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CENTRALIZED RESOURCE ALLOCATION TO CREATE NEW MOST PRODUCTIVE SCALE SIZE (MPSS) DMUS

KAMYAR NOJOUMI[1], SABER SAATI[2] AND LEILA KHOSHANDAM[3]

Abstract. Data envelopment analysis (DEA) is a mathematical programming - based technique to evaluate the performance of a homogeneous group of decision-making units (DMUs) with multiple inputs and outputs. One of the DEA applications involves aggregating input resources and reallocating them to create efficient DMUs.The present study employs the centralized resource allocation (CRA) approach to develop a model for creating new DMUs. These new DMUs are the most productive scale size (MPSS), and all new DMUs lie on a strong supporting hyperplane. In this case, a dual model is used to obtain the strong supporting hyperplane which all new DMUs lie on. This hyperplane is derived by solving the dual model and generating a common set of weights. Then, it is shown that all new DMUs lie on a strong supporting hyperplane, and an MPSS facet is the intersection of this hyperplane with the production possibility set (PPS).

Keywords: Data envelopment analysis, Centralized resource allocation, Most productive scale size

Mathematics Subject Classification. ???, ???

INTRODUCTION

Data envelopment analysis (DEA) is a non-parametric approach to evaluate the relative performance of a set of DMUs. It consumes multiple inputs to produce multiple outputs. Farrell [1] first introduced a method to assess the performance of DMUs. One of the substantial shortcomings of the Farrell method was related to consideronly one output. Charnes et al. [2] developed Farrell's work for the case of multiple inputs and outputs.

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[1] Department of Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran; e-mail: kamyar.nojoumi@gmail.com

[2] Department of Mathematics, North Tehran Branch, Islamic Azad University, Tehran, Iran; e-mail: s_saatim@iau-tnb.ac.ir

[3] Department of Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran; e-mail: L.khoshandam@lziau.ac.ir

Their work, called the CCR model, introduced the data envelopment analysis concept. Then, Banker et al. [3] extended the CCR model by incorporating the concept of variable returns to scale and thus introduced the BCC model. Afterward, DEA was widely used in different performance evaluation problems, and various models were developed to measure the efficiency and assess the performance of DMUs. Several researchers have also proposed ways to improve the efficiency of inefficient DMUs. In this regard, Golany et al. [4–6] developed the centralized resource allocation (CRA) approach for achieving this purpose. Then, Lozano and Villa [7–9] suggested a radial and non-radial model for CRA. In this approach, the total input consumed by all DMUs is reallocated to them, so that all DMUs become efficient. Also, it is assumed that DMUs are controlled by a centralized common decision-maker to manage inputs of all DMUs and reallocate them to DMUs. A point related to the CRA is an equitable allocation of the resources in which the decision-maker tends to allocate resources or fixed costs to the controlled DMUs fairly.

In addition, Cook and Kress [10] proposed a DEA method for the equitable allocation of fixed costs based on the two principles of invariance and Pareto-optimality. Athanassopoulos [11] suggested a technique based on the DEA and goal programming method. In this study, some bounds of the total consumption have been estimated for each input using a series of LP models. Although these bounds may not be reachable, their deviation can be minimized to find a resource reallocation. Then, Athanassopoulos [12] developed a goal programming approach using multiplier DEA models. In this case, the weights were pre-calculated by an additive model to prevent the model's nonlinearity. Beasley [13] proposed a non-linear model for allocating resources. Based on this study, the input and output levels of DMUs have been determined to maximize the average efficiency. Thus, a non-linear fractional model was developed considering some bounds for inputs and outputs. However, the deficiency of the proposed method was related to the issue that the projection point of DMUs was not necessarily at the efficient frontier. Asmild et al. [14] recommended a model for CRA based on the BCC model. This model applies the process only for inefficient DMUs, and thus efficient and inefficient DMUs are first supposed to be determined.

Cook and Zhu [15] extended the approach proposed by Cook and Kress [10] and found a practical method for CRA problems. Jahanshahloo et al. [16] evolved a method for the CRA problem and obtained a solution without solving a linear programming problem and only with a simple formula. This approach was based on the invariance principle presented by Cook and Kress [10]. Korhonen et al. [17] suggested a multi-objective linear programming method for the CRA problem. In this case, it was assumed that a central unit controlled all DMUs simultaneously. Toloo [18] demonstrated the role of non-Archimedean epsilon in determining the most-efficient DMU. Also, Hatami-Marbini and Toloo [19] proposed several models for the ratio data. Toloo et al. [20] published an article entitled "Robust optimization and its duality in data envelopment analysis". Also, Salahi et al. [21] published an article entitled "A new robust optimization approach to common weights formulation in DEA". Salahi et al. [22] published an article entitled "Robust Russell and enhanced Russell measures in DEA".

Hadi-Vencheh et al. [23] proposed a CRA approach using inverse DEA. In addition, the authors can recommend other studies on CRA for further review, such as Fang [24] and Tone [25]. DU et al. [26] used the cross-efficiency concept in DEA to address resource allocation problems. The literature survey showed that the most significant shortcoming of the previous studies ignored the growing potential of the DMU, while it plays a crucial role in allocating extra resources. Indeed, the extra-resource allocation to a DMU must affect its output. Amirteimori and Kordrostami [27] developed a method for implementing demand and supply changes in a centralized decision-making process. Finally, Wu et al. [28] suggested a CRA model based on MOLP considering desirable and undesirable inputs and outputs. Another effort to cope with the CRA problem was performed by Zhang et al. [29]. In this study, a linear model with bounded variables was proposed for extra resource allocation. Tao et al. [30] employed the theories of network flows to examine CRA for all individual units in a DEA framework. Zhang et al. [31] suggested a two-stage scheme for CRA and distributed power control regarding NR V2X integrated with NOMA technology. Fang et al. [32] introduced the Nash bargaining game theory to develop a new CRA model. The authors considered the overall goals of the organization and competition among DMUs.

Also, many researchers have investigated the development and management of centralized resource reallocation as one of the classic research topics in the management and economy sciences. In particular, Managers strive to maximize productivity by consuming rare and expensive input resources and preventing waste of these valuable resources. In this regard, the present study proposes a new approach to obtaining new units with maximum productivity through the reallocation of centralized resources. Thus, a fractional model is first developed to derive the most productive scale size (MPSS) units. Then, these are converted into a linear model by transforming the Charnes-Cooper variable. Afterward, a dual model is solved, and it was shown that all new units are located on the BCC-CCR joint frontiers. The proposed model has several advantages over other centralized resource reallocation models. First, the new units can be obtained by solving only one linear and non-radial model. Second, the new units are all MPSS ones. By consuming valuable and expensive equipment, it is especially important to achieve maximum productivity. This paper is organized into five sections. Section 1 includes an introduction to DEA and a literature review. Then, Section 2 presents the basic definitions and proposed models to identify MPSS units and CRA schemes. In Section 3, the proposed CRA model aims to constitute new MPSS units. Also, this section discusses some properties of the dual model. Section 4 gives some numerical examples. Finally, Section 5 represents conclusions and suggestions for future research.

