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SUFFICIENT CONDITIONS FOR GRAPHS WITH $\{P_2, P_5\}$-FACTORS*

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Abstract. For a graph G, a spanning subgraph F of G is called an $\{P_2, P_5\}$-factor if every component of F is isomorphic to P_2 or P_5, where P_k denotes the path of order k. It was proved by Egawa and Furuya that if G satisfies $3c_1(G-S)+2c_3(G-S) \leq 4|S|+1$ for all $S \subseteq V(G)$, then G has a $\{P_2, P_5\}$-factor, where $c_k(G-S)$ denotes the number of components of $G-S$ with order k. By this result, we give some other sufficient conditions for a graph to have a $\{P_2, P_5\}$-factor by various graphic parameters such as toughness, binding number, degree sums, etc. Moreover, we obtain some regular graphs and some $K_{1,r}$-free graphs having $\{P_2, P_5\}$-factors.

Keywords: $\{P_2, P_5\}$-factor, degree sum, binding number, toughness, regular graph, $K_{1,r}$-free graph.

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1. Introduction

In this paper, we consider only finite and undirected graph without loops or multiple edges. Other basic graph-theoretic terminologies not defined here can be found in [4]. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A spanning subgraph of G is a subgraph H of G such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, $G - X$ denotes the graph obtained from G by deleting all the vertices of X and $G[X]$ denotes the subgraph of G induced by X. For $v \in V(G)$, we use $d_G(v)$ and $N_G(v)$ to denote the degree of v and the set of vertices adjacent to v in G, respectively. For $S \subseteq V(G)$, we write $N_G(S) = \cup_{v\in S} N_G(v)$. A graph G is said to be r-regular if every vertex of G has degree r. We denote the minimum degree and the number of connected components of a graph G by $\delta(G)$ and $\omega(G)$, respectively. Define $\sigma_2(G) = \min\{d_G(u) + d_G(v) : \{u, v\} \subseteq V(G)$ is an independent set of $G\}$.

For a connected graph G, its *toughness*, denoted by $\tau(G)$, was first introduced by Chvátal [5] as follows. If G is complete, then $\tau(G) = +\infty$; otherwise,

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G), \omega(G-S) \geq 2\right\}.$$

The *binding number* is introduced by Woodall [21] and defined as

$$bind(G) = \min\left\{\frac{|N_G(S)|}{|S|} : \emptyset \neq S \subseteq V(G), N_G(S) \neq V(G)\right\}.$$

The complete bipartite graph $K_{1,r}$ is called the *star* of order $r + 1$. We call a graph G is $K_{1,r}$-free if G does not contain an induced subgraph isomorphic to $K_{1,r}$. In particular, a graph is said to be claw-free if it is $K_{1,3}$-free.

For a family of connected graphs $\mathcal{F}$, a spanning subgraph H of a graph G is called an $\mathcal{F}$-factor of G if each component of H is isomorphic to some graph in $\mathcal{F}$. Let P_k denote the path of order k. A spanning subgraph of a graph G is called a $\{P_2, P_5\}$-factor of G if its each component is isomorphic to P_2 or P_5. Similarly, $\{P_2, P_3\}$-factor means a graph factor in which every component is a path of order exactly two or three.

Since Tutte proposed the well-known Tutte 1-factor theorem [20], path-factors of graphs [3, 7, 8, 13, 17] and path-factor covered graphs [6, 9, 23] have attracted a great deal of attention. More results on graph factors are referred to the survey papers and books [2, 19, 22].

Akiyama, Avis and Era [1] demonstrated the following classical result, which is a criterion for graphs with $\{P_2, P_3\}$-factors. We denote by $i(G)$ the number of isolated vertices of a graph G.

