - f_j(x^*) + \epsilon_j)\,. \end{aligned}$$

Since α can be chosen arbitrarily close to 1, we obtain

$$f_t(x^*) - f_t(\tilde{x}) - \epsilon_t \leqslant M_t(f_j(\tilde{x}) - f_j(x^*) + \epsilon_j),$$

where $M_t = (1/\gamma_t)\,(K/\varphi + (1-\gamma_t))$. Therefore, relation (4.6) is satisfied. It follows that x^* is an ϵ-properly efficient solution of IMOP (2.2). □

A similar analysis as that of Theorem 3.2 shows that if $|I| = \infty$, then $\lim \gamma_i = \inf \gamma_i = 0$. This concludes $\lim M_i = \inf M_i = \infty$. Thus, a general upper bound for M_i does not exist, in general. Thus, Theorem 4.2 does not hold for an infinite dimensional problem with a constant M. A sufficient condition for ϵ-properly efficient solutions of the multicriteria problem (2.1) via a modification of the augmented weighted Tchebycheff problem was presented by Ghaznavi-ghosoni and Khorram (see Theorem 3.16 in [18]). The proof of Theorem 3.16 in [18] is applicable only for the finite-dimensional case, since, by selecting $M = \max_{k\in I}\{(1 + \sum_{t\in I}\gamma_t)/\gamma_k\}$, for an infinite dimensional problem the value of γ_i tends to zero. It causes that M tends to infinity. However, the proof presented for Theorem 4.2 is general and holds for any number of objective functions in the optimization problem.

The next theorem yields a necessary optimality condition for ϵ-properly efficient solution of IMOP (2.2).

Theorem 4.3. *Let $\epsilon \in D_{\geqq}$. If $x^* \in X$ is an ϵ-properly efficient solution of IMOP (2.2) with $M_i < \infty$ for each $i \in I$. Then, there exist $\lambda \in D_{\geq}$, $\gamma \in D_{>}$ and $\varphi > 0$ such that x^* is a δ-optimal solution of the scalarized problem (4.1) with $\delta = \varphi \sum_{i \in I} \gamma_i \epsilon_i + \sup_{i \in I}\{\lambda_i \epsilon_i\}$.*

Proof. We define $\lambda_i = 1/(f_i(x^*) - u_i)$ for all $i \in I$. Therefore $0 < \lambda_i \leqslant 1/\beta < \infty$, due to relation (4.2). Thus

$$\sup_{i \in I}\{\lambda_i (f_i(x^*) - u_i)\} = 1. \tag{4.11}$$

Since the vector $x^* \in X$ is an ϵ-properly efficient solution, there exists at least one index $i \in I$ such that $f_i(x) \geqslant f_i(x^*) - \epsilon_i$ for every x. Therefore $f_i(x) - u_i + \epsilon_i \geqslant f_i(x^*) - u_i$ and hence

$$\lambda_i (f_i(x) - u_i + \epsilon_i) \geqslant 1. \tag{4.12}$$

We distinguish two cases.

Consider $\varphi \sum_{i \in I} \gamma_i (f_i(x) + \epsilon_i) \geqslant \varphi \sum_{i \in I} \gamma_i f_i(x^*)$, in the first case. Thus, from relations (4.11) and (4.12) it follows that

$$\begin{aligned}
\sup_{i \in I}\{\lambda_i (f_i(x^*) - u_i)\} + \varphi \sum_{i \in I} \gamma_i f_i(x^*) &\leqslant 1 + \varphi \sum_{i \in I} \gamma_i (f_i(x) + \epsilon_i) \\
&\leqslant \sup_{i \in I}\{\lambda_i (f_i(x) - u_i + \epsilon_i)\} + \varphi \sum_{i \in I} \gamma_i (f_i(x) + \epsilon_i) \\
&\leqslant \sup_{i \in I}\{\lambda_i (f_i(x) - u_i)\} + \varphi \sum_{i \in I} \gamma_i f_i(x) \\
&\quad + \varphi \sum_{i \in I} \gamma_i \epsilon_i + \sup_{i \in I}\{\lambda_i \epsilon_i\}.
\end{aligned}$$

