REMARKS ON COMPONENT FACTORS IN GRAPHS

GUOWEI DAI

Abstract. For a family of connected graphs $\mathcal{F}$, a spanning subgraph $H$ of a graph $G$ is called an $\mathcal{F}$-factor of $G$ if its each component is isomorphic to an element of $\mathcal{F}$. In particular, $H$ is called an $S_k$-factor of $G$ if $\mathcal{F} = \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$, where integer $k \geq 2$; $H$ is called a $P_{\geq 3}$-factor of $G$ if every component in $\mathcal{F}$ is a path of order at least three. As an extension of $S_k$-factors, the induced star-factor (i.e., $IS_k$-factor) is a spanning subgraph each component of which is an induced subgraph isomorphic to some graph in $\mathcal{F} = \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$. In this paper, we firstly prove that a graph $G$ has an $S_k$-factor if and only if its isolated toughness $I(G) \geq \frac{1}{k}$. Secondly, we prove that a planar graphs $G$ has an $S_2$-factors if its minimum degree $\delta(G) \geq 3$. Thirdly, we give two sufficient conditions for graphs with $IS_k$-factors by toughness and minimum degree, respectively. Additionally, we obtain three special classes of graphs admitting $P_{\geq 3}$-factors.

Keywords: Star-factor, Induced star-factor, $P_{\geq 3}$-factor, Toughness, Minimum degree

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1. INTRODUCTION

The graphs considered here are finite and simple, unless explicitly stated. Let $G = (V(G), E(G))$ be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For $v \in V(G)$, we use $d_G(v)$ and $N_G(v)$ to denote the degree of $v$ and the set of vertices adjacent to $v$ in $G$, respectively. For $S \subseteq V(G)$, we write $N_G(S) = \bigcup_{v \in S} N_G(v)$. A graph $G$ is called an $r$-regular graph if $d_G(v) = r$ for each $v \in V(G)$. We use $\delta(G)$ to denote the minimum degree of a

1 School of Mathematical Sciences, Nanjing Normal University, Nanjing, Jiangsu 210023, P.R. China. (E-mail: guowei_dai@aliyun.com)
The number of connected components and isolated vertices of a graph \( G \) is denoted by \( \omega(G) \) and \( i(G) \), respectively. We refer to [3] for the notation and terminologies not defined here.

The complete bipartite graph \( K_{1,r} \) is called the star of order \( r+1 \), where \( r \) is a positive integer. We use \( S_k \) to denote the set \( \{K_{1,1}, K_{1,2}, K_{1,3}, \ldots, K_{1,k}\} \), where integer \( k \geq 2 \).

Let \( \mathcal{F} \) be a family of connected graphs. Then a spanning subgraph \( H \) of \( G \) is called an \( \mathcal{F} \)-factor of \( G \) if each component of \( H \) is isomorphic to an element of \( \mathcal{F} \). In particular, for an integer \( k \geq 2 \), a \( \{K_{1,1}, K_{1,2}, K_{1,3}, \ldots, K_{1,k}\} \)-factor is briefly called an \( S_k \)-factor. Similarly, a \( \{P_k, P_{k+1}, \ldots\} \)-factor is called a \( P_{\geq k} \)-factor.

In 1947, Tutte [10] presented a criterion for the existence of 1-factors (perfect matchings), which is one of the classical results in graph theory. Denote by \( o(G) \) the number of odd components of \( G \), whose orders are odd.

**Theorem 1.** (Tutte [10]) A graph \( G \) has a 1-factor if and only if \( o(G - S) \leq |S| \) for any \( S \subseteq V(G) \).

Since the well-known Tutte 1-factor theorem [10] was proposed, there are many results about component-factors, see [5,9,13,14], etc.

Akiyama, Avis and Era [1] demonstrated the following classical result, which is a characterization for the existence of \( P_{\geq 2} \)-factors in a graph.

**Theorem 2.** (Akiyama, Avis and Era [1]) A graph \( G \) has a \( P_{\geq 2} \)-factor if and only if \( i(G - S) \leq 2|S| \) for any \( S \subseteq V(G) \).

Amahashi & Kano [2] and Las Vergnas [11] gave independently a characterization for graphs with \( S_k \)-factors, which is a generalization of Theorem 2.