Theorem 1.1. *(Akiyama, Avis and Era [1]) A graph G has a $\{P_2, P_3\}$-factor if and only if $i(G - S) \leq 2|S|$ for all $S \subseteq V(G)$.*

For an integer $k \geq 2$, a $\{P_i : i \geq k\}$-factor is briefly denoted by $\mathcal{P}_{\geq k}$-factor. Note that a graph has $\mathcal{P}_{\geq 2}$-factors if and only if it has $\{P_2, P_3\}$-factors. Kaneko [14] gave a necessary and sufficient condition for the existence of $\mathcal{P}_{\geq 3}$-factors. For $k \geq 4$, it is not known that whether the existence problem of $\mathcal{P}_{\geq k}$-factors is polynomially solvable or not, though some results about such factors on special classes of graphs have been obtained (see, for example, Kano et al. [16], Ando et al. [3], and Kawarabayashi et al. [17]).

A graph H is *hypomatchable* if $H-x$ has a perfect matching for every $x \in V(H)$. A graph is a *propeller* if it is obtained from a hypomatchable graph H by adding new vertices u, v and edge uv, and joining u to some vertices of H. Loebal and Poljak [18] proved the following theorem

Theorem 1.2. *(Loebal and Poljak [18]) Let H be a connected graph. If H has a perfect matching, H is hypomatchable, or H is a propeller, then the existence problem of a $\{P_2, H\}$-factor is polynomially solvable. The problem is **NP**-complete for all other graphs H.*

In particular, the existence problem of a $\{P_2, P_{2k+1}\}$-factor is **NP**-complete for $k \geq 2$. As $\{P_2, P_{2k+1}\}$-factor is a useful tool for finding large matchings, Egawa, Furuya and Ozeki [12] investigated the existence of $\{P_2, P_{2k+1}\}$-factors and obtained the following theorem.

For $S \subseteq V(G)$, let $\mathcal{C}_i(G-S)$ be the set of components of order i in $G-S$, where integer $i \geq 1$. Write $c_i(G-S) = |\mathcal{C}_i(G-S)|$. For $0 \leq i \leq k-1$, we use $c^o_{<2k}(G-S)$ to denote the number of odd components of $G-S$ with order less than $2k$, that is, $c^o_{<2k}(G-S) = \sum_{1 \leq i \leq k} c_{2i-1}(G-S)$.

Theorem 1.3. *(Egawa, Furuya and Ozeki [12]) Let $k \geq 3$ be an integer, and let G be a graph. If $c^o_{<2k}(G-S) \leq \frac{5}{6k^2}|S|$ for all $S \subseteq V(G)$, then G has a $\{P_2, P_{2k+1}\}$-factor.*

Recently, Egawa and Furuya [10,11] obtained stronger sufficient conditions for $\{P_2, P_{2k+1}\}$-factors with $k = 2, 3, 4$. In particular, they proved the following theorem.

Theorem 1.4. *(Egawa and Furuya [10]) A graph G has a $\{P_2, P_5\}$-factor if $3c_1(G-S) + 2c_3(G-S) \leq 4|S| + 1$ for all $S \subseteq V(G)$.*

Although a sufficient condition for the existence of $\{P_2, P_5\}$-factors was proposed by Egawa and Furuya, to check the condition in Theorem 1.4 is a nontrivial task. This paper is attempted to find more sufficient conditions for the existence of $\{P_2, P_5\}$-factors using various graphic parameters, or to determine special classes of graphs to have $\{P_2, P_5\}$-factors such as r-regular graphs, planar graphs and $K_{1,r}$-free graphs. The graphic parameters been studied in this paper include minimum degree, toughness, binding number, etc.

Theorem 1.5. *Let G be a connected graph of order $n \geq 4$. Then G has a $\{P_2, P_5\}$-factor if one of the following statements holds:* (i) $\tau(G) \geq \frac{3}{4}$; (ii) $bind(G) \geq \frac{5}{2}$; (iii) $n \geq 9$ *and* $\sigma_2(G) \geq \frac{6n}{7}$; (iv) $i(G-S) \leq \frac{2}{5}|S|$ *for all* $S \subseteq V(G)$.