The claim above implies that

$$\sup_{i \in I}\{\lambda_i (f_i(x^*) - u_i)\} + \varphi \sum_{i \in I} \gamma_i f_i(x^*) \leqslant \sup_{i \in I}\{\lambda_i (f_i(x) - u_i)\} + \varphi \sum_{i \in I} \gamma_i f_i(x) + \delta, \tag{4.13}$$

where $\delta = \varphi \sum_{i \in I} \gamma_i \epsilon_i + \sup_{i \in I}\{\lambda_i \epsilon_i\}$.

Now, introduce the second case. Assume that $\tilde{x} \in X$ exists such that

$$\varphi \sum_{i \in I} \gamma_i (f_i(\tilde{x}) + \epsilon_i) < \varphi \sum_{i \in I} \gamma_i f_i(x^*).$$

Define $\sigma = \sup_{i\in I}\{f_i(x^*) - u_i\} \geqslant \beta > 0$. We consider $\varphi = \frac{1}{M_i\sigma}$ and $\psi = \frac{\sum_{i\in I}\gamma_i(f_i(x^*)-f_i(\tilde{x})-\epsilon_i)}{\sigma} > 0$. Hence,

$$\psi\sigma = \sum_{i\in I}\gamma_i f_i(x^*) - \sum_{i\in I}\gamma_i(f_i(\tilde{x}) + \epsilon_i) = \sum_{i\in I}\gamma_i(f_i(x^*) - f_i(\tilde{x}) - \epsilon_i). \tag{4.14}$$

Because $\gamma > 0$ and $\sum_{i\in I}\gamma_i = 1$, so some index $i \in I$ exists such that

$$f_i(x^*) - f_i(\tilde{x}) - \epsilon_i \geqslant \psi\sigma > 0. \tag{4.15}$$

Because the vector x^* is an ϵ-properly efficient solution of IMOP (2.2), there exists at least one index $j \in I$ such that $f_j(x^*) - \epsilon_j < f_j(\tilde{x})$ and

$$f_i(x^*) - f_i(\tilde{x}) - \epsilon_i \leqslant M_i(f_j(\tilde{x}) - f_j(x^*) + \epsilon_j).$$

Therefore

$$\frac{f_i(x^*) - f_i(\tilde{x}) - \epsilon_i}{M_i} \leqslant f_j(\tilde{x}) - u_j - f_j(x^*) + u_j + \epsilon_j.$$

Dividing the latter inequality by $f_j(x^*) - u_j > 0$ we obtain

$$\frac{f_j(x^*) - u_j}{f_j(x^*) - u_j} + \frac{f_i(x^*) - f_i(\tilde{x}) - \epsilon_i}{M_i(f_j(x^*) - u_j)} \leqslant \frac{f_j(\tilde{x}) - u_j + \epsilon_j}{f_j(x^*) - u_j}.$$

That is

$$1 + \frac{f_i(x^*) - f_i(\tilde{x}) - \epsilon_i}{M_i(f_j(x^*) - u_j)} \leqslant \frac{f_j(\tilde{x}) - u_j + \epsilon_j}{f_j(x^*) - u_j}. \tag{4.16}$$

Based on relations (4.15)-(4.16), $\sigma = \sup_{i\in I}\{f_i(x^*) - u_i\}$ and $\varphi = \frac{1}{M_i\sigma}$, we have

$$1 + \varphi\psi\sigma \leqslant 1 + \varphi(f_i(x^*) - f_i(\tilde{x}) - \epsilon_i) \leqslant 1 + \frac{f_i(x^*) - f_i(\tilde{x}) - \epsilon_i}{M_i(f_j(x^*) - u_j)} \leqslant \frac{f_j(\tilde{x}) - u_j + \epsilon_j}{f_j(x^*) - u_j}. \tag{4.17}$$