**Theorem 3.** (Amahashi and Kano [2]; Las Vergnas [11]) Let \( k \) be an integer with \( k \geq 2 \). Then a graph \( G \) has an \( S_k \)-factor if and only if \( i(G - S) \leq k|S| \) for any \( S \subseteq V(G) \).

A connected graph is called a cactus if each block of the graph is a complete subgraph. A cactus of odd order is called an odd-cactus. As an extension of \( S_k \)-factors, the induced star factor, denoted by \( IS_k \)-factor, is a spanning subgraph each component of which is an induced subgraph isomorphic to some graph in \( \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\} \). Denote by \( oc(G - S) \) the number of odd-cactus of \( G - S \). The criterion for \( IS_k \)-factors was obtained by Egawa, Kano and Kelmans as following.

**Theorem 4.** (Egawa, Kano and Kelmans [6]) Let \( k \geq 2 \) be an integer. A graph \( G \) has an \( IS_k \)-factor if and only if \( oc(G - S) \leq k|S| \) for any \( S \subseteq V(G) \).

The toughness of a connected graph \( G \), denoted by \( \tau(G) \), was first introduced by Chvátal [4] as follows. If \( G \) is complete, then \( \tau(G) = +\infty \); otherwise,

\[
\tau(G) = \min \left\{ \frac{|S|}{\omega(G - S)} : S \subseteq V(G), \omega(G - S) \geq 2 \right\}.
\]

Kaneko [7] introduced the concept of a sun and gave a characterization for the existence of \( P_{\geq 3} \)-factors in a graph. It is perhaps the first criterion of graphs
admitting path factors not including $P_2$. Additionally, Kano et al. [8] obtained a simpler proof for Kaneko’s result [7].

A graph $H$ is said to be a factor-critical graph if for each $v \in V(H)$, $H - \{v\}$ has a 1-factor. Let $H$ be a factor-critical graph such that $V(H) = \{v_1, v_2, ..., v_n\}$. A graph is called a sun if it is obtained from $H$ by adding new vertices $\{u_1, u_2, ..., u_n\}$ together with new edges $\{v_i u_i : 1 \leq i \leq n\}$ to $H$. Note that, according to Kaneko [7], $K_1$ and $K_2$ are also regarded as a sun, respectively. Usually, the suns other than $K_1$ are called big suns. We use $\text{sun}(G - X)$ to denote the number of sun components of $G - X$.

**Theorem 5.** (Kaneko [7]) A graph $G$ has a $P_{\geq 3}$-factor if and only if $\text{sun}(G - S) \leq 2|S|$ for any $S \subseteq V(G)$.

**Corollary 6.** (Kaneko [7]) A graph $G$ has a $P_{\geq 3}$-factor if one of the following holds: (i) $G$ is $r$-regular where $r \geq 2$; (ii) $\tau(G) = 1$; (iii) $\tau(G) = \frac{1}{2}$ and $\delta(G) \geq 2$; (iv) $G$ is 3-connected planar; (v) $G$ is claw-free with $\delta(G) \geq 2$.

This paper attempts to find more sufficient conditions for the existence of these component factors by different graphic parameters including minimum degree, toughness, isolated toughness, binding number, etc.

### 2. Star-factor

The isolated toughness of a connected graph $G$ denoted by $I(G)$. If $G$ is complete, then $I(G) = +\infty$; otherwise,

$$I(G) = \min \left\{ \frac{|S|}{i(G - S)} : S \subseteq V(G), i(G - S) \geq 2 \right\}.$$

**Lemma 7.** [15] Let $G$ be a graph and $k \geq 1$ be a real number. Then the following three statements are equivalent.

(i) $i(G - S) \leq k|S|$ for all $S \subseteq V(G)$.
(ii) $|U| \leq k|N_G(U)|$ for all independent set $U$ of $G$.

**Theorem 8.** A connected nontrivial graph $G$ has an $S_k$-factor if and only if $I(G) \geq \frac{1}{k}$, where integer $k \geq 2$.