1. Preliminaries

1.1. Production Possibility Set and MPSS in DEA

Consider n DMUs with coordinates $(\mathbf{x}_j, \mathbf{y}_j)$ in which the jth DMU consumes m inputs as the components of the vector $\mathbf{x}_j = (x_{1j}, .., x_{mj})$ to produce s outputs as the components of the vector $\mathbf{y}_j = (y_{1j}, .., y_{sj})$. Hence, the PPS can be defined as $T = \{\,(\mathbf{x}, \mathbf{y})\,|\mathbf{x}\ can\ produce\ \mathbf{y}\}$.

The PPS with constant returns to scale is as follows:

$$T_C = \left\{ (\mathbf{x},\mathbf{y}) \mid \sum_{j=1}^{n} \lambda_j \mathbf{x}_j \leq \mathbf{x},\ \sum_{j=1}^{n} \lambda_j \mathbf{y}_j \geq \mathbf{y},\ \lambda_j \geq 0 \qquad j = 1, 2, ..., n \right\}$$

Also, the PPS for the case of variable returns to scale is as follows:

$$T_V = \left\{ (\mathbf{x},\mathbf{y}) \mid \sum_{j=1}^{n} \lambda_j \mathbf{x}_j \leq \mathbf{x},\ \sum_{j=1}^{n} \lambda_j \mathbf{y}_j \geq \mathbf{y},\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0 \qquad j = 1, 2, ..., n \right\}$$

Fare, Groskopf, and Lovell [33] and Cooper et al. [34] introduced non-increasing returns to scale (NIRS) and non-decreasing returns to scale (NDRS) models and called FG and ST models, respectively. DMUs can be assessed by different DEA models. This procedure is accomplished using various production technologies discussed above. Recently, the CCR and BCC models have broadly been used by researchers. The CCR and BCC models for evaluating DMU_p are as below, respectively:

$$\begin{array}{ll} \min & \theta - \varepsilon \left(\mathbf{1t}^- + \mathbf{1t}^+ \right) \\ \text{s.t} & \sum_{j=1}^{n} \lambda_j \mathbf{x}_j + \mathbf{t}^- = \theta \mathbf{x}_p \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}_j - \mathbf{t}^+ = \mathbf{y}_p \\ & \lambda_j \geq 0 \quad \forall j, \quad \mathbf{t}^- \geq \mathbf{0}, \quad \mathbf{t}^+ \geq \mathbf{0} \end{array} \tag{1}$$

$$\begin{array}{ll} \min & \theta - \varepsilon \left(\mathbf{1s}^- + \mathbf{1s}^+ \right) \\ \text{s.t} & \sum_{j=1}^{n} \lambda_j \mathbf{x}_j + \mathbf{s}^- = \theta \mathbf{x}_p \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}_j - \mathbf{s}^+ = \mathbf{y}_p \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0 \quad \forall j, \quad \mathbf{s}^- \geq \mathbf{0}, \quad \mathbf{s}^+ \geq \mathbf{0} \end{array} \tag{2}$$

where t^- and $t^+$are slack variables for input and output CCR models. Also $\mathbf{s}^-$ and $\mathbf{s}^+$ are slack variables for input and output BCC models, respectively.

In this research, the symbol $\varepsilon > 0$ utilized in models (1) and (2) signifies that the variable θ is minimized in the first stage, and then the sum of slack variables is maximized in the second stage while assuming $\theta = \theta^*$, where θ^* is the optimal value in the first stage.

Assume that (θ^*, λ^*,$\mathbf{s}^{-*}$,$\mathbf{s}^{+*}$) is an optimal solution to model (2). Then, DMU_p is efficient if and only if the following conditions are satisfied in each optimal solution:

$$i) \quad \theta^* = 1 \qquad ii \quad) \, \mathbf{s}^{-*} = \mathbf{0}, \ \ \mathbf{s}^{+*} = \mathbf{0}$$

Nowadays, analysis of economic scales is of particular importance. The MPSS concept in DEA was first produced by Banker [35]. The MPSS region shows that points in PPS whose ratio of their output to input are the highest values. Also, DMUs in the MPSS region have the best size. Due to variations in rare, expensive, or valuable inputs (e.g., large equipment), these DMUs are important to optimize outputs to inputs ratios.

Definition 1.1. *The unit* $(\mathbf{x}_p,\mathbf{y}_p) \in T_V$ *is an MPSS if and only if:*

$$\forall\alpha,\quad \forall\beta,\quad \alpha > 0,\quad \beta > 0,\quad (\alpha\mathbf{x}_p,\quad \beta\mathbf{y}_p) \in T_V \Rightarrow \frac{\beta}{\alpha} \leq 1$$

.

Based on the above definition, Banker [36] proposed the following model to identify whether or not the point $(\mathbf{x}_p,\mathbf{y}_p)$ is an MPSS:

$$\begin{array}{ll} max & \frac{\beta}{\alpha} \\ \text{s.t} & \sum\limits_{j=1}^{n} \lambda_j \mathbf{x}_j + \mathbf{s}^- = \alpha\mathbf{x}_p \\ & \sum\limits_{j=1}^{n} \lambda_j \mathbf{y}_j - \mathbf{s}^+ = \beta\mathbf{y}_p \\ & \sum\limits_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0 \forall j, \mathbf{s}^- \geq \mathbf{0}, \quad \mathbf{s}^+ \geq \mathbf{0} \end{array} \tag{3}$$

Assume that $(\alpha^*, \beta^*, \lambda^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$ is an optimal solution to model (3). Then, the unit $(\mathbf{x}_p,\mathbf{y}_p)$ is an MPSS if and only if the following conditions are satisfied in each optimal solution:

$$i)\, \alpha^* = \beta^* \qquad ii)\, \mathbf{s}^{-^*} = \mathbf{0},\;\; \mathbf{s}^{+^*} = \mathbf{0}$$

Consider the following output-oriented CCR model to evaluate DMU_p:

$$\begin{array}{ll} max & \varphi + \varepsilon\left(1\,\mathrm{s}^- + 1\,\mathrm{s}^+\right) \\ \text{s.t} & \sum_{j=1}^{n} \lambda_j \mathbf{x}_j + \mathbf{s}^- = \mathbf{x}_p \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}_j - \mathbf{s}^+ = \varphi\mathbf{y}_p \\ & \lambda_j \geq 0 \quad \forall j, \quad \mathbf{s}^- \geq \mathbf{0}, \;\; \mathbf{s}^+ \geq \mathbf{0} \end{array} \tag{4}$$