By the definition of λ_i for $i = j$, and relations (4.11), (4.14) and (4.17), we obtain

$$\begin{aligned}
\varphi\sum_{i\in I}\gamma_i f_i(\tilde{x}) + \sup_{i\in I}\{\lambda_i(f_i(\tilde{x}) - u_i + \epsilon_i)\} &\geqslant \varphi\sum_{i\in I}\gamma_i f_i(\tilde{x}) + \frac{f_j(\tilde{x}) - u_j + \epsilon_j}{f_j(x^*) - u_j} \\
&\geqslant \varphi\sum_{i\in I}\gamma_i f_i(\tilde{x}) + \varphi\psi\sigma + 1 \\
&= \varphi\sum_{i\in I}\gamma_i f_i(\tilde{x}) + \varphi(\sum_{i\in I}\gamma_i(f_i(x^*) - f_i(\tilde{x}) - \epsilon_i)) + 1 \\
&= \varphi\sum_{i\in I}\gamma_i f_i(x^*) - \varphi\sum_{i\in I}\gamma_i\epsilon_i + \sup_{i\in I}\{\lambda_i(f_i(x^*) - u_i)\}.
\end{aligned}$$

Hence,

$$\varphi \sum_{i\in I} \gamma_i f_i(\tilde{x}) + \sup_{i\in I}\{\lambda_i(f_i(\tilde{x}) - u_i + \epsilon_i)\} \geqslant \varphi \sum_{i\in I} \gamma_i f_i(x^*) - \varphi \sum_{i\in I} \gamma_i \epsilon_i + \sup_{i\in I}\{\lambda_i(f_i(x^*) - u_i)\}. \quad (4.18)$$

Moreover,

$$\varphi \sum_{i\in I} \gamma_i f_i(\tilde{x}) + \sup_{i\in I}\{\lambda_i(f_i(\tilde{x}) - u_i + \epsilon_i)\} \leqslant \varphi \sum_{i\in I} \gamma_i f_i(\tilde{x}) + \sup_{i\in I}\{\lambda_i(f_i(\tilde{x}) - u_i)\} + \sup_{i\in I}\{\lambda_i \epsilon_i\}. \quad (4.19)$$

The assumption of the theorem, implies that $\delta = \sup_{i\in I}\{\lambda_i \epsilon_i\} + \varphi \sum_{i\in I} \gamma_i \epsilon_i$. Finally, due to (4.18) and (4.19), it follows that

$$\varphi \sum_{i\in I} \gamma_i f_i(\tilde{x}) + \sup_{i\in I}\{\lambda_i(f_i(\tilde{x}) - u_i)\} \geqslant \varphi \sum_{i\in I} \gamma_i f_i(x^*) + \sup_{i\in I}\{\lambda_i(f_i(x^*) - u_i)\} - \delta. \quad (4.20)$$

Based on relations (4.13) and (4.20), the vector x^* is a δ-optimal solution of the scalarized problem (4.1) with $\delta = \varphi \sum_{i\in I} \gamma_i \epsilon_i + \sup_{i\in I}\{\lambda_i \epsilon_i\}$. □

Remark 4.4. Theorem 4.3 extends Theorem 3.18 in [18] to infinitely many criteria. Also, by letting $\epsilon = 0$, this theorem reduces to Theorem 3.3 in [13].

5 Numerical experiment

The following example verifies the results obtained in this article. Consider the index set $I = \{0, 1, 2, \dots\}$. Consider the following optimization problem for infinitely many objectives, investigated in Example 2.7:

$$\min_{x\in[0,4]} \quad F(x) = (f_0(x), f_1(x), \dots) \quad (5.1)$$

where $F : [0, 4] \to D$, $f_0(x) = \frac{16}{49}(x - 2)$ and $f_i(x) = \min\{2, 2^i(2 - x)\}$ for all $i \geqslant 1$. We consider the feasible point $x^* = 2$. Since investigating all of the feasible solutions is rather time-consuming, we only consider the case $x^* = 2$, here. Let $\epsilon_0 = \frac{3}{7}$ and $\epsilon_i = \frac{1}{7}$ for all $i \geqslant 1$. In Example 2.7, we showed that $x^* = 2$ is an ϵ-properly efficient solution with respect to Definition 2.5, but Definition 2.3 does not hold for this point.