Theorem 1.6. *A connected graph G has a $\{P_2, P_5\}$-factor if G is one of the following two special classes of graphs:* (i) *r-regular graphs with $r \geq 3$;* (ii) *$K_{1,r}$-free graphs with $\delta(G) \geq \frac{3r+5}{4}$.*

2. Proof of Theorem 1.5

Suppose, to the contrary, that G is a connected graph of order $n \geq 4$ and contains no $\{P_2, P_5\}$-factor. By Theorem 1.4, there exists $S \subseteq V(G)$ such that $3c_1(G-S) + 2c_3(G-S) > 4|S| + 1$. Due to the integrality, we obtain

$$c_1(G-S) + c_3(G-S) \geq c_1(G-S) + \frac{2}{3}c_3(G-S) \geq \frac{4}{3}|S| + \frac{2}{3}. \tag{1}$$

Claim 2.1. *$S \neq \emptyset$.*

Proof. Suppose that $S = \emptyset$, then by (1), we have $c_1(G) + c_3(G) = c_1(G-S) + c_3(G-S) \geq \frac{2}{3}$. According to the integrality, $c_1(G) + c_3(G) \geq 1$. Note that $c_1(G)+c_3(G) \leq \omega(G) = 1$ since G is connected. Then, $G \in \{K_1, K_3, P_3\}$ and thus $|G| \leq 3$, a contradiction. □

(i) If G is complete, then G has a Hamilton path P and $|P| \geq 4$. Obviously, P has a $\{P_2, P_5\}$-factor which is also a $\{P_2, P_5\}$-factor of G, a contradiction. In the following, we assume that G is not complete.

By Claim 2.1 and (1), we have that

$$\begin{aligned} |S| &\leq \frac{3 \times (c_1(G-S) + c_3(G-S))}{4} - \frac{1}{2} \\ &< \frac{3}{4} \times (c_1(G-S) + c_3(G-S)). \end{aligned}$$

By the definition of $\tau(G)$, it follows that

$$\tau(G) \leq \frac{|S|}{\omega(G-S)} < \frac{\frac{3}{4} \times (c_1(G-S) + c_3(G-S))}{c_1(G-S) + c_3(G-S)} = \frac{3}{4}.$$

This contradiction completes the proof of Statement (i) of Theorem 1.5.

(ii) We choose one vertex from each component of $G-S$ with order 3, and denote by S' the set of such vertices. Let S'' be the set of isolated vertices of $G-S$. By (1), we have that

$$\begin{aligned} |S| &\leq \frac{3}{4} \times \left(c_1(G-S) + \frac{2}{3}c_3(G-S) - \frac{2}{3}\right) \\ &= \frac{3}{4}c_1(G-S) + \frac{1}{2}c_3(G-S) - \frac{1}{2}. \end{aligned}$$

Then,

$$\begin{aligned}|N_G(S'\cup S'')| &\leq |S| + 2\times c_3(G-S)\\ &\leq \frac{3}{4}c_1(G-S) + \frac{1}{2}c_3(G-S) - \frac{1}{2} + 2\times c_3(G-S)\\ &= \frac{3}{4}c_1(G-S) + \frac{5}{2}c_3(G-S) - \frac{1}{2}\\ &< \frac{3}{4}c_1(G-S) + \frac{5}{2}c_3(G-S)\end{aligned}$$

It follows that

$$\frac{5}{2} \leq bind(G) \leq \frac{|N_G(S'\cup S'')|}{|S'\cup S''|} < \frac{\frac{3}{4}c_1(G-S) + \frac{5}{2}c_3(G-S)}{c_1(G-S)+c_3(G-S)} \leq \frac{5}{2}.$$

This contradiction completes the proof of Statement (ii) of Theorem 1.5.

(iii) By Claim 2.1 and (1), we have that

$$c_1(G-S) + c_3(G-S) \geq \frac{4}{3}|S| + \frac{2}{3} \geq 2. \tag{2}$$

Case 1. $c_1(G-S) \geq 2$.