Assume that $(\varphi^*, \lambda^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$ is an optimal solution to model (4). Let:

$$\hat{\mathbf{x}}_p = \frac{\mathbf{x}_p - \mathbf{s}^{-*}}{\hat{\lambda}} \quad , \quad \hat{\mathbf{y}}_p = \frac{\varphi^*\mathbf{y}_p + \mathbf{s}^{+*}}{\hat{\lambda}}$$

where $\hat{\lambda} = \sum\limits_{j=1}^{n} \lambda_j^*$. Then, $(\hat{\mathbf{x}}_p, \hat{\mathbf{y}}_p) \in \partial T_V \cap \partial T_C$ so that $\partial T_V \cap \partial T_C$ is the intersection between the T_V and T_C boundaries. Figure 1 shows that Model (4) projects DMU $(\mathbf{x}_p,\mathbf{y}_p)$. It is first performed to an efficient point $(\mathbf{x}_p - \mathbf{s}^{-*}, \varphi^*\mathbf{y}_p + \mathbf{s}^{+*})$ in T_C, and then it is accomplished to an MPSS point according to equations $\hat{\mathbf{x}}_p = \frac{\mathbf{x}_p - \mathbf{s}^{-*}}{\hat{\lambda}}$ and $\hat{\mathbf{y}}_p = \frac{\varphi^*\mathbf{y}_p + \mathbf{s}^{+*}}{\hat{\lambda}}$.

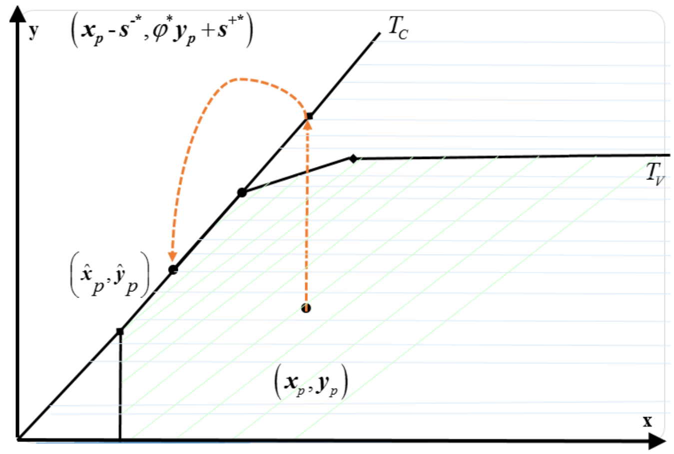

FIGURE 1. Projection $(\mathbf{x}_p,\mathbf{y}_p)$to the Intersection of BCC and CCR Frontiers

Definition 1.2. *Unit $(\mathbf{x}_p,\mathbf{y}_p) \in T_V$ is an MPSS only if $(\mathbf{x}_p,\mathbf{y}_p) \in \partial T_V \cap \partial T_C$. Jahanshahloo and Khodabakhshi [37] proposed the following model to identify MPSS DMUs:*

$$\begin{array}{ll} \max & \varphi - \theta \\ \text{s.t} & \sum_{j=1}^{n} \lambda_j \mathbf{x}_j + \mathbf{s}^- = \theta \mathbf{x}_p \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}_j - \mathbf{s}^+ = \varphi \mathbf{y}_p \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0 \quad \forall j, \quad \mathbf{s}^- \geq \mathbf{0}, \quad \mathbf{s}^+ \geq \mathbf{0} \end{array} \tag{5}$$

Assume that $(\theta^*, \varphi^*, \lambda^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$ is an optimal solution to model (5). Then, the unit $(\mathbf{x}_p,\mathbf{y}_p)$ is an MPSS if and only if the following conditions are satisfied in each optimal solution:

$$i)\, \theta^* = \varphi^* \qquad ii)\, \mathbf{s^-}^* = \mathbf{0}, \;\; \mathbf{s^+}^* = \mathbf{0}.$$

1.2. RESOURCE ALLOCATION PROBLEM

Consider a situation in which an organization consists of homogeneous units, and each unit consumes several input resources to produce output resources. If the organization faces extra resources in the subsequent period of activity, it requires reallocating input resources to constitute new efficient units. In this regard, Lozano and Villa [7] first addressed the centralized resource allocation (CRA) problem, and their model became a basis for subsequent research on CRA. They aggregated inputs and outputs as $\sum\limits_{j=1}^{n} x_{ij}$ $(i = 1, 2, ..., m)$ and $\sum\limits_{j=1}^{n} y_{rj}$ $(r = 1, 2, ..., s)$, respectively. Then, the authors proposed the following radial model:

$$\begin{array}{lll} \min & \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) & \\ \text{s.t} & \sum_{k=1}^{n}\sum_{j=1}^{n} \lambda_{kj} x_{ij} + s_i^- = \theta \sum_{j=1}^{n} x_{ij} & i = 1,2,\dots,m \\ & \sum_{k=1}^{n}\sum_{j=1}^{n} \lambda_{kj} y_{rj} - s_r^+ = \sum_{j=1}^{n} y_{rj} & r = 1,2,\dots,s \\ & \sum_{j=1}^{n} \lambda_{kj} = 1 & \\ & \lambda_{kj} \geq 0, & \\ & s_i^- \geq 0, s_r^+ \geq 0 & \forall i,r,j,k \end{array} \tag{6}$$

Here, $\mathbf{s}_i^-$ and $\mathbf{s}_r^+$ are slack variables for aggregated inputs and outputs, respectively. Also, ε is the non-Archimedean infinitesimal. In addition, $\lambda_{kj}$is the proportion of new DMU_k from the old DMU_j. Now, if it is assumed that $(\theta^*, \lambda^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$ is an optimal solution to model (5), the projection points represent n new efficient DMUs, which are formulated as follows:

$$\mathbf{x}_k^* = \sum_{j=1}^{n} \lambda_{kj}^* \mathbf{x}_j, \quad \mathbf{y}_k^* = \sum_{j=1}^{n} \lambda_{kj}^* \mathbf{y}_j \qquad k = 1,2,...,n$$

2. Proposed model

Consider n DMUs with coordinates $(\mathbf{x}_j, \mathbf{y}_j)$, where:

$$\begin{array}{lll} \mathbf{x}_j \in \mathbb{R}^m, & \mathbf{x}_j > 0 & j = 1,2,\dots,n \\ \mathbf{y}_j \in \mathbb{R}^s, & \mathbf{y}_j > 0 & j = 1,2,\dots,n \end{array}$$

The inputs and outputs are aggregated as $\sum_{j=1}^{n} x_{ij}$ $(i = 1,2,...,m)$ and $\sum_{j=1}^{n} y_{rj}$ $(r = 1,2,...,s)$, respectively. Then, the inputs and outputs are reallocated using the proposed model so that the new DMUs are MPSS. The new MPSS DMUs are obtained by defining the decision variable λ_{kj} and the proportion of new DMU $(\bar{\mathbf{x}}_k, \bar{\mathbf{y}}_k)$ from the old unit $(\mathbf{x}_j, \mathbf{y}_j)$. Thus, $\sum_{j=1}^{n} \lambda_{kj} = 1$ is supposed to be fulfilled for each k. Now, the following fractional model is solved to obtain new MPSS DMUs.