Now, we show that the feasible point $x^* = 2$ is a δ-optimal solution of the weighted sum scalarization problem (3.1) for a vector $\gamma > 0$ which verifies that if the number of criteria is infinite, a δ-optimal solution of the weighted sum problem (3.1) with positive weights is not necessarily an ϵ-properly efficient solution in the sense of Definition 2.3. In particular, we select $\gamma_0 = \frac{7}{8}$ and $\gamma_i = (\frac{1}{9})^i$ for all $i \geqslant 1$. Hence

$$\sum_{i=0}^{\infty} \gamma_i = \gamma_0 + \sum_{i=1}^{\infty} \gamma_i = \frac{7}{8} + \frac{1}{8} = 1.$$

If $2^i(2-x) < 2$ then $f_i = 2^i(2-x)$ for every $i \geqslant 1$, and we have

$$\begin{aligned} \sum_{i\in I} \gamma_i f_i(x) &= \frac{7}{8}\Big(\frac{16}{49}(x-2)\Big) + \sum_{i=1}^{\infty} (\frac{2}{9})^i (2-x) \\ &= \frac{2}{7}x - \frac{4}{7} + \frac{4}{7} - \frac{2}{7}x = 0. \end{aligned} \tag{5.2}$$

If $2 < 2^i(2-x)$ then $f_i(x) = 2$ for each $i \geqslant 1$, and only $x < 2$ can occur (since if $x > 2$ then $2^i(2-x) < 0 < 2$). Therefore

$$\begin{aligned} \sum_{i\in I} \gamma_i f_i(x) &= \frac{7}{8}\Big(\frac{16}{49}(x-2)\Big) + 2\sum_{i=1}^{\infty} (\frac{1}{9})^i \\ &= \frac{2}{7}x - \frac{4}{7} + \frac{1}{4}. \end{aligned} \tag{5.3}$$

Hence, from relations (5.2) and (5.3) it implies that if $\delta \leqslant \sum_{i\in I} \gamma_i \epsilon_i = \frac{22}{25}$, then

$$\sum_{i\in I} \gamma_i f_i(x) \geqslant \sum_{i\in I} \gamma_i f_i(2) - \delta \geqslant \sum_{i\in I} \gamma_i f_i(2) - \sum_{i\in I} \gamma_i \epsilon_i \geqslant 0 - \frac{22}{56}.$$

This statement shows that $x^* = 2$ is a δ-optimal solution of the scalarized problem (3.1) with $\delta \leqslant (22/56)$.

Thus, Definition 2.3 of ϵ-proper efficiency, may produce solutions that for a vector γ some of which may be ϵ-properly efficient and some of them may not be ϵ-properly efficient for an infinite dimensional space.

By Theorem 3.2 we conclude that for each ϵ-properly efficient point there exists some vector $\gamma > 0$ such that this point is a δ-optimal solution of the scalarized problem (3.1) with $\delta \leqslant \sum_{i\in I} \gamma_i \epsilon_i$. Therefore, every ϵ-properly efficient points for the problem (5.1) can be generated by Theorem 3.2.

6 Conclusions

In this article, we have redefined the definition of ϵ-proper efficiency in an infinite dimensional space and presented a modified concept of Definition 2.3, which appears well suitable to generate efficient solutions with bounded trade-offs and to gain an appropriate theoretical property, when the number of objective functions is finite or infinite. Also, utilizing this definition, we obtained the relations between δ-optimal solutions of the scalarized problems and ϵ-properly efficient solutions of the vector minimization problem. We proved these conditions for the weighted sum problem under a kind of subconvexity assumption. The proofs of these facts in [19, 38] concerning the Definition 2.3 of ϵ-proper efficiency become invalid in an infinite dimensional space. These conditions have been proved under a convexity assumption in [13] for $\epsilon = 0$. Whereas, we utilize a weaker hypothesis in the proof of this result. We established necessary and sufficient optimality conditions via the augmented weighted Tchebycheff norm problem without any convexity assumption. The results extend the results of [18] to multiobjective problems with infinitely many criteria. The derived results for the weighted sum scalarization approach and the augmented weighted Tchebycheff approach are summarized in Tables 1 and 2, respectively.