**Proof.** Sufficiency: If $G$ is complete and nontrivial, then $G$ has an $S_k$-factor obviously. Thus we may assume that $G$ is a graph of order at least two and not complete. Suppose, by way of contradiction, that $G$ has no $S_k$-factor, then by Theorem 3, there is a subset $S \subseteq V(G)$ such that $i(G - S) > k|S|$. Then, by the integrality of $i(G - S)$, we obtain that

$$i(G - S) \geq k|S| + 1. \quad (1)$$

If $|S| = 0$, then $i(G) = i(G - S) \geq k|S| + 1 = 1$, which contradicts the fact that $G$ is connected.
If $|S| = 1$, then $i(G - S) > k|S| = k$. By the definition of $I(G)$, we have that

$$I(G) \leq \frac{|S|}{i(G - S)} < \frac{1}{k},$$

a contradiction.

If $|S| \geq 2$, then by (1), we have

$$|S| \leq \frac{i(G - S) - 1}{k}.$$

By the definition of $I(G)$, we have

$$I(G) \leq \frac{|S|}{i(G - S)} \leq \frac{i(G - S) - 1}{k \times i(G - S)} \leq \frac{1}{k} - \frac{1}{k \times i(G - S)} < \frac{1}{k},$$

a contradiction.

Necessity: Suppose that $G$ has an $S_k$-factor and $I(G) < \frac{1}{k}$. Then by Theorem 3 and Lemma 7, for each independent set $U \subseteq V(G)$, we have

$$|U| \leq k|N_G(U)|.$$  \hspace{1cm} (2)

Since $I(G) < \frac{1}{k}$, there is a subset $S \subseteq V(G)$ such that $\frac{|S|}{i(G - S)} < \frac{1}{k}$. Let $U$ be the set of isolated vertices of $G - S$, then $N_G(U) \subseteq S$. Obviously, $U$ is independent and

$$|N_G(U)| \leq |S| < \frac{i(G - S)}{k} = \frac{|U|}{k},$$

which contradicts (2). \hfill \Box

**Lemma 9.** \cite{3} Let $G$ be a simple connected planar graph of order at least three. If $G$ does not contain triangles, then $|E(G)| \leq 2|V(G)| - 4$.

**Theorem 10.** Let $G$ be a connected planar graph. If $\delta(G) \geq 3$, then $G$ has an $S_2$-factor.

**Proof.** Suppose that $G$ is a connected planar graph with no $S_2$-factor. By Theorem 3, there exists a subset $S \subseteq V(G)$ such that $i(G - S) > 2|S|$. According to the integrality of $i(G - S)$, we obtain that

$$i(G - S) \geq 2|S| + 1.$$  \hspace{1cm} (3)
Claim 2.1. \( S \neq \emptyset \).

Proof. Suppose \( S = \emptyset \), by (3), \( i(G - S) \geq 2|S| + 1 = 1 \). On the other hand, \( i(G) \leq \omega(G) = 1 \) since \( G \) is a connected graph. So, we obtain that \( G \) is an isolated vertex, which contradicts that \( \delta(G) \geq 3 \). \( \square \)

By Claim 2.1, \( S \neq \emptyset \). Set \( |S| = s \). Then by (3), \( i(G - S) \geq 2s + 1 \). The set of isolated vertices in \( G - S \) is denoted by \( I(G - S) \). Then we construct a simple bipartite graph \( H = H[X,Y] \) as follows. Let \( X = S \) and \( Y \subseteq I(G - S) \) such that \( |Y| = 2s + 1 \). For any \( s \in X \) and \( y \in Y \), \( sy \in E(H) \) if and only if \( sy \in E(G) \). Since \( \delta(G) \geq 3 \), it is clear that for each \( y \in Y \), we have \( |N_H(y)| \geq 3 \). Hence, \( |H| = s + (2s + 1) = 3s + 1 \geq 4 \) and

\[
|E(H)| \geq 3 \times (2s + 1) = 6s + 3 > 6s.
\] (4)

As \( G \) is a connected planar graph, it is easy to see that \( H \) is also a connected planar graph. According to the fact that a bipartite graph does not contain any odd cycles, Lemma 9 implies that

\[
|E(H)| \leq 2|H| - 4 = 2 \times (3s + 1) - 4 = 6s - 2 < 6s,
\]

which is a contradiction to (4). \( \square \)

Remark 11. Now, we explain that the condition of minimum degree \( \delta(G) \geq 3 \) in Theorem 10 is the best possible. Let \( G = 2K_1 \lor 5K_1 \) be a complete bipartite graph, where \( \lor \) means “join”. We know that \( G \) is a connected planar graph with \( \delta(G) = 2 < 3 \). Choose \( X =: V(2K_1) \) with \( |X| = 2 \), then we have that

\[
i(G - X) = 5 > 2|X| = 4.
\]

In view of Theorem 3, \( G \) has no \( S_2 \)-factor.