$$\begin{array}{lll} P^* = & \min \frac{\frac{1}{m}\sum_{i=1}^{m} \theta_i}{\frac{1}{s}\sum_{r=1}^{s} \varphi_r} & \\ \text{s.t} & \sum_{k=1}^{n}\sum_{j=1}^{n} \lambda_{ij} x_{ij} \leq \theta_i \sum_{j=1}^{n} x_{ij} & i = 1,2,\dots,m \\ & \sum_{k=1}^{n}\sum_{j=1}^{n} \lambda_{kj} y_{rj} \geq \varphi_r \sum_{j=1}^{n} y_{rj} & r = 1,2,\dots,s \\ & \sum_{j=1}^{n} \lambda_{kj} = 1 & k = 1,2,\dots,n \\ & \lambda_{kj} \geq 0, \quad \theta_i \geq 0, \quad \varphi_r \geq 0 & \forall i,r,j,k \end{array} \tag{7}$$

Theorem 2.1. *Model (7) is always feasible, and $P^* \leq 1$.*

Proof. See Appendix. □

Theorem 2.2. *All input and output constraints are binding in each optimal solution to model (7).*

Proof. See Appendix. □

Definition 2.3. *Assume that $(\lambda^*, \theta^*, \varphi^*)$ is an optimal solution to model (7). Then, the coordinates of new DMUs, $(\bar{\mathbf{x}}_k, \bar{\mathbf{y}}_k)$ $(k = 1, 2, ..., n)$, is as follows:*

$$\bar{\mathbf{x}}_k = \sum_{j=1}^{n} \lambda^*_{kj} \mathbf{x}_j, \bar{\mathbf{y}}_k = \sum_{j=1}^{n} \lambda^*_{kj} \mathbf{y}_j \qquad k = 1, 2, ..., n.$$

Afterward, it is proved that these new DMUs are MPSS. In this regard, model (7) is first linearized using the Charnes-Cooper transformation. We set $\frac{1}{s} \sum_{r=1}^{s} \varphi_r = \frac{1}{t}$ Therefore, model (8) can be transformed into the following formulation:

$$\begin{array}{lll} \min & \frac{1}{m} \sum_{i=1}^{m} \hat{\theta}_i & \\ \text{s.t} & \sum_{k=1}^{n} \sum_{j=1}^{n} \mu_{kj} x_{ij} \leq \hat{\theta}_i \sum_{j=1}^{n} x_{ij} & i = 1, 2, \ldots, m \\ & \sum_{k=1}^{n} \sum_{j=1}^{n} \mu_{kj} y_{rj} \geq \hat{\varphi}_r \sum_{j=1}^{n} y_{rj} & r = 1, 2, \ldots, s \\ & \frac{1}{s} \sum_{r=1}^{s} \hat{\varphi}_r = 1 & \\ & \sum_{j=1}^{n} \mu_{kj} = t & k = 1, 2, \ldots, n \\ & \mu_{kj} \geq 0, \quad \hat{\theta}_i \geq 0, \quad \hat{\varphi}_r \geq 0 & \forall i, j, k, r \end{array} \tag{8}$$

Here, $\hat{\varphi}_r = t\varphi_r \forall r, \;\; \hat{\theta}_i = t\theta_i \forall i, \;\; \mu_{kj} = t\lambda_{kj} \forall k, \; j.$

Theorem 2.4. *Assume that $\left(\mu^*, \hat{\theta}^*, \hat{\varphi}^*, t^*\right)$ is an optimal solution to model (8). Then, $t^* > 0$.*

Proof. See Appendix. □

Theorem 2.5. *If $(\lambda^*, \theta^*, \varphi^*, P^*)$ is an optimal solution to model (7), then*

$$\begin{array}{ll} A)\, \theta^*_i > 0 & i = 1, 2, ..., m \\ B)\, \varphi^*_r > 0 & r = 1, 2, ..., s \end{array}$$

Now, the dual expression of the model (8) can be given as follows:

$$\begin{array}{llll} \max & w & & \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + d_k \leq 0 & k = 1, 2, \ldots, n & j = 1, 2, \ldots, n \\ \text{s.t} & v_i \sum_{j=1}^{n} x_{ij} \leq \frac{1}{m} & i = 1, 2, \ldots, m & \\ & u_r \sum_{j=1}^{n} y_{rj} \geq \frac{w}{s} & r = 1, 2, \ldots, s & \\ & u_r, v_i, d_k \geq 0, \quad \forall i, k, r, \quad w & \text{free} & \end{array} \tag{9}$$

Now, if $\left(\mu^*, \hat{\theta}^*, \hat{\varphi}^*, t^*\right)$ is the optimal solution to model (8), considering $t^* > 0$, then the optimal solution to model (7) is as follows.

$$\theta_i^* = \frac{\hat{\theta}_i^*}{t^*} \quad \forall i, \quad \lambda_{kj}^* = \frac{\mu_{kj}^*}{t^*} \quad \forall k, j, \quad \varphi_r^* = \frac{\hat{\varphi}_r^*}{t^*} \quad \forall r$$

The coordinates of the new DMUs are as follows.

$$\begin{cases} \bar{\mathbf{x}}_k = \sum_{j=1}^{n} \lambda_{kj}^* \mathbf{x}_j \\ \bar{\mathbf{y}}_k = \sum_{j=1}^{n} \lambda_{kj}^* \mathbf{y}_j \end{cases} \qquad k = 1, 2, ..., n \qquad (a)$$

Proof. See Appendix. □

Theorem 2.6. *If* $(u^*, v^*, w^*, \mathbf{d}^*)$ *is an optimal solution to model* (9)*, it is concluded that* $\mathbf{d}^* = \mathbf{0}$.

Proof. See Appendix. □

Theorem 2.7. *Suppose that* $(u^*, v^*, w^*, \mathbf{d}^*)$ *is the optimal solution to model* (9)*, then is the* $H = \{(\mathbf{x}, \mathbf{y}) \,|u^*\mathbf{y} - v^*\mathbf{x} \leq \mathbf{0}\}$ *supporting hyperplane of* T_V.

Proof. See Appendix. □

Theorem 2.8. *All new DMUs* (a) *lie on the supporting hyperplane* $H = \{(\mathbf{x}, \mathbf{y}) \,|u^*\mathbf{y} - v^*\mathbf{x} \leq \mathbf{0}\}$, *which* (u^*, v^*) *represents the optimal solution to model* (9).