Table 1: Summary of the weighted sum scalarization approach for IMOP (2.2)

Parameters	Conditions	Result	Reference
$\gamma > 0$	$\delta \leqslant \sum_{i\in I} \gamma_i \epsilon_i$	δ-opt. $\Rightarrow$ ϵ-eff.	Lemma 3.1
$\gamma > 0$	$\delta \leqslant \sum_{i\in I} \gamma_i \epsilon_i$	δ-opt. $\Rightarrow$ ϵ-proper eff.	Theorem 3.2
$\exists \gamma > 0$	$\delta = \sum_{i\in I} \gamma_i \epsilon_i$ and generalized subconvexity assumption	ϵ-proper eff. $\Rightarrow$ δ-opt.	Theorem 3.5

Table 2: Summary of the augmented weighted Tchebycheff approach for IMOP (2.2)

Parameters	Conditions	Result	Reference
$\lambda \geq 0$ and $\gamma > 0$	$\delta \leqslant \inf_{i\in I}\{\lambda_i \epsilon_i\} + \varphi \sum_{i\in I} \gamma_i \epsilon_i$	δ-opt. $\Rightarrow$ ϵ-eff.	Lemma 4.1
$\lambda > 0$ and $\gamma > 0$	$\delta \leqslant \inf_{i\in I}\{\lambda_i \epsilon_i\} + \varphi \sum_{i\in I} \gamma_i \epsilon_i$	δ-opt. $\Rightarrow$ ϵ-proper eff.	Theorem 4.2
$\exists \lambda \geq 0$, $\gamma > 0$ and $\varphi > 0$	$\delta = \varphi \sum_{i\in I} \gamma_i \epsilon_i + \sup_{i\in I}\{\lambda_i \epsilon_i\}$	ϵ-proper eff. $\Rightarrow$ δ-opt.	Theorem 4.3

References

[1] Adali, S., Bruch, J. C., Sloss, J. M., Sadek, I. S.: Structural control of a variable cross-section beam by distributed forces. Mech. Based Des. Struct. Mach., 16(3), 313-333 (1988)

[2] Adan, M., Novo, V.: Proper efficiency in vector optimization on real linear spaces. J. Optim. Theory Appl., 121(3), 515-540 (2004)

[3] Annamdas, K. K., Rao, S. S.: Multi-objective optimization of engineering systems using game theory and particle swarm optimization. Optim. Eng., 41(8), 737-752 (2009)

[4] Beldiman, M., Panaitescu, E., Dogaru, L.: Approximate quasi efficient solutions in multi-objective optimization. Bull. Math. Soc. Sci. Math. Roumanie Tome, 99, 109-121 (2008)

[5] Benson, H.: An improved definition of proper efficiency for vector maximization with respect to cones. J. Optim. Theory Appl., 71, 232-241 (1979)

[6] Borwein, J.M.: Proper efficient points for maximization with respect to cones. SIAM J. Control Optim. 15, 57-63 (1977)

[7] Choo, E.U., Atkins, D.R.: Proper efficiency in nonconvex multicriteria programming. Math. Operat. Res., 8(3), 467-470 (1983)

[8] Dauer, J.P., Stadler, W.: A survey of vector optimization in infinite-dimensional spaces. II. J. Optim. Theory Appl. 51(2), 205–241 (1986)

[9] Dutta, J., Vetrivel, V.: On approximate minima in vector optimization. Numer. Funct. Anal. Optim., 22(7-8), 845-859 (2001)

[10] Ehrgott, M.: Multicriteria Optimization. Springer, Berlin (2005)

[11] Eichfelder, G.: Adaptive Scalarization Methods in Multiobjective Optimization. Springer, Berlin (2008)