3. Induced star-factor

Theorem 12. Let \( G \) be a connected graph of order at least three. If \( G \) is not an odd cactus and \( \tau(G) \geq \frac{1}{k} \), then \( G \) has an \( IS_k \)-factor.

Proof. Suppose, to the contrary, that \( G \) is a connected graph with no \( IS_k \)-factor. If \( G \) is a complete graph, then \( G \) has a Hamilton cycle, denoted by \( C \). Since \( G \) is not an odd cactus, \( C \) is an even cycle and thus \( G \) has a 1-factor. Hence, \( G \) has an \( IS_k \)-factor, a contradiction. Thus, we may assume that \( G \) is not a complete graph.
By Theorem 4, there is a subset $S \subseteq V(G)$ such that $oc(G - S) > k|S|$. Due to the integrality, we obtain
\[ oc(G - S) \geq k|S| + 1. \] (5)

**Claim 3.1.** $S \neq \emptyset$.

**Proof.** Suppose that $S = \emptyset$, then by (5), we have $oc(G) = oc(G - S) \geq kS + 1 = 1$. Note that $oc(G) \leq \omega(G) = 1$ since $G$ is connected. Thus $G$ is an odd cactus, a contradiction. \hfill \square

By Claim 3.1, we have $|S| \geq 1$.

If $|S| = 1$, then by (5), we have $oc(G - S) \geq k|S| + 1 = k + 1$. Then due to the definition of $\tau(G)$, we obtain that
\[
\frac{1}{k} \leq \tau(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{|S|}{oc(G - S)} \leq \frac{1}{k + 1} < \frac{1}{k},
\]
a contradiction.

If $|S| \geq 2$, then by (5), we have
\[
|S| \leq \frac{oc(G - S) - 1}{k}.
\] (6)

Then by (6) and the definition of $\tau(G)$, we obtain that
\[
\frac{1}{k} \leq \tau(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{|S|}{oc(G - S)} \leq \frac{oc(G - S) - 1}{k \times oc(G - S)} = \frac{1}{k} - \frac{1}{k \times oc(G - S)} < \frac{1}{k},
\]
a contradiction. \hfill \square

**Theorem 13.** Let $G$ be a connected graph of order $n \geq 3$ which is not an odd cactus. Then $G$ has an $\mathcal{IS}_k$-factor if $\delta(G) \geq \max\{\frac{n}{k + 1}, \frac{4n}{3k + 1} - 1\}$.

**Proof.** Suppose, to the contrary, that $G$ is a connected graph having no $\mathcal{IS}_k$-factor. By Theorem 4, there exists $S \subseteq V(G)$ such that $oc(G - S) > k|S|$. Due to the integrality, we obtain
\[ oc(G - S) \geq k|S| + 1. \] (7)

**Claim 3.2.** $S \neq \emptyset$. 
Proof. Suppose that $S = \emptyset$, then by (7), we have $oc(G) = oc(G - S) \geq k|S| + 1 = 1$. Note that $oc(G) \leq \omega(G) = 1$ since $G$ is connected. Thus $G$ is an odd cactus, a contradiction. □

By Claim 3.2 and (7), we have that

$$oc(G - S) \geq k|S| + 1 \geq k + 1. \quad (8)$$

Let $C_1, C_2, ..., C_m$ be the odd cactus components of $G - S$, where $m = oc(G - S)$. Choose an odd cactus component $C_i$ of $G - S$ such that $|C_i|$ is as small as possible, where $1 \leq i \leq m$. Without loss of generality, we assume that $C_1$ is such an odd cactus component and $|C_1| = t$.

Case 1. $t = 1$.

In this case, let $C_1 = \{x\}$. Since $N_G(x) \subseteq S$, we have that

$$|S| \geq d_G(x) \geq \delta(G) \geq \frac{n}{k + 1}.$$ 

It follows from (7) that

$$|G| \geq |S| + \sum_{i=1}^{m} |C_i| \geq |S| + (k|S| + 1) = (k + 1)|S| + 1 \geq (k + 1) \times \frac{n}{k + 1} + 1 = n + 1,$$

a contradiction.