Proof. See Appendix. □

Corollary 2.9. *All new DMUs* (a) *are MPSS.*

3. Numerical Examples

Example 1

A set of DMUs is considered with a single input and output. This set of DMUs is presented in Table 1. In Table 1, $\theta_{CCR}$and θ_{BCC} are efficiency scores obtained from the input-oriented CCR and BCC models, respectively.

Table 1 provides 10 DMUs denoted as A to J. DMUs of A, B, F, H, and I are BCC efficient, and DMUs of F and H are CCR efficient. Therefore, DMUs of F and H and all points lying on line segment FH are MPSS.

According to Table 2, the approach proposed by Lozano and Villa [7] projects all DMUs onto DMU_B. Figure 2 depicts the projection of all DMUs with Lozano and Villa's model. Also,DMU_B is BCC efficient, but it is not CCR efficient. Therefore, $DMU_B \notin \partial T_V \cap \partial T_C$. This situation demonstrated that Lozano and Villa's approach could not provide MPSS projections for DMUs.

TABLE 1. Data of the Original DMUs (Zhang et al. [29])

DMU	A	B	C	D	E	F	G	H	I	J
x	3	3.5	4	4.5	4.5	5	5.5	6	8	9
y	1	2.5	2.4	2.7	3.6	5	4.4	6	7	7
θ_{CCR}	0.333	0.714	0.6	0.6	0.8	1	0.8	1	0.875	0.778
θ_{BCC}	1	1	0.867	0.804	0.924	1	0.844	1	1	0.889

TABLE 2. Data of New DMUs Generated From the Models Proposed by Lozano and Villa [7], Zhang et al. [29], and the Proposed Approach

	Lozano and Villa's approach		Zhang et al.'s approach		Proposed approach	
DMU	x'	y'	x''	y''	x	y
A	3.5	2.5	5.33	5.33	6	6
B	3.5	2.5	5.25	5.25	6	6
C	3.5	2.5	5.41	3.90	6	6
D	3.5	2.5	5.43	3.52	6	6
E	3.5	2.5	5.32	4.60	6	6
F	3.5	2.5	5.24	5.24	6	6
G	3.5	2.5	5.31	4.25	6	6
H	3.5	2.5	5.22	5.22	6	6
I	3.5	2.5	5.25	5.25	6	6
J	3.5	2.5	5.37	5.37	6	6

Also, the proposed model by Zhang et al. [29] projects the efficient DMUs of A, B, F, H, and I and the weak efficient DMU_J onto the MPSS facet. But it does not project inefficient DMUs of C, D, E, and G onto the frontier of the MPSS region. Figure 3 shows the projection of all DMUs using Zhang et al.'s model.

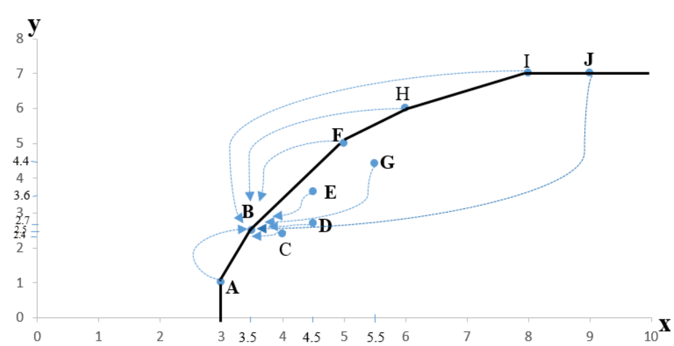

FIGURE 2. Distribution of the New DMUs Generated From Lozano and Villa's Model

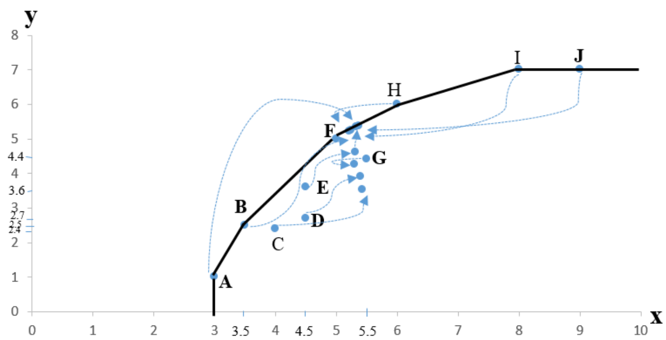

FIGURE 3. Distribution of the New DMUs Generated From Zhang et al.'s Model

As shown in Figure 4, the proposed model in this research (model 8) projects all DMUs to point H. Also, this figure indicates that $DMU_H \in \partial T_V \cap \partial T_C$, and thus DMU_H is an MPSS point.

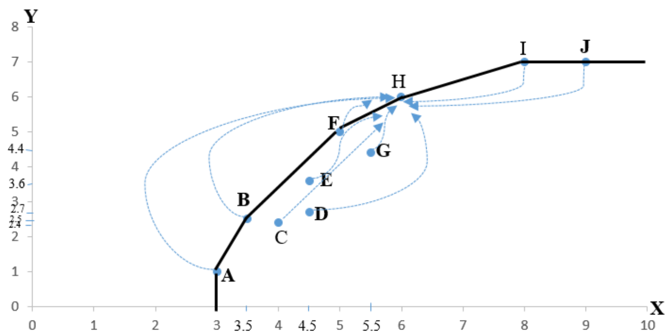

FIGURE 4. Distribution of the New DMUs Generated by the Proposed Approach

The optimal solution of model (9) with the DMUs of Table 1 is expressed as follows:

$$u^* = 0.019 \quad , \quad v^* = 0.019 \quad , \quad d_k^* = 0 \quad k = 1, 2, ..., 10$$

Therefore, the strong supporting hyperplane including new DMUs with the proposed approach is given by the following equation: $H = \{(x, y) \mid 0.019y - 0.019x = 0\}$.

EXAMPLES 2

Consider two DMUs with two inputs and a single output (Lozano and Villa [7]) as presented in Table 3.

TABLE 3. Data of DMUs (Lozano and Villa [7])

DMU	A	B	C	D	E	F	G	H	I	J
x_1	9	12	7	6	10	8	12	14	12	8
x_2	9	8	12	10	5	10	10	6	12	8
y_1	2	3	2	5	4	3	6	8	1	3
y_2	1	1	2	3	4	3	6	2	6	5

The results of the CCR and BCC models for DMUs in Table 3 are presented in Table 4.

TABLE 4. Optimal Solutions of CCR and BCC Models

DMU	A	B	C	D	E	F	G	H	I	J
θ_{CCR}	0.33	0.402	0.5	1	1	0.663	1	1	0.8	1
θ_{BCC}	0.86	0.76	0.85	1	1	0.87	1	1	0.9	1

According to the solutions of CCR and BCC models, it is implied that DMUs D, E, G, H, and J are CCR and BCC efficient. This shows that $\{D, E, G, H, J\} \in \partial T_C \cap \partial T_V$. Thus, by Definition 2, DMUs D, E, G, H, and J are MPSS. Now, solving Model (6) proposed by Lozano and Villa [7] and Model (8) for DMUs in Table 3, we obtain new DMUs according to Table 5.