[12] Engau. A.: Proper efficiency and tradeoffs in multiple criteria and stochastic optimization. Math. Oper. Res., 42(1), 119-134 (2017)

[13] Engau, A.: Definition and characterization of Geoffrion proper efficiency for real vector optimization with infinitely many criteria. J. Optim. Theory Appl., 165, 439-457 (2015)

[14] Engau, A., Wiecek, M.M.: Cone characterizations of approximate solutions in real vector optimization. J. Optim. Theory Appl., 134, 499-513 (2007)

[15] Engau, A., Wiecek, M.M.: Generating ϵ-efficient solutions in multi-objective programming. Eur. J. Oper. Res., 177, 1566-1579 (2007)

[16] Gao, Y., Yang, X., Teo, K.L.: Optimality conditions for approximate solutions in vector optimization problems. J. Ind. Manag. Optim., 7, 483-496 (2011)

[17] Geoffrion, A.M.: Proper efficiency and the theory of vector maximization. J. Math. Anal. Appl., 22, 618-630 (1968)

[18] Ghaznavi-ghosoni, B.A. (Ghaznavi, M.), Khorram, E.: On approximating weakly/properly efficient solutions in multi-objective programming. Math. Comput. Model., 54, 3172–-3181 (2011)

[19] Ghaznavi-ghosoni, B.A. (Ghaznavi, M.), Khorram, E., Soleimani-damaneh, M.: Scalarization for characterization of approximate strong/ weak/ proper efficiency in multiobjective optimization. Optimization, 62(6), 703-720 (2013)

[20] Ghaznavi, M., Akbari, F., Khorram, E.: Optimality conditions via a unified direction approach for (approximate) efficiency in multiobjective optimization. Optim. Methods Softw., 36(2-3), 627-652 (2021)

[21] Ghaznavi, M.: Optimality conditions via scalarization for approximate quasi efficiency in multiobjective optimization. Filomat, 31(3), 671-680 (2017)

[22] Ginchev, I., Guerraggio A., Rocca M.: Geoffrion type characterization of higher-order properly efficient points in vector optimization. J. Math. Anal. Appl., 328(2), 780-788 (2007)

[23] Gutierrez, C., Jimenez, B., Novo, V.: A unified approach and optimality conditions for approximate solutions of vector optimization problems. SIAM J. Optim., 17, 688-710 (2006)

[24] Gutierrez, C., Jimenez, B., Novo, V.: Optimality conditions via scalarization for a new ϵ-efficiency concept in vector optimization problems. Eur. J. Oper. Res., 201(1), 11-22 (2010)

[25] Henig, M.I.: Proper efficiency with respect to cones. J. Optim. Theory Appl., 36(3), 387-407 (1982)

[26] Hozzar, B., Tohidi, G., Daneshian, B.: Two methods for determining properly efficient solutions with a minimum upper bound for trade-offs. Filomat, 33(6), 1551-1559 (2019)

[27] Ide, J., Köbis, E., Kuroiwa, D., Schöbel, A., Tammer, C.: The relationship between multi-objective robustness concepts and set-valued optimization. J. Fixed Point Theory Appl., 1-20 (2014)

[28] Jahn, J.: Vector Optimization, 2nd edn. Springer, Berlin (2011)

[29] Kaliszewski, I.: A modified weighted Tchebycheff metric for multiple objective programming. Comput. Operat. Res., 14(4), 315-323 (1987)

[30] Karimi, M., Karimi, B.: Linear and conic scalarizations for obtaining properly efficient solutions in multiobjective optimization. J. Math. Sci., 11(4), 319-325 (2017)

[31] Keeney, R.L., Raiffa, H.: Decisions with multiple objectives: preferences and value tradeoffs. Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, New York, NY (1976)

[32] Kesarwani, P., Shukla, P.K., Dutta, J., Deb, K.: Approximations for Pareto and proper Pareto solutions and their KKT conditions. Math. Methods Oper. Res., 1-26 (2022)

[33] Khaledian, K., Khorram, E., Karimi, B.: Characterizing ε-properly efficient solutions. Optim. Methods Softw., 30, 583-593 (2014)