Case 2. $t \geq 2$.

Since $C_1$ is an odd cactus and $t \geq 2$, we find that $|C_1| = t \geq 3$. On the other hand, according to the minimality property, we have that

$$t \leq \frac{|G|}{oc(G - S)} \leq \frac{n}{k|S| + 1} \leq \frac{n}{k + 1} < \frac{3n}{3k + 1}. \quad (9)$$

Let $u$ be the vertex with maximum degree in $C_1$, then $d_{C_1}(u) \leq t - 1$. It follows that

$$|S| \geq d_G(u) \geq \delta(G) - d_{C_1}(u) \geq \frac{4n}{3k + 1} - 1 - (t - 1) \geq \frac{4n}{3k + 1} - t.$$
This together with (7), (9) and \( t \geq 3 \) implies that

\[
|G| \geq |S| + \sum_{i=1}^{m} |C_i|
\]
\[
\geq |S| + (k|S| + 1) \times t
\]
\[
> (kt + 1)|S|
\]
\[
\geq (kt + 1) \times \left( \frac{4n}{3k + 1} - t \right)
\]
\[
= (kt + 1) \times \left( \frac{n}{3k + 1} + \left( \frac{3n}{3k + 1} - t \right) \right)
\]
\[
> (kt + 1) \times \frac{n}{kt + 1} = n,
\]
a contradiction. \( \square \)

**Remark 14.** Now, we explain that the condition of toughness \( \tau(G) \geq \frac{1}{k} \) in Theorem 12 and minimum degree \( \delta(G) \geq \max \{ \frac{n}{k+1}, \frac{4n}{3k+1} - 1 \} \) in Theorem 13 are all the best possible. Let \( H_1, H_2, \ldots, H_{k+1} \) be \( k+1 \) odd complete graphs, each of which contains exactly \( \frac{n}{k+1} \) vertices, where integer \( k \geq 2 \) and \( \frac{n}{k+1} \) is an integer. We construct a connected graph \( G = K_1 \vee (\bigcup_{i=1}^{k+1} H_i) \), the order of which is \( n \). It is obviously that \( \tau(G) = \frac{1}{k+1} < \frac{1}{k} \), and \( \delta(G) = \frac{n-1}{k+1} < \frac{n}{k+1} \). Choose \( X = V(K_1) \) with \( |X| = 1 \), then we have that

\[
\text{oc}(G - X) = k + 1 > k|X| = k.
\]

It follows from Theorem 4 that \( G \) has no \( IS_k \)-factor.

### 4. Path-factor

In this section, we obtain some sufficient conditions for the existence of graphs admitting \( P_{\geq 3} \)-factors.

The *binding number* is introduced by Woodall [12] and defined as

\[
\text{bind}(G) = \min \left\{ \frac{|N_G(S)|}{|S|} : \emptyset \neq S \subseteq V(G), N_G(S) \neq V(G) \right\}.
\]

**Theorem 15.** Let \( G \) be a connected graph of order \( n \geq 3 \). Then \( G \) has a \( P_{\geq 3} \)-factor if one of the following statements holds:

(i) \( I(G) \geq \frac{3}{2} \);

(ii) \( \text{bind}(G) \geq \frac{3}{4} \);

(iii) \( n \geq 8 \) and for all three independent vertices \( u,v,w \in V(G) \),

\[
\max\{d_G(u), d_G(v), d_G(w)\} \geq \frac{n}{3}.
\]
Proof. By way of contradiction, suppose that $G$ is a connected graph with no $P_{\geq 3}$-factor. Then by Theorem 5, there is a subset $S \subseteq V(G)$ such that $\text{sun}(G - S) > 2|S|$. Due to the integrality of $\text{sun}(G - S)$, we obtain

$$\text{sun}(G - S) \geq 2|S| + 1. \quad (10)$$

(i) Obviously $G$ has a $P_{\geq 3}$-factor if $G$ is complete, a contradiction. Thus, we may assume that $G$ is not complete. We shall consider two cases by the value of $|S|$ and derive a contradiction in each case.

Case 1. $|S| = 0$.