Let $\{x, y\}$ be two distinct isolated vertices of $G-S$. Since $\sigma_2(G) \geq \frac{6n}{7}$ and $N_G(x) \cup N_G(y) \subseteq S$, we have that

$$|S| \geq \frac{1}{2}\sigma_2(G) \geq \frac{3n}{7}.$$

It follows from (2) that

$$c_1(G-S) + c_3(G-S) \geq \frac{4}{3}\times\frac{3n}{7} + \frac{2}{3} = \frac{4n}{7} + \frac{2}{3}$$

and thus

$$n \geq |S| + c_1(G-S) + 3\times c_3(G-S) \geq \frac{3n}{7} + \frac{4n}{7} + \frac{2}{3} > n,$$

a contradiction.

Case 2. $c_1(G-S) \leq 1$.

In this case, by (2), we have $c_3(G-S) \geq 1$. Let $C_1, C_2, ..., C_t$ be the components of $G-S$ such that $|C_1| = 1$ or 3 and $|C_i| = 3$ for $2 \leq i \leq t$. We take a vertex $c_i \in V(C_i)$ for every $1 \leq i \leq t$. Obviously, $c_1c_2 \notin E(G)$. Then $d_G(c_1) + d_G(c_2) \geq \sigma_2(G) \geq \frac{6n}{7}$. Here we assume $d_G(c_2) \geq \frac{d_G(c_1)+d_G(c_2)}{2} \geq \frac{3n}{7}$. Note that in the case were $d_G(c_2) \geq \frac{3n}{7}$, the following argument can be applied. Then $d_{C_2}(c_2) \leq 2$ and so

$$|S| \geq d_G(c_2) - d_{C_2}(c_2) \geq \frac{3n}{7} - 2.$$

Since $n \geq 9$ and (2),

$$\begin{aligned} n &\geq |S| + c_1(G-S) + 3 \times c_3(G-S) \\ &= |S| + 3 \times (c_1(G-S) + c_3(G-S)) - 2 \times c_1(G-S) \\ &\geq |S| + 3 \times \left(\frac{4}{3}|S| + \frac{2}{3}\right) - 2 \\ &= 5|S| \\ &\geq \frac{15n}{7} - 10 > n. \end{aligned}$$

This contradiction completes the proof of Statement (iii) of Theorem 1.5.

(iv) We choose two vertex from each nontrivial component of $G-S$ with order 3, and denote the set of such vertices by X. Let $S' = S \cup X$, then

$$i(G-S') = c_1(G-S) + c_3(G-S).$$

It follows from (1) that $3c_1(G-S) + 2c_3(G-S) \geq 4|S| + 2 > 4|S|$. Thus we have

$$|S| < \frac{3}{4}c_1(G-S) + \frac{1}{2}c_3(G-S).$$

Furthermore, it follows that

$$\begin{aligned} |S'| &= |S| + |X| \\ &< \frac{3}{4}c_1(G-S) + \frac{1}{2}c_3(G-S) + 2c_3(G-S) \\ &= \frac{3}{4}c_1(G-S) + \frac{5}{2}c_3(G-S) \\ &\leq \frac{5}{2}\left(c_1(G-S) + c_3(G-S)\right) \\ &= \frac{5}{2}i(G-S'). \end{aligned}$$

This contradicts the condition that $i(G-S') \leq \frac{2}{5}|S'|$ for all $S' \subseteq V(G)$. This completes the proof of Statement (iv) of Theorem 1.5.

3. Proof of Theorem 1.6

Suppose, to the contrary, that G is a connected graph and contains no $\{P_2, P_5\}$-factor. By Theorem 1.4, there exists $S \subseteq V(G)$ such that $3c_1(G-S)+2c_3(G-S) >$

$4|S|+1$. Due to the integrality, we obtain

$$c_1(G-S)+c_3(G-S) \geq c_1(G-S)+\frac{2}{3}c_3(G-S) \geq \frac{4}{3}|S|+\frac{2}{3}. \tag{3}$$

It follows immediately that

$$|S| \leq \frac{3}{4}(c_1(G-S)+c_3(G-S))-\frac{1}{2}. \tag{4}$$

(i) We first argue that $|S| \geq 1$.