TABLE 5. Data of New DMUs Generated From the Models Proposed by Lozano and Villa [7], Zhang et al. [29], and the Proposed Approach

	Lozano and Villa's approach				Proposed approach			
DMU	x_1	x_2	y_1	y_2	x_1	x_2	y_1	y_2
	6	10	5	3	10	5	4	4
	6.71	9.12	4.82	3.18	10	5	4	4
	6	10	5	3	10	5	4	4
	6	10	5	3	10	5	4	4
	6	10	5	3	10	5	4	4
	10	5	4	4	10	5	4	4
	10	5	4	4	10	5	4	4
	10	5	4	4	10	5	4	4
	10	5	4	4	10	5	4	4
	10	5	4	4	10	5	4	4
Total	80.71	74.12	44.82	35.18	100	50	40	40

As seen, Lozano and Villa's model [7] projects DMU B to the point $(\bar{x}, \bar{y} = (6.71, 9.12, 4.82, 3.18)$, which is BCC efficient but CCR inefficient. Therefore, $(\bar{x}, \bar{y}) \notin \partial T_C \cap \partial T_V$, and, by Definition 2, the point $(\bar{x}, \bar{y})$ is not an MPSS, while the new DMUs built by the proposed approach are not MPSS. As a result, Lozano and Villa's model [7] cannot construct MPSS units, whereas the proposed approach provides new MPSS units.

4. Conclusions

In the present study, a new method has been proposed for the centralized resource reallocation to construct new MPSS units. Since the input indices include rare and high-value equipment and resources, it is necessary to achieve maximum productivity by reallocating and re-aggregating these resources. This situation prevents wasting these resources and makes the best use of the potential of these resources. Thus, the re-aggregation and reallocation of the centralized resources have been performed to construct new MPSS units. In this regard, we proposed a method that yielded new MPSS units that would be homogeneous. Afterward, a dual model was used to show that all the newly constructed units were located at the CCR-BCC joint frontiers. Also, the joint boundary between CCR and BCC included the points with maximum productivity. The proposed model has several advantages over conventional models, including (I) the new efficient units can be obtained by solving only one linear and non-radial model and (II) unlike other prevalent used centralized resource reallocation models, all the new units are homogeneous MPSS ones. Therefore, these units can provide the ground for full utilization of the available potential of the inputs.

In this research, two numerical examples were presented. The first example, including a single input and a single output and taken from Zhang et al. [29], aimed to compare the proposed approach with other centralized resource reallocation approaches. Because the sum of inputs for the new MPSS units determined by Model (8) is greater than that of DMUs, future studies may consider centralized resource reallocation focusing on the lowest or highest point on the MPSS facet.

Also, the second example, with two inputs and two outputs taken from Lozano and Villa [7], cannot construct MPSS units, while the proposed approach can to provide new MPSS units using centralized resource reallocation.

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APPENDIX

PROOF OF THEOREM 1

Let $\bar{\theta}_i = 1$ $(i = 1, 2, ..., n)$ and $\bar{\varphi}_r = 1$ $(r = 1, 2, ..., s)$, and assume that $\bar{\lambda}$ is the unit matrix of order $n \times n$. Then, $\left(\bar{\lambda}, \bar{\theta}, \bar{\varphi}\right)$ is a feasible solution to model (7). Further, we have:

$$\bar{P} = min \frac{\frac{1}{m}\sum_{i=1}^{m}\bar{\theta}_i}{\frac{1}{s}\sum_{r=1}^{s}\bar{\varphi}_r} = 1$$

If $(\lambda^*, \theta^*, \varphi^*, P^*)$ is an optimal solution to model (7), we can conclude that $P^* \leq \bar{P} = 1$.

PROOF OF THEOREM 2

Assume that $(\lambda^*, \theta^*, \varphi^*, P^*)$ is an optimal solution to model (7). By contradiction, assume that

$$\exists t \quad 1 \leq t \leq m \quad , \quad \sum_{k=1}^{n}\sum_{j=1}^{n}\lambda^*_{kj}x_{tj} < \theta^*_t \sum_{j=1}^{n} x_{tj}$$

Let:

$$\begin{aligned}
&\bar{\theta}_t = \frac{\sum_{k=1}^{n}\sum_{j=1}^{n}\lambda^*_{kj}x_{tj}}{\sum_{j=1}^{n}x_{tj}} \\
&\bar{\theta}_i = \theta^*_i \qquad\qquad i = 1, 2, ..., m \quad i \neq t \\
&\bar{\varphi}_r = \varphi^*_r \qquad\qquad r = 1, 2, ..., s \\
&\bar{\mu} = \lambda^*
\end{aligned}$$

Now, $\left(\bar{\mu}, \bar{\theta}, \bar{\varphi}\right)$ is a feasible solution to model (7). Since $\bar{\theta}_t < \theta_t^*$we have:

$$\bar{P} = \frac{\frac{1}{m}\sum_{i=1}^{m}\bar{\theta}_i}{\frac{1}{s}\sum_{r=1}^{s}\bar{\varphi}_r} < \frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i^*}{\frac{1}{s}\sum_{r=1}^{s}\varphi_r^*} = P^*.$$

This situation contradicts the optimality of $(\lambda^*, \theta^*, \varphi^*)$ in the model (7).

PROOF OF THEOREM 3

By contradiction, assume that $t^* = 0$. Then, it is concluded from $\sum_{j=1}^{n}\mu_{kj}^* = t^* = 0 (k = 1, 2, ..., n)$ that $\mu_{kj}^* = 0$ $(\forall k, j)$. Also, if $\sum_{k=1}^{n}\sum_{j=1}^{n}\mu_{kj}^* y_{rj} \geq \hat{\varphi}_r^* \sum_{j=1}^{n} y_{rj} (\forall r)$, we can conclude that $\hat{\varphi}_r^* \sum_{j=1}^{n} y_{rj} \leq 0$ $(\forall r)$. Since $\sum_{j=1}^{n} y_{rj} > 0 (\forall r)$ $\hat{\varphi}^*{}_r = 0$ $(\forall r)$. This issue contradicts the constraint $\frac{1}{s}\sum_{r=1}^{s}\hat{\varphi}_r^* = 1$. Thus, the proof is complete, and it is concluded that $t^* > 0$.

PROOF OF THEOREM 4

Proof (A): From the constraint $\sum_{j=1}^{n}\lambda_{kj}^* = 1$ $(\forall k)$, it is observed that $\lambda^* \neq 0$ and $\mathbf{x}_j > \mathbf{0}$ $(\forall j)$. Therefore, we have:

$$\sum_{k=1}^{n}\sum_{j=1}^{n}\lambda_{kj}^* x_{ij} \leq \theta_i^* \sum_{j=1}^{n} x_{ij} \qquad \forall i$$

$$\theta_i^* \geq \frac{\sum_{k=1}^{n}\sum_{j=1}^{n}\lambda_{kj}^* x_{ij}}{\sum_{j=1}^{n} x_{ij}} > 0 \qquad \forall i$$

Proof (B): By contradiction, assume that $\varphi_t^* = 0$ for some t $(1 \leq t \leq s)$. Let:

$$\bar{\varphi}_t = \frac{\sum_{k=1}^{n}\sum_{j=1}^{n}\lambda_{kj}^* y_{tj}}{\sum_{j=1}^{n} y_{tj}} > 0$$

$$\bar{\varphi}_r = \varphi_r^* \qquad r = 1, 2, ..., s \qquad r \neq t$$

$$\bar{\theta}_i = \theta_i^* \qquad i = 1, 2, ..., m$$

$$\bar{\lambda} = \lambda^*$$

Then, $\left(\bar{\lambda}, \bar{\theta}, \bar{\varphi}\right)$ is a feasible solution to model (7), and we have:

$$\bar{P} = \frac{\frac{1}{m}\sum\limits_{i=1}^{m}\bar{\theta}_i}{\frac{1}{s}\sum\limits_{r=1}^{s}\bar{\varphi}_r} < \frac{\frac{1}{m}\sum\limits_{i=1}^{m}\theta_i^*}{\frac{1}{s}\sum\limits_{r=1}^{s}\varphi_r^*}$$

This situation contradicts the optimality of $(\lambda^*, \theta^*, \varphi^*, P^*)$ for model (7). Thus, our claim is proved, that is, $\varphi_r^* > 0$ $(r = 1, 2, ..., s)$.

PROOF OF THEOREM 5

Suppose that $(\lambda^*, \theta^*, \varphi^*, P^*)$ is an optimal solution to model (7). Then:

$$P^* = w^* = \frac{\frac{1}{m}\sum\limits_{i=1}^{m}\theta_i^*}{\frac{1}{s}\sum\limits_{r=1}^{s}\varphi_r^*}.$$

Also,(u^*, v^*, w^*, d^*) is an optimal solution to model (9), and then we have:

$$\begin{aligned}
& u^*\mathbf{y}_j - v^*\mathbf{x}_j + d_k^* \le \mathbf{0} && k, j = 1, 2, ..., n \\
& u^*\lambda_{kj}^*\mathbf{y}_j - v^*\lambda_{kj}^*\mathbf{x}_j + d_k^*\lambda_{kj}^* \le \mathbf{0} && k, j = 1, 2, ..., n \\
& u^*\sum_{j=1}^{n}\lambda_{kj}^*\mathbf{y}_j - v^*\sum_{j=1}^{n}\lambda_{kj}^* + d_k^*\sum_{j=1}^{n}\lambda_{kj}^* \le \mathbf{0} && k = 1, 2, ..., n \\
& u^*\sum_{k=1}^{n}\sum_{j=1}^{n}\lambda_{kj}^*\mathbf{y}_j - v^*\sum_{k=1}^{n}\sum_{j=1}^{n}\lambda_{kj}^*\mathbf{x}_j + \sum_{k=1}^{n}d_k^* \le \mathbf{0} && (a) \\
& \sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{s}u_r^*\lambda_{kj}^*y_{rj} - \sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{m}v_i^*\lambda_{kj}^*x_{ij} + \sum_{k=1}^{n}d_k^* \le 0
\end{aligned}$$

Also, we have:

$$v_i^*\sum_{j=1}^{n}x_{ij} \le \frac{1}{m} i = 1, 2, ..., m \qquad \Rightarrow \qquad \theta_i^* v_i^*\sum_{j=1}^{n}x_{ij} \le \frac{\theta_i^*}{m} \qquad i = 1, 2, ..., m$$

It is proved in Theorem 1 that $\sum_{k=1}^{n}\sum_{j=1}^{n}\lambda_{kj}^{*}x_{ij} = \theta_i^{*}\sum_{j=1}^{n}x_{ij}$ $(\forall i)$. Therefore, we can get:

$$v_i^{*}\sum_{k=1}^{n}\sum_{j=1}^{n}\lambda_{kj}^{*}x_{ij} \leq \frac{\theta_i^{*}}{m} \qquad \forall i \qquad (b)$$

$$\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{m}v_i^{*}\lambda_{kj}^{*}x_{ij} \leq \frac{1}{m}\sum_{i=1}^{m}\theta_i^{*}$$

Multiplying the two sides of the inequality $u_r^{*}\sum_{j=1}^{n}y_{rj} \geq \frac{w^{*}}{s}$ $(\forall r)$ by φ_r^{*}, we have:

$$\varphi_r^{*}u_r^{*}\sum_{j=1}^{n}y_{rj} \geq \frac{w^{*}}{s}\varphi_r^{*} \qquad \forall r$$

It was proved in equation (7) that $\sum_{k=1}^{n}\sum_{j=1}^{n}\lambda_{kj}^{*}y_{rj} = \varphi_r^{*}\sum_{j=1}^{n}y_{rj}$ $(\forall r)$. Therefore, we have:

$$u_r^{*}\sum_{k=1}^{n}\sum_{j=1}^{n}\lambda_{kj}^{*}y_{rj} \geq \frac{w^{*}}{s}\varphi_r^{*} \qquad \forall r \qquad (c)$$

$$\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{s}u_r^{*}\lambda_{kj}^{*}y_{rj} \geq \frac{w^{*}}{s}\sum_{r=1}^{s}\varphi_r^{*}$$

From Equations (a), (b), and (c), we can conclude that:

$$\frac{w^{*}}{s}\sum_{r=1}^{s}\varphi_r^{*} - \frac{1}{m}\sum_{i=1}^{m}\theta_i^{*} + \sum_{k=1}^{n}d_k^{*} \leq \sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{s}u_r^{*}\lambda_{kj}^{*}y_{rj}$$

$$-\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{m}\lambda_{kj}^{*}v_i^{*}x_{ij} + \sum_{k=1}^{n}d_k^{*} \leq 0$$

$$\frac{w^{*}}{s}\sum_{r=1}^{s}\varphi_r^{*} \leq \frac{1}{m}\sum_{i=1}^{m}\theta_i^{*} - \sum_{k=1}^{n}d_k^{*}$$

$$w^{*} = \frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i^{*} - \sum_{k=1}^{n}d_k^{*}}{\frac{1}{s}\sum_{r=1}^{s}\varphi_r^{*}}$$

Since $w^{*} = \frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i^{*}}{\frac{1}{s}\sum_{r=1}^{s}\varphi_r^{*}}$ we have $\sum_{k=1}^{n}d_k^{*} = 0$. On the other hand, we have $d_k^{*} \geq 0$ $(\forall k)$. Therefore, it is concluded that $d_k^{*} = 0$ $(k = 1, 2, ..., n)$.

PROOF OF THEOREM 6

First, we prove that $T_v \subseteq \bar{H} = \{(\mathbf{x}, \mathbf{y}) \,|\, u^*\mathbf{y} - v^*\mathbf{x} \le \mathbf{0}\}$. In this regard, it is assumed that $(\hat{x}, \hat{y}) \in T_v$. Then, we have:

$$\exists \bar{\mu} \ge 0, \quad \sum_{j=1}^{n} \bar{\mu}_j \mathbf{x}_j \le \hat{\mathbf{x}}, \quad \sum_{j=1}^{n} \bar{\mu}_j \mathbf{y}_j \ge \hat{\mathbf{y}}, \quad \sum_{j=1}^{n} \bar{\mu}_j = 1$$

$$v^* \sum_{j=1}^{n} \bar{\mu}_j \mathbf{x}_j \le v^* \hat{\mathbf{x}}, \quad u^* \sum_{j=1}^{n} \bar{\mu}_j \mathbf{y}_j \ge u^* \hat{\mathbf{y}}$$

$$u^* \hat{\mathbf{y}} - v^* \hat{\mathbf{x}} \le u^* \sum_{j=1}^{n} \bar{\mu}_j \mathbf{y}_j - v^* \sum_{j=1}^{n} \bar{\mu}_j \mathbf{x}_j = \sum_{j=1}^{n} \bar{\mu}_j \left(u^* \mathbf{y}_j - v^* \mathbf{x}_j\right) \le \mathbf{0}$$

Thus, $(\hat{x}, \hat{y}) \in \bar{H}$. Now, we show that $H \cap T_v \ne \phi$. Suppose that $\left(\mu^*, \hat{\theta}^*, \hat{\varphi}^*, t^*\right)$ is an optimal solution to model (8), Since $t^* > 0$, therefore, $\forall k \exists p \;\; \mu^*_{kp} > 0$. The complementary slackness condition yields $u^*\mathbf{y}_p - v^*\mathbf{x}_p = \mathbf{0}$, and consequently, $H \cap T_v \ne \phi$. Now, it is shown that $u^*{>}0$ and $v^* > 0$. Since $t^* > 0$, therefore, an optimal solution to model (7) is obtained as follows:

$$\theta^*_i = \frac{\hat{\theta}^*_i}{t^*} \qquad \forall i \quad , \varphi^*_r = \frac{\hat{\varphi}^*_r}{t^*} \qquad \forall r \quad , \lambda^*_{kj} = \frac{\mu^*_{kj}}{t^*} \qquad \forall k, j$$

In Theorem (4), it was proved that $\theta^*_i > 0 \; (\forall i)$ and $\varphi^*_r > 0 \; (\forall r)$. Thus, $\hat{\theta}^*_i > 0 \; (\forall i)$ and $\hat{\varphi}^*_r > 0 \; (\forall r)$. Therefore, the complementary slackness condition gives the following expression:

$$v^*_i \sum_{j=1}^{n} x_{ij} = \frac{1}{m} \quad \Rightarrow v^*_i = \frac{1}{m \sum\limits_{j=1}^{n} x_{ij}} > 0 \qquad\qquad \forall i$$

Also, $w^* = \frac{1}{s} \sum\limits_{r=1}^{s} \varphi^*_r > 0$. Thus,

$$u^*_r \sum_{j=1}^{n} y_{rj} = \frac{w^*}{s} \quad \Rightarrow \quad u^*_r = \frac{w^*}{s \sum\limits_{j=1}^{n} y_{rj}} > 0 \qquad \forall r$$

PROOF OF THEOREM 7

Suppose that $\left(\mu^*, \hat{\theta}^*, \hat{\varphi}^*, t^*\right)$ is an optimal solution to model (8). Let $E = \left\{ j \,\middle|\, \mu^*_{kj} > 0 \right\}$. An optimal solution to model (7) is obtained as follows:

$$\theta^*_i = \frac{\hat{\theta}^*_i}{t^*} \qquad \forall i, \quad \varphi^*_r = \frac{\hat{\varphi}^*_r}{t^*} \qquad \forall r, \quad \lambda^*_{kj} = \frac{\mu^*_{kj}}{t^*} \qquad \forall k, j$$

Also,(u^*, v^*, w^*) is an optimal solution to model (9). Thus, the complementary slackness condition gives the following expressions:

$$\begin{array}{lll} u^*\mathbf{y}_j - v^*\mathbf{x}_j = 0 & j \in E & \\ u^*\lambda^*_{kj}\mathbf{y}_j - v^*\lambda^*_{kj}\mathbf{x}_j = \mathbf{0} & j \in E & k = 1, 2, ..., n \end{array}$$

Considering the equation $\mu^*_{kj} = t\lambda^*_{kj}$ $(\forall k, j)$, we have:

$$\begin{array}{lll} u^*\lambda^*_{kj}\mathbf{y}_j - v^*\lambda^*_{kj}\mathbf{x}_j = \mathbf{0} & j \in E, & k = 1, 2, ..., n \\ u^*\sum_{j\in E}\lambda^*_{kj}\mathbf{y}_j - v^*\sum_{j\in E}\lambda^*_{kj}\mathbf{x}_j = \mathbf{0} & & k = 1, 2, ..., n \end{array}$$

The coordinates of new DMUs $(\bar{\mathbf{x}}_k, \bar{\mathbf{y}}_k)$ are as follows:

$$\overline{\mathbf{x}}_k = \sum_{j\in E}\lambda^*_{kj}\mathbf{x}_j \quad , \quad \overline{\mathbf{y}}_k = \sum_{j\in E}\lambda^*_{kj}\mathbf{y}_j \quad k = 1, 2, \ldots, n$$

Therefore, it is implied that $u^*\bar{\mathbf{y}}_k - v^*\bar{\mathbf{x}}_k = \mathbf{0}$ $(k = 1, 2, ..., n)$. This issue shows that $(\bar{\mathbf{x}}_k, \bar{\mathbf{y}}_k) \in H$ $(k = 1, 2, ..., n)$.

PROOF OF COROLLARY

Suppose that (u^*, v^*, w^*, d^*) is an optimal solution to model (8). Also, it was proved in Theorem (5) that $\mathbf{d}^*= \mathbf{0}$, thus the hyperplane H was obtained as $H = \{(\mathbf{x}, \mathbf{y})\,|u^*\mathbf{y} - v^*\mathbf{x} = \mathbf{0}\}$. Besides, it was shown in Theorem (6) that H was a strong supporting hyperplane. Also, Theorem (7) proves that all new units lie on the strong supporting hyperplane H. Since $\mathbf{d}^*= \mathbf{0}$, therefore, $H \cap T_v$ is an MPSS facet. Thus, all new units obtained from model (7) are MPSS.