By (10), we have $\text{sun}(G) = \text{sun}(G - S) \geq 2|S| + 1 = 1$. Note that $\text{sun}(G) \leq \omega(G) = 1$ since $G$ is connected. Then, $\text{sun}(G) = 1$ and thus $G$ is a big sun. Of course, $G$ is not an isolated edge since its order at least three. Let $R$ be the factor-critical subgraph of $G$ and set $U = V(R)$. It is clear that $G - U$ is an independent set and $|G - U| = |U|$. By the definition of $I(G)$ and $I(G) \geq \frac{3}{2}$, we have that

$$\frac{3}{2} \leq I(G) \leq \frac{|U|}{i(G - U)} = 1,$$

a contradiction.

Case 2. $|S| \geq 1$.

By (10), we have that

$$|S| \leq \frac{\text{sun}(G - S) - 1}{2}. \quad (11)$$

Assume that $\text{sun}(G - S) - i(G - S) = m$, i.e., there are $m$ big sun components of $G - S$, denoted by $C = \{C_1, C_2, \ldots, C_m\}$. For each $i \in [1, m]$, let $R_i$ be the factor-critical subgraph of $C_i$ if $C_i$ is not an isolated edge, and choose vertices $c_i \in V(R_i)$. If $C_i$ is an isolated edge, then choose arbitrarily $c_i \in V(R_i)$ where $1 \leq i \leq m$. Let $S' = \{c_i : 1 \leq i \leq m\}$. Then by (11), we have that

$$|S \cup S'| = |S| + \text{sun}(G - S) - i(G - S)$$
$$\leq |S| + \text{sun}(G - S)$$
$$\leq \frac{\text{sun}(G - S) - 1}{2} + \text{sun}(G - S)$$
$$= 3 \times \frac{\text{sun}(G - S) - 1}{2}.$$
By the definition of $I(G)$, it follows that
\[
\frac{3}{2} \leq I(G) \leq \frac{|S \cup S'|}{i(G - S - S')} \\
\leq \frac{3 \times \text{sun}(G - S) - 1}{2 \times i(G - S - S')} \\
= \frac{3 \times \text{sun}(G - S) - 1}{2 \times \text{sun}(G - S)} < \frac{3}{2},
\]
a contradiction.

The statement (i) in Theorem 15 is proved.

(ii) Let $S' = V(G - S)$. By the definition of $\text{bind}(G)$, we have that
\[
|N_G(S')| \geq \frac{5}{4}|S'|.
\] (12)

**Case 1.** $|S| \geq \frac{n}{5}$.

In this case, $|S'| = |G| - |S| \leq \frac{4n}{5}$. By (12),
\[
\frac{5}{4}(n - |S|) = \frac{5}{4}|S'| \leq |N_G(S')| \\
= |N_G(G - S)| \leq n - i(G - S).
\]

It follows immediately that
\[
i(G - S) \leq \frac{5}{4}|S| - \frac{n}{4}.\] (13)

Hence, by (13),
\[
n \geq |S| + i(G - S) + 2 \times (\text{sun}(G - S) - i(G - S)) \\
= |S| + 2 \times \text{sun}(G - S) - i(G - S) \\
> |S| + 4|S| - (\frac{5}{4}|S| - \frac{n}{4}) \\
= \frac{15}{4}|S| + \frac{n}{4} \geq n,
\]
a contradiction.

**Case 2.** $|S| < \frac{n}{5}$.

In this case, $|S'| = |G| - |S| > \frac{4n}{5}$. Let $S_0 \subseteq S'$ such that $|S_0| = \frac{4n}{7}$. By (12), we have that $|N_G(S_0)| \geq \frac{5}{4}|S_0| = n$ and so $V(G) \subseteq N_G(S')$. Consequently, there exists no singleton component of $G - S$, i.e.,
\[
i(G - S) = 0.\] (14)
Consider all the sun components in $G - S$ and let $S'' = V(Sun(G - S))$. Since $sun(G - S) > 2|S|$, by (14), $|S''| > 2 \times sun(G - S) > 4|S|$. Hence,

$$\text{bind}(G) \leq \frac{|N_G(S'')|}{|S''|} \leq \frac{|S''| + |S|}{|S''|} = 1 + \frac{|S|}{|S''|} < 1 + \frac{1}{4} = \frac{5}{4},$$

a contradiction.