Claim 3.1. *$S \neq \emptyset$.*

Proof. Suppose that $S=\emptyset$, then by (3), we have $c_1(G)+c_3(G)=c_1(G-S)+c_3(G-S) \geq \frac{2}{3}$. According to the integrality, $c_1(G)+c_3(G) \geq 1$. Note that $c_1(G)+c_3(G) \leq \omega(G)=1$ since G is connected. Then, $G \in \{K_1, K_3, P_3\}$, which contradicts that G is r-regular where $r \geq 3$. □

Let $\mathcal{C}$ be the set of component of $G-S$ with order 1 or 3, and let $X := \cup_{C\in\mathcal{C}} V(C)$. Let $H := [X,S]$ be a bipartite graph such that $V(H)=X\cup S$ and $xs \in E(H)$ if and only if $xs \in E(G)$ for any $x \in X$ and $s \in S$.

Let $a=c_1(G-S)$ and $b=c_3(G-S)$. Since X is an independent set of H and $N_H(X) \subseteq S$, we have that $|E(H)| \leq r|S|$. Then, by (4),

$$r\times a+3(r-2)\times b \leq |E(H)| \leq r|S| \leq r\times\left(\frac{3}{4}(a+b)-\frac{1}{2}\right).$$

That is

$$\frac{1}{4}ra+\frac{9}{4}rb+\frac{1}{2}r \leq 6b. \tag{5}$$

It follows from (5) and $r \geq 3$ that

$$6b \geq \frac{1}{4}ra+\frac{9}{4}rb+\frac{1}{2}r \geq \frac{3}{4}a+\frac{27}{4}b+\frac{3}{2} > 6b+1,$$

a contradiction. This completes the proof of Statement (i) of Theorem 1.6.

(ii) We distinguish two cases below to show that G has a $\{P_2, P_5\}$-factor, which is a contradiction.

Case 1. $S=\emptyset$.

In this case, by (3), we have $c_1(G)+c_3(G)=c_1(G-S)+c_3(G-S) \geq \frac{2}{3}$. According to the integrality, $c_1(G)+c_3(G) \geq 1$. On the other hand, $c_1(G)+c_3(G) \leq \omega(G)=1$ since G is connected. So, we obtain that $G \in \{K_1, K_3, P_3\}$, which contradicts the minimum degree of G.

Case 2. $|S| \geq 1$.

Let $S=\{x_1,x_2,...,x_k\}$, where $|S|=k\ge 1$. Let $\{C_1,C_2,...,C_t\}$ be the set of components of $G-S$ with order 1 or 3. Then, by (3), we have that

$$t=c_1(G-S)+c_3(G-S)\ge \frac{4k+2}{3}. \tag{6}$$

For any $i\in[1,t]$, there exists $y_i\in V(C_i)$ such that $d_{C_i}(y_i)\le 2$. Let $Y=\{y_1,y_2,\ldots,y_t\}$. Then we construct a bipartite subgraph $H\subseteq G$ such that $V(H)=S\cup Y$ and $x_iy_j\in E(H)$ if and only if $x_iy_j\in E(G)$ for any $i\in[1,k]$, $j\in[1,t]$. Since for any $j\in[1,t]$, $d_{C_j}(y_j)\le 2$, we have

$$d_H(y_j)=d_G(y_j)-d_{C_j}(y_j)\ge \frac{3r+5}{4}-2=\frac{3r-3}{4}. \tag{7}$$

It follows from (7) that

$$|X|\ge d_H(y_j)\ge \frac{3r-3}{4}.$$

Then, by (6) and (7), we have that

$$\begin{aligned}|E(H)| &= \sum_{j=1}^{t} d_H(y_j)\\ &\ge t\times\frac{3r-3}{4}\\ &\ge \frac{4k+2}{3}\times\frac{3r-3}{4}\\ &= k(r-1)+\frac{r-1}{2}.\end{aligned}$$

Since $k(r-1)+\frac{r-1}{2}\le |E(H)|=\sum_{i=1}^{k} d_H(x_i)$, there exists $x_b\in S$ such that $d_H(x_b)\ge r$. Then $H[\{x_b\}\cup N_H(x_b)]=G[\{x_b\}\cup N_H(x_b)]$ includes $K_{1,r}$. This is a contradiction and completes the proof of Statement (ii) of Theorem 1.6.