[34] Klamroth, K., Köbis, E., Schöbel, A., Tammer, C.: A unified approach for different concepts of robustness and stochastic programming via non-linear scalarizing functionals. Optimization, 62(5), 649– 671 (2013)

[35] Kuhn, H.W., Tucker, A.W.: Nonlinear programming. In: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 481-492. University of California Press, Berkeley (1951)

[36] Kutateladze, S.S.: Convex ϵ-programming. Sov. Math. Dokl., 20, 391-393 (1979)

[37] Lee, G.M., Kim, G.S., Dinh, N.: Optimality conditions for approximate solutions of convex semi-infinite vector optimization problems. In: Ansari, Q.H., Yao, J.-C. (eds.) Recent Developments in Vector Optimization, Vector Optimization, vol. 1, pp. 275–295. Springer, Berlin (2012)

[38] Liu, J.C.: ϵ-Properly efficient solutions to nondifferentiable multi-objective programming problems. Appl. Math. Lett., 12, 109-113 (1999)

[39] Li, Z. Wang, S.: ϵ-Approximation solutions in multiobjective optimization. Optimization, 44, 161-174 (1998)

[40] Loridan, P.: ε-Solutions in vector minimization problems. J. Optim. Theory Appl., 43, 265-276 (1984)

[41] Ogryczak, W: Multiple criteria optimization and decisions under risk. Control Cybern., 31, 975-1003 (2002)

[42] Pourkarimi, L., Soleimani-Damaneh, M.: Existence, Proper Pareto reducibility, and connectedness of the nondominated set in multi-objective optimization, J. Nonlinear Convex Anal., 19(7), 1287-1295 (2018)

[43] Qiu, Q., Yang, X.: Some properties of approximate solutions for vector optimization problem with set-valued functions. J. Global Optim., 47, 1-12 (2010)

[44] Rastegar, N., Khorram, E.: A combined scalarizing method for multiobjective programming problems. Eur. J. Oper. Res., 236(1), 229-237 (2014)

[45] Shao, L., Ehrgott, M.: Approximately solving multiobjective linear programmes in objective space and an application in radiotherapy treatment planning. Math. Methods Oper. Res., 68, 257-276 (2008)

[46] Shao, L., Ehrgott, M.: Approximating the nondominated set of an MOLP by approximately solving its dual problem. Math. Methods Oper. Res., 68, 469-492 (2008)

[47] Shitkovskaya, T., Kim, D.S.: ϵ-efficient solutions in semi-infinite multiobjective optimization. RAIRO Oper. Res., 52, 1397–1410 (2018)

[48] Shukla, P.K., Dutta, J., Deb, K., Kesarwani, P.: On a practical notion of Geoffrion proper optimality in multicriteria optimization. Optimization, 1-27 (2019)

[49] Steuer, R.E., Choo, E.U.: An interactive weighted Tchebycheff procedure for multiple objective programming. Math. Program., 26, 326-344 (1983)

[50] Steuer, R.E.: Multiple Criteria Optimization. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. Wiley, New York (1986)

[51] Steuer, R. E. Na., P.: Multiple criteria decision making combined with finance: a categorized bibliographic study. Eur. J. Oper. Res., 150(3), 496-515 (2003)

[52] Yang, X.: Alternative theorems and optimality conditions with weakened convexity. Opscarch, 29, 125-135 (1992)

[53] Yang, X.: Generalized subconvexlike functions and multiple objective optimization. Syst. sci. math. sci., 8(6), 254-259 (1995)

[54] Winkler, K.: Geoffrion proper efficiency in an infinite dimensional space. Optimization, 53(4), 355-368 (2004)

[55] Zarepisheh, M., Pardalos, P.M.: An equivalent transformation of multi-objective optimization problems. Ann. Oper. Res., 249, 5-15 (2017)

[56] Zhao, K.Q., Yang, X.M.: E-Benson proper efficiency in vector optimization. Optimization. 64(4), 739-752 (2015).