The statement (ii) in Theorem 15 is proved.

(iii) We first give the argument as following.

**Claim 4.1.** $S \neq \emptyset$.

*Proof.* Suppose $S = \emptyset$, by (10), $sun(G) = sun(G - S) \geq 1$. On the other hand, $sun(G) \leq \omega(G) = 1$. So, we obtain that $G$ is a big sun containing at least 8 vertices. It follows that there exist three vertices of degree one, denoted by $\{u, v, w\}$, which contradicts that $\max\{d_G(u), d_G(v), d_G(w)\} \geq \frac{n}{3} > 2$. □

By Claim 4.1 and (10), we have $sun(G - S) \geq 2|S| + 1 \geq 3$.

**Case 1.** $i(G - S) \geq 3$.

Let $\{x, y, z\}$ be three distinct isolated vertices of $G - S$. Since $\max\{d_G(x), d_G(y), d_G(z)\} \geq \frac{n}{3}$ and $N_G(x) \cup N_G(y) \cup N_G(z) \subseteq S$, we have that

$$|S| \geq \max\{d_G(x), d_G(y), d_G(z)\} \geq \frac{n}{3}.$$

It follows from (10) that $sun(G - S) \geq 2|S| + 1 \geq \frac{2n}{3} + 1$ and thus

$$n \geq |S| + sun(G - S) \geq \frac{n}{3} + \frac{2n}{3} + 1 = n + 1,$$

a contradiction.

**Case 2.** $i(G - S) \leq 2$.

In this case, by (10), there exist at least three suns of $G - S$, denoted by $C_1, C_2, ..., C_t$ where $t \geq 3$. Then we choose $c_i \in V(C_i)$ such that $d_{C_i}(c_i) \leq 1$, where $i = 1, 2, 3$. Obviously, $\{c_1, c_2, c_3\}$ is an independent set of $G$. Then $\max\{d_G(c_1), d_G(c_2), d_G(c_3)\} \geq \frac{n}{3}$. Without loss of generality, we assume $d_G(c_1) \geq \frac{n}{3}$. Since $d_S(c_1) = d_G(c_1) - d_{C_1}(c_1) \geq \frac{n}{3} - 1$, we have that $|S| \geq d_S(c_1) \geq \frac{n}{3} - 1$. It follows from (10) that

$$sun(G - S) \geq 2|S| + 1 \geq \frac{2n}{3} - 1,$$
and thus
\[
\begin{align*}
n \geq & \ |S| + 2 \times \text{sun}(G - S) - i(G - S) \\
& \geq \frac{n}{3} - 1 + 2 \times \left(\frac{2n}{3} - 1\right) - 2 \\
& = \frac{5n}{3} - 5 > n,
\end{align*}
\]
a contradiction. The statement (iii) in Theorem 15 is proved. \(\square\)

**Remark 16.** Now, we claim that the conditions of isolated toughness \(I(G) \geq \frac{2}{3}\) and binding number \(\text{bind}(G) \geq \frac{5}{4}\) in Theorem 15 are all the best possible. Let \(P_5\) be a path of order 5, the center vertex of which is denoted by \(u\). We construct a connected graph \(G = P_5 \cup \{v\} \cup e\), where \(e = uv\). It is obvious that \(I(G) = 1 < \frac{2}{3}\), and \(\text{bind}(G) = 1 < \frac{5}{4}\). Choose \(X = \{u\}\), then we have that \(\text{sun}(G - X) = 3 < 2 = 2|X|\). It follows from Theorem 5 that \(G\) has no \(P_{\geq 3}\)-factor.

**Remark 17.** Now, we explain that the degree condition in the statement (iii) of Theorem 15 is the best possible. Let \(G = 2K_1 \lor 7K_1\) be a connected complete bipartite graph of order \(n = 9\). We know there exists three independent vertices \(\{u, v, w\} \subseteq V(7K_1)\) such that \(\max\{d_G(u), d_G(v), d_G(w)\} = 2 < 3 = \frac{n}{3}\). Choose \(X = V(2K_1)\) with \(|X| = 2\), then we have that \(\text{sun}(G - X) = 7 > 2|X| = 4\). Using Theorem 5, \(G\) has no \(P_{\geq 3}\)-factor.

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