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References

[1] J. Akiyama, D. Avis, H. Era, On a {1,2}-factor of a graph, TRU Math. **16** (1980) 97-102.
[2] J. Akiyama, M. Kano, Factors and factorizations of graphs−a survey, J. Graph Theory. **9** (1985) 1-42.
[3] K. Ando, Y. Egawa, A. Kaneko, K.I. Kawarabayashi, H. Matsuda, Path factors in claw-free graphs, Discrete Math. **243** (2002) 195-200.

[4] J.A. Bondy, U.S.R. Murty, Graph theory with applications. North-Holland, NewYork-Amsterdam-Oxford, (1982).
[5] V. Chvátal, Tough graphs and Hamiltonian Circuits, Discrete Math. **5** (1973) 215-228.
[6] G. Dai, The existence of path-factor covered graphs, Discussiones Mathematicae Graph Theory, https://doi.org/10.7151/dmgt.2353.
[7] G. Dai, Remarks on component factors in graphs, RAIRO-Operations Research, **56**, pp. 721-730, 2022.
[8] G. Dai, Z. Hu, P_3-factors in the square of a tree, Graphs and Combinatorics, **36** (2020) 1913-1925.
[9] G. Dai, Z. Zhang, Y. Hang, X. Zhang, Some degree conditions for $P_{\geq 3}$-factor covered graphs, RAIRO-Operations Research, **55**, pp. 2907-2913, 2021.
[10] Y. Egawa, M. Furuya, The existence of a path-factor without small odd paths, Electron. J. Combin. **25** (2018) 1-40.
[11] Y. Egawa, M. Furuya, Path-factors involving paths of order seven and nine, Theory and Applications of Graphs, **3(1)** (2016), https://doi.org/10.20429/tag.2016.030105.
[12] Y. Egawa, M. Furuya, K. Ozeki, Sufficient conditions for the existence of a path-factor which are related to odd components, J. Graph Theory, **89** (2018) 327-340.
[13] A. Kaneko, A. Kelmans, T. Nishimura, On packing 3-vertex paths in a graph. J. Graph Theory. **36** (2001) 175-197.
[14] A. Kaneko, A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two. J. Combin. Theory Ser. B. **88** (2003) 195-218.
[15] M. Kano, G.Y. Katona, Z. Király, Packing paths of length at least two, Discrete Math. **283** (2004) 129-135.
[16] M. Kano, C. Lee, K. Suzuki, Path and cycle factors of cubic bipartite graphs, Discuss. Math. Graph Theory. **28** (2008) 551-556.
[17] K. Kawarabayashi, H. Matsuda, Y. Oda, K. Ota, Path factors in cubic graphs. J. Graph Theory. **39** (2002) 188-193.
[18] M. Loebl, S. Poljak, Efficient subgraph packing, J. Combin. Theory Ser. B **59** (1993) 106-121.
[19] M.D. Plummer, Perspectives: Graph factors and factorization: 1985-2003: A survey, Discrete Math. **307** (2007) 791-821.
[20] W.T. Tutte, The factors of graphs, Canad. J. Math. **4** (1952) 314-328.
[21] D.R. Woodall, The binding number of a graph and its Anderson number, J. Combin. Theory Ser. B, **15** (1973) 225-255.
[22] Q.R. Yu, G.Z. Liu, Graph Factors and Matching Extensions. Higher Education Press, Beijing (2009).
[23] P. Zhang, S. Zhou, Characterizations for $\mathcal{P}_{\geq 2}$-factor and $\mathcal{P}_{\geq 3}$-factor covered graphs, Discrete Math. **309** (2009) 2067-